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Establishments Dynamics and Matching Frictions in Classical Competitive Equilibrium*

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VERY PRELIMINARY AND INCOMPLETE

Abstract: This paper develops a Walrasian equilibrium theory of establishment level dynamics and matching frictions and uses it to evaluate the effects of congestion externalities in the matching process and determine the government interventions that are needed to implement a Pareto optimal allocation. The optimal policy, which involves a tax on the creation of help-wanted ads and an unemployment subsidy, is highly contractionary. However, it leads to large welfare gains. The policy also plays an important role in dampening the response of the economy to aggregate productivity changes.

*This paper originated in a conversation with Randall Wright and was heavily influenced by it. I have also benefited from the comments of seminar participants at the Federal Reserve Bank of Philadelphia. All remaining errors are solely mine. The views express here do not necessarily reflect the position of the Federal Reserve Bank of Chicago or the Federal Reserve System. Address: Federal Reserve Bank of Chicago, Research Department, 230 South LaSalle Street, Chicago, IL 60604. E-mail: mveracierto@frbchi.org. Phone: (312) 322-5695.

1. Introduction

This paper develops a general equilibrium theory of establishment level dynamics and matching frictions and uses it to evaluate the effects of congestion externalities in the matching process and determine the government interventions needed to implement a Pareto optimal allocation. The theory blends together three important strands in the literature: 1) the Hopenhayn and Rogerson [6] model of establishment dynamics, 2) the Mortensen and Pissarides [8] matching model, and 3) the Lucas and Prescott [7] islands model. Establishments in my model are similar to those in Hopenhayn and Rogerson [6] except that the amount of hiring that establishments can undertake is determined by the number of help-wanted ads that they posted during the previous period. Similarly to Mortensen and Pissarides [8] unemployed workers and help-wanted ads meet according to a matching function. However, wages are determined by Walrasian markets instead of bilateral Nash bargaining and productions units are constituted by establishments instead of individual jobs. From the Lucas and Prescott [7] model I retain the directed search specification but assume that the search costs are incurred by the establishments instead of the unemployed workers. These modelling choices give rise to a unified framework for analyzing establishment dynamics, vacancies, unemployment and matching frictions. Moreover, the model completely relies on classical competitive theory. An advantage of this approach is the considerable discipline that it imposes: Wages are determined by preferences and technology instead of free bargaining parameters.

The economy is populated by a representative household constituted by a continuum of members that value consumption and leisure. Household members differ in their employment histories but obtain full insurance within the household. The consumption good is produced by a large number of spatially separated establishments that operate a decreasing returns to scale technology using labor as the only input of production. The establishments are subject to idiosyncratic productivity shocks that induce them to expand and contract over time. In order to hire a worker an establishment needs to have a hiring opportunity. In the same vein, an unemployed workers needs to have a job opportunity in order to become employed. Hiring opportunities and job opportunities are jointly produced: They are obtained when an unemployed worker meets the help-wanted ad posted by an establishment. The rate at which a help-wanted ad meets unemployed workers is determined by the total number of help-wanted ads and unemployed workers that enter a constant returns to scale matching technology. In order to generate a Walrasian market structure I assume that any of the unemployed workers with a job opportunity can be hired by any of the establishments with a

hiring opportunity.

I analyze this economy in a number of ways. First, I solve the social planner's problem and characterize the efficient allocation. Second, I describe a competitive equilibrium that attains the Pareto optimum allocation. The decentralization requires a large number of competitive matching companies operating the constant returns to scale matching technology at a reduced scale. The equilibrium is efficient because the matching companies fully internalize the congestion effects that unemployed workers and help-wanted ads generate. The competitive equilibrium has some undesirable features, though. At equilibrium the matching companies charge a fee for the hiring opportunities that they create but give the job opportunities for free. Moreover, the matching companies pay unemployed workers for entering the matching technology that they operate. This is not only an unrealistic feature of the competitive equilibrium but makes it very fragile, since workers have the incentive of entering as many matching technologies as possible while the matching companies demand their exclusivity. Third, I describe a competitive equilibrium with matching externalities. In this equilibrium the search technology can only be operated at the aggregate level and establishments and unemployed workers have free access to it. Since no decision maker regulates the operation of the matching technology, congestion externalities arise and the competitive equilibrium is inefficient. However, the undesirable features of the efficient competitive equilibrium disappear. In particular, unemployed workers now receive zero payments for entering the aggregate matching technology. Fourth, I characterize the government interventions needed to attain the Pareto optimal allocation in the competitive equilibrium with matching externalities. I find that the optimal policy involves a subsidy to unemployed workers and a tax to the creation of help-wanted ads. In addition, I find that the optimal policy regime is self-financing.

Given its more realistic features the steady state equilibrium with matching externalities is calibrated to the U.S. economy. Model parameters are chosen to reproduce establishment level observations (e.g. the size distribution of establishments, job creation and destruction rates, etc.) as well as aggregate labor market statistics (e.g. the unemployment rate, the hazard rate of unemployment, the elasticity of the hazard rate of unemployment with respect to market tightness, etc.). Once the steady state equilibrium with matching externalities is calibrated to U.S. data, I introduce the optimal policy regime and evaluate its effects. I find that the optimal policy intervention is quite significant: It entails a tax on help-wanted ads of 267% and a subsidy to unemployed workers equal to 4% of wages. I also find that the optimal policy has sizable effects: It reduces output by 6.2%, consumption by 5.5% and employment by 9.9%. Although the policy is contractionary,

the increase in leisure more than compensates the drop in consumption and a steady state welfare gain of 0.8% in terms of permanent consumption is obtained. When transitionary dynamics are considered the welfare gain increases to 1.1% in terms of consumption. I also find that the optimal policy has sizable business cycle effects, dampeninng the response of the economy to aggregate productivity shocks quite substantially.

The paper is organized as follows. Section 2 describes the economy. Section 3 describes the set of feasible allocations. Section 4 characterizes a Pareto optimal allocation. Section 5 describes the competitive equilibrium that decentralizes the optimal allocation. Section 6 describes a competitive equilibrium with matching externalities. Section 7 characterizes the government intervention that attains the Pareto optimal allocation in a competitive equilibrium with matching externalities. Section 8 calibrates the steady state competitive equilibrium with matching externalities. Section 9 presents the results. Finally, Section 10 concludes the paper.

2. The economy

The economy is populated by a measure one of households, each composed by a unit interval of members called workers. Labor is indivisible: Workers can be either employed or unemployed, and employed workers must spend a larger amount of time in market activities than unemployed workers. The preferences of the representative household are given by:

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma} - 1}{1-\sigma} + \varphi U_t \right], \tag{2.1}$$

where c_t is consumption, U_t is unemployment, $\varphi > 0$, $\sigma > 0$ and $0 < \beta < 1$. The parameter φ is positive since the household is assumed to value leisure.

The consumption good is produced by a large number of spatially separated establishments. Each establishment has a production function given by

$$y_t = s_t F(n_t),$$

where $s_t \in S = \{0, s_1, s_2, ..., s_{\max}\}$ is an idiosyncratic productivity shock, n_t is labor, and F is a twice continuously differentiable, strictly increasing, and strictly concave function with $\lim_{n \to 0} F'(n) = +\infty$ and $\lim_{n \to \infty} F'(n) = 0$. The idiosyncratic productivity shock s_t follows a finite Markov process with monotone transition matrix Q. Realizations of s_t are assumed independent across all establish-

ments and $s_t = 0$ is assumed to be an absorbing state. Since there are no fixed costs of operation, exit takes place only when the idiosyncratic productivity level becomes zero. In every period, a measure ς of new establishments is exogenously born. Their distribution over initial productivity shocks is given by ψ.

Establishments can hire workers in a central island called the *hiring market*. However, the number of workers that an establishment can hire is constrained by the number of *hiring opportunities* m_t that the establishment has available at the beginning of the period. An establishment receives a hiring opportunity when one of its posted *help-wanted ads* meets an unemployed worker. Since workers are assumed to quit at the exogenous rate δ, an establishment's employment level n_t is constrained as follows:

$$n_t \leq (1 - \delta)\, n_{t-1} + m_t.$$

All workers that separate from an establishment (either because of quits or firings) join the pool of unemployment.

Unemployed workers have the possibility of becoming employed in the hiring market, but only if they have a *job opportunity*. An unemployed worker receives a job opportunity when he meets a help-wanted ad posted by some establishment. Once in the hiring market, any unemployed worker with a job opportunity can be hired by any establishment with a hiring opportunity.

The *matching technology*, which jointly produces job and hiring opportunities, is described as follows. Let A_t be the total number of help-wanted ads in the matching technology, and let U_t be the total number of unemployed workers in the matching technology. The number of hiring opportunities obtained by an establishment that created a_t help-wanted ads is given by:

$$m_{t+1} = a_t m(A_t, U_t), \tag{2.2}$$

where m is assumed to be strictly concave, strictly decreasing in A and strictly increasing in U. The total number of job opportunities obtained by unemployed workers is then equal to

$$M_{t+1} = M\left(A_t, U_t\right),$$

i.e. it is given by the total number of hiring opportunities

3. Feasibility

In what follows, it will be convenient to index establishments by the date j of their creation, by their history of idiosyncratic shocks $s_j^t = (s_j, s_{j+1}..., s_{t-1}, s_t) \in S^{t-j+1}$ since the date of their creation and by the number of workers e that they were endowed with at the time of their creation. Only establishments created at $t = 0$ are allowed to have a positive initial endowment of workers e. Establishments created after $t = 0$ have a zero initial endowment of workers.[1]

The number of establishments $\mu_0(0, s_0, e)$ created at $t = 0$ with initial productivity s_0 and initial endowment of workers e, is given. For $t \geq 1$, the number of establishments $\mu_t\left[j, s_j^t, e\right]$ with creation date $j \leq t$, history s_j^t and initial endowment of workers e satisfies the following equations:

$$\begin{aligned}
\mu_t(t, s_t, 0) &= \varsigma\psi(s_t), \\
\mu_t(t, s_t, e) &= 0, \text{ for } e > 0, \\
\mu_t[j, (s_j, s_{j+1}..., s_{t-1}, s_t), e] &= \mu_{t-1}[j, (s_j, s_{j+1}..., s_{t-1}), e] Q(s_{t-1}, s_t), \text{ for } 0 \leq j < t.
\end{aligned}$$

Aggregate consumption is given by

$$c_t \leq \sum_{j=0}^{t} \sum_{s_j^t} \sum_{e} \left[s_t F\left[n_t\left(j, s_j^t, e\right)\right] - k a_t\left(j, s_j^t, e\right)\right] \mu_t\left(j, s_j^t, e\right), \tag{3.1}$$

where $a_t\left(j, s_j^t, e\right)$ are the help-wanted ads created by an establishment of type $\left(j, s_j^t, e\right)$. This equation states that aggregate consumption is equal to the sum of output net of help-wanted ad costs across all types of establishments.

Aggregate help-wanted ads are given by:

$$\sum_{j=0}^{t} \sum_{s_j^t} \sum_{e} a_t\left(j, s_j^t, e\right) \mu_t\left(j, s_j^t, e\right) \leq A_t, \tag{3.2}$$

i.e. it is the sum of help-wanted ads across all type of establishments. Unemployment is given by:

$$U_t \leq 1 - \sum_{j=0}^{t} \sum_{s_j^t} \sum_{e} n_t\left(j, s_j^t, e\right) \mu_t\left(j, s_j^t, e\right), \tag{3.3}$$

[1] The "initial endowment" variable e is introduced to avoid carrying separate notation for incumbent establishments at $t = 0$ and establishments created after $t = 0$.

i.e. it is the total number of agents that do not work.

Employment feasibility at the establishment level is described as follows. For $j = t$,

$$n_t\left(j, s_j^t, e\right) \leq e, \tag{3.4}$$

since at the time of their creation establishments cannot employ more than their initial endowment of workers. For $j < t$:

$$n_t\left(j, s_j^t, e\right) \leq (1-\delta) n_{t-1}\left(j, s_j^{t-1}, e\right) + a_{t-1}\left(j, s_j^{t-1}, e\right) m\left[A_{t-1}, U_{t-1}\right]. \tag{3.5}$$

That is, the employment level of an establishment cannot exceed the sum of its previous employment level net of quits and the number of hiring opportunities that it has at the beginning of the period.

4. Pareto optimal allocations

The social planner's problem is to maximize (2.1) subject to equations (3.1)-(3.5). Since the utility function in equation (2.1), the production function F and the matching function M are concave, this is a standard problem. An analysis of its first order conditions leads to the following characterization.

An efficient allocation can be described by a sequence $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t\}_{t=0}^{\infty}$. The elements v_t, n_t, and a_t are functions of the state of an establishment (x, s), where x are the units of labor available for employment at the beginning of the period and s is the current productivity level. The number $v_t(x, s)$ is the date t shadow value of a worker at an establishment of type (x, s), the number $n_t(x, s)$ is the date t employment level at an establishment of type (x, s), and the number $a_t(x, s)$ are the date t help-wanted ads created by an establishment of type (x, s). The elements θ_t and γ_t are numbers: θ_t is the date t shadow value of a worker in the pool of aggregate unemployment and γ_t is the shadow value of a next-period hiring opportunity.[2] Finally, the element μ_t is the date t measure of establishments across states (x, s). To represent an efficient allocation, the sequence must satisfy the following conditions.

The shadow value of a worker at an establishment of type (x, s) must satisfy the following

[2] All shadow values are expressed in terms of the consumption good.

equation:

$$\begin{aligned} v_t(x,s) &= \max\left\{\theta_t,\ sF'(x) + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\delta\theta_{t+1}\right. \\ &\quad \left. + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}(1-\delta)\sum_{s'} v_{t+1}\left[(1-\delta)x + a_t(x,s)\,m\left[A_t,U_t\right],s'\right]Q\left(s,s'\right)\right\}. \end{aligned} \tag{4.1}$$

This equation is quite intuitive. The shadow value $v_t(x,s)$ cannot be less than the social value of sending an additional worker to the pool of unemployment θ_t. Otherwise, welfare could be improved by reducing the employment level of the establishment. If the shadow value $v_t(x,s)$ exceeds the value of unemployment, the planner retains all workers available to the establishment x. In this case, the flow shadow value of a worker is given by the marginal productivity $sF'(x)$. With probability δ the worker quits and his continuation value is the discounted shadow value θ_{t+1} of having an additional worker in the pool of unemployment at the beginning of the following period. With probability $(1-\delta)$ the worker does not quit the following period, and his continuation value is equal to the expected discounted shadow value v_{t+1} of beginning the following period in an establishment with $(1-\delta)x + a_t(x,s)\,m\left[A_t,U_t\right]$ workers available for employment. Observe that the social planner discounts next period values using the product of the discount factor β and the intertemporal marginal rate of substitution $(c_t/c_{t+1})^{\sigma}$.

The optimal number of help-wanted ads created by an establishment of type (x,s) is characterized by the following condition:

$$\begin{aligned} \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\,n_t(x,s) + a_t(x,s)\,m\left(A_t,U_t\right),s'\right]Q\left(s,s'\right) &\leq \gamma_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}, \\ \text{with equality if } a_t(x,s) &> 0. \end{aligned} \tag{4.2}$$

That is, if the establishment creates a positive number of help-wanted ads it must be the case that the expected discounted shadow value of a worker at the establishment v_{t+1} is equal to the shadow value of a next-period hiring opportunity γ_t plus the discounted shadow value of a next-period unemployed worker θ_{t+1}. This conditions is also intuitive. If $\beta\left(c_t/c_{t+1}\right)^{\sigma}E\left[v_{t+1}\right] > \gamma_t + \beta\left(c_t/c_{t+1}\right)^{\sigma}\theta_{t+1}$, the value of increasing the establishment's next-period hiring opportunities by one unit exceeds the opportunity cost. As a consequence, welfare could be improved by realizing this increase. If $\beta\left(c_t/c_{t+1}\right)^{\sigma}E\left[v_{t+1}\right] < \gamma_t + \beta\left(c_t/c_{t+1}\right)^{\sigma}\theta_{t+1}$ even when zero hiring opportunities are created, the social planner stays at the corner solution of $a_t(x,s) = 0$.

Substitution equation (4.2) in equation (4.1) gives

$$\begin{aligned} v_t(x,s) &= \max\left\{\theta_t, sF'(x) + \delta\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}\right. \\ &\quad \left. + (1-\delta)\min\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)x, s'\right]Q\left(s,s'\right), \gamma_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}\right]\right\}. \end{aligned}$$

The optimal employment rule $n_t(x,s)$ is then easily obtained. It is characterized by a threshold level $\bar{x}_t(s)$ that satisfies the following condition:

$$\begin{aligned} \theta &= sF'(\bar{x}_t(s)) + \delta\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1} \\ &\quad + (1-\delta)\min\left\{\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\bar{x}_t(s), s'\right]Q\left(s,s'\right), \gamma_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}\right\}. \end{aligned} \tag{4.3}$$

That is, $\bar{x}_t(s)$ is the unique value of x at which the social planner is indifferent between leaving the marginal worker at the establishment and sending him to the pool of unemployment.

The optimal employment rule is then given by:

$$n_t(x,s) = \min\left\{\bar{x}_t(s), x\right\}. \tag{4.4}$$

That is, if $x > \bar{x}_t(s)$, the shadow value of a worker in an establishment of type (x,s) is less than the shadow value of unemployment. As a consequence, the social planner reduces the employment level of the establishment to the point at which he is indifferent between making a further reduction in employment or not.

The optimal help-wanted ads creation rule $a_t(x,s)$ is also easily obtained. It is characterized by a threshold level $\underline{x}_t(s)$ that satisfies the following condition:

$$\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\underline{x}_t(s), s'\right]Q\left(s,s'\right) = \gamma_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}. \tag{4.5}$$

The help-wanted ads creation rule is then given by

$$a_t(x,s)\,m(A_t,U_t) = \left\{\begin{array}{c} 0, \text{ if } \min\left\{\bar{x}_t(s), x\right\} > \underline{x}_t(s) \\ (1-\delta)\left[\underline{x}_t(s) - \min\left\{\bar{x}_t(s), x\right\}\right], \text{ otherwise} \end{array}\right\}. \tag{4.6}$$

The shadow value of a next-period hiring opportunity γ_t satisfies the following condition:

$$\gamma_t = \frac{k}{m\left(A_t, U_t\right) + A_t m_A\left(A_t, U_t\right)}. \tag{4.7}$$

Observe that $m + A.m_A > 0$ is the marginal product of a help-wanted ad in creating next-period hiring opportunities. Since k is the cost of creating a help-wanted ad, equation (4.7) states that the shadow value of a next-period hiring opportunity equals its cost of production.

The shadow value of an unemployed worker θ_t is given by

$$\theta_t = c_t^{\sigma}\varphi + \gamma_t A_t m_U\left(A_t, U_t\right) + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}. \tag{4.8}$$

Observe that Am_U is the marginal product of an unemployed worker in creating next-period hiring opportunities, which are valued at the shadow price γ_t. Thus equation (4.8) states that the shadow value of an unemployed worker θ_t is equal to his value of leisure expressed in consumption units $c_t^{\sigma}\varphi$, plus his shadow value in creating next period hiring opportunities $\gamma_t A_t m_U$, plus the discounted shadow value of being an unemployed worker during the following period.

Consumption c_t is given by

$$c_t = \sum_s \left[s_t F\left[n_t\left(x, s\right)\right] - k a_t\left(x, s\right)\right]\mu_t\left(dx, s\right), \tag{4.9}$$

aggregate help-wanted ads A_t are given by

$$A_t = \sum_s a_t\left(x, s\right)\mu_t\left(dx, s\right), \tag{4.10}$$

and aggregate unemployment U_t is

$$U_t = 1 - \sum_s n_t\left(x, s\right)\mu_t\left(dx, s\right). \tag{4.11}$$

Finally, the sequence of measures μ_t must satisfy

$$\mu_{t+1}\left(X', s'\right) = \sum_s \int_{B_t(s)} Q(s, s')d\mu_t + \varsigma\psi\left(s'\right) I\left(X'\right), \tag{4.12}$$

where $I\left(X'\right)$ is an indicator function that takes a value equal to one if $0 \in X'$ and a value of zero, otherwise, and $B_t\left(s\right)$ is the set of all x such that $\left(1-\delta\right)x + a_t\left(x, s\right)m\left(A_t, U_t\right)$ lies in the Borel set

X'.

An *efficient allocation* is a sequence $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t\}_{t=0}^{\infty}$ such that equations (4.1)-(4.12) hold, with μ_0 given. Observe that, from the concavity of the planner's problem, a unique efficient allocation exists.

5. A competitive equilibrium

In what follows I specify a competitive equilibrium in which workers are bought and sold as capital goods.[3] Establishments buy workers in the hiring market, sell them in the firing market and buy hiring opportunities from matching companies. Households buy workers in the firing market, sell them in the hiring market, rent unemployed workers to the matching companies, and receive job-opportunities for free. Since at every period of time, there is always some positive number of establishments that do not use all the hiring opportunities available to them (because their productivity shocks have changed), the price of a worker in the hiring market is always brought down to the price in the firing market. Hereon, I will refer to this single price as the *price of a worker* p_t^w in the *labor market*.

The date t problem of an establishment that has e workers carried over from the previous period, m hiring opportunities and current productivity level s is given by:

$$\begin{aligned} J_t(e,m,s) &= \max_{n_t, m_{t+1}, h_t, f_t} \Big\{ sF(n_t) + p_t^w (f_t - h_t) - q_t^h m_{t+1} \\ &\quad + \frac{1}{1+i_t} \sum_{s'} J_{t+1}\left[(1-\delta) n_t, m_{t+1}, s'\right] Q(s,s') + \frac{1}{1+i_t} p_{t+1}^w \delta n_t \Big\} \end{aligned} \tag{5.1}$$

subject to

$$\begin{aligned} n_t &= e + h_t - f_t \\ h_t &\leq m \end{aligned}$$

where p_t^w is the price of a worker, q_t^h is the price of a hiring opportunity in the next period, n_t is the employment level of the establishment, h_t is the amount of hiring, f_t is the amount of firing, m_{t+1}

[3] I choose this unusual specification because it easy to describe. It turns out that it is equivalent to a much more complicated specification in which households and establishments trade binding state contingent employment contracts (see Alvarez and Veracierto [1] for this alternative formulation).

is the number of hiring opportunities that the establishment purchases, i_t is the interest rate, and $J_t(e,m,s)$ is the present expected discounted value at date t of an establishment of type (e,m,s) after quits have taken place. Observe that an establishment maximizes the present discounted value of its profits, which are given by the sum of its output and its net sale of workers, minus the value of the hiring opportunities that the establishment buys in the matching market. Also observe that the amount of hiring cannot exceed the number of hiring opportunities available at the beginning of the period.

It is straightforward to show that $J_{et} = J_{mt} + p_t^w$. Moreover, J_{et} and J_{mt} depend on (e,m) only through the sum $x = e+m$. Using these properties, the first order conditions for the establishment's problem can be written as follows:

$$\begin{aligned} J_{et}(x,s) &= \max\left\{ p_t^w, sF'(x) + \frac{1}{1+i_t} p_{t+1}^w \delta \right. \\ &\quad \left. + (1-\delta)\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)x + m_{t+1}(x,s), s'\right] Q\left(s,s'\right) \right\}, \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)n_t(x,s) + m_{t+1}(x,s), s'\right] Q\left(s,s'\right) &\leq q_t^h + \frac{1}{1+i_t} p_{t+1}^w, \\ \text{with equality if } m_{t+1}(x,s) &> 0. \end{aligned} \tag{5.3}$$

The establishment's employment rule $n_t(x,s)$ is similar to that of the social planner. It is characterized by a threshold level $\bar{x}_t(s)$ that satisfies the following condition:

$$\begin{aligned} p_t^w &= sF'(\bar{x}_t(s)) + \frac{1}{1+i_t} p_{t+1}^w \delta \\ &\quad + (1-\delta)\min\left[\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\bar{x}_t(s), s'\right] Q\left(s,s'\right), q_t^h + \frac{1}{1+i_t} p_{t+1}^w \right]. \end{aligned} \tag{5.4}$$

The optimal employment rule is then given by

$$n_t(x,s) = \min\left\{\bar{x}_t(s), x\right\}. \tag{5.5}$$

The optimal hiring opportunities creation rule is also easily obtained. It is characterized by a

threshold level $\underline{x}_t(s)$ that satisfies the following condition:

$$\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\,\underline{x}_t(s), s'\right] Q\left(s, s'\right) = q_t^h + \frac{1}{1+i_t} p_{t+1}^w. \tag{5.6}$$

The hiring opportunities creation rule is given by

$$m_{t+1}(x,s) = \left\{ \begin{array}{c} 0, \text{ if } \min\left\{\bar{x}_t(s), x\right\} > \underline{x}_t(s) \\ (1-\delta)\left[\underline{x}_t(s) - \min\left\{\bar{x}_t(s), x\right\}\right], \text{ otherwise} \end{array} \right\}. \tag{5.7}$$

The problem of a household that had u_{t-1} unemployed workers during the previous period and that during the current period has j_t job opportunities and b_t bond holdings is the following:

$$H_t(u_{t-1}, j_t, b_t) = \max\left\{\frac{c_t^{1-\sigma}-1}{1-\sigma} + \varphi u_t + \beta H_{t+1}(u_t, j_{t+1}, b_{t+1})\right\}$$

subject to

$$\begin{aligned} c_t + b_{t+1} &\leq p_t^m u_t + p_t^w (h_t - f_t) + (1+i_{t-1}) b_t + \Pi_t \\ u_t &= (u_{t-1} - j_t) + f_t + (j_t - h_t) \\ h_t &\leq j_t \\ f_t &\leq 1 - u_{t-1} \\ j_{t+1} &= u_t \frac{M_{t+1}}{U_t} \end{aligned}$$

where u_t is the number of unemployed members during the current period, h_t are the household members sold in the hiring market, f_t are the household members bought in the firing market and Π_t are the profits of all the establishments in the economy. The first constraint is the household's budget constraint. Observe that the household receives income from renting unemployed members to the matching companies $p_t^m u_t$, from the selling of workers in the hiring market $p_t^w h_t$, from interest and principal payments on its bond holdings $(1+i_{t-1})b_t$ and from the establishments' profits Π_t. The household spends its income in purchasing workers in the firing market $p_t^w f_t$, on consumption c_t and on next period bond holdings b_{t+1}.The second constraint states that the number of unemployed members during the current period u_t is given by the number of unemployed members in the previous period that were left unmatched $u_{t-1} - j_t$, by the household members bought in the firing market f_t, and by the household members left unsold in the hiring market

$j_t - h_t$. The third constraint states that the number of household members sold in the hiring market h_t cannot exceed the job-opportunities available to the household at the beginning of the period j_t. The fourth constraint states that the number of household members bought in the firing market f_t cannot exceed the number of household members that were employed during the previous period $1-u_{t-1}$. The last constraint states that the number of job opportunities that the household will have the following period is given by the number of unemployed members during the current period times the rate at which the representative firm creates job-opportunities.

Observe that the third and fourth constraints must hold with strict inequality at equilibrium. For these constraints not to bind, the household must view its problem as being independent of its job-opportunities j_t. This requires that

$$p_t^w = c_t^\sigma \varphi + p_t^m + \beta \left(\frac{c_t}{c_{t+1}}\right)^\sigma p_{t+1}^w. \tag{5.8}$$

This condition states that the household must be indifferent between selling workers in the hiring market and keeping them unemployed. The left hand side is the price p_t^w that the household can get by selling an unemployed worker in the hiring market. The right hand side is the sum of the value of leisure expressed in consumption units $c_t^\sigma \varphi$, the rental price of an unemployed worker in the matching industry p_t^m, and the savings from buying one less unemployed worker in the firing market at the beginning of the following period (expressed in current consumption units) $\beta \left(\frac{c_t}{c_{t+1}}\right)^\sigma p_{t+1}^w$. Also observe that the solution to the household's problem requires that

$$\frac{1}{1+i_t} = \beta \left(\frac{c_t}{c_{t+1}}\right)^\sigma, \tag{5.9}$$

i.e. that the interest rate be equal to the marginal rate of substitution between current consumption and future consumption.

The date t problem of the representative matching company is given by:

$$\max \left\{ q_t^h M_{t+1} - p_t^m U_t - k A_t \right\}$$

subject to

$$M_{t+1} = A_t m(A_t, U_t).$$

That is, the matching company obtains revenues $q_t^h M_{t+1}$ from selling next period hiring opportunities, pays $p_t^m U_t$ for renting unemployed workers and pays kA_t for creating help-wanted ads. The

first order conditions to this problem are the following:

$$q_t^h \left[m(A_t, U_t) + A_t m_A \left(A_t, U_t \right) \right] = k, \tag{5.10}$$

$$q_t^h A_t m_U \left(A_t, U_t \right) = p_t^m. \tag{5.11}$$

The first equation states that the marginal revenue product of a help-wanted ad equals its marginal cost k. The second equation states that the marginal revenue product of an unemployed worker equals its marginal cost p_t^m.

A *competitive equilibrium* is a sequence $\left\{ c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, m_{t+1}, p_t^w, q_t^h, p_t^m, i_t, \mu_t \right\}_{t=0}^{\infty}$ such that $b_t = 0$,

$$a_t \left(x, s \right) = \frac{m_{t+1} \left(x, s \right)}{m \left(A_t, U_t \right)},$$

and equations (5.2)-(5.11) and (4.9)-(4.12) are satisfied every period t, with μ_0 given.

The following proposition states that every competitive equilibrium allocation is efficient.

Proposition 5.1. *(First Welfare Theorem) Let* $\left\{ c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, m_{t+1}, p_t^w, q_t^h, p_t^m, i_t, \mu_t \right\}_{t=0}^{\infty}$ *be a competitive equilibrium. Let*

$$\begin{aligned} \theta_t &= p_t^w, \\ v_t &= J_{et}, \\ \gamma_t &= q_t^h. \end{aligned}$$

Then, $\left\{ c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t \right\}_{t=0}^{\infty}$ *is an efficient allocation.*

Since there is a unique efficient allocation, a corollary of this proposition is that there is a unique competitive equilibrium. The following proposition states that the Second Welfare Theorem also holds.

Proposition 5.2. *(Second Welfare Theorem) Let* $\left\{ c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t \right\}_{t=0}^{\infty}$ *be an efficient*

allocation. Let

$$\begin{aligned} p_t^w &= \theta_t, \\ q_t^h &= \gamma_t, \\ p_t^m &= \gamma_t A_t m_U\left(A_t, U_t\right) \\ J_{et} &= v_t \\ m_{t+1}\left(x,s\right) &= a_t\left(x,s\right) m\left(A_t, U_t\right) \\ b_t &= 0 \\ \frac{1}{1+i_t} &= \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \end{aligned}$$

Then, $\left\{c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, m_{t+1}, p_t^w, q_t^h, p_t^m, i_t, \mu_t\right\}_{t=0}^{\infty}$ *is a competitive equilibrium.*

Both propositions follow from comparing first order conditions for the social planner's problem and the competitive equilibrium.

6. A competitive equilibrium with congestion externalities

In the competitive equilibrium described in the previous section the matching companies charge establishments a positive fee q_t^h for obtaining a hiring opportunity but charge nothing to unemployed workers for obtaining a job-opportunity. This is a realistic feature of that type of equilibrium. In the actual world private employment agencies, web-based job-search engines, staffing companies and executive search firms charge a positive fee to potential employers (their clients) but nothing to the potential employees that they place. The same is true with advertisement companies that provide no screening at all, such as newspapers, television and radio stations.[4] However, a difficulty with identifying these type of firms with the matching companies of the previous section is that none of them pay unemployed workers a positive fee p^m for attempting to place them. This leads me to consider an alternative competitive equilibrium specification.

In this section, the matching technology is assumed to operate at the aggregate level. Moreover, all help-wanted ads and all unemployed workers in the economy are assumed to be inputs to it. Since no decision maker internalizes the operation of the matching technology, standard congestion

[4]Newspapers appear to violate this claim, since unemployed workers must pay for the newspapers that they read. However, the price that they pay is the same as the price that people not looking for a job pay. This suggests that newspapers charge for the general-news that they provide and not for theirs help-wanted adds.

externalities arise. Establishments buy workers in the hiring market, sell them in the firing market and buy help-wanted ads from job-posting companies. Households buy workers in the firing market and sell them in the hiring market. Since not all hiring opportunities are used in equilibrium, the price of a worker p_t^w must be the same in the hiring market and in the firing market.

The date t problem of an establishment that has e workers carried over from the previous period, m hiring opportunities and current productivity level s is given by

$$\begin{aligned} J_t(e,m,s) &= \max_{n_t,h_t,f_t,a_t} \{sF(n_t) + p_t^w(f_t - h_t) - p_t^a a_t \\ &\quad + \frac{1}{1+i_t}\sum_{s'} J_{t+1}\left[(1-\delta)n_t, m_{t+1}, s'\right] Q(s,s') + \frac{1}{1+i_t} p_{t+1}^w \delta n_t \Bigg\} \end{aligned}$$

subject to

$$\begin{aligned} n_t &= e + h_t - f_t \\ h_t &\leq m \\ m_{t+1} &= a_t m(A_t, U_t) \end{aligned}$$

where h_t is the number of workers bought in the hiring market, f_t is the number of workers sold in the firing market, a_t is the number of help-wanted ads purchased, p_t^a is the price of a help-wanted ad, i_t is the interest rate and m_{t+1} is the number of hiring opportunities that the establishment generates. Observe that the number of hiring opportunities that the establishment generates m_{t+1} is proportionate to the number of help-wanted ads purchased a_t, with constant of proportionality given by the aggregate effectiveness of help-wanted ads in generating hiring opportunities $m(A_t, U_t)$. The establishment takes A_t and U_t as given. Also, observe that the amount of hiring h_t cannot exceed the hiring opportunities available to the establishment at the beginning of the period.

It is straightforward to show that $J_{et} = J_{mt} + p_t^w$. Moreover, J_{et} and J_{mt} depend on (e,m) only through the sum $x = e+m$. Using these properties, the first order conditions for the establishment's problem can be written as follows:

$$\begin{aligned} J_{et}(x,s) &= \max\Bigg\{ p_t^w, sF'(x) + \frac{1}{1+i_t} p_{t+1}^w \delta \\ &\quad + (1-\delta)\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)x + a_t(x,s)m(A_t,U_t), s'\right] Q(s,s') \Bigg\} \end{aligned} \tag{6.1}$$

and

$$\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\, n_t(x,s) + a_t(x,s)\, m(A_t,U_t), s'\right] Q\left(s,s'\right) \leq \frac{p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w$$
$$\text{with equality if } a_t(x,s) > 0. \tag{6.2}$$

The establishment's employment rule $n_t(x,s)$ has the same form as in the previous sections. It is characterized by a threshold level $\bar{x}_t(s)$ that satisfies the following condition:

$$\begin{aligned} p_t^w &= sF'\left(\bar{x}_t(s)\right) + \frac{1}{1+i_t} p_{t+1}^w \delta \\ &\quad + (1-\delta)\min\left[\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\,\bar{x}_t(s), s'\right] Q\left(s,s'\right), \frac{p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w\right] \end{aligned} \tag{6.3}$$

The optimal employment rule is then given by

$$n_t(x,s) = \min\left\{\bar{x}_t(s), x\right\}. \tag{6.4}$$

The optimal hiring opportunities creation rule also has the same form as before. It is characterized by a threshold level $\underline{x}_t(s)$ that satisfies the following condition:

$$\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\,\underline{x}_t(s), s'\right] Q\left(s,s'\right) = \frac{p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w. \tag{6.5}$$

The help-wanted ads creation rule is given by

$$a_t(x,s)\, m(A_t,U_t) = \left\{ \begin{array}{c} 0, \text{ if } \min\left\{\bar{x}_t(s), x\right\} > \underline{x}_t(s) \\ (1-\delta)\left[\underline{x}_t(s) - \min\left\{\bar{x}_t(s), x\right\}\right], \text{ otherwise} \end{array} \right\}. \tag{6.6}$$

The representative job-posting company solves the following static profit maximization problem:

$$\max\left\{p_t^a A_t - k A_t\right\}.$$

where p_t^a is the price of a help-wanted ad, k is the cost of producing a help-wanted ad, and A_t are the total help-wanted ads produced. Thus, at equilibrium we must have that

$$p_t^a = k \tag{6.7}$$

The date t problem of a household that had u_{t-1} unemployed workers during the previous period and that during the current period has j_t job opportunities and b_t bond holdings is given by

$$H_t\left(u_{t-1}, j_t, b_t\right) = \max\left\{\frac{c_t^{1-\sigma}-1}{1-\sigma} + \varphi u_t + \beta H_{t+1}\left(u_t, j_{t+1}, b_{t+1}\right)\right\}$$

subject to

$$\begin{aligned} c_t + b_{t+1} &\leq p_t^w\left(h_t - f_t\right) + \left(1 + i_{t-1}\right) b_t + \Pi_t \\ u_t &= \left(u_{t-1} - j_t\right) + f_t + \left(j_t - h_t\right), \\ h_t &\leq j_t, \\ f_t &\leq 1 - u_{t-1}, \\ j_{t+1} &= u_t \frac{M_{t+1}}{U_t} \end{aligned}$$

where u_t is the number of unemployed members during the current period, h_t are the household members sold in the hiring market, f_t are the household members bought in the firing market, and Π_t are the profits of all the establishments in the economy. The household takes the aggregate effectiveness at which unemployed workers reach the hiring market.M_{t+1}/U_t as given. Observe that the third and fourth constraints must hold with strict inequality at equilibrium. For these constraints to be non-binding at every period t, the household must view its problem as being independent of its job-opportunities j_t. This requires that

$$p_t^w = c_t^\sigma \varphi + \beta\left(\frac{c_t}{c_{t+1}}\right)^\sigma p_{t+1}^w \tag{6.8}$$

for every t. This condition states that the household must be indifferent between selling workers in the hiring market and keeping them unemployed. Also observe that the solution to the household's problem requires that

$$\frac{1}{1+i_t} = \beta\left(\frac{c_t}{c_{t+1}}\right)^\sigma. \tag{6.9}$$

A *competitive equilibrium with congestion externalities* is a sequence $\{c_t,\, b_t,\, A_t,\, U_t,\, J_{et},\, n_t,\, a_t,\, p_t^w,\, p_t^a,\, i_t,\, \mu_t\}_{t=0}^{\infty}$ such that $b_t = 0$ and equations (6.1)-(6.9) and (4.9)-(4.12) are satisfied every period t, with μ_0 given.

6.1. A myopic-planner characterization

Consider the problem of a social planner that maximizes (2.1) subject to equations (3.1)-(3.4) and the following employment feasibility condition at the establishment level:

$$n_t\left(j, s_j^t, e\right) \leq (1-\delta) n_{t-1}\left(j, s_j^{t-1}, e\right) + a_{t-1}\left(j, s_j^{t-1}, e\right) \bar{m}_{t-1}, \text{ for } j < t. \tag{6.10}$$

The planner takes the sequence $\{\bar{m}_t\}_{t=0}^{\infty}$ as given. Observe that equation (6.10) is identical to equation (3.5) except that that the arguments of m are taken as given. Given that this social planner does not take into account how his decisions affect m, I will refer to him as a *myopic-planner*.

A solution to the myopic planner's problem can be described by a sequence $\{c_t,\, A_t,\, U_t,\, v_t,\, n_t,\, a_t,\, \theta_t,\, \mu_t\}_{t=0}^{\infty}$ that satisfies the following conditions.

The shadow value of a worker at an establishment of type (x, s) must satisfy the following equation:

$$\begin{aligned} v_t(x,s) \;=\; & \max\left\{\theta_t, s_t F'(x) + \delta\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}\right. \\ & \left. + (1-\delta)\,\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\,x + a_t(x,s)\,\bar{m}_t, s'\right] Q\left(s, s'\right)\right\} \end{aligned} \tag{6.11}$$

The optimal number of help-wanted ads created for an establishment of type (x, s) is characterized by the following condition:

$$\begin{aligned} \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\,n_t(x,s) + a_t(x,s)\,\bar{m}_t, s'\right] Q\left(s, s'\right) \;&\leq\; \frac{k}{\bar{m}_t} + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}, \\ \text{with equality if } a_t(x,s) \;&>\; 0. \end{aligned} \tag{6.12}$$

The optimal employment rule $n_t(x, s)$ is characterized by a threshold level $\bar{x}_t(s)$ that satisfies the following condition:

$$\begin{aligned} \theta_t \;=\; & s_t F'\left(\bar{x}_t(s)\right) + \delta\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1} \\ & + (1-\delta)\min\left[\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\,\bar{x}_t(s)\right] Q\left(s, s'\right), \frac{k}{\bar{m}_t} + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}\right]. \end{aligned} \tag{6.13}$$

The optimal employment rule is then given by

$$n_t(x,s) = \min\{\bar{x}_t(s), x\}. \tag{6.14}$$

The optimal hiring opportunities creation rule $a_t(x,s)$ is characterized by a threshold level $\underline{x}_t(s)$ that satisfies the following condition:

$$\beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\sum_{s'} v_{t+1}\left[(1-\delta)\,\underline{x}_t(s), s'\right]Q(s,s') = \frac{k}{\bar{m}_t} + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}. \tag{6.15}$$

The help-wanted ads creation rule is given by

$$a_t(x,s)\,\bar{m}_t = \left\{\begin{array}{c} 0, \text{ if } \min\{\bar{x}_t(s), x\} > \underline{x}_t(s) \\ (1-\delta)\left[\underline{x}_t(s) - \min\{\bar{x}_t(s), x\}\right], \text{ otherwise} \end{array}\right\}. \tag{6.16}$$

The shadow value of an unemployed worker satisfies that

$$\theta_t = c_t^{\sigma}\varphi + \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\theta_{t+1}, \text{ for } t \geq 0 \tag{6.17}$$

A *myopically-efficient allocation with respect to* $\{\bar{m}_t\}_{t=0}^{\infty}$ is a sequence $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \mu_t\}_{t=0}^{\infty}$ such that equations (6.11)-(6.17) and (4.9)-(4.12) hold, with μ_0 given.

In what follows, I provide a modified version of the Welfare Theorems, which I will refer to as the *Myopic Welfare Theorems*.

Proposition 6.1. *(First Myopic Welfare Theorem) Let* $\{c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, p_t^w, p_t^a, i_t, \mu_t\}_{t=0}^{\infty}$ *be a competitive equilibrium with congestion externalities. Let*

$$\begin{aligned} \bar{m}_t &= m(A_t, U_t) \\ \theta_t &= p_t^w, \\ v_t &= J_{et} \end{aligned}$$

Then, $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \mu_t\}_{t=0}^{\infty}$ *is a myopically-efficient allocation with respect to* $\{\bar{m}_t\}_{t=0}^{\infty}$.

Proposition 6.2. *(Second Myopic Welfare Theorem) Let* $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \mu_t\}_{t=0}^{\infty}$ *be a myopically-*

efficient allocation with respect to $\{\bar{m}_t\}_{t=0}^{\infty}$. *Suppose that*

$$\bar{m}_t = m(A_t, U_t), \text{ for every } t.$$

Let

$$\begin{aligned} p_t^w &= \theta_t, \\ J_{et} &= v_t \\ p_t^a &= k \\ b_t &= 0 \\ \frac{1}{1+i_t} &= \beta \left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \end{aligned}$$

Then, $\{c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, p_t^w, p_t^a, i_t, \mu_t\}_{t=0}^{\infty}$ *is a competitive equilibrium with congestion externalities*

The characterization of a competitive equilibrium with congestion externalities as a myopic-efficient allocation will turn out to be extremely usefule in computations.

7. The optimal policy regime

In this section I introduce government policies to the competitive equilibrium with congestion externalities of the previous section. In particular, I introduce a tax τ_t to help-wanted ads and a subsidy ρ_t to unemployed workers. Any negative (positive) difference between the tax revenues and the subsidy payments associated with those tax rates are rebated (taxed) to households in a lump-sum way. The purpose is to look for government interventions that will attain the first best allocation described in Section 4.

The date t problem of an establishment that has e workers carried over from the previous period, m hiring opportunities and current productivity level s is now given by

$$\begin{aligned} J_t(e, m, s) = \max_{n_t, h_t, f_t, a_t} \{ & sF(n_t) + p_t^w (f_t - h_t) - (1+\tau_t) p_t^a a_t \\ & + \frac{1}{1+i_t} \sum_{s'} J_{t+1} \left[(1-\delta) n_t, m_{t+1}, s' \right] Q(s, s') + \frac{1}{1+i_t} p_{t+1}^w \delta n_t \Bigg\} \end{aligned}$$

subject to

$$\begin{aligned} n_t &= e + h_t - f_t \\ h_t &\leq m \\ m_{t+1} &= a_t m(A_t, U_t) \end{aligned}$$

where the purchases of help-wanted ads are now taxed at the rate τ_t.

The first order conditions for the establishment's problem are the following:

$$\begin{aligned} J_{et}(x,s) &= \max\Bigg\{ p_t^w, sF'(x) + \frac{1}{1+i_t} p_{t+1}^w \delta \\ &\quad + (1-\delta)\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)x + a_t(x,s)\, m(A_t,U_t), s'\right] Q(s,s') \Bigg\} \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} \frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\, n_t(x,s) + a_t(x,s)\, m(A_t,U_t), s'\right] Q(s,s') &\leq \frac{(1+\tau_t)\, p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w \\ \text{with equality if } a_t(x,s) &> 0 \end{aligned} \tag{7.2}$$

The establishment's employment rule $n_t(x,s)$ has the same form as in the previous sections. It is characterized by a threshold level $\bar{x}_t(s)$ that satisfies the following condition:

$$\begin{aligned} p_t^w &= sF'(\bar{x}_t(s)) + \frac{1}{1+i_t} p_{t+1}^w \delta \\ &\quad + (1-\delta)\min\left[\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\bar{x}_t(s), s'\right] Q(s,s'), \frac{(1+\tau_t)\, p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w \right] \end{aligned} \tag{7.3}$$

The optimal employment rule is then given by

$$n_t(x,s) = \min\left\{\bar{x}_t(s), x\right\}. \tag{7.4}$$

The optimal hiring opportunities creation rule also has the same form as before. It is characterized by a threshold level $\underline{x}_t(s)$ that satisfies the following condition:

$$\frac{1}{1+i_t}\sum_{s'} J_{e,t+1}\left[(1-\delta)\underline{x}_t(s), s'\right] Q(s,s') = \frac{(1+\tau_t)\, p_t^a}{m(A_t,U_t)} + \frac{1}{1+i_t} p_{t+1}^w \tag{7.5}$$

The help-wanted ads creation rule is given by

$$a_t(x,s)\,m(A_t,U_t) = \left\{ \begin{array}{c} 0, \text{ if } \min\{\bar{x}_t(s), x\} > \underline{x}_t(s) \\ (1-\delta)\left[\underline{x}_t(s) - \min\{\bar{x}_t(s), x\}\right], \text{ otherwise} \end{array} \right\}. \tag{7.6}$$

The representative job-posting company solves the following static profit maximization problem:

$$\max\left\{p_t^a A_t - kA_t\right\}.$$

where p_t^a is the price of a help-wanted ad, k is the cost of producing a help-wanted ad, and A_t are the total help-wanted ads produced. Thus, at equilibrium we must have that

$$p_t^a = k \tag{7.7}$$

The date t problem of a household that had u_{t-1} unemployed members during the previous period and that during the current period has j_t job opportunities and b_t bond holdings is given by

$$H_t(u_{t-1}, j_t, b_t) = \max\left\{\frac{c_t^{1-\sigma}-1}{1-\sigma} + \varphi u_t + \beta H_{t+1}(u_t, j_{t+1}, b_{t+1})\right\}$$

subject to

$$\begin{aligned} c_t + b_{t+1} &\leq \rho_t u_t + p_t^w(h_t - f_t) + (1+i_{t-1})\,b_t + \Pi_t + T_t \\ u_t &= (u_{t-1} - j_t) + f_t + (j_t - h_t), \\ h_t &\leq j_t, \\ f_t &\leq 1 - u_{t-1}, \\ j_{t+1} &= u_t \frac{M_{t+1}}{U_t} \end{aligned}$$

where unemployed workers are now subsidized at the rate ρ_t and the household now faces a lump-sum tax T_t. Since the second and third constraints must hold with strict inequality at equilibrium, the price of a worker must satisfy the following condition:

$$p_t^w = c_t^\sigma \varphi + \rho_t + \beta\left(\frac{c_t}{c_{t+1}}\right)^\sigma p_{t+1}^w. \tag{7.8}$$

Also observe that the solution to the household's problem requires that

$$\frac{1}{1+i_t} = \beta \left(\frac{c_t}{c_{t+1}} \right)^{\sigma}. \tag{7.9}$$

The government balances its budget period-by-period. This means that the lump-sum taxes T_t must be given by

$$T_t = \rho_t U_t - \tau_t p_t^a A_t. \tag{7.10}$$

A *competitive equilibrium with externalities and policy intervention* is a sequence $\{c_t,\, b_t,\, A_t,\, U_t,\, J_{et},\, n_t,\, a_t,\, p_t^w,\, p_t^a,\, i_t,\, \mu_t,\, \tau_t,\, \rho_t,\, T_t\}_{t=0}^{\infty}$ such that $b_t = 0$ and equations (7.1)-(7.10) and (4.9)-(4.12) are satisfied every period t, with μ_0 given.

The following proposition characterizes the optimal policy intervention, i.e., the policy regime that attains the first best allocation.

Proposition 7.1. *(Necessity of optimal policy regime) Let $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t\}_{t=0}^{\infty}$ be the unique efficient allocation (as defined in Section 4). Define,*

$$\begin{aligned}
\tau_t &= -\frac{A_t m_A\left(A_t, U_t\right)}{m\left(A_t, U_t\right) + A_t m_A\left(A_t, U_t\right)}, \\
\rho_t &= \frac{k A_t m_U\left(A_t, U_t\right)}{m\left(A_t, U_t\right) + A_t m_A\left(A_t, U_t\right)}, \\
T_t &= 0 \\
p_t^w &= \theta_t \\
p_t^a &= k \\
J_{et} &= v \\
b_t &= 0 \\
\frac{1}{1+i_t} &= \beta \left(\frac{c_t}{c_{t+1}} \right)^{\sigma}
\end{aligned}$$

Then, $\{c_t, b_t, A_t, U_t, J_{et}, n_t, a_t, p_t^w, p_t^a, i_t, \mu_t, \tau_t, \rho_t, T_t\}_{t=0}^{\infty}$ is a competitive equilibrium with externalities and policy intervention.

The converse is also true.

Proposition 7.2. *(Sufficiency of optimal policy regime) Let $\{c_t,\, b_t,\, A_t,\, U_t,\, J_{et},\, n_t,\, a_t,\, p_t^w,\, p_t^a,\, i_t,$*

μ_t, τ_t, ρ_t, $T_t\}_{t=0}^{\infty}$ be a competitive equilibrium with externalities and policy intervention such that

$$\begin{aligned} \tau_t &= -\frac{A_t m_A(A_t, U_t)}{m(A_t, U_t) + A_t m_A(A_t, U_t)}, \\ \rho_t &= \frac{k A_t m_U(A_t, U_t)}{m(A_t, U_t) + A_t m_A(A_t, U_t)}, \\ T_t &= 0 \end{aligned} \tag{7.11}$$

Let

$$\begin{aligned} \theta_t &= p_t^w \\ \gamma_t &= \frac{k}{m(A_t, U_t) + A_t m_A(A_t, U_t)} \\ v_t &= J_{et} \end{aligned}$$

Then, $\{c_t, A_t, U_t, v_t, n_t, a_t, \theta_t, \gamma_t, \mu_t\}_{t=0}^{\infty}$ is the unique efficient allocation

Observe that the optimal policy regime is fully funded: It does not require lump-sum taxes or subsidies to implement it. It is also straightforward to verify that when the matching function has the following Cobb-Douglas functional form

$$M_t = \Omega U_t^{\phi} A_t^{1-\phi}, \tag{7.12}$$

the optimal policy simplifies to:

$$\begin{aligned} \tau_t &= \frac{\phi}{1-\phi}, \\ \rho_t &= k\frac{\phi}{1-\phi}\frac{A_t}{U_t}. \end{aligned}$$

That is, while the optimal tax rate on help-wanted ads is constant, the unemployment subsidy varies inversely with the degree of market tightness.

8. Calibration

In this section I calibrate the steady state of the competitive equilibrium with externalities and no policy interventions described in Section 6 to long-run U.S. observations. I believe that this type of equilibria provides a better representation of the U.S. economy than an efficient equilibrium

because in the real world we do not observe that unemployed workers receive payments from matching companies for entering the technologies that they operate (as the equilibrium in Section 5 requires) nor we observe that the government imposes taxes on the creation of help-wanted ads (as the optimal policy of Section 7 specifies). The next section will explore the effects of introducing an optimal policy regime to the inefficient economy calibrated in this section.

I choose the model time period to be two-weeks to accommodate for the relatively short average durations of unemployment and vacancies in the U.S. economy. Calibrating to an annual interest rate of 4 percent, which is a standard value in the macro literature, then requires a time discount factor β equal to 0.99835.

The preference parameter σ, which determines the elasticity of intertemporal substitution, is taken as a free parameter. However, I will restrict attention to two values: $\sigma = 0$ and $\sigma = 1$. The first value is often used in the search literature based on the Mortensen and Pissarides [8] model (e.g. Hall [5], Hagedorn and Manovskii [4] and Shimer [10]). The second value, which is consistent with the stylized growth facts, is generally used in the macro literature. In what follows, I describe the calibration strategy assuming that $\sigma = 0$. At the end of the section, I discuss how to accommodate to the other case.

It is straightforward to verify that doubling the utility of leisure φ, the values of all idiosyncratic productivity levels z, and the cost of creating a help-wanted ad k doubles the units in which consumption and output are measured but leaves all other equilibrium variables unchanged. As a consequence, I normalize the utility of leisure φ to one.

I assume that the production function has the following functional form:

$$y_t = s_t n_t^{\alpha},$$

where $0 < \alpha < 1$. Following the macro literature I choose the curvature parameter α to reproduce a labor share in National Income $(1 - \beta)p^w N/Y$ equal to 0.64.

The values for the idiosyncratic productivity levels s, the distribution over initial productivity levels ψ and the transition matrix Q are key determinants of the job-flows generated by the model. As a consequence I choose them to reproduce observations from the Business Employment Dynamics (BED) data set, which is a virtual census of establishments level dynamics. Since BED data across establishment sizes can be found for the nine employment ranges shown in the first column of Table 1, I restrict the idiosyncratic productivity levels s to take nine positive values and choose them so

that all establishments with a same idiosyncratic productivity level choose employment levels in the same range.

The average size of new entrants can be obtained by dividing the total gross job gains at opening establishments by the total number of opening establishments. Using data between 1992:Q3 and 2005:Q4, I find that the average size of new entrants is equal to 5.3 employees. Since this is a small number, I restrict the distribution over initial productivity levels ψ to put positive mass on only the two lowest values of s and choose $\psi(s_1)$ to reproduce that average size.

Similarly, the average size at exit can be obtained by dividing the total gross job losses at closing establishments by the total number of closing establishments. Using data for the same time period as above, I find that the average size at exit is equal to 5.2 employees. Since this is also a small number, I restrict the probabilities of transiting to a zero productivity level $Q(s,0)$ to take positive values only at the two lowest values of s and choose $Q(s_1,0)/Q(s_2,0)$ to reproduce that average size. The level for $Q(s_1,0)$ is then chosen to reproduce the average quarterly rate of gross job losses due to closing establishments (JLD) over the same time period, which is equal to 1.6%.[5]

The rest of the transition matrix Q is parameterized with enough flexibility to reproduce important establishment level observations. The only restriction that I impose is that $Q(s_i, s_j) > 0$ only if $j = i-1$, $j = i$ or $j = i+1$. Since the rows of Q add to one this introduces 16 parameters (2 parameters each, for $i = 2, ..., 8$, and 1 parameter each, for $i = 1, 9$). Eight of these parameters are chosen to reproduce the shares in total employment across size classes (which provide eight independent observations). The other eight parameters are chosen to reproduce the shares in total gross job gains across size classes (which also provide eight independent observations). I must point out that the BED does not tabulate statistics across size classes in its regular reports. However, these statistics can be found in Okolie [9] (Tables 1 and 3) for the first two quarters of 2000. These statistics together with the corresponding model statistics are shown in the first panel of Table 1. The second panel reports the average sizes at entry and exit both in the model and the data. We see that the model does a good job at reproducing these observations. As a test of the model, Table 1 also includes the shares in total gross job losses across size classes for the first two quarters of 2000 in Okolie [9], and the average quarterly rates of gross job gains due to expanding establishments (JGE), gross job gains due to opening establishments (JGB), gross job

[5] Since the model is bi-weekly, monthly and quarterly statistics are constructed following establishments over two and six consecutive time periods, respectively.

losses due to contracting establishments (JLC), and exit rates reported by the BED for the period 1992:Q3-2005:Q4. Although the fit is not perfect, we see that the model also does a good job at reproducing these statistics.[6]

The exogenous separation rate δ and the number of establishments created every period ς are important determinants of the worker flows in and out of unemployment, so I calibrate them to reproduce this type of observations. In particular, I target an average monthly separation rate from employment equal to 3.5% and an average monthly hazard rate from unemployment equal to 46%, which were estimated by Shimer [11] using CPS data between 1948 and 2004. Since the separation rate of 3.5% is significantly larger than the rate of job losses experienced by establishments, I select a positive value of δ to reproduce the excess worker reallocation.[7] Also, observe that the separation and hazard rates estimated by Shimer [11] imply a steady state unemployment rate equal to 7.1%. The average size of establishments implied by the shares in total employment in Table 1 thus determine the entry rate of establishments ς needed to generate an aggregate employment level N equal to 0.929.

I assume that the matching function takes the Cobb-Douglas form in equation (7.12). The matching parameters Ω and ϕ together with the cost of posting a help-wanted ad k are important determinants of the role that vacancies play in the model economy. In fact, one of these parameters entails a mere normalization: It is easy to show that dividing k by a factor λ, and dividing Ω by a factor $\lambda^{1-\phi}$ leaves the equilibrium unchanged except that the units in which vacancies A are measured are multiplied by the factor λ. Given this result, I normalize Ω so that the units in which vacancies are measured are such that the vacancy rate $A/(A+N)$ of continuing establishments equals 2.2%, which is the the average between 2000 and 2005 in the JOLTS dataset.[8] This normalization allows me to identify k with the cost of creating a vacancy. Since Hagedorn and Manovskii [4] estimate that the flow cost of posting a vacancy is 60% of the wage rate and since the wage rate in my model economy is $\varphi = 1$, I calibrate k to 0.60. In turn, I calibrate the curvature parameter ϕ in the matching function (7.12) to 0.72 which is the elasticity estimated by Shimer [10].

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[6] The main discrepancy is with the shares in total gros job losses for the size ranges $(5, 10)$ and $(10, 20)$, which are too large in the first case and too small in the second. This could be remedied by allowing for a postivive $Q(s_3, 0)$ and by lowering $Q(s_2, 0)$, since the range $(5, 10)$ accounts for a large fraction of the establishments closings. However, I do not expect that such modification would affect the main results in the paper.

[7] Not surprisingly, my calibrated value of δ is smaller than the quit rate of workers measured by JOLTS, since many of those separations entail job-to-job transitions that the model abstracts from.

[8] I restrict attention to continuing establishments because this is the only type of establishments included in JOLTS.

As a test of the model Table 2 reports a set of basic monthly statistics both for JOLTS and the model economy that were not used as calibration targets.[9] We see that the model does a reasonable job at reproducing the hiring and separation rates for continuing establishments, the vacancy yield rate (i.e. the hires per vacancy), the fraction of vacancies with zero hirings and the fraction of hires with zero vacancies. Time aggregation plays an important role in generating a vacancy yield rate greater than one and the large fraction of hires with zero vacancies since, following JOLTS, vacancies are measured at the end of a period and hirings are measured over the two subsequent periods. The low rate of exogenous separations δ explains the model's success in reproducing the fraction of vacancies with zero hirings. The reason is that a significant number of establishments reach the lower thresholds $\underline{x}(s)$ and start hiring just enough workers to replenish the exogenous separation of workers. Since the monthly rate of exogenous separation is less than 1%, following Davis et al. [**?**, **?**], I classify these establishments (and their corresponding vacancies) as having zero hirings. Observe that the model performs less satisfactorily in reproducing the fraction of establishments with zero vacancies and the fraction of establishments with zero hiring. The small number of idiosyncratic productivity levels that I allow for explains this result since they lead to a large number of inactive establishments. Introducing more idiosyncratic productivity level would generate smaller and more frequent adjustments and improve the performance of the model in this dimension. However, I do not expect that this modification to change the main results in the paper.

Table 3 displays all calibrated parameter values for the case of linear preferences, i.e. for the case $\sigma = 0$. The other case is easily handled since at steady state the curvature parameter σ only enters equation 6.8. In particular, when σ is greater than zero I leave all parameter values unchanged except for the utility of leisure $\varphi(\sigma)$ which I set to

$$\phi(\sigma) = \varphi(0)\, c(0)^{-\sigma}, \tag{8.1}$$

where $c(0)$ and $\varphi(0)$ are the steady state consumption level and the utility of leisure in the case $\sigma = 0$. This choice of $\varphi(\sigma)$ leaves the calibrated steady state allocation unchanged across the different values of σ. The values that satisfy equation (8.1) turn out to be $\varphi(1) = 0.6846$ when σ is equal to one.

[9] JOLTS statistics are from Davis et al. [**?**, **?**].

9. Results

To be written

10. Conclusions

To be written

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Table 1

Quarterly observations

Panel A: BED data, March 2000 to June 2000						
Size	Data			Model		
Classes*	Shares in	Shares in	Shares in	Shares in	Shares in	Shares in
(employees)	Employment	Job Gains	Job Losses	Employment	Job Gains	Job Losses
$[1, 5)$	6.4%	16.9%	9.7%	7.6%	15.0%	7.5%
$[5, 10)$	8.1%	13.1%	11.6%	6.6%	15.7%	20.5%
$[10, 20)$	10.7%	14.9%	13.7%	11.0%	16.3%	4.2%
$[20, 50)$	16.6%	18.3%	18.2%	17.1%	17.1%	16.9%
$[50, 100)$	13.1%	11.6%	12.6%	12.5%	11.7%	14.4%
$[100, 250)$	16.5%	11.9%	14.6%	16.8%	12.5%	13.6%
$[250, 500)$	9.8%	5.9%	8.5%	9.5%	5.0%	11.5%
$[500, 1000)$	7.3%	3.5%	5.4%	7.5%	6.9%	5.6%
$[1000, \infty)$	11.6%	4.2%	5.9%	11.3%	0.0%	5.9%

Panel B: BED data, 1992:Q3 to 2005:Q4		
	Data	Model
size at entry	5.3	4.7
size at exit	5.2	4.8
JGB	1.7%	1.6%
JGE	6.2%	6.2%
JLD	1.6%	1.6%
JLC	6.0%	6.2%
Exit Rate	5.2%	6.9%

$(*)$: The classification of establishments into size classes is as follows: Continuing establishments between t and $t+1$ are classified according to their size at t, opening establishments at $t+1$ are classified according to their size at $t+1$, closing establishments at $t+1$ are classified according to their size at t.

Table 2

Monthly observations

Panel A: CPS data, 1948-2004		
	Data	Model
Separation rate	3.5%	3.6%
Hazard rate	46%	46%
Panel B: JOLTS data, 2000-2005		
	Data	Model
Vacancy rate	2.2%	2.2%
Hiring rate	3.2%	3.0%
Separation rate	3.1%	3.0%
Vacancies yield rate	1.3	1.3
% Vacancies with zero hiring	18.7%	19.0%
% Hiring with zero vacancies	42.3%	58.1%
% Establishments with zero hiring	81.6%	90.0%
% Establishments with zero vacancies	87.6%	95.0%

Table 3

Parameter values ($\sigma = 0$)

General Parameters:

β	φ	α	δ	ς	k	Ω	ϕ
0.9984	1.0	0.64	0.0045	0.0006	0.61	1.493	0.219

Productivity levels:

s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
0.0	2.19	3.33	4.0	5.71	7.16	10.24	13.33	16.38	20.0

Distribution over initial productivity levels:

$\psi(s_0)$	$\psi(s_1)$	$\psi(s_2)$	$\psi(s_3)$	$\psi(s_4)$	$\psi(s_5)$	$\psi(s_6)$	$\psi(s_7)$	$\psi(s_8)$	$\psi(s_9)$
0.0	0.017	0.983	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Transition matrix Q:

	s'_0	s'_1	s'_2	s'_3	s'_4	s'_5	s'_6	s'_7	s'_8	s'_9
s_0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
s_1	0.0007	0.8677	0.1317	0.0	0.0	0.0	0.0	0.0	0.0	0.0
s_2	0.038	0.1384	0.82	0.0036	0.0	0.0	0.0	0.0	0.0	0.0
s_3	0.0	0.0	0.0071	0.98	0.0129	0.0	0.0	0.0	0.0	0.0
s_4	0.0	0.0	0.0	0.0223	0.953	0.0247	0.0	0.0	0.0	0.0
s_5	0.0	0.0	0.0	0.0	0.0529	0.94	0.0071	0.0	0.0	0.0
s_6	0.0	0.0	0.0	0.0	0.0	0.0165	0.971	0.0125	0.0	0.0
s_7	0.0	0.0	0.0	0.0	0.0	0.0	0.0414	0.95	0.0086	0.0
s_8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.022	0.955	0.023
s_9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.024	0.976