ML-5
7/22/11

Heterogeneous Agents Models

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July 11, 2011

Introduction

- Often, we want to deal with model with heterogeneous agents.
- Examples:
  1. Heterogeneity in age: OLG models.
  2. Heterogeneity in preferences: risk sharing.
  3. Heterogeneity in abilities: job market.
  4. Heterogeneity in policies: progressive marginal tax rates.
- Why General Equilibrium?
  1. It imposes discipline: relation between β and r is endogenous.
  2. It generates an endogenous consumption and wealth distribution.
  3. It enables meaningful policy experiments.
- The following slides borrow extensively from Dirk Krueger's lecture notes.

Models without Aggregate Uncertainty I

- Continuum of measure 1 of individuals, each facing an income fluctuation problem.
- Labor income: $w_t y_t$.
- Same labor endowment process $\{y_t\}_{t=0}^{\infty}$, $y_t \in Y = \{y_1, y_2, \ldots y_N\}$.
- Labor endowment process follows stationary Markov chain with transitions $\pi(y'|y)$.
- Law of large numbers: $\pi(y'|y)$ also the deterministic fraction of the population that has this transition.
- Π: stationary distribution associated with π, assumed to be unique.
- At period 0 income of all agents, y_0, is given. Population distribution given by Π.

Models without Aggregate Uncertainty II

- Total labor endowment in the economy at each point of time

$$\bar{L} = \sum_{y} y\Pi(y)$$

- Probability of event history y^t, given initial event y_0

$$\pi_t(y^t|y_0) = \pi(y_t|y_{t-1}) * \ldots * \pi(y_1|y_0)$$

- Note use of Markov structure.

- Substantial idiosyncratic uncertainty, but no aggregate uncertainty.

- Thus, there is hope for stationary equilibrium with constant w and r.

Models without Aggregate Uncertainty III

- Preferences

$$u(c) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Budget constraint

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

- Borrowing constraint

$$a_{t+1} \geq 0$$

- Initial conditions of agent (a_0, y_0) with initial population measure $\Phi_0(a_0, y_0)$.

- Allocation: $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}$.

Models without Aggregate Uncertainty IV

- Technology

$$Y_t = F(K_t, L_t)$$

with standard assumptions.

- Capital depreciates at rate $0 < \delta < 1$.

- Aggregate resource constraint

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t)$$

- The only asset in economy is the physical capital stock. No state-contingent claims (a form of incomplete markets).

Sequential Markets Competitive Equilibrium I

Definition

Given Φ_0, a sequential markets competitive equilibrium is allocations for households $\{c_t(a_0, y^t), a_{t+1}(a_0, y^t)\}$ allocations for the representative firm $\{K_t, L_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t\}_{t=0}^{\infty}$ such that:

1. Given prices, allocations maximize utility subject to the budget constraint and subject to the nonnegativity constraints on assets and consumption.

$$r_t = F_k(K_t, L_t) - \delta$$
$$w_t = F_L(K_t, L_t)$$

Sequential Markets Competitive Equilibrium II

Definition (cont.)

2. For all t

$$
\begin{aligned}
K_{t+1} &= \int \sum_{y^t \in Y^t} a_{t+1}(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) \\
L_t &= \bar{L} = \int \sum_{y^t \in Y^t} y_t \pi(y^t|y_0)d\Phi_0(a_0, y_0) \\
&\int \sum_{y^t \in Y^t} c_t(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) \\
&+ \int \sum_{y^t \in Y^t} a_{t+1}(a_0, y^t)\pi(y^t|y_0)d\Phi_0(a_0, y_0) \\
&= F(K_t, L_t) + (1-\delta)K_t
\end{aligned}
$$

Recursive Equilibrium

- Individual state (a, y).
- Aggregate state variable $\Phi(a, y)$.
- $A = [0, \infty)$: set of possible asset holdings.
- Y : set of possible labor endowment realizations.
- $\mathcal{P}(Y)$ is power set of Y.
- $\mathcal{B}(A)$ is Borel σ-algebra of A.
- $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$.
- $\mathcal{M}$: set of all probability measures on the measurable space $M = (Z, \mathcal{B}(Z))$.

Household Problem in Recursive Formulation

$$v(a, y; \Phi) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v(a', y'; \Phi')$$

$$\text{s.t. } c + a' = w(\Phi)y + (1 + r(\Phi))a$$
$$\Phi' = H(\Phi)$$

- Function $H : \mathcal{M} \rightarrow \mathcal{M}$ is called the aggregate "law of motion".

Definition

A RCE is value function $v : Z \times M \rightarrow R$*, policy functions for the household* $a' : Z \times M \rightarrow R$ *and* $c : Z \times M \rightarrow R$*, policy functions for the firm* $K : M \rightarrow R$ *and* $L : M \rightarrow R$, *pricing functions* $r : M \rightarrow R$ *and* $w : M \rightarrow R$ *and law of motion* $H : M \rightarrow M$ *s.t.*

1. v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies Bellman equation and a', c are the policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$

$$r(\Phi) = F_K(K(\Phi), L(\Phi)) - \delta$$
$$w(\Phi) = F_L(K(\Phi), L(\Phi))$$

Definition (cont.)

3. For all $\Phi \in \mathcal{M}$

$$K'(\Phi') = K(H(\Phi)) = \int a'(a, y; \Phi) d\Phi$$

$$L(\Phi) = \int y d\Phi$$

$$\int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi = F(K(\Phi), L(\Phi)) + (1 - \delta) K(\Phi)$$

4. Aggregate law of motion H is generated by π and a'.

Transition Functions I

- Define transition function $Q_\Phi : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ by

$$Q_\Phi((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a'(a,y;\Phi) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

  for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$

- $Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))$ is the probability that an agent with current assets a and income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow.

- Hence

$$\begin{aligned} \Phi'(\mathcal{A},\mathcal{Y}) &= (H(\Phi))(\mathcal{A},\mathcal{Y}) \\ &= \int Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy) \end{aligned}$$

Stationary RCE I

Definition

A stationary RCE is value function $v : Z \longrightarrow R$, policy functions for the household $a' : Z \longrightarrow R$ and $c : Z \longrightarrow R$, policies for the firm K, L, prices r, w and a measure $\Phi \in M$ such that

1. *v, a', c are measurable with respect to $B(Z)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given r and w.*
2. *K, L satisfy, given r and w*

$$\begin{aligned} r &= F_k(K, L) - \delta \\ w &= F_L(K, L) \end{aligned}$$

Stationary RCE II

Definition (cont.)

3.

$$K = \int a'(a, y) d\Phi$$

$$L(\Phi) = \int y d\Phi$$

$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta) K$$

4. For all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

where Q is transition function induced by π and a'.

Example: Discrete State Space

- Suppose $A = \{a_1, \ldots, a_M\}$. Then Φ is $M * N \times 1$ column vector and $Q = (q_{ij,kl})$ is $M * N \times M * N$ matrix with

$$q_{ij,kl} = \Pr\left((a', y') = (a_k, y_l) \middle| (a, y) = (a_i, y_l)\right)$$

- Stationary measure Φ satisfies matrix equation

$$\Phi = Q^T \Phi.$$

- Φ is (rescaled) eigenvector associated with an eigenvalue $\lambda = 1$ of Q^T.
- Q^T is a stochastic matrix and thus has at least one unit eigenvalue. If it has more than one unit eigenvalue, continuum of stationary measures.

Existence, Uniqueness, and Stability

- Existence (and Uniqueness) of Stationary RCE boils down to one equation in one unknown.
- Asset market clearing condition

$$K = K(r) = \int a'(a, y) d\Phi \equiv Ea(r)$$

- By Walras' law forget about goods market.
- Labor market equilibrium $L = \bar{L}$ and $\bar{L}$ is exogenously given.
- Capital demand of firm $K(r)$ is defined implicitly as

$$r = F_k(K(r), \bar{L}) - \delta$$

- Existence is usually easy to show.
- Uniqueness is more complicated.
- Stability is not well-understood.

Computation

1. Fix an $r \in (-\delta, 1/\beta - 1)$.
2. For a fixed r, solve household's recursive problem. This yields a value function v_r and decision rules a'_r, c_r.
3. The policy function a'_r and π induce Markov transition function Q_r.
4. Compute the unique stationary measure Φ_r associated with this transition function.
5. Compute excess demand for capital

$$d(r) = K(r) - Ea(r)$$

If zero, stop, if not, adjust r.

Qualitative Results

- Complete markets model: $r^{CM} = 1/\beta - 1$.
- This model: $r^* < 1/\beta - 1$.
- Overaccumulation of capital and oversaving (because of precautionary reasons: liquidity constraints, prudence, or both).
- Question: How big a difference does it make?
- Policy implications?

Calibration I

- *CRRA* with values $\sigma = \{1, 3, 5\}$.
- $r^{CM} = 0.0416$ ($\beta = 0.96$).
- Cobb-Douglas production function with $\alpha = 0.36$.
- $\delta = 8\%$.
- Earning profile:

$$\log(y_{t+1}) = \theta \log(y_t) + \sigma_\varepsilon \left(1 - \theta^2\right)^{\frac{1}{2}} \varepsilon_{t+1}$$

s.t.

$$corr(\log(y_{t+1}), \log(y_t)) = \theta$$
$$Var(\log(y_{t+1})) = \sigma_\varepsilon^2$$

- Consider $\theta \in \{0, 0.3, 0.6, 0.9\}$ and $\sigma_\varepsilon \in \{0.2, 0.4\}$.

Calibration II

- Discretize, using Tauchen's method.
  - Set $N = 7$.
  - Since $\log(y_t) \in (-\infty, \infty)$ subdivide in intervals

$$\left(-\infty, -\tfrac{5}{2}\sigma_\varepsilon\right) \quad \left[-\tfrac{5}{2}\sigma_\varepsilon, -\tfrac{3}{2}\sigma_\varepsilon\right) \quad \ldots \quad \left[\tfrac{3}{2}\sigma_\varepsilon, \tfrac{5}{2}\sigma_\varepsilon\right) \quad \left[\tfrac{5}{2}\sigma_\varepsilon, \infty\right)$$

  - State space for log-income: "midpoints"

$$Y^{\log} = \{-3\sigma_\varepsilon, -2\sigma_\varepsilon, -\sigma_\varepsilon, 0, \sigma_\varepsilon, 2\sigma_\varepsilon, 3\sigma_\varepsilon\}$$

  - Matrix π: fix $s_i = \log(y) \in Y^{\log}$ today and the conditional probability of $s_j = \log(y') \in Y^{\log}$ tomorrow is

$$\pi(\log(y') = s_j | \log(y) = s_i) = \int_{I_j} \frac{e^{-\frac{(x-\theta s_i)^2}{2\sigma_y}}}{(2\pi)^{0.5}\sigma_y} dx$$

where $\sigma_y = \sigma_\varepsilon \left(1-\theta^2\right)^{\frac{1}{2}}$.

Calibration III

- Find the stationary distribution of π, hopefully unique, by solving

$$\Pi = \pi^T \Pi$$

- Take $\tilde{Y} = e^{Y^{\log}}$

$$\tilde{Y} = \{e^{-3\sigma_\varepsilon}, e^{-2\sigma_\varepsilon}, e^{-\sigma_\varepsilon}, 1, e^{\sigma_\varepsilon}, e^{2\sigma_\varepsilon}, e^{3\sigma_\varepsilon}\}$$

- Compute average labor endowment $\bar{y} = \sum_{y \in \tilde{Y}} y\Pi(y)$.
- Normalize all states by $\bar{y}$

$$\begin{aligned} Y &= \{y_1, \ldots, y_7\} \\ &= \{\frac{e^{-3\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{-\sigma_\varepsilon}}{\bar{y}}, \frac{1}{\bar{y}}, \frac{e^{\sigma_\varepsilon}}{\bar{y}}, \frac{e^{2\sigma_\varepsilon}}{\bar{y}}, \frac{e^{3\sigma_\varepsilon}}{\bar{y}}\} \end{aligned}$$

- Then:

$$\sum_{y \in Y} y\Pi(y) = 1$$

Results

- Cobb-Douglas production function and $\bar{L} = 1$ we have $Y = K^{\alpha}$ and

$$r + \delta = \alpha K^{\alpha - 1}$$

- s is the aggregate saving rate:

$$r + \delta = \frac{\alpha Y}{K} = \frac{\alpha \delta}{s} \Rightarrow s = \frac{\alpha \delta}{r + \delta}$$

- Benchmark of complete markets: $r^{CM} = 4.16\%$ and $s = 23.7\%$.
- Keeping σ and σ_{ε} fixed, an increase in θ leads to more precautionary saving and more overaccumulation of capital.
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- Keeping σ and θ fixed, an increase in σ_{ε} leads to more precautionary saving and more overaccumulation of capital.

Unexpected Aggregate Shocks and Transition Dynamics

- Hypothetical thought experiment:
  - Economy is in stationary equilibrium, with a given government policy.
  - Unexpectedly government policy changes. Exogenous change may be either transitory or permanent.
  - Want to compute transition path induced by the exogenous change, from the old stationary equilibrium to a new stationary equilibrium.
- Example: permanent introduction of a capital income tax at rate τ. Receipts are rebated lump-sum to households as government transfers.
- Key: *assume* that after T periods the transition from old to new stationary equilibrium is completed.

Algorithm I

1. Fix T.
2. Compute stationary equilibrium $\Phi_0, v_{0,} r_0, w_0, K_0$ associated with $\tau = \tau_0 = 0$.
3. Compute stationary equilibrium $\Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$ associated with $\tau_\infty = \tau$. Assume:

$$\Phi_T, v_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$$

4. Guess sequence $\{\hat{K}_t\}_{t=1}^{T-1}$ Note that $\hat{K}_1$ is determined by decisions at time 0, $\hat{K}_1 = K_0$, and $L_t = L_0 = \bar{L}$ is fixed. Also:

$$\hat{w}_t = F_L(\hat{K}_t, \bar{L})$$
$$\hat{r}_t = F_K(\hat{K}_t, \bar{L}) - \delta$$
$$\hat{T}_t = \tau_t \hat{r}_t \hat{K}_t.$$

Algorithm II

5. Since we know $v_T(a, y)$ and $\{\hat{r}_t, \hat{w}_t, \hat{T}_t\}_{t=1}^{T-1}$, we can solve for $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$ backwards.
6. With $\{\hat{a}_{t+1}\}$ define transition laws $\{\hat{\Gamma}_t\}_{t=1}^{T-1}$. Given $\Phi_0 = \Phi_1$ from the initial stationary equilibrium, iterate forward:

$$\hat{\Phi}_{t+1} = \hat{\Gamma}_t(\hat{\Phi}_t)$$

   for $t = 1, \ldots, T-1$.
7. With $\{\hat{\Phi}_t\}_{t=1}^{T}$, compute $\hat{A}_t = \int a d\hat{\Phi}_t$ for $t = 1, \ldots, T$.
8. Check whether:

$$\max_{1 \leq t < T} \left|\hat{A}_t - \hat{K}_t\right| < \varepsilon$$

   If yes, go to 9. If not, adjust your guesses for $\{\hat{K}_t\}_{t=1}^{T-1}$ in 4.
9. Check whether $\left|\hat{A}_T - K_T\right| < \varepsilon$. If yes, we are done and should save $\{\hat{v}_t, \hat{a}_{t+1}, \hat{c}_t, \hat{\Phi}_t, \hat{r}_t, \hat{w}_t, \hat{K}_t\}$. If not, go to 1. and increase T.

Welfare Consequences of the Policy Reform I

- Previous procedure determines aggregate variables such as r_t, w_t, Φ_t, K_t, decision rules c_r, a_{t+1}, and value functions v_t.
- We can use v_0, v_1, and v_T to determine the welfare consequences from the reform.
- Suppose that

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Optimal consumption allocation in initial stationary equilibrium, in sequential formulation, $\{c_s\}_{s=0}^{\infty}$.

$$v_0(a, y) = \mathbb{E}_0 \sum_{s=0}^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma}$$

Welfare Consequences of the Policy Reform II

- Define g implicitly as:

$$\begin{aligned} v_1(a,y) &= v_0(a,y;g) = (1+g)^{1-\sigma}\mathbb{E}_0 \sum_{s=0}^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma} \\ &= (1+g)^{1-\sigma} v_0(a,y) \end{aligned}$$

- Then:

$$g(a,y) = \left[\frac{v_1(a,y)}{v_0(a,y)}\right]^{\frac{1}{1-\sigma}} - 1$$

- Steady state welfare consequences:

$$g_{ss}(a,y) = \left[\frac{v_T(a,y)}{v_0(a,y)}\right]^{\frac{1}{1-\sigma}} - 1$$

- $g(a,y)$ and $g_{ss}(a,y)$ may be quite different.
- Example: social security reform.

Aggregate Uncertainty and Distributions as State Variables

- Why complicate the model? Want to talk about economic fluctuations and its interaction with idiosyncratic uncertainty.
- But now we have to characterize and compute entire recursive equilibrium: distribution as state variable.
- Infinite-dimensional object.
- Very limited theoretical results about existence, uniqueness, stability, goodness of approximation

The Model I

- Aggregate production function:

$$Y_t = s_t F(K_t, L_t)$$

- Let

$$s_t \in \{s_b, s_g\} = S$$

with $s_b < s_g$ and conditional probabilities $\pi(s'|s)$.

- Idiosyncratic labor productivity y_t correlated s_t.

$$y_t \in \{y_u, y_e\} = Y$$

where $y_u < y_e$ stands for the agent being unemployed and y_e stands for the agent being employed.

- Probability of being unemployed is higher during recessions than during expansions.

The Model II

- Probability of individual productivity tomorrow of y' and aggregate state s' tomorrow, conditional on states y and s today:

$$\pi(y', s'|y, s) \geq 0$$

π is 4×4 matrix.

- Law of large numbers: idiosyncratic uncertainty averages out and only aggregate uncertainty determines $\Pi_s(y)$, the fraction of the population in idiosyncratic state y if aggregate state is s.
- Consistency requires:

$$\sum_{y' \in Y} \pi(y', s'|y, s) = \pi(s'|s) \text{ all } y \in Y, \text{ all } s, s' \in S$$

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

Recursive Formulation

- Individual state variables (a, y).
- Aggregate state variables (s, Φ).
- Recursive formulation:

$$v(a, y, s, \Phi) = \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', \Phi') \right.$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s, \Phi) y + (1 + r(s, \Phi)) a \\ \Phi' &= H(s, \Phi, s') \end{aligned}$$

Definition

A RCE is value function $v : Z \times S \times \mathcal{M} \rightarrow R$, policy functions for the household $a' : Z \times S \times \mathcal{M} \rightarrow R$ and $c : Z \times S \times \mathcal{M} \rightarrow R$, policy functions for the firm $K : S \times \mathcal{M} \rightarrow R$ and $L : S \times \mathcal{M} \rightarrow R$, pricing functions $r : S \times \mathcal{M} \rightarrow R$ and $w : S \times \mathcal{M} \rightarrow R$ and an aggregate law of motion $H : S \times \mathcal{M} \times S \rightarrow \mathcal{M}$ such that

1. v, a', c are measurable with respect to $\mathcal{B}(S)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$

$$r(s, \Phi) = F_K(K(s, \Phi), L(s, \Phi)) - \delta$$
$$w(s, \Phi) = F_L(K(s, \Phi), L(s, \Phi))$$

Definition (cont.)

3. For all $\Phi \in \mathcal{M}$ and all $s \in S$

$$\begin{aligned} K(H(s,\Phi)) &= \int a'(a,y,s,\Phi)d\Phi \\ L(s,\Phi) &= \int yd\Phi \\ & \int c(a,y,s,\Phi)d\Phi + \int a'(a,y,s,\Phi)d\Phi \\ &= F(K(s,\Phi),L(s,\Phi)) + (1-\delta)K(s,\Phi) \end{aligned}$$

4. The aggregate law of motion H is generated by the exogenous Markov process π and the policy function a'.

Transition Function and Law of Motion

- Define $Q_{\Phi,s,s'} : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ by

$$Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y',s'|y,s) \text{ if } a'(a,y,s,\Phi) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

- Aggregate law of motion

$$\begin{aligned} \Phi'(\mathcal{A},\mathcal{Y}) &= \left(H(s,\Phi,s')\right)(\mathcal{A},\mathcal{Y}) \\ &= \int Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy) \end{aligned}$$

Computation of the Recursive Equilibrium I

- Key computational problem: aggregate wealth distribution Φ is an infinite-dimensional object.
- Agents need to keep track of the aggregate wealth distribution to forecast future capital stock and thus future prices. But for K' need entire Φ since

$$K' = \int a'(a, y, s, \Phi) d\Phi$$

- If a' were linear in a, with same slope for all $y \in Y$, exact aggregation would occur and K would be a sufficient statistic for K'.
- Trick: Approximate the distribution Φ with a finite set of moments.
- Let the n-dimensional vector m denote the first n moments of the asset distribution

Computation of the Recursive Equilibrium II

- Agents use an approximate law of motion

$$m' = H_n(s, m)$$

- Agents are boundedly rational: moments of higher order than n of the current wealth distribution may help to more accurately forecast prices tomorrow.
- We choose the number of moments and the functional form of the function H_n.
- Krusell and Smith pick $n = 1$ and pose

$$\log(K') = a_s + b_s \log(K)$$

for $s \in \{s_b, s_g\}$. Here (a_s, b_s) are parameters that need to be determined.

Computation of the Recursive Equilibrium III

- Household problem

$$v(a,y,s,K) = \max_{c,a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi(y', s' | y, s) v(a', y', s', K') \right\}$$

$$\begin{aligned} \text{s.t. } c + a' &= w(s,K)y + (1 + r(s,K))a \\ \log(K') &= a_s + b_s \log(K) \end{aligned}$$

- Reduction of the state space to a four dimensional space $(a, y, s, K) \in \mathbf{R} \times Y \times S \times \mathbf{R}$.

Algorithm I

1. Guess (a_s, b_s).
2. Solve households problem to obtain $a'(a, y, s, K)$.
3. Simulate economy for large number of T periods for large number N of households:
    - Start with initial conditions for the economy (s_0, K_0) and for each household (a_0^i, y_0^i).
    - Draw random sequences $\{s_t\}_{t=1}^{T}$ and $\{y_t^i\}_{t=1,i=1}^{T,N}$ and use $a'(a, y, s, K)$ and perceived law of motion for K to generate sequences of $\{a_t^i\}_{t=1,i=1}^{T,N}$.
    - Aggregate:

$$K_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i$$

Algorithm II

4. Run the regressions

$$\log(K') = \alpha_s + \beta_s \log(K)$$

to estimate (α_s, β_s) for $s \in S$.

5. If the R^2 for this regression is high and $(\alpha_s, \beta_s) \approx (a_s, b_s)$ stop. An approximate equilibrium was found.

6. Otherwise, update guess for (a_s, b_s). If guesses for (a_s, b_s) converge, but R^2 remains low, add higher moments to the aggregate law of motion and/or experiment with a different functional form for it.

Calibration I

- Period 1 quarter.
- *CRRA* utility with $\sigma = 1$ (log-utility)
- $\beta = 0.99^4 = 0.96$.
- $\alpha = 0.36$.
- $\delta = (1 - 0.025)^4 - 1 = 9.6\%$.
- Aggregate component: two states (recession, expansion)

$$S = \{0.99, 1.01\} \Rightarrow \sigma_s = 0.01$$

- Symmetric transition matrix $\pi(s_g | s_g) = \pi(s_b | s_b)$.
- Expected time in each state: 8 quarters, hence $\pi(s_g | s_g) = \frac{7}{8}$ and

$$\pi(s' | s) = \begin{pmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

Calibration II

- Idiosyncratic component: two states (employment and unemployment):

$$Y = \{0.25, 1\}$$

  Unemployed person makes 25% of the labor income of an employed person.

- Transition probabilities:

$$\pi(y', s'|y, s) = \pi(y'|s', y, s) * \pi(s'|s)$$

- Specify four 2×2 matrices $\pi(y'|s', y, s)$.

Calibration III

- Expansion: average time of unemployment equal to 1.5 quarters

$$\pi(y' = y_u | s' = s_g, y = y_u, s = s_g) = \frac{1}{3}$$
$$\pi(y' = y_e | s' = s_g, y = y_u, s = s_g) = \frac{2}{3}$$

- Recession: average time of unemployment equal to 2.5 quarters

$$\pi(y' = y_u | s' = s_b, y = y_u, s = s_b) = 0.6$$
$$\pi(y' = y_e | s' = s_b, y = y_u, s = s_b) = 0.4$$

Calibration IV

- Switch from g to b: probability of remaining unemployed 1.25 higher

$$\begin{aligned}\pi(y' &= y_u \| s' = s_b, y = y_u, s = s_g) = 0.75 \\ \pi(y' &= y_e \| s' = s_b, y = y_u, s = s_g) = 0.25\end{aligned}$$

- Switch from b to g: probability of remaining unemployed 0.75 higher

$$\begin{aligned}\pi(y' &= y_u \| s' = s_g, y = y_u, s = s_b) = 0.25 \\ \pi(y' &= y_e \| s' = s_g, y = y_u, s = s_b) = 0.75\end{aligned}$$

- Idea: best times for finding a job are when the economy moves from a recession to an expansion, the worst chances are when the economy moves from a boom into a recession.

Calibration V

- In recessions unemployment rate is $\Pi_{s_b}(y_u) = 10\%$ and in expansions it is $\Pi_{s_g}(y_u) = 4\%$. Remember:

$$\Pi_{s'}(y') = \sum_{y \in Y} \frac{\pi(y', s'|y, s)}{\pi(s'|s)} \Pi_s(y) \text{ for all } s, s' \in S$$

- This gives

$$\begin{aligned}
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_g) = 0.028 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_g) = 0.972 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_b) = 0.04 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_b) = 0.96 \\
\pi(y' &= y_u|s' = s_b, y = y_e, s = s_g) = 0.079 \\
\pi(y' &= y_e|s' = s_b, y = y_e, s = s_g) = 0.921 \\
\pi(y' &= y_u|s' = s_g, y = y_e, s = s_b) = 0.02 \\
\pi(y' &= y_e|s' = s_g, y = y_e, s = s_b) = 0.98
\end{aligned}$$

Calibration VI

- In summary:

$$\pi = \begin{pmatrix} 0.525 & 0.035 & 0.09375 & 0.0099 \\ 0.35 & 0.84 & 0.03125 & 0.1151 \\ 0.03125 & 0.0025 & 0.292 & 0.0245 \\ 0.09375 & 0.1225 & 0.583 & 0.8505 \end{pmatrix}$$

Numerical Results

Model delivers

1. Aggregate law of motion

$$m' = H_n(s, m)$$

2. Individual decision rules

$$a'(a, y, s, m)$$

3. Time-varying cross-sectional wealth distributions

$$\Phi(a, y)$$

Aggregate Law of Motion I

- Agents are boundedly rational: aggregate law of motion perceived by agents may not coincide with actual law of motion.
- Only thing to forecast is K'. Hence try $n = 1$.
- Converged law of motion:

$$\begin{aligned} \log(K') &= 0.095 + 0.962 \log(K) \text{ for } s = s_g \\ \log(K') &= 0.085 + 0.965 \log(K) \text{ for } s = s_b \end{aligned}$$

- How irrational are agents? Use simulated time series $\{(s_t, K_t\}_{t=0}^{T}$, divide sample into periods with $s_t = s_b$ and $s_t = s_g$, and run

$$\log(K_{t+1}) = \alpha_j + \beta_j \log(K_t) + \varepsilon_{t+1}^{j}$$

Aggregate Law of Motion II

- Define

$$\hat{\varepsilon}^{j}_{t+1} = \log(K_{t+1}) - \hat{\alpha}_j - \hat{\beta}_j \log(K_t) \text{ for } j = g, b$$

- Then:

$$\sigma_j = \left(\frac{1}{T_j} \sum_{t \in \tau_j} \left(\hat{\varepsilon}^{j}_{t} \right)^2 \right)^{0.5}$$

$$R_j^2 = 1 - \frac{\sum_{t \in \tau_j} \left(\hat{\varepsilon}^{j}_{t} \right)^2}{\sum_{t \in \tau_j} \left(\log K_{t+1} - \log \bar{K} \right)^2}$$

- If $\sigma_j = 0$ for $j = g, b$ (if $R_j^2 = 1$ for $j = g, b$), then agents do not make forecasting errors

Aggregate Law of Motion III

- Results

$$\begin{aligned} R_j^2 &= 0.999998 \text{ for } j = b, g \\ \sigma_g &= 0.0028 \\ \sigma_b &= 0.0036 \end{aligned}$$

- Maximal forecasting errors for interest rates 25 years into the future is 0.1%.
- Corresponding utility losses?
- Approximated equilibria may be arbitrarily far away from exact one.

Why Quasi-Aggregation?

- If all agents have linear savings functions with same marginal propensity to save

$$a'(a, y, s, K) = a_s + b_s a + c_s y$$

- Then:

$$\begin{aligned} K' &= \int a'(a, y, s, K) d\Phi = a_s + b_s \int a d\Phi + c_s \bar{L} \\ &= \tilde{a}_s + b_s K \end{aligned}$$

- Exact aggregation: K sufficient statistic for Φ for forecasting K'.
- In this economy: savings functions almost linear with same slope for $y = y_u$ and $y = y_e$.
- Only exceptions are unlucky agents ($y = y_u$) with little assets. But these agents hold a negligible fraction of aggregate wealth and do not matter for K dynamics.
- Hence quasi-aggregation!!!

Why is Marginal Propensity to Save Close to 1? I

- PILCH model with certainty equivalence and $r = 1/\beta$

$$c_t = \frac{r}{1+r}\left(\mathbb{E}_t \sum_{s=0}^{T-t} \frac{y_{t+s}}{(1+r)^s} + a_t\right)$$

- Agents save out of current assets for tomorrow

$$\frac{a_{t+1}}{1+r} = \left(1 - \frac{r}{1+r}\right) a_t + G(y)$$

- Thus under certainty equivalence

$$a_{t+1} = a_t + H(y)$$

Why is Marginal Propensity to Save Close to 1? II

- In this economy agents are prudent and face liquidity constraints, but almost act as if they are certainty equivalence consumers. Why?

1. With $\sigma = 1$ agents are prudent, but not all that much.
2. Unconditional standard deviation of individual income is roughly 0.2, at the lower end of the estimates.
3. Negative income shocks (unemployment) are infrequent and not very persistent.