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Search and Rest Unemployment*

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Abstract

The paper extends Lucas and Prescott's (1974) search model to develop a notion of rest unemployment. The economy consists of a continuum of labor markets, each of which produces a heterogeneous good. There is a constant returns to scale production technology in each labor market, but labor productivity is continually hit by idiosyncratic shocks, inducing the costly reallocation of workers across labor markets. Under some conditions, some workers may be rest-unemployed, waiting for local labor market conditions to improve, rather than engage in time consuming search. The model has distinct notions of unemployment (moving to a new labor market or waiting for labor market conditions to improve) and inactivity (enjoying leisure while disconnected from the labor market). We obtain closed-form expressions for key aggregate variables and use them to evaluate the model quantitatively. We find that rest unemployment may be far more important than search unemployment in the U.S. economy.

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1 Introduction

This paper distinguishes "search" and "rest" unemployment. Search unemployment is a costly reallocation activity in which workers look for the best available employment opportunities. Rest unemployment is a less costly activity where a worker waits for her current labor market conditions to improve. While one might naturally think of temporary layoffs as the empirical counterpart of rest unemployment, we believe it corresponds to a more common phenomenon. We view any worker who is not working but still loosely attached to an industry as rest-unemployed.

We construct a model where there is a role for both search and rest unemployment. We then use the model to ask whether economies with different amounts of search and rest unemployment would behave differently and whether the life of a worker in search unemployment is substantially different from that of one in rest unemployment. We then use the answers to evaluate the importance of search and rest unemployment in the U.S. economy.

Our model is an extension of Lucas and Prescott's (1974) search model. The economy consists of a continuum of sectors, each of which produces a heterogeneous intermediate good which aggregates into the final consumption good using a Dixit-Stiglitz technology with constant elasticity of substitution θ. Each intermediate good is produced with a constant returns to scale technology using labor only. Labor productivity is continually hit by idiosyncratic shocks whose growth rate has a constant expected value and a constant variance per unit of time. Households have standard time additive preferences. In any time period, households can use their time endowment to engage in four mutually exclusive activities, from which they derive different amounts of leisure: work (no leisure), search unemployment, rest unemployment, and inactivity, i.e. out of the labor force.

We assume that reallocation of workers across intermediate good sectors requires a period of search unemployment. Because of that we refer interchangeably to intermediate good sectors as labor markets. A worker in a given labor market can either work, engage in rest unemployment, or leave her current labor market. A rest-unemployed worker is available to return to work in that labor market, and that labor market only, at no cost. If a worker leaves her labor market she can either be inactive or engage in search unemployment. A search-unemployed worker finds a job after a random, exponentially distributed amount of time, upon which she can locate in the market of her choosing. Thus, search is directed as in Lucas and Prescott (1974). Finally, workers can costlessly move between unemployment and inactivity.

We study stationary competitive equilibria with complete markets. Equivalently, we assume that the household is composed of a large number of members. This implies that a household values the contribution of the earnings of their members in terms of their expected discounted values. Firms producing a given intermediate good take as given the aggregate output of the final good and the price of all intermediate goods. Taking the final good as numeraire, labor demand in each market has elasticity θ, due to the effect of the sector output on its relative price. Idiosyncratic productivity shocks shift the demand for labor, unless $\theta = 1$.[1] Wages are determined competitively in each labor market and so depend on the number of workers in the labor market and on labor productivity.

To characterize the equilibrium, let ω denote the log of the wage that would prevail in a particular labor market if all workers in the market were employed, i.e. if there were no rest unemployment; we measure wages in utility-equivalent units. The behavior of workers in different labor markets is characterized by three threshold values $\underline{\omega} \leq \hat{\omega} < \bar{\omega}$. Workers who have successfully concluded their search process arrive in the best labor markets, which keeps ω below $\bar{\omega}$ in all labor markets. Workers in depressed labor markets leave to become search unemployed, which keeps ω above $\underline{\omega}$. In markets with $\omega > \hat{\omega}$ there is full employment and the log wage is ω. For $\omega < \hat{\omega}$, wages stay at $\hat{\omega}$ and the rest unemployment rate in the market increases. Workers engaged in rest unemployment stay in depressed labor markets waiting for conditions to improve. If conditions get bad enough, $\omega = \underline{\omega}$, they leave the market. Depending on parameter values, there may be no rest unemployment, $\hat{\omega} = \underline{\omega}$.

We solve our model in continuous time, with log productivity following a Brownian motion. This implies that ω is a regulated Brownian Motion, with barriers given by the endogenously determined thresholds $\underline{\omega}$ and $\bar{\omega}$. Any barriers $\underline{\omega}$ and $\bar{\omega}$ imply an invariant distribution of ω across workers. An equilibrium also requires that the value of final output, which every individual takes as given, is consistent with the value of final output obtained by aggregating output across markets using this invariant distribution. This consistency condition, which does not appear in Lucas and Prescott (1974), is due to our assumption that wages in individual markets depend on aggregate output, i.e. that there is Dixit-Stiglitz technology for producing the final goods.

We obtain simple characterizations of key endogenous variables, including a pair of equations for the two thresholds $\underline{\omega}$ and $\bar{\omega}$ and closed-form solutions for the labor force participation rate, unemployment rate, and share of searchers in the unemployment pool. This facilitates comparative statics and a quantitative evaluation of the model. In particular, we find a tight relationship between the search unemployment rate and the autocorrelation of wages at the labor market level. Using data for four- and five-digit North American Industry Classification System (NAICS) industries, we show that average weekly earnings at the industry level are

[1] When $\theta = 1$ expenditure in each intermediate good is constant, and hence in equilibrium a one percent more productive sector will have a one percent lower price and pay the same wage.

essentially a random walk. According to our model, this implies that labor markets rarely hit the barriers $\bar{\omega}$ and $\underline{\omega}$ that regulate wages. But since a labor market must move from the upper barrier to the lower barrier in order for a newly hired worker to enter search unemployment, it follows that the model cannot simultaneously generate strongly autocorrelated wages and significant amounts of search unemployment. Our calibrations suggest that the search unemployment rate—the ratio of search unemployment to the labor force—is less than 0.5%. One tension with our conclusion that the search unemployment rate is small is that it requires relatively large cost of search. We conjecture that the introduction of human capital accumulation will substantially reduce this tension, a topic that we leave for future research.

Rest provides a complementary source of unemployment. We find that the model is able to generate significant levels of rest unemployment with plausible parameters. In particular, although rest provides almost leisure as inactivity, the transition rate from rest unemployment back to employment is high, approximately $1/2t$ per unit of time at short unemployment durations t. We also find that rest unemployment can explain why measures of job creation and job destruction are a concave function of the time horizon. For labor markets with rest unemployment, creation and destruction are frequently reversed, inducing the observed concavity. With only search unemployment, creation and destruction would be nearly linear functions of elapsed time. Again, our closed-form solutions help to illustrate these points.

Our model is closely related to Lucas and Prescott (1974). There are three significant differences between the models. First, we introduce rest unemployment to the framework. Second, we make particular assumptions on the stochastic process for productivity which enable us to obtain closed-form solutions; however, we believe our insights, e.g. on the link between search unemployment and the autocorrelation of wages, carry over to alternative productivity processes. Finally, in Lucas and Prescott (1974), all labor markets produce a homogeneous good but there are diminishing returns to scale in each labor market. In our model, each labor market produce a heterogeneous good and has constant returns to scale. We believe this approach is more attractive because the extent of diminishing returns is determined by the elasticity of substitution between goods, which is potentially more easily measurable than the degree of decreasing returns on variable inputs (Atkeson, Khan, and Ohanian, 1996). An online Appendix H tightens the connections between these models by a solving a market social planner's problem and proving that the equilibrium is efficient.

Our concept of rest unemployment is closely related to the one used in Jovanovic (1987), from whom we borrow the term.[2] While in both his model and ours search and rest unem-

[2]Our decision to use this name was not easy. An obvious alternative is "wait unemployment," but the literature uses this to refer to workers who wait for a job in a high wage primary labor market rather than accept a readily available job in a low wage secondary labor market. Although workers in rest unemployment are waiting for a job, their behavior is quite different from those referred to in this literature. Our con-

ployment coexist, the aims of both papers and hence the setup of the models are different. Jovanovic (1987) focuses on the cyclical behavior of unemployment and productivity, and so allows for both idiosyncratic and aggregate productivity shocks. But to be able to analyze the model with aggregate shocks, Jovanovic (1987) assumes that at the end of each period, there is exactly one worker in each location. This implies that search unemployment is socially wasteful. Our attempts to illustrate how search unemployment may play an important role in reallocating workers away from severely depressed labor markets, while rest unemployment may be an efficient use of workers' time in marginal labor markets.

Still, we believe that it would be interesting to analyze aggregate fluctuations in our framework. First, as in Jovanovic's (1987) model, rest unemployment is likely to be countercyclical, consistent with the empirical evidence but not with many models of reallocation. Second, we could use a version of our model with aggregate shocks to evaluate the implications that the mismatch of rest-unemployed workers across labor markets has for aggregate labor market dynamics. Our model is a natural laboratory for revisiting the findings of Shimer (2007), that mismatch can generate realistic labor market dynamics. In contrast to that earlier paper, the extent of mismatch in our model is determined by the costs and benefits of reallocation. We leave the introduction of aggregate shocks for future research.

In Section 2, we describe the economic environment. We analyze a special case where workers can immediately move to the best labor market in Section 3. Without any search cost, there no rest unemployment, since either working in the best labor market or dropping out of the labor force dominates this activity. Instead, idiosyncratic productivity shocks lead to a continual reallocation of workers across labor markets.

Section 4 characterizes the stationary equilibrium of the economy. Although the microeconomic structure of our model is rich, we can characterize the equilibrium as the solution to a system of two equations in two endogenous variables and various model parameters. The system is simple in the sense that we can express the equations in closed form. We prove that the equilibrium is unique and show that it is easy to compute and perform simple comparative statics. In particular, we find that there is rest unemployment only if the cost in terms of foregone leisure is low relative to the cost of search. We also provide closed form expressions for the employment, search unemployment, and rest unemployment rates. While the unemployment rates depend a comparison of the relative advantage of different leisure activities—search, rest, and inactivity—the employment rate depends on a comparison of market versus nonmarket activity.

Section 5 uses our model to understand the extent of search and rest unemployment in

cept of rest unemployment corresponds closely to one notion of structural unemployment; see, for example Abel and Bernanke (2001, p. 95).

the U.S. economy. The model has trouble simultaneous generating significant levels of search unemployment and strongly autocorrelated wages; however, there is no tension between rest unemployment and wages. We also show how the presence of rest unemployment affects the behavior of job creation and destruction and the hazard rate of finding a job. The model with rest unemployment is broadly in line with the data, while the model with only search is not.

Section 6 uses our model to ask whether economies with different amounts of search and rest unemployment can be distinguished and whether the life of a worker in search unemployment is substantially different from that of one in rest unemployment. Our preliminary conclusions are affirmative and support our conclusion that rest unemployment may be an empirically important phenomenon.

2 Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.

2.1 Intermediate Goods

There is a continuum of intermediate goods indexed by $j \in [0,1]$. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market j at time t, there is a measure $l(j,t)$ workers. Of these, $e(j,t)$ are employed, each producing $Ax(j,t)$ units of good j, while the remaining $l(j,t) - e(j,t)$ are rest-unemployed. The price of good j, $p(j,t)$, and the wage in labor market j, $w(j,t)$, are determined competitively at each instant t.

A is the aggregate component in productivity while $x(j,t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(j,t) = \mu_x dt + \sigma_x dz(j,t), \tag{1}$$

where μ_x measures the drift of log productivity, $\sigma_x > 0$ measures the standard deviation, and $z(j,t)$ is a standard Wiener process with increments that are independent over time and across labor markets.

To keep a well-behaved distribution of labor productivity, we assume that labor market j shuts down according to a Poisson process with arrival rate λ, independent across labor markets and independent of labor market j's productivity. When this shock hits, all the workers are forced out of the market. A new labor market, also named j, enters with initial productivity $x \sim F(x)$, keeping the total measure of labor markets constant.[3] We assume a law of large numbers, so the share of labor markets experiencing any particular sequence of shocks is deterministic.

2.2 Final Goods

A competitive final goods producing sector combines the intermediate goods using the constant returns to scale technology

$$Y(t) = \left(\int_0^1 q(j,t)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \tag{2}$$

where $q(j,t)$ is the input of good j at time t and $\theta > 0$ is the elasticity of substitution across goods. In the special case with a unitary elasticity of substitution, $\theta = 1$, this becomes $Y(t) = e^{\int_0^1 \log q(j,t) dj}$.

The final goods sector takes the price of the intermediate goods as given, and hence chooses $q(j,t)$ to maximize profits:

$$\max_{\{q(j,t)\}} \left(\int_0^1 q(j,t)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} - \int_0^1 q(j,t)p(j,t)dj,$$

where we have normalized the price of the final good to 1 at each point in time. It follows the final goods sector sets

$$q(j,t) = \frac{Y(t)}{p(j,t)^{\theta}}. \tag{3}$$

Given constant returns to scale in final goods production, an interior solution to the final-goods-producing firms' profit maximization problem requires

$$\int_0^1 p(j,t)^{1-\theta} dj = 1,$$

or $\int_0^1 \log p(j,t)dj = 0$ when $\theta = 1$. Equation (3) implies that this is equivalent to equation (2).

2.3 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for study-

[3] Because the drift in the stochastic process for productivity μ_x only hits existing intermediate good vintages, it determines the rate that existing vintages fall behind new vintages and so is analogous to embodied technological change.

ing complete markets allocations.

At each moment in time t, each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the intermediate goods (or equivalently labor) markets.
  - $E(t)$ of these workers are employed at the prevailing wage.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest-unemployed, waiting in their local labor market until conditions improve and enjoying leisure b_r.
- $U_s(t)$ household members are search-unemployed, looking for a new labor market and enjoying leisure b_s.
- The remaining $1 - E(t) - U_r(t) - U_s(t)$ household members are inactive, enjoying leisure b_i. We assume $b_i \geq b_r$ and $b_i > b_s$.

Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch intermediate goods markets without going through a spell of search unemployment. Workers may exit their intermediate goods market at no cost and must do so if it shuts down. Finally, a worker in search unemployment finds a job according to a Poisson process with arrival rate α. When this happens, she may enter the intermediate goods market of her choice.

We can represent the household's preferences via the utility function

$$\int_0^\infty e^{-\rho t}\big(u(C(t)) + b_i\big(1 - E(t) - U_r(t) - U_s(t)\big) + b_r U_r(t) + b_s U_s(t)\big)dt, \tag{4}$$

where $\rho > 0$ is the discount rate, u is increasing, differentiable, strictly concave, and satisfies the Inada conditions $u'(0) = \infty$ and $\lim_{C\to\infty} u'(C) = 0$, and $C(t)$ is the household's consumption of the final good. The household finances its consumption using its labor income. Following Hansen (1985) and Rogerson (1988), the linear utility of search unemployment, rest unemployment, and inactivity reflects the microeconomic indivisibility constraint. Note that $u(C) = \log C$ is consistent with balanced growth.

2.4 Equilibrium

We look for a competitive equilibrium of this economy. At each instant, each household chooses how much to consume and how to allocate its members between employment in each labor market, rest unemployment in each labor market, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across labor markets, taking as given the stochastic process for wages in each labor market; each final goods producer maximizes profits by choosing inputs taking as given the price for all the intermediate goods; and each intermediate goods producer j maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from intermediate goods producers is equal to the supply from households in each intermediate goods market; the demand for intermediate goods from the final goods producers is equal to the supply from intermediate goods producers; and the demand for final goods from the households is equal to the supply from the final goods producers.

Standard arguments imply that for given initial conditions, there is at most one competitive equilibrium of this economy. The first welfare implies that any equilibrium is Pareto optimal. Since there is only one type of household, if there were multiple equilibria, household utility would be equal in each. But a convex combination of the equilibrium allocations would be feasible and Pareto superior, a contradiction.

We look for a stationary equilibrium where all aggregate quantities and the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets are constant. With identical households and complete markets, consumption is equal to current labor income and hence we ignore financial markets in the remainder of this paper.

3 Costless Mobility

To understand the mechanics of the model, we start with a version where nonworkers can instantaneously become workers; formally, this is equivalent to the limit of the model when $\alpha \to \infty$. In this limit, the household does not need to devote any workers to search unemployment. Moreover, there is no rest unemployment since with costless mobility, resting is dominated by inactivity. Thus the household divides its time between employment and inactivity. Finally, all workers must earn a common, constant wage w, since otherwise households would costlessly reallocate workers to higher-wage markets.

The household therefore solves

$$\max_{E(t)} \int_0^\infty e^{-\rho t}\big(u(wE(t)) + b_i(1 - E(t))\big)dt,$$

where $C(t) = wE(t)$. The first order conditions imply that at each date t, $E \leq 1$ and $b_i \leq wu'(Ew)$ with complementary slackness; note that the Inada condition $u'(0) = \infty$ rules out the possibility of zero employment. In the special case of logarithmic utility, this simplifies further. If $b_i < 1$, the household is fully employed, $E = 1$. Otherwise the household works a fraction $E = 1/b_i$ of its members at each instant.

To close the model, we compute the equilibrium wage. Consider an intermediate goods market with productivity x and l workers. Output is $Q(l, x) = lAx$ and so equation (3) implies the price of the good is $P(l, x) = \left(\frac{Y}{lAx}\right)^{1/\theta}$. Then since workers are paid their marginal revenue product, the wage is

$$W(l, x) = \left(\frac{Y(Ax)^{\theta-1}}{l}\right)^{\frac{1}{\theta}}. \tag{5}$$

Since wages are equalized across markets, $W(l, x) = w$, this pins down the relationship between productivity and employment across labor markets. When $\theta > 1$, more productive labor markets employ more workers, while if goods are poor substitutes, $\theta < 1$, an increase in labor productivity lowers employment so as to keep output relatively constant. In the special case of $\theta = 1$, employment is constant and equal to Y/w in all labor markets.

Using equation (5), substitute $Q(l, x) = lAx = \frac{Y(Ax)^\theta}{w^\theta}$ into equation (2) and simplify to show that the wage is a weighted average of productivity across markets,

$$w = A\left(\int_0^1 x(j, t)^{\theta-1} dj\right)^{\frac{1}{\theta-1}}, \tag{6}$$

or $w = Ae^{\int_0^1 \log x(j,t)dj}$ if $\theta = 1$. The right hand side of this expression is constant due to the law of large numbers.

Alternatively, consider a labor market with initial productivity x_0. Let $f_x(\tilde{x}; x_0)$ denote the steady state density of log productivity, $\tilde{x} \equiv \log x$, across all such labor markets. In Appendix A we show that this density is given by

$$f_x(\tilde{x}; x_0) = \begin{cases} \frac{\tilde{\eta}_1\tilde{\eta}_2}{\tilde{\eta}_1-\tilde{\eta}_2} e^{\tilde{\eta}_2(\tilde{x}-\log x_0)} & \text{if } \tilde{x} < \log x_0 \\ \frac{\tilde{\eta}_1\tilde{\eta}_2}{\tilde{\eta}_1-\tilde{\eta}_2} e^{\tilde{\eta}_1(\tilde{x}-\log x_0)} & \text{if } \tilde{x} > \log x_0, \end{cases} \tag{7}$$

where $\tilde{\eta}_1 < 0 < \tilde{\eta}_2$ are the two real roots of the characteristic equation

$$\lambda = -\mu_x\tilde{\eta} + \frac{\sigma_x^2}{2}\tilde{\eta}^2. \tag{8}$$

With this notation, and recalling that initial productivity x_0 is distributed as $F(x_0)$ across labor markets, we can rewrite equation (6) as

$$w = A\left(\int_0^\infty \int_{-\infty}^\infty e^{(\theta-1)\tilde{x}} f_x(\tilde{x}; x_0) d\tilde{x} dF(x_0)\right)^{\frac{1}{\theta-1}}. \tag{9}$$

Using equation (7), the interior integral converges if $\tilde{\eta}_1 + \theta - 1 < 0 < \tilde{\eta}_2 + \theta - 1$. The definition of $\tilde{\eta}_i$ in equation (8) implies these inequalities both hold if and only if

$$\lambda > (\theta - 1)\left(\mu_x + (\theta - 1)\frac{\sigma_x^2}{2}\right). \tag{10}$$

In words, the exit rate of markets must be sufficiently large, the drift and variance of productivity sufficiently small, or the elasticity of substitution sufficiently close to 1. If this parameter restriction were to fail, the equilibrium wage and income would be unbounded. In the case of $\theta > 1$, the extremely productive firms would produce an enormous amount of easily-substitutable goods; with $\theta < 1$, the very unproductive firms would use a huge amount of labor to produce enough of the poorly substitutable goods. We impose condition (10) through the remainder of the paper. With this restriction, equation (9) reduces to

$$w = AX_0 \left(\frac{\lambda}{\lambda - (\theta - 1)\left(\mu_x + (\theta - 1)\frac{\sigma_x^2}{2}\right)}\right)^{\frac{1}{\theta-1}}, \tag{11}$$

when $\theta \neq 1$, where $X_0 = \left(\int_0^\infty x_0^{\theta-1} dF(x_0)\right)^{\frac{1}{\theta-1}}$. When $\theta = 1$, $w = AX_0 e^{\mu_x/\lambda}$, where $X_0 = e^{\int_0^\infty \log x_0 dF(x_0)}$. Condition (10) ensures the fraction is strictly positive. Then the integral converges, and so wages and income are finite, if and only if the $\theta - 1^{\text{st}}$ moment of the x_0 distribution is finite. Again, we impose this restriction throughout the remainder of the paper.

For future reference, we note two important properties of the frictionless wage. First, if $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$, $w = AX_0$, independent of λ. We introduced the assumption that intermediate goods markets are destroyed at a positive rate mainly for technical convenience, so imposing the restriction $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$ allows to focus on the limit as λ converges to zero; we do so in Section 5. Second, because of constant returns to scale, the wage depends on technology but not on preference parameters. This implies that a proportional increase in the leisure value of inactivity b_i raises the marginal utility of consumption $u'(C)$ by the same proportion, while employment E decreases in proportion to C. Similar properties holds in the frictionless, which we turn to now.

4 Characterization of Equilibrium

We now return to the model where it takes time to find a new labor market, $\alpha < \infty$. We look for a steady state equilibrium where the household maintains constant consumption, obtains a constant income stream, and keeps a positive and constant fraction of its workers in each of the activities, employment, rest unemployment, search unemployment, and inactivity. In equilibrium, in each labor market, which is characterized by productivity x and the number of workers l, the ratio $x^{\theta-1}/l$ follows a Markov process. Workers enter labor markets when the ratio exceeds a threshold and exit labor markets when it falls below a strictly smaller threshold. We prove that there is a unique stationary equilibrium.

4.1 The Marginal Value of Household Members

We start by computing the marginal value of an additional household member engaged in each of the three activities. These are related by the possibility of reallocating household members.

Consider first a household member who is permanently inactive. It is immediate from equation (4) that he contributes

$$\underline{v} = \frac{b_i}{\rho} \tag{12}$$

to household utility. Since the household may freely shift workers between inactivity and search unemployment, this must also be the incremental value of a searcher, assuming some members are engaged in each activity. A searcher gets flow utility b_s and the possibility of finding a labor market at rate α. This gives a standard Hamilton-Jacobi-Bellman equation:

$$\rho \underline{v} = b_s + \alpha \left(\bar{v} - \underline{v} \right).$$

The left hand side is the flow value of a searcher to the household, while the right hand side is the sum of her flow utility value and product of the rate at which a searcher finds a labor market and the ensuing capital gain, with $\bar{v}$ representing the value to the household of having a worker in the best labor market. This pins down the value of having a worker in the best labor market,

$$\bar{v} = b_i \left(\frac{1}{\rho} + \kappa \right) \tag{13}$$

where

$$\kappa \equiv \frac{b_i - b_s}{b_i \alpha}$$

is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment $1/\alpha$. Conversely, a worker may freely exit the labor market, and so the lower bound on the value to the household of a household member in a labor market, either employed or search unemployed, is $\underline{v}$. If the household values a worker at some intermediate amount, it will be willing to keep her in her labor market rather than having her search for a new one.

Finally, consider the margin between employment and resting for a worker in a labor market paying a wage w. An employed worker generates income valued at $u'(C)w$, while a resting worker generates b_r utils. Since switching between employment and resting is costless, all workers are employed in any labor market with $w > b_r/u'(C)$, and all workers are resting in any market with $w < b_r/u'(C)$. In the intermediate case, some may be employed and some resting.

4.2 Wage and Labor Force Dynamics

Consider a labor market with l workers and productivity x. Let $P(l,x)$ denote the price of its good, $Q(l,x)$ denote the amount of the good produced, $W(l,x)$ denote the wage rate, and $E(l,x)$ denote the number of workers who are employed. Competition ensures that the wage is equal to the marginal product of labor, $W(l,x) = P(l,x)Ax$, while the production function implies $Q(l,x) = E(l,x)Ax$. Combining these conditions with the intermediate good demand curve from equation (3) and the labor supply curve, that a worker works if $W(l,x) > b_r/u'(C)$ and rests if $W(l,x) < b_r/u'(C)$, gives

$$W(l,x) = \frac{1}{u'(C)} \max\{b_r, e^{\omega}\} \tag{14}$$

$$P(l,x) = \frac{1}{Axu'(C)} \max\{b_r, e^{\omega}\} \tag{15}$$

$$Q(l,x) = lAx \min\{1, e^{\omega}/b_r\}^{\theta} \tag{16}$$

$$E(l,x) = l \min\{1, e^{\omega}/b_r\}^{\theta}, \tag{17}$$

where

$$\omega \equiv \frac{\log Y + (\theta - 1)\log(Ax) - \log l}{\theta} + \log u'(C) \tag{18}$$

is the logarithm of the "full-employment wage" measured in utils, the wage that would prevail if there were full employment in the labor market.

e^{ω} is the marginal rate of transformation of labor into utils when productivity in the market is Ax, employment is l, aggregate output is Y, and the marginal utility of consumption is $u'(C)$. This is $R(Ax, l)$ in Lucas and Prescott's (1974) notation. Their production technology implies that Y does not affect R, while risk-neutrality ensures that $u'(C)$ is constant.

Lucas and Prescott (1974) also assume that $R_x > 0$—see their equation (1)—which in our set-up is equivalent to $\theta > 1$.

To understand these expressions, consider any labor market with $e^{\omega} > b_r$, because labor supply is relatively small given productivity. Equations (14)–(17) imply there is full employment and the wage exceeds the value of leisure. Output is equal to lAx and the output price exceeds $b_r/Axu'(C)$. Conversely, in a labor market with $e^{\omega} < b_r$, there are $l\big(1 - (e^{\omega}/b_r)^{\theta}\big)$ workers in rest unemployment and the wage is b_r, so all workers are indifferent between employment and rest unemployment. Output is less than Axl while the output price is driven down to $b_r/Axu'(C)$.

4.3 Labor Force Participant Value Function

We return now to the value of a worker in a labor market with productivity x and with l workers. Equation (14) shows that the wage depends only on ω, a simple function of these variables. This suggests looking for an equilibrium in which workers immediately enter any labor market with $\omega > \bar{\omega}$ and exit any labor market with $\omega < \underline{\omega}$, where the thresholds $\underline{\omega} \leq \bar{\omega}$ are determined endogenously. There is neither exit nor entry from labor markets with $\omega \in (\underline{\omega}, \bar{\omega})$. We allow for the possibility that $\underline{\omega} = -\infty$ so workers never exit labor markets.

If there is an equilibrium with this property, its definition in equation (18) implies $\omega(j,t)$ is a regulated Brownian motion in each market j. When $\omega(j,t) \in (\underline{\omega}, \bar{\omega})$,

$$d\omega(j,t) = \frac{\theta - 1}{\theta} d \log x(j,t) = \mu dt + \sigma dz(j,t), \tag{19}$$

where

$$\mu \equiv \frac{\theta - 1}{\theta}\mu_x \quad \text{and} \quad \sigma \equiv \frac{|\theta - 1|}{\theta}\sigma_x,$$

since only productivity shocks change ω. When the thresholds $\underline{\omega}$ and $\bar{\omega}$ are finite, they act as reflecting barriers, since productivity shocks that would move ω outside the boundaries are offset by the entry and exit of workers. If $\theta = 1$, $\omega(j,t) = \log Y - \log l(j,t) + \log u'(C)$ is constant when employment is constant.

Given arbitrary values for the barriers $\underline{\omega} \leq \bar{\omega}$, we can define the incremental value to the household of having a worker in a market with current productivity ω_0 as

$$v(\omega_0; \underline{\omega}, \bar{\omega}) = \mathbb{E}\left(\int_0^{\infty} e^{-(\rho+\lambda)t} \left(\max\{b_r, e^{\omega(t)}\} + \lambda \underline{v} \right) dt \;\middle|\; \omega(0) = \omega_0 \right), \tag{20}$$

where expectations are taken with respect to future values of the random variable $\omega(t)$. The time-t payoff is the prevailing wage; this holds whether the worker is employed or rest-unemployed because when there is rest unemployment, the worker is indifferent between the two states. We discount this back to the current date at rate $\rho + \lambda$, accounting both for impatience and for the possibility that the labor market ends exogenously before time t. In addition, if the labor market ends exactly at time t, at a hazard rate λ, the household gets a terminal value $\underline{v}$ discounted back to time 0.

In an equilibrium, we require that

$$\begin{aligned} v(\omega; \underline{\omega}, \bar{\omega}) &\in [\underline{v}, \bar{v}] \text{ for all } \omega \\ v(\bar{\omega}; \underline{\omega}, \bar{\omega}) &= \bar{v} \\ v(\underline{\omega}; \underline{\omega}, \bar{\omega}) &= \underline{v} \text{ if } \underline{\omega} > -\infty \end{aligned} \tag{21}$$

The utility of a worker in any market must be between $\underline{v}$ and $\bar{v}$. If she is in the best possible market, her utility must be $\bar{v}$ so searchers are willing to take a job there. If she is in the worst possible market, her utility must be $\underline{v}$ so she is indifferent about exiting her labor market; such a market exists only if the lower threshold is finite.

If $\theta = 1$, $\omega(t) = \omega_0$ for any $\omega_0 \in [\underline{\omega}, \bar{\omega}]$, making it trivial to solve equation (20) analytically. Equation (21) implies the thresholds $\underline{\omega}^*$ and $\bar{\omega}^*$ solve

$$\frac{e^{\underline{\omega}^*} + \lambda \underline{v}}{\rho + \lambda} = \underline{v} \text{ and } \frac{e^{\bar{\omega}^*} + \lambda \underline{v}}{\rho + \lambda} = \bar{v}. \tag{22}$$

Since $\bar{v} > \underline{v}$, these equations imply $\bar{\omega}^* > \underline{\omega}^*$. In addition, using equation (12) we get $\underline{\omega}^* = \log b_i \geq \log b_r$.

When $\theta \neq 1$, it is still possible to characterize the thresholds analytically. Define

$$\Pi(\omega; \omega_0; \underline{\omega}, \bar{\omega}) \equiv \mathbb{E}\left(\int_0^\infty e^{-(\rho+\lambda)t} I_\omega(\omega(t)) dt \,\middle|\, \omega(0) = \omega_0 \right),$$

where $I_\omega(\omega(t))$ is an indicator function, equal to 1 if $\omega(t) < \omega$ and equal to zero otherwise. This *discounted occupancy function* evaluates to zero at $\omega \leq \underline{\omega}$ and to $\frac{1}{\rho+\lambda}$ at $\omega \geq \bar{\omega}$. We use $\Pi_\omega(\omega; \omega_0; \underline{\omega}, \bar{\omega})$ for the density of ω or the *discounted local time function*. Then switching the order of integration in equation (20), which is permissible since for $-\infty \leq \underline{\omega} \leq \bar{\omega} < \infty$ and $\rho + \lambda > 0$, the function $\max\{b_r, e^\omega\} + \lambda \underline{v}$ is integrable, we get

$$v(\omega_0; \underline{\omega}, \bar{\omega}) = \int_{\underline{\omega}}^{\bar{\omega}} \left(\max\{b_r, e^\omega\} + \lambda \underline{v} \right) \Pi_\omega(\omega; \omega_0; \underline{\omega}, \bar{\omega}) d\omega. \tag{23}$$

The value of being in a market with current log full-employment wage ω_0 is equal to the expected value of future ω weighted by the appropriate discounted local time function.

Equation (23) is convenient because $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ is a known function. Stokey (2006) proves in Proposition 10.4 that for all $\omega_0 \in [\underline{\omega},\bar{\omega}]$,

$$\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \begin{cases} \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - \zeta_1 e^{\zeta_1\bar{\omega}+\zeta_2\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \underline{\omega} \le \omega < \omega_0 \\ \\ \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\underline{\omega}} - \zeta_1 e^{\zeta_1\underline{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\bar{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\bar{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \omega_0 \le \omega \le \bar{\omega}, \end{cases} \tag{24}$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation

$$\rho + \lambda = \mu\zeta + \frac{\sigma^2}{2}\zeta^2. \tag{25}$$

For $\omega_0 < \underline{\omega}$, $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \Pi_\omega(\omega;\underline{\omega};\underline{\omega},\bar{\omega})$ and for $\omega_0 > \bar{\omega}$, $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega}) = \Pi_\omega(\omega;\bar{\omega};\underline{\omega},\bar{\omega})$. This implies that

Lemma 1. *v is continuous and nondecreasing in ω_0, $\underline{\omega}$, and $\bar{\omega}$. It is strictly increasing in each argument if $\omega_0 \in (\underline{\omega},\bar{\omega})$, $\bar{\omega} > \log b_r$, and $\theta \neq 1$.*

Proof. When $\theta = 1$, the result follows trivially from equation (22). Otherwise, that v is continuous follows immediately from equations (23) and (24). In particular, the latter equation defines Π_ω as a continuous function.

We next prove that the distribution $\Pi(\cdot;\omega_0;\underline{\omega},\bar{\omega})$ is increasing in each of ω_0, $\underline{\omega}$, and $\bar{\omega}$ in the sense of first order stochastic dominance. This follows from differentiating equation (24) with respect to each variable and using simple algebra. One can verify that an increase in $\underline{\omega}$ strictly increases $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for all $\omega \in (\underline{\omega},\bar{\omega})$. This therefore strictly reduces $\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\bar{\omega})$. Similarly, an increase in $\bar{\omega}$ strictly reduces $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for all $\omega \in (\underline{\omega},\bar{\omega})$, which also strictly reduces $\Pi(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\bar{\omega})$. Finally, an increase in ω_0 when $\omega_0 \in (\underline{\omega},\bar{\omega})$ reduces $\Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})$ for $\omega \in (\underline{\omega},\omega_0)$ and raises it for $\omega \in (\omega_0,\bar{\omega})$. Once again, this implies a stochastic dominating shift in Π.

Since the return function $\max\{b_r, e^\omega\} + \lambda\underline{v}$ is nondecreasing in ω, weak monotonicity of v in each argument follows immediately from equation (23). In addition, the return function is strictly increasing when $\omega > \log b_r$, and so we obtain strict monotonicity when the support of the integral includes some $\omega > \log b_r$, i.e. when $\bar{\omega} > \log b_r$. $\square$

It is worth noting that equations (21) and (23) imply some familiar conditions:

$$(\rho+\lambda)v(\omega_0;\underline{\omega},\bar{\omega}) = \max\{b_r, e^{\omega}\} + \lambda\underline{v} + \mu v_{\omega_0}(\omega_0;\underline{\omega},\bar{\omega}) + \frac{\sigma^2}{2}v_{\omega_0,\omega_0}(\omega_0;\underline{\omega},\bar{\omega}), \tag{26}$$

$$\text{and } v_{\omega_0}(\bar{\omega};\underline{\omega},\bar{\omega}) = v_{\omega_0}(\underline{\omega};\underline{\omega},\bar{\omega}) = 0, \tag{27}$$

where subscripts denote partial derivatives with respect to the first argument. The first condition, the Hamilton-Jacobi-Bellman equation, can be verified directly by differentiating equation (23) using the definition of Π_ω in equation (24). The interested reader can consult the online Appendix F for the details of the algebra. The second pair of conditions, "smooth-pasting," follow from equation (23) because equation (24) implies $\frac{\partial \Pi_\omega(\omega;\omega_0;\underline{\omega},\bar{\omega})}{\partial \omega_0} = 0$ when $\omega_0 = \underline{\omega}$ or $\omega_0 = \bar{\omega}$. Together with the "value-matching" conditions $v(\bar{\omega};\underline{\omega},\bar{\omega}) = \bar{v}$ and $v(\underline{\omega};\underline{\omega},\bar{\omega}) = \underline{v}$ in equation (21), this is an equivalent representation of the labor force participant's value function.

We can now prove that there exist unique thresholds consistent with equilibrium.

Proposition 1. *There exists a unique $\underline{\omega}$ and $\bar{\omega}$ such that equation (21) holds when v satisfies equation (23). A proportional increase in b_i, b_r, and b_s raises $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ by the same proportion. Moreover, $\underline{\omega} < \log b_i < \bar{\omega} < \infty$, with $\underline{\omega} > -\infty$ if and only if $b_r < b_i$.*

The Proposition establishes that $\underline{\omega}$ is finite when $b_r < b_i$. Intuitively, if a market is hit by sufficiently adverse shocks, workers will leave since rest unemployment is costly and has low expected payoffs. In contrast, when $b_r = b_i$, rest unemployment is costless and hence workers only leave labor markets when they shut down. In addition, the Proposition shows that there is no rest unemployment in the best labor markets, $\bar{\omega} > \log b_i \geq \log b_r$. If there were rest unemployment in all labor markets, no would be willing to search for a job. For $b_r < b_i$, this Proposition does not establish whether this is rest unemployment in the worst labor markets, $\underline{\omega} \gtrless \log b_r$. Proposition 2 addresses this issue.

Proof. We have already established this result using equation (22) when $\theta = 1$ and so assume $\theta \neq 1$ in what follows. We start by proving the result when $b_r < b_i$ and defer $b_r = b_i$ until the end.

Lemma 1 ensures v is continuous and strictly increasing in its first three arguments. Moreover, for any $\underline{\omega} < \bar{\omega}^*$ defined in equation (22), we can make $v(\bar{\omega};\underline{\omega},\bar{\omega})$ unboundedly large by increasing $\bar{\omega}$, while we can make it smaller than $\bar{v}$ by setting $\bar{\omega} = \bar{\omega}^*$. Then by the intermediate value theorem, for any $\underline{\omega} < \bar{\omega}^*$, there exists a $\bar{\Omega}(\underline{\omega}) > \bar{\omega}^*$ solving $v(\bar{\Omega}(\underline{\omega});\underline{\omega},\bar{\Omega}(\underline{\omega})) \equiv \bar{v}$.

Continuity of v ensures $\bar{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. In addition, because the period return function $s(\omega) \equiv \max\{b_r, e^{\omega}\} + \lambda \underline{v}$ is bounded below but not above, $\bar{\omega}^{**} \equiv \lim_{\underline{\omega} \to -\infty} \bar{\Omega}(\underline{\omega})$ is finite. Thus $\bar{\Omega}(\underline{\omega}) \in (\bar{\omega}^*, \bar{\omega}^{**})$ for any $\underline{\omega} < \bar{\omega}^*$. Figure 1 illustrates this function.

Similarly, for any $\bar{\omega} > \underline{\omega}^*$ defined in equation (22), we can make $v(\underline{\omega}; \underline{\omega}, \bar{\omega})$ approach $\frac{\rho b_r + \lambda b_i}{\rho(\rho+\lambda)} < \underline{v}$ by making $\underline{\omega}$ arbitrarily small, while we can make it bigger than $\underline{v}$ by setting $\underline{\omega} = \underline{\omega}^*$. Then by the intermediate value theorem, for any $\bar{\omega} > \underline{\omega}^*$, there exists a $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ solving $v(\underline{\Omega}(\bar{\omega}); \underline{\Omega}(\bar{\omega}), \bar{\omega}) \equiv \underline{v}$. Continuity of v ensures $\underline{\Omega}$ is continuous while monotonicity of v ensures it is decreasing. Thus $\underline{\Omega}(\bar{\omega}) < \underline{\omega}^*$ for any $\bar{\omega} > \underline{\omega}^*$.

An equilibrium is simply a value of $\bar{\omega}$ such that $\bar{\Omega}(\underline{\Omega}(\bar{\omega})) = \bar{\omega}$, i.e. a fixed point of the composition of the functions $\bar{\Omega} \circ \underline{\Omega}$. The preceding argument implies that this composition maps $[\bar{\omega}^*, \bar{\omega}^{**}]$ into itself and is continuous, and hence has a fixed point.

To prove the uniqueness of the fixed point when $b_r < b_i$, we prove that the composition of the two functions has a slope less than 1, i.e. $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$. To start, simple transformations of equation (24) imply that the cross partial derivatives of the discounted occupancy function satisfy

$$\Pi_{\omega_0,\bar{\omega}}(\omega; \omega_0; \underline{\omega}, \bar{\omega}) = \frac{\zeta_1\zeta_2 e^{(\zeta_1+\zeta_2)\bar{\omega}} \left(e^{-\zeta_1(\omega-\underline{\omega})} - e^{-\zeta_2(\omega-\underline{\omega})}\right)\left(e^{\zeta_1\underline{\omega}+\zeta_2\omega_0} - e^{\zeta_1\omega_0+\zeta_2\underline{\omega}}\right)}{\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)^2(\rho+\lambda)} < 0$$

$$\Pi_{\omega_0,\underline{\omega}}(\omega; \omega_0; \underline{\omega}, \bar{\omega}) = \frac{-\zeta_1\zeta_2 e^{(\zeta_1+\zeta_2)\underline{\omega}} \left(e^{\zeta_2(\bar{\omega}-\omega)} - e^{\zeta_1(\bar{\omega}-\omega)}\right)\left(e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\omega_0}\right)}{\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)^2(\rho+\lambda)} > 0,$$

where the inequalities use the fact that all the terms in parenthesis are positive. Then use integration-by-parts on equation (23) to write

$$v(\omega_0; \underline{\omega}, \bar{\omega}) = \frac{s(\bar{\omega})}{\rho+\lambda} - \int_{\underline{\omega}}^{\bar{\omega}} s'(\omega)\Pi(\omega; \omega_0; \underline{\omega}, \bar{\omega})d\omega,$$

where the period return function $s(\omega)$ is nondecreasing and strictly increasing for $\omega > \log b_r$, and Π is the discounted occupancy function. Taking the cross partial derivatives of this expression gives $v_{\omega_0,\bar{\omega}}(\omega_0; \underline{\omega}, \bar{\omega}) > 0 > v_{\omega_0,\underline{\omega}}(\omega_0; \underline{\omega}, \bar{\omega})$. In particular,

$$v_{\bar{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega}) > v_{\bar{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega}) \text{ and } v_{\underline{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega}) > v_{\underline{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega}).$$

Now since $v_{\omega_0}(\omega_0; \underline{\omega}, \bar{\omega}) > 0$ from Lemma 1, these inequalities imply

$$\frac{v_{\underline{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega})}{v_{\omega_0}(\bar{\omega}; \underline{\omega}, \bar{\omega}) + v_{\bar{\omega}}(\bar{\omega}; \underline{\omega}, \bar{\omega})} \frac{v_{\bar{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega})}{v_{\omega_0}(\underline{\omega}; \underline{\omega}, \bar{\omega}) + v_{\underline{\omega}}(\underline{\omega}; \underline{\omega}, \bar{\omega})} < 1.$$

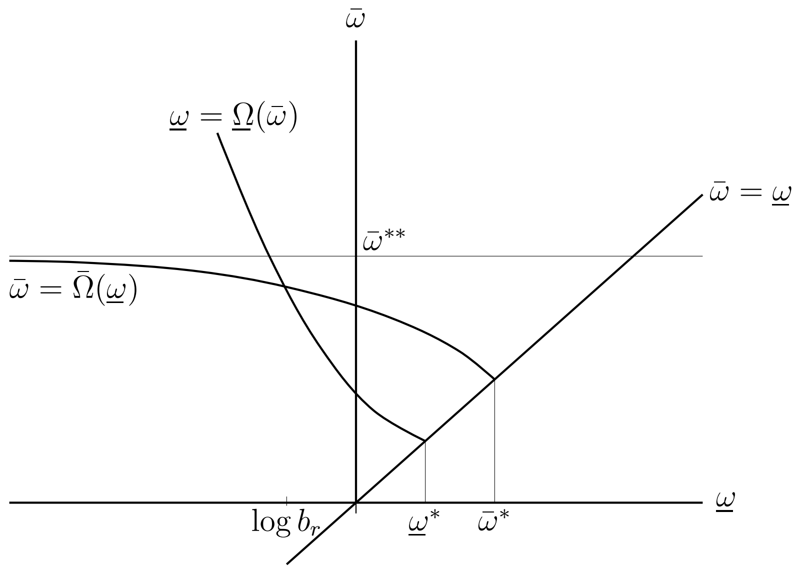

Figure 1: Illustration of the proof of Proposition 1 when $b_r < b_i$.

In particular, this is true when evaluated at any point $\{\underline{\omega}, \bar{\omega}\}$ where $\bar{\omega} = \bar{\Omega}(\underline{\omega})$ and $\underline{\omega} = \underline{\Omega}(\bar{\omega})$. Implicit differentiation of the definitions of these functions shows that the first term in the above inequality is $-\bar{\Omega}'(\underline{\omega})$ and the second term is $-\underline{\Omega}'(\bar{\omega})$, which proves $\bar{\Omega}'(\underline{\Omega}(\bar{\omega}))\underline{\Omega}'(\bar{\omega}) < 1$.

Next we prove proportionality of the thresholds $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$ to the leisure values b_r, b_i, and b_s. From equations (12) and (13), $\underline{v}$ and $\bar{v}$ are homogeneous of degree one in the three leisure values. The function $\max\{b_r, e^{\omega}\} + \lambda \underline{v}$ is also homogeneous of degree 1 in the leisure values and e^{ω}. By inspection of equation (24), Π_ω is unaffected by an equal absolute increase in each of its arguments. Then the integral in equation (23) is homogeneous of degree one in the b's and $e^{\bar{\omega}}$ and $e^{\underline{\omega}}$. The result follows from equation (21).

Finally we consider $b_r = b_i$, so the period return function $s(\omega) \geq b_i + \lambda \underline{v} = (\rho + \lambda)\underline{v}$ for all ω. This implies $v(\omega; -\infty, \bar{\omega}) \geq \underline{v}$ for all ω and $\bar{\omega}$. Then an equilibrium is defined by $v(\bar{\omega}; -\infty, \bar{\omega}) = \bar{v}$. As discussed above, the solution of this equation is $\bar{\omega}^{**} \in (\bar{\omega}^*, \infty)$. $\square$

Our uniqueness proof relies on two key properties of the model. First, the period return function is monotonic. Second, an increase in the upper (lower) bound affects the discounted occupancy function more when ω_0 is closer to the upper (lower) bound. Although our proof relies on the exact functional form of the discounted occupancy function, it seems like a plausible property of such functions more generally when ω is persistent. An alternative proof of the uniqueness of the solution to equations (21) and (23) relies on solving an "island planner's problem" developed in the online Appendix H.

We now characterize whether there is rest unemployment.

Proposition 2. Assume $\theta \neq 1$. There exists a $\bar{b}_r$ such that in an equilibrium, $b_r \gtreqless e^{\underline{\omega}}$ if and only if $b_r \gtreqless \bar{b}_r$, with $\bar{b}_r = B(\kappa, \rho + \lambda, \mu, \sigma) b_i$ for some function B, positive-valued and decreasing in κ with $B(0, \rho + \lambda, \mu, \sigma) = 1$.

In particular, there is rest unemployment if the leisure value of resting b_r is sufficiently close to the leisure of inactivity b_i given any $\kappa > 0$; or equivalently if search costs κ are sufficiently high given any $b_r \in (0, b_i)$. If a searcher finds a job sufficiently fast or resting gives too little leisure, there is no reason to wait for labor market conditions to improve.

Proof. First, set $b_r = 0$. By Proposition 1, there exists a unique equilibrium characterized by thresholds $\underline{\omega}_0$ and $\bar{\omega}_0$. We now prove that $\bar{b}_r \equiv e^{\underline{\omega}_0}$. To see why, observe that for all $b_r \leq \bar{b}_r$, the equations characterizing equilibrium are unchanged from the case of $b_r = 0$ because $\log b_r \leq \underline{\omega}_0$, and hence the equilibrium is unchanged. Conversely, for all $b_r > \bar{b}_r$, the equations characterizing equilibrium necessarily are changed, and so the equilibrium must have $\log b_r > \underline{\omega}_{b_r}$.

Next we prove that $\bar{b}_r / b_i = B(\kappa, \rho + \lambda, \mu, \sigma)$. Again with $b_r = 0$, combine equations (21) and (23), noting the discounted local time function Π_ω integrates to $\frac{1}{\rho+\lambda}$, and use the definitions of $\underline{v}$ and $\bar{v}$ in equations (12) and (13):

$$\frac{b_i}{\rho + \lambda} = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_\omega(\omega; \underline{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega \text{ and}$$
$$b_i \left(\frac{1}{\rho + \lambda} + \kappa \right) = \int_{\underline{\omega}_0}^{\bar{\omega}_0} e^{\omega} \Pi_\omega(\omega; \bar{\omega}_0; \underline{\omega}_0, \bar{\omega}_0) d\omega.$$

Since Π_ω is homogeneous of degree zero in the exponentials of its arguments (see equation 24), this implies $e^{\underline{\omega}_0}$ and $e^{\bar{\omega}_0}$ are homogeneous of degree 1 in b_i. Moreover, ζ_i depends on $\rho + \lambda$, μ, and σ by equation (25) and so the density Π_ω in equation (24) depends on these same parameters. It follows that the solution to these equations can depend only on these parameters and the parameters on the left hand side of the above equations. In particular, this proves

$$e^{\underline{\omega}_0} = b_i B(\kappa, \rho + \lambda, \mu, \sigma).$$

Since $\bar{b}_r = e^{\underline{\omega}_0}$, that establishes the dependence of $\bar{b}_r$ on this limited set of parameters.

Obviously B is positive-valued. By Proposition 1, $\underline{\omega}_0 < \log b_i$ and so $B < 1$. We finally prove it is decreasing in κ. Since κ affects $\underline{\omega}$ and $\bar{\omega}$ only through $\bar{v}$, to establish that B is decreasing in κ it suffices to show that the $\underline{\omega}$ and $\bar{\omega}$ that solve equations (21) and (23) is decreasing in $\bar{v}$. This follows because $\bar{\Omega}(\underline{\omega})$ is increasing in $\bar{v}$ and $\underline{\Omega}(\bar{\omega})$ is unaffected, where these functions are defined in the proof of Proposition 1. A decrease in $\bar{v}$ then reduces the composition $\bar{\Omega} \circ \underline{\Omega}$. Since the slope of this function is less than 1, it reduces the location of

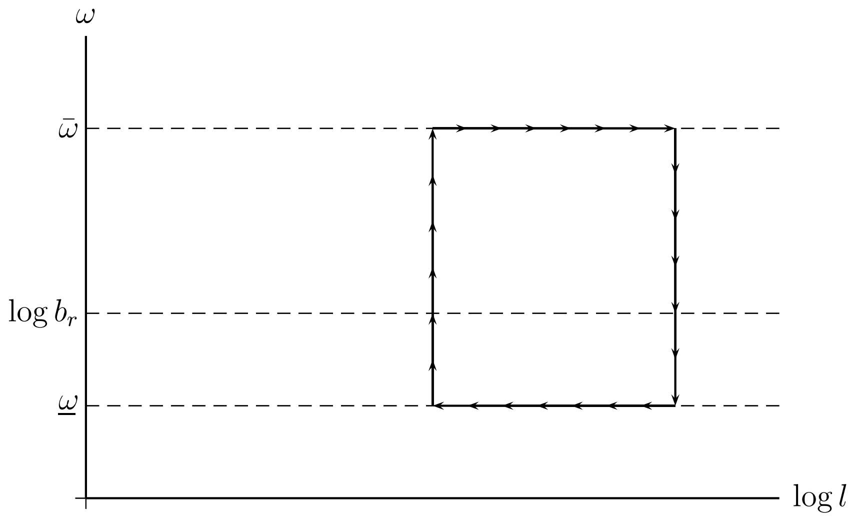

Figure 2: The dynamics of ω and $\log l$ when ω is regulated above at $\bar{\omega}$ and below at $\underline{\omega}$. All new markets enter at $\bar{\omega}$. Markets with $\omega > \log b_r$ have no unemployment, while markets with $\omega < \log b_r$ have unemployment.

the fixed point $\bar{\omega}$ and hence raises $\underline{\omega} = \underline{\Omega}(\bar{\omega})$. □

In terms of Figure 1, the monotonicity of B follows because a reduction in κ reduces $\bar{v}$ which shifts the function $\bar{\Omega}$ down and moves the fixed point down and to the right.

When $\theta = 1$, equation (22) implies $\underline{\omega} = \log b_i \geq \log b_r$, and so there cannot be any rest unemployment. Workers never wait for labor markets to improve because in equilibrium labor market conditions never change.

4.4 The Distribution of ω across Workers

We now characterize the density of ω among workers who are in a labor market, employed or rest-unemployed. We assume $\theta \neq 1$ and so $\sigma > 0$ throughout this section. We take as given that the log full-employment wage in each labor market j is a Brownian motion regulated on $[\underline{\omega}, \bar{\omega}]$. When a positive shock hits a labor market with $\omega(j,t) = \bar{\omega}$, ω stays constant and the labor force l increases (see Figure 2). Conversely, negative shocks reduce ω without affecting l. At $\underline{\omega} < \omega(j,t) < \bar{\omega}$, both positive and negative shocks affect ω but leave l unchanged. When $\omega(j,t) = \underline{\omega}$, a negative shock reduces l without affecting ω, while a positive shock raises ω and keeps l unchanged. We allow for $\underline{\omega} = -\infty$, as Proposition 1 shows happens if $b_r = b_i$, in which case the process for ω is not regulated below. In addition, each labor market shuts down at rate λ, independent of current conditions, and reappears as a labor market with productivity $x_0 \sim F(x_0)$ and log wage $\bar{\omega}$.

Proposition 3. *Assume $\theta \neq 1$. In steady state, the distribution of the fraction of workers* of L that are located in markets with ω has density $f(\omega)$ given by:

$$f(\omega)=\frac{\sum_{i=1}^{2}|\eta_i+\theta|e^{\eta_i(\omega-\underline{\omega})}}{\sum_{i=1}^{2}|\eta_i+\theta|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}}, \tag{28}$$

where $\eta_1<0<\eta_2$ solve the characteristic equation $\lambda=-\mu\eta+\frac{\sigma^2}{2}\eta^2$. Under condition (10), $\eta_1<-\theta$.

If aggregate consumption and output is Y, the share of workers in labor markets is

$$L=\frac{-2\lambda Y u'(Y)^{\theta}(AX_0)^{\theta-1}e^{-\theta\bar{\omega}}}{\sigma^2(\theta+\eta_1)(\theta+\eta_2)\left(e^{\eta_2(\bar{\omega}-\underline{\omega})}-e^{\eta_1(\bar{\omega}-\underline{\omega})}\right)}\sum_{i=1}^{2}|\theta+\eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}. \tag{29}$$

Our proof of this Proposition in Appendix B derives the appropriate differential equation and boundary condition for f.[4]

4.5 Consistency of Wages and Output

Equation (16) implies output in a market with l workers and log full-employment wage ω is

$$Y^{\frac{-1}{\theta-1}}\left(\frac{e^{\omega}l}{u'(C)}\right)^{\frac{\theta}{\theta-1}}\min\{1,e^{\omega}/b_r\}^{\theta}.$$

Substitute this into equation (2) and simplify to get

$$\begin{aligned} Y &= \frac{L}{u'(C)}\int_{\underline{\omega}}^{\bar{\omega}} e^{\omega}\min\{1,e^{\omega}/b_r\}^{\theta-1}f(\omega)d\omega \\ &= \frac{Le^{\bar{\omega}}}{u'(C)}\frac{\sum_{i=1}^{2}|\theta+\eta_i|e^{-(\bar{\omega}-\hat{\omega})}\left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})}-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta+\eta_i}+e^{\eta_i(\bar{\omega}-\underline{\omega})}\frac{e^{(\bar{\omega}-\hat{\omega})}-e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1+\eta_i}\right)}{\sum_{i=1}^{2}|\theta+\eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}}, \end{aligned} \tag{30}$$

where $\hat{\omega}\equiv\max\{\underline{\omega},\log b_r\}$. Substituting for L from equation (29), we can eliminate $Y=C$ and solve for $u'(Y)$:

[4] In a previous version of this paper, we derived the joint distribution of ω and l across markets. Since we no longer use the joint density, we have relegated the derivation to online Appendix G.

Proposition 4. For $\theta \neq 1$, output solves

$$u'(Y)^{1-\theta} = \frac{-2\lambda(AX_0 e^{-\bar{\omega}})^{\theta-1}}{\sigma^2(\theta+\eta_1)(\theta+\eta_2)\left(e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})}\right)} \times \\ \sum_{i=1}^{2} |\theta+\eta_i| e^{-(\bar{\omega}-\hat{\omega})} \left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})} - e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta+\eta_i} + e^{\eta_i(\bar{\omega}-\underline{\omega})} \frac{e^{(\bar{\omega}-\hat{\omega})} - e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1+\eta_i} \right). \quad (31)$$

Proof. This follows immediately from equations (29) and (30). □

Since Proposition 1 determined the thresholds as functions of model parameters, this consistency condition pins down output. Suppose there were two equilibria with different levels of output and consumption. In the high consumption, high output equilibrium, the marginal utility of consumption is lower. If nothing else changed, ω would be lower in all markets. But we know from Proposition 1 that the distribution of ω is the same in all equilibria, so the number of workers in each market must be lower, which is inconsistent with higher equilibrium consumption.

There is no condition analogous to this in Lucas and Prescott (1974). In that model, all goods are perfect substitutes in consumption but there are diminishing returns in each labor market because of a fixed factor of production. Labor demand in each labor market depends on productivity and wages but not on aggregate output.

4.6 Equilibrium: Definition, Existence, and Uniqueness

It is now straight-forward to provide a precise definition of an equilibrium, prove it exists and is unique, and give a simple algorithm for finding it. Our definition focuses on the case when $\theta \neq 1$, but we comment below on how it changes when $\theta = 1$.

Definition 1. An equilibrium is a value function v, thresholds $\underline{\omega}$ and $\bar{\omega}$, and consumption $C > 0$ such that (i) v, $\underline{\omega}$, and $\bar{\omega}$ solve equations (21) and (23); and (ii) individual labor market and aggregate output are consistent, i.e. equation (31) holds.

Proposition 5. There exists a unique equilibrium.

Proof. When $\theta \neq 1$, Proposition 1 implies there are unique thresholds $\underline{\omega}$ and $\bar{\omega}$ solving equations (21)–(23). Equation (31) then uniquely defines $Y = C$.

Now consider $\theta = 1$. Workers enter labor markets until the wage falls to $\bar{\omega} = \bar{\omega}^*$ defined in equation (22). It then remains constant since productivity shocks do not affect wages and so there is neither entry nor exit as long as the labor market survives. In an economy with a degenerate wage distribution, the wage is the ratio of output to employment and so employment in all labor markets is $l = Yu'(Y)e^{-\bar{\omega}^*}$ and output in market j at time t is $q(j,t) = lAx(j,t) = Yu'(Y)e^{-\bar{\omega}^*}Ax(j,t)$. Since the distribution of $x(j,t)$ is known from equation (7), we can use equation (2) for the special case of $\theta = 1$ to get

$$\frac{e^{\bar{\omega}^*}}{u'(Y)} = Ae^{\int_0^1 \log x(j,t)dj} = AX_0 e^{\mu_x/\lambda}.$$

This is the same wage as prevailed in the frictionless economy. Finally, use equation (22) to recover consumption and income. We have thus constructed the unique equilibrium.

4.7 Measurement of Employment and Unemployment

Once we have found an equilibrium, we can measure the key labor market variables. To find employment, rewrite equation (30) as

$$L = \frac{Yu'(Y)e^{-\bar{\omega}}\sum_{i=1}^{2}|\theta+\eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}}{\sum_{i=1}^{2}|\theta+\eta_i|e^{-(\bar{\omega}-\hat{\omega})}\left(\frac{e^{\eta_i(\hat{\omega}-\underline{\omega})}-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta+\eta_i} + e^{\eta_i(\bar{\omega}-\underline{\omega})}\frac{e^{(\bar{\omega}-\hat{\omega})}-e^{-\eta_i(\bar{\omega}-\hat{\omega})}}{1+\eta_i}\right)}, \tag{32}$$

where as before $\hat{\omega} = \max\{\log b_r, \underline{\omega}\}$. Since Y, $\bar{\omega}$, and $\underline{\omega}$ are known functions of model parameters, this equation determines L. In the special case of log utility, employment depends on preferences and search frictions but not on productivity.

Next, recall U_r is the fraction of household members who are rest-unemployed. If $\log b_r \leq \underline{\omega}$, this is zero. Otherwise, in a market with $\omega \in [\underline{\omega}, \hat{\omega}]$, the rest unemployment rate is $1 - e^{\theta(\omega-\hat{\omega})}$. Integrating across such markets using equation (28) gives

$$\frac{U_r}{L} = \int_{\underline{\omega}}^{\hat{\omega}} \left(1 - e^{\theta(\omega-\hat{\omega})}\right) f(\omega)\, d\omega = \theta\frac{\frac{e^{\eta_2(\hat{\omega}-\underline{\omega})}-1}{\eta_2} - \frac{e^{\eta_1(\hat{\omega}-\underline{\omega})}-1}{\eta_1}}{\sum_{i=1}^{2}|\theta+\eta_i|\left(\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}\right)}. \tag{33}$$

The remaining household members who are in labor markets are employed, $E = L - U_r$.

Now we turn to the search unemployed. Let N_s be the number of workers among L that leave their labor market per unit of time, either because conditions are sufficiently bad or because their labor market has exogenously shut down. Appendix C takes limits of the discrete time, discrete state space model to show that this rate is given by

$$N_s = \frac{\theta\sigma^2}{2}f(\underline{\omega})L + \lambda L. \tag{34}$$

The first term gives the fraction of workers who leave their labor market to keep ω above

$\underline{\omega}$. The second term is the fraction of workers who leave when their market shuts down. In steady state, the fraction of workers who leave labor markets must balance the fraction of workers who arrive in labor markets. The latter is given by the fraction of workers engaged in search unemployment U_s, times the rate at which they arrive to the labor market α, so $\alpha U_s = N_s$. Solve equation (34) using equation (28) to obtain an expression for the ratio of search unemployment to workers in labor markets:

$$\frac{U_s}{L} = \frac{1}{\alpha}\left(\frac{\theta\sigma^2}{2}\frac{\eta_2 - \eta_1}{\sum_{i=1}^{2}|\theta + \eta_i|\frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}} + \lambda\right) \tag{35}$$

To have an interior equilibrium we require that $U_s + U_r + E \leq 1$ so that the labor force is smaller than the total population.[5]

We close this section by noting some homogeneity properties of employment, rest unemployment, search unemployment, and consumption.

Proposition 6. Assume $\theta \neq 1$. Let $b_r = \beta\bar{b}_r$, $b_s = \beta\bar{b}_s$, $b_i = \beta\bar{b}_i$ for fixed $\bar{b}_r$, $\bar{b}_s$, and $\bar{b}_i$. The equilibrium value of the unemployment rate $\frac{U_s+U_r}{U_s+U_r+E}$ and the share of rest-unemployed $\frac{U_r}{U_r+U_s}$ do not depend on β, the level of productivity A, the distribution of productivity in new labor markets F, or the utility function u. The equilibrium value of $u'(Y)$ is proportional to $\frac{\beta}{AX_0}$.

Proof. By inspection, the unemployment rate and share of rest-unemployed are functions of the difference in thresholds $\bar{\omega} - \underline{\omega}$ and $\bar{\omega} - \hat{\omega}$ and the parameters α, λ, θ, μ (or μ_x), and σ (or σ_x), either directly or indirectly through the roots η_i. From Proposition 1, the thresholds depend on the same parameters and on the discount rate ρ. This completes the first part of the proof.

Next, recall from Proposition 1 that $e^{\underline{\omega}}$ and $e^{\bar{\omega}}$ are proportional to β. Then Proposition 4 implies $u'(Y)$ inherits the same proportionality. On the other hand, Proposition 1 implies AX_0 does not affect any of the thresholds and so Proposition 4 implies $u'(Y)$ is inversely proportional to AX_0. □

This proposition shows that the unemployment rate and composition of unemployment is determined by the relative advantage of different leisure activities, while output, and hence consumption and employment, depends on an absolute comparison of leisure versus market production. Indeed, the finding that $u'(C)$ is proportional to β/AX_0 holds in the frictionless benchmark, where an interior solution for the employment rate requires $b_i = u'(C)w$,

[5] If this condition fails, all household members participate. The equilibrium is equivalent to one with a higher leisure value of inactivity, the value of b_i such that $U_s + U_r + E = 1$. In any case, Proposition 6 implies that for b_r, b_s, and b_i large enough, the equilibrium has $U_s + U_r + E < 1$.

while the wage is proportional to AX_0 (see equation 11). Whether higher productivity lowers or raises equilibrium employment depends on whether income or substitution effects dominate in labor supply. With $u(C) = \log C$, an increase in productivity raises consumption proportionately without affecting employment or labor force participation.

5 Quantitative Evaluation

The goal of this section is to use our model to understand the role of search and rest unemployment in the U.S. economy. Because we have closed-form expressions for the unemployment and labor force participation rates, comparative statics are relatively straightforward. It is also straightforward to see how various parameters that affect other variables of interest, including the stochastic process for wages, measures of job creation and destruction, and the hazard rate of exiting unemployment.

We focus throughout this section on a convenient limit of the model, when the exogenous shut-down rate λ is zero. We introduced the assumption that intermediate goods markets are destroyed at a positive rate mainly for technical convenience, to ensure an invariant distribution of productivity and employment. Still, with the natural parameter restriction, $\mu = -\theta\sigma^2/2$, discussed before, the economy is well behaved even when λ limits to zero.

It is clear from Proposition 1 that $\underline{\omega}$ and $\bar{\omega}$ converge nicely for any value of μ as long as the discount rate ρ is positive. More problematic is whether aggregate employment and consumption converge. Recall from the frictionless benchmark that if $\mu_x + (\theta - 1)\sigma_x^2/2 = 0$, the wage is independent of λ (see equation 11). The same parameter restriction, which is equivalent to $\mu + \theta\sigma^2/2 = 0$, ensures a well-behaved limit of the frictional economy. We assume this restriction throughout this section.

5.1 The Limiting Economy

When $\mu + \theta\sigma^2/2 = 0$ and $\lambda \to 0$, the roots of the characteristic equation in Proposition 3 converge to $\eta_1 = -\theta$ and $\eta_2 = 0$. Taking limits in equation (28), we find that the density of ω across workers in labor markets is uniform on $[\underline{\omega}, \bar{\omega}]$, $f(\omega) = \frac{1}{\bar{\omega}-\underline{\omega}}$. With this density, the remaining expressions simplify considerably. Equation (31) implies that output is positive and finite in the limiting economy:

$$u'(Y)^{1-\theta} = \frac{(AX_0 e^{-\bar{\omega}})^{\theta-1}}{1 - e^{-\theta(\bar{\omega}-\underline{\omega})}} \Big(e^{-(\bar{\omega}-\hat{\omega})}\big(1 - e^{-\theta(\hat{\omega}-\underline{\omega})}\big) + \theta\big(1 - e^{-(\bar{\omega}-\hat{\omega})}\big)\Big).$$

We can similarly compute limits of the key measures of employment and unemployment. From equation (32), the fraction of household members in labor markets is

$$L = \frac{Y u'(Y) e^{-\hat{\omega}} (\bar{\omega} - \underline{\omega})}{e^{\bar{\omega} - \hat{\omega}} - 1 + \frac{1 - e^{-\theta(\hat{\omega} - \underline{\omega})}}{\theta}},$$

while from equations (33) and (35), rest and search unemployment converge to

$$\frac{U_r}{L} = \frac{\theta(\hat{\omega} - \underline{\omega}) + e^{-\theta(\hat{\omega} - \underline{\omega})} - 1}{\theta(\bar{\omega} - \underline{\omega})} \quad \text{and} \quad \frac{U_s}{L} = \frac{\theta \sigma^2}{2\alpha \left(\bar{\omega} - \underline{\omega}\right)}. \tag{36}$$

Thus, although productivity and employment do not have an invariant distribution, aggregate output, employment, and unemployment are well-behaved in this limit.

5.2 No Rest Unemployment

To connect the model to the data, we find it useful first to examine the reduced-form relationships in equation (36), without worrying about the determinants of the three thresholds.[6] Focus first on U_s/L and assume there is no rest unemployment, $\hat{\omega} = \underline{\omega}$. One can think of the search unemployment rate depending on three forces. $1/\alpha$ is the mean duration of a spell of search unemployment and so not surprisingly U_s/L is decreasing in α. $\frac{\bar{\omega} - \underline{\omega}}{\sigma}$ determines the average time it takes a labor market to move from the hiring threshold to the firing threshold, and so is related to the duration of employment. And $\theta\sigma$ determines how many workers must exit a labor market at the lower threshold following a one standard deviation productivity shock, and so is also important for determining the duration of employment.

To determine reasonable values for these variables, we map our model to the data. We do not believe our model is well-suited for explaining the job search patterns of young workers, who move frequently from employer to employer while trying to find a good career (Jovanovic, 1979; Neal, 1999). To be conservative, we focus instead on 45 to 54 year old workers. From 1995 to 2006, the unemployment rate for this group averaged 3.3 percent. The mean duration of an in-progress unemployment spell was 0.4 years, which is generated by $\alpha = 2.5$, while the median was about 0.2 years, consistent with $\alpha = 3.5$.[7]

Next, we need to take a stand on the nature of a labor market. A labor market has two defining characteristics. First, the goods produced within a labor market are homoge-

[6]The thresholds are determined by the discount rate ρ and the three leisure values, b_i, b_r, and b_s, in addition to the parameters that directly enter equation (36), θ, σ, and α.

[7]With a constant exit hazard α, the mean duration of unemployment is $\int_0^\infty t\alpha e^{-\alpha t} dt$ and the median duration is $\log 2/\alpha$. The empirical duration numbers were constructed by the Bureau of Labor Statistics from the Current Population Survey and may be obtained from http://www.bls.gov/cps/.

neous while the goods produced in different labor markets are heterogeneous, as captured by the elasticity of substitution θ. This suggests modeling a labor market as an industry. Second, it is free to move within a labor market but costly to move between labor markets, presumably both because of some specificity of human capital and because of geographic mobility costs. To the extent that human capital is occupation, not industry, specific (Kambourov and Manovskii, 2007), this suggests that a labor market may be a cross between an occupation and a geographic location. In the end, our definition of a labor market is governed by data availability: we measure a labor market as a four- or five-digit NAICS industry. Using international trade data, Broda and Weinstein (2006) report median estimates of the elasticity of substitution between fairly homogeneous goods of 2.2 to 3.7 (see their Table IV). We therefore consider values of θ between 2 and 4.

Rather than take a stand on values of σ and $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$, we observe that σ is critical for the volatility and $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ for the autocorrelation of wages at the industry level. That σ affects the volatility is immediate from equation (19). When $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ is large, wages are nearly a random walk and so this year's wage level is very informative about next year's, i.e. the autocorrelation of wages is large. For small values of $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$, wages hit the bounds frequently within a short period of time and so are nearly uncorrelated.[8] We measure industry-level average weekly earnings for 233 four-digit and 312 five-digit industries from 1990–2006 from the Current Employment Statistics (`http://www.bls.gov/ces/`), all the industries with available data.[9] We deflate the nominal annual average of industry earnings by the nominal annual average of private sector earnings to construct $y_{j,t}$, average weekly earnings in industry j and year t.[10] We then measure the standard deviation of $\log y_{j,t} - \log y_{j,t-1}$ and the correlation between $\log y_{j,t}$ and $\log y_{j,t-1}$ and compare with model-generated data.

To do this, we generate data from a discrete time version of the model, where a time period is 1 week. Each week, we add a normal innovation to log wages. At the end of each year, we average the data and we compute the standard deviation of the growth rate and the autocorrelation of the level of earnings over a 17 year period. We repeat the simulation 10,000 times to obtain accurate estimates of these two moments.

To begin, we match the empirical search unemployment rate of $\frac{U_s}{U_s+L} = 0.033$ and set $\alpha = 2.5$ and $\theta = 4$. This is consistent with many different values of σ and $\bar{\omega} - \underline{\omega}$, each

[8] Ball and Roma (1998) find an exact formula for the autocorrelation of annual observations from a reflected Brownian motion without drift and prove it depends only on $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$.

[9] For example, motor vehicles are a four-digit industry while automobiles and light trucks are a five-digit industry. Data for at the six-digit level (e.g. automobiles) are available for 175 industries and look similar to the five-digit data that we report here.

[10] Fluctuations in productivity A may cause fluctuations in average earnings. With log utility, such fluctuations cause proportional changes in wages but do not affect the unemployment rate. We can therefore perfectly control for aggregate fluctuations through this deflator.

associated with a different standard deviation of the growth rate and autocorrelation of the level of earnings. Figure 3 shows that for any value of σ, the average autocorrelation of the level of earnings is smaller than 0.58, which we obtain by setting $\sigma = 0.17$. In contrast, the empirical average autocorrelation of the level of earnings is 0.76 at the four-digit level and 0.73 at the three digit level.[11]

To obtain the highest feasible autocorrelation wages must be very volatile—the standard deviation of the annual growth rate is 0.10 when $\sigma = 0.17$. In contrast, the empirical average standard deviation of the annual growth rate of earnings is 0.03 at both the four-digit and the five-digit level. When $\sigma = 0.07$, we obtain a theoretical standard deviation of 0.03, but the autocorrelation is far lower, just 0.34. Reasonable changes in α and θ only exacerbate this problem. From equation (36), an increase in α or a decrease in θ reduces the value of $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ or raises the value σ consistent with a given unemployment rate. Although this does not affect the maximum possible theoretical autocorrelation, it raises the standard deviation of the annual growth rate of earnings consistent with any autocorrelation.

Alternatively, we can reduce the amount of search unemployment that we ask the model to generate. If $\frac{U_s}{U_s+L} = 0.01$ and $\alpha = 2.5$ and $\theta = 4$, the maximum possible autocorrelation is 0.68. With $\frac{U_s}{U_s+L} = 0.005$, setting $\sigma = 0.055$ (and hence $\bar{\omega} - \underline{\omega} = 0.447$) generates an autocorrelation of 0.72 and a standard deviation of the growth rate of 0.04. With such low search unemployment rates, average weekly earnings are nearly a random walk, bringing the model broadly in line with the data.

Of course, by allowing λ to be positive, the model can generate search unemployment despite having persistent earnings. Keeping the same values of the other parameters as in the previous paragraph—$\alpha = 2.5$, $\theta = 4$, $\sigma = 0.053$, and $\bar{\omega} - \underline{\omega} = 0.447$—we can raise the unemployment rate to 3.3% by setting $\lambda = 0.081$, so labor markets 12 years on average. With these parameters, 95% of unemployment spells start when a labor market shuts down. We find it implausible that shocks that shut down industries are so common, and we find a model where exogenous parameters govern both unemployment incidence (λ) and duration (α) to be uninteresting.

Finally, we note that for younger age groups, the unemployment rate is higher and unemployment duration is shorter. Both forces imply stronger mean reversion in wages and so the model with only search unemployment can generate an even smaller share of the unemployment experienced by these cohorts while maintaining a plausible stochastic process for wages at the industry level.

[11] These differences are highly significant. Kendall's (1954) formula for the small-sample bias indicates that if log wages were a random walk, we would estimate an autocorrelation of 0.76 using 17 years of data. Thus empirically wages are close to a random walk, while this calibration of the model delivers strong mean reversion.

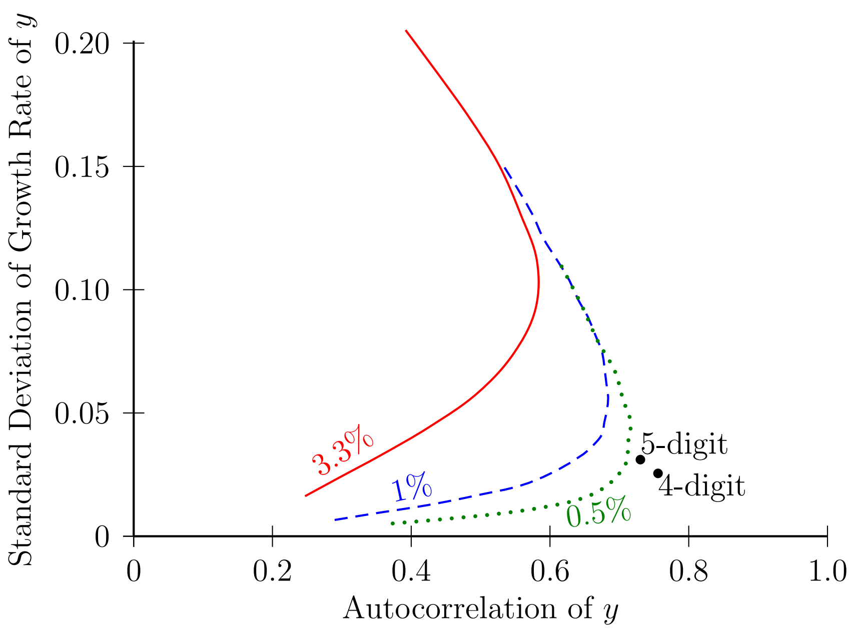

Figure 3: The solid red line shows the correlation between $\log y_{j,t}$ and $\log y_{j,t-1}$ and the standard deviation of $\log y_{j,t} - \log y_{j,t-1}$ consistent with $\alpha = 2.5$, $\theta = 4$, and a 3.3% unemployment rate with no rest unemployment. The dashed blue line shows the combinations consistent with 1% search unemployment while the dotted green line shows 0.5% search unemployment. The two larger dots show the empirical averages at the four-digit and five-digit NAICS levels.

5.3 Reintroducing Rest Unemployment

Reintroducing rest unemployment to the model improves its fit for several reasons: it creates another source of unemployment; it reduces the standard deviation of wages by creating an interval $[\underline{\omega}, \hat{\omega}]$ where wages are constant from year-to-year; and it raises the autocorrelation of wages because of the persistence generated by labor markets that spend time in this interval. The last two forces imply that the presence of rest unemployment may permit more search unemployment without generating unrealistically low autocorrelations or high standard deviations of growth rates of average weekly earnings.

To be concrete, we again assume $\lambda = 0$ and keep $\alpha = 2.5$ and $\theta = 4$. For arbitrary σ, we find the value of $\bar{\omega} - \underline{\omega}$ consistent with $\frac{U_s}{U_s+L} = 0.005$. We then set $\hat{\omega} - \underline{\omega}$ to ensure that the overall unemployment rate is 3.3%, i.e. $\frac{U_r}{U_s+L} = 0.028$. When $\sigma = 0.053$, this procedure sets $\bar{\omega} - \underline{\omega} = 0.447$ and $\hat{\omega} - \underline{\omega} = 0.084$. The standard deviation of the growth rate of earnings is 0.03 and the autocorrelation of earnings is 0.71, both in line with the data at the five-digit NAICS level. Quantitatively, the possibility of rest unemployment does not much affect the behavior of wages for a given search unemployment rate.

5.4 Structural Parameters

We can back out the structural parameters consistent with this search and rest unemployment. To be consistent with balanced growth, we assume $u(\cdot) = \log(\cdot)$. Still thinking of a unit of a time as a year, we set $\rho = 0.05$. The remaining parameters are three leisure values. We set $b_i = 1.16$, $b_r = 1.1$, and $b_s = -6$ in order to match three numbers: $\frac{U_r}{U_s+L} = 0.028$, $\frac{U_s}{U_s+L} = 0.005$, and $U_s + L = 0.82$, where the last number is the labor force participation rate for 45 to 54 year olds from 1995 to 2006.

These numbers reveal two regularities. First, in order to generate rest unemployment, b_r/b_i must be close to 1. In this case, we require $b_r/b_i > 0.896$ to have any rest unemployment (see Proposition 2). This suggests that, while the rest unemployed must pay some cost to remain in contact with their labor market, the cost is small. Put differently, rest unemployment and inactivity may look quite similar to an outsider who observes individuals' time use, even though the rest unemployed may be much more likely to return to work.

Second, to generate a low search unemployment rate, we need b_s to be small; here it is actually negative. The reason is that a low search unemployment rate requires a large gap region of inaction $\bar{\omega} - \underline{\omega}$. But this implies the wage in the most productive labor markets is much higher than the wage in less productive markets. In order for workers to be willing to endure such an unproductive market, the cost of moving, $\kappa = \frac{b_i - b_s}{\alpha b_i}$, must be large. This is essentially the contrapositive of Hornstein, Krusell, and Violante's (2006) finding that when search costs are small, search models cannot generate much wage dispersion. Introducing other mobility costs, such as market-specific human capital, may alleviate this issue.

5.5 Job Creation and Destruction

We can use the model to measure gross job creation and job destruction rate at the industry level. Following Davis, Haltiwanger, and Schuh's (1996) analysis of firms, we define the number of jobs destroyed in a labor market between t_0 and t_1 as the decrease in the number of employed workers in that labor market between those dates, or zero if the number of employed workers increased. Job creation is defined symmetrically. The gross job destruction (creation) rate is then defined as the total number of jobs destroyed (created) across all labor markets divided by employment E.

The job creation and destruction rates are easily computed numerically. Consider a labor market that has a log full-employment wage ω at t_0. Using Monte Carlo, we find the job creation and destruction rates at t_1; these rates depend on ω but are independent of the number of workers in the market. We then take a weighted average of job creation and destruction rates, weighting by the fraction of employed workers at each value of ω, $e^{\theta \min\{\omega-\hat{\omega},0\}} f(\omega)$, where $e^{\theta \min\{\omega-\hat{\omega},0\}}$ accounts for rest unemployment in markets with $\omega < \hat{\omega}$.

When there is no rest unemployment, jobs are destroyed only to keep ω regulated above $\underline{\omega}$. The instantaneous job destruction rate is then given by N_s in equation (34). Moreover, once a job is destroyed, it is only recreated if the market experiences a series of shocks that brings ω back to $\bar{\omega}$. Over annual frequencies, this probability is negligible if $\frac{\bar{\omega}-\underline{\omega}}{\sigma}$ is large enough to give a plausible autocorrelation of earnings. Thus we would expect the job destruction rate over a time horizon t to be roughly equal to $N_s t$. Unreported simulations support this intuition.

In the full model, jobs are also destroyed when workers become rest unemployed, which is easily reversed. We argue in Appendix D that when the rest unemployment rate is not too small, the job destruction rate over an interval of length t is approximately

$$JD(t) = \frac{\frac{\theta\sigma^2}{4} t + \frac{1}{\sqrt{2\pi}} \sigma \left(1 - e^{-\theta(\hat{\omega}-\underline{\omega})}\right) \sqrt{t}}{\bar{\omega} - \hat{\omega} + \frac{1-e^{-(\hat{\omega}-\underline{\omega})\theta}}{\theta}}, \tag{37}$$

In practice, this approximation is extremely accurate. To see this, we again set $\alpha = 2.5$, $\theta = 4$, $\sigma = 0.053$, $\bar{\omega} - \underline{\omega} = 0.447$, and $\hat{\omega} - \underline{\omega} = 0.084$, which generates 0.5% search unemployment and 2.8% rest unemployment. The solid blue line in Figure 4 shows that the job creation and job destruction rates, which are the same since the economy is in an aggregate steady state, are concave functions of time. The dashed red line shows $JD(t)$, indistinguishable from the

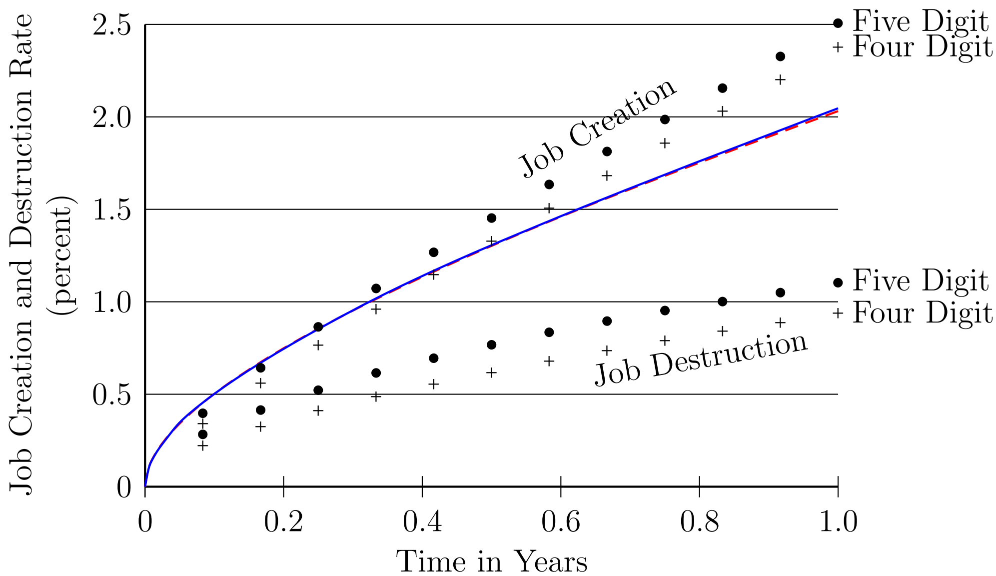

Figure 4: The solid blue line shows the job creation and destruction rate using Monte Carlo. The dashed red line shows $JD(t)$ in equation (37). The parameter values are in the text. The pluses show show job creation and destruction for four-digit industries at different frequencies. The dots show it for five-digit industries.

Monte Carlo results at this time horizon. Thus we can usefully view the job destruction rate as the sum of a term that is linear in t and a term that is linear in $\sqrt{t}$. With these parameter values, the weight on the linear term is about $\frac{1}{3}$ and so the curvature in $JD(t)$ is visible even after one year.

We can compare this finding with data on job creation and job destruction at the industry level. We use the Current Employment Statistics measure of monthly employment for 258 four-digit and 387 five-digit NAICS industries from 1990 to 2006 and adjust the data to eliminate multiplicative seasonal factors.[12] Figure 4 shows a clear concave pattern in these measures as well. In the data, the ratio of job destruction at annual and quarterly frequencies is 2.3 (four-digit) or 2.1 (five-digit). For job creation, the ratio is somewhat higher, 3.1 (four-digit) or 2.9 (five-digit). According to the calibrated model, the ratio of job creation or destruction at annual and quarterly frequencies should be 2.4, in line with the data.

The finding that job creation and destruction increase less than linearly with time is consistent with other datasets. Davis, Haltiwanger, and Schuh (1996) report a quarterly job destruction rate for manufacturing establishments of 5.5% and an annual destruction rate

[12]The data shows net employment growth, which is easily introduced to the model by making households increase in size over time. We find the existence of significant seasonal factors intriguing. It would be hard to get workers moving in and out of the labor force seasonally when doing so entails time-consuming search. It is easier instead to move between employment and rest unemployment. Incorporating seasonal fluctuations in the value of leisure into the model goes beyond the scope of this paper.

of 10.3%.[13] Using the same data set, Schuh and Triest (2000) report 27.4% job destruction at a five year horizon. Faberman (2003) studies all private sector establishments in 53 Metropolitan Statistical Areas in five states. He reports 6.7% job destruction at quarterly frequencies and 11.4% at annual frequencies. Using microdata from the Job Openings and Labor Turnover Survey, Davis, Faberman, and Haltiwanger (2006) report a monthly job destruction rate of 1.5% and quarterly job destruction rate of 3.1%. From the Business Employment Dynamics survey of all private establishments, they report 7.6% quarterly job destruction and 13.7% annual. Each of these papers finds similar curvature for job creation. The strong reversibility of job creation and job destruction, consistent with firms having easy access to a pool of rest-unemployed workers, appears to be a robust empirical fact.

5.6 Hazard Rate of Exiting Unemployment

When there is no rest unemployment, the hazard of exiting unemployment is simply α. This section characterizes the hazard of exiting unemployment when there is rest unemployment, $\hat{\omega} > \underline{\omega}$. Observe that our model determines the number of workers who are employed and rest unemployed in each labor market j and the number of workers who are in search unemployment. It does not determine which of the workers in a labor market with rest unemployment is employed, nor who exits a market when ω hits $\underline{\omega}$, nor which workers are in search unemployment rather than inactivity. To discuss labor market flows, we need to determine these variables.

First, we impose that some workers are permanently inactive. This means that workers who leave their labor market become search unemployed. This can be interpreted as the outcome of heterogeneity in preferences, in which case our analysis considers the limit as the heterogeneity vanishes.

To determine who works and who is unemployed, suppose that there are many different tasks that can be performed within a labor market, some more productive than others. Each task can be performed by only one worker. Workers are assigned to a task when they enter the market, so the market assigns newly arrived workers to the most productive vacant task. If there is any cost of reallocating workers across tasks, variation in market-wide productivity x induces the worker assigned to the least productive task to be the first to enter rest unemployment and the first to exit the labor market. Again, our analysis correspond to the limit as this heterogeneity in tasks vanishes.

To see the implications for unemployment duration, name the l workers in a labor market

[13] Our model is inconsistent with annual job destruction less than half as big as quarterly job destruction. There is a significant seasonal component in Davis, Haltiwanger, and Schuh's (1996) data which may explain this finding.

$n \in [0, l]$. Workers keep their name as long as they remain in that market. If ω hits $\bar{\omega}$, new entrants to the market are assigned correspondingly higher names. The task assignment problem implies worker n works if

$$(\theta - 1)\log(Ax) \geq \log n + \theta(\hat{\omega} - \log u'(C)) - \log Y \equiv (\theta - 1)\log(A\hat{x}(n)),$$

quits the market if

$$(\theta - 1)\log(Ax) < \log n + \theta(\underline{\omega} - \log u'(C)) - \log Y \equiv (\theta - 1)\log(A\underline{x}(n)),$$

and is rest-unemployed otherwise. Note that $\log \hat{x}(n) - \log \underline{x}(n) = \frac{\theta}{\theta-1}(\hat{\omega} - \underline{\omega})$, independent of n. The adverse productivity shock required to a move a worker from the margin between working and being rest unemployed to the margin between rest and search unemployed is the same for all workers. This implies that the hazard rate of exiting unemployment conditional on unemployment duration is the same for all workers.

Let $H(t, \Delta)$ be the probability that a worker who has had a (rest or search) unemployment spell of duration t will transit from unemployment to employment at least once in the next Δ units of time and let $h(t) = \lim_{\Delta \to 0} H(t, \Delta)/\Delta$ be the associated hazard rate. Given values for the thresholds $\underline{\omega}$, $\hat{\omega}$, and $\bar{\omega}$, we can compute the hazard rate $h(t)$ as the weighted average of two hazard rates:

$$h(t) = \hat{h}_r(t)\frac{u_r(t)}{u_r(t) + u_s(t)} + \alpha\frac{u_s(t)}{u_r(t) + u_s(t)}$$

where $\frac{u_r(t)}{u_r(t)+u_s(t)}$ is the probability that a worker with unemployment duration t is rest-unemployed. For a search-unemployed worker, spells end at rate α, independent of the duration of the spell. For a rest-unemployed worker, her spell ends when local labor market conditions improve enough for her to reenter employment. We let $\hat{h}_r(t)$ denote that hazard rate of this event. It is also useful to let $\underline{h}_r(t)$ denote the hazard of exiting rest unemployment for search unemployment. We prove in Appendix E that

$$\hat{h}_r(t) = \frac{\sum_{n=1}^{\infty} n^2 e^{-t\psi_n}}{\sum_{n=1}^{\infty} \frac{n^2}{\psi_n} e^{-t\psi_n}\left(1 - (-1)^n e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)} \tag{38}$$

$$\underline{h}_r(t) = \frac{-\sum_{n=1}^{\infty} n^2 e^{-t\psi_n}(-1)^n e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{\sum_{n=1}^{\infty} \frac{n^2}{\psi_n} e^{-t\psi_n}\left(1 - (-1)^n e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)},$$

where

$$\psi_n \equiv \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{(\hat{\omega} - \underline{\omega})^2}\right).$$

These sums are easily calculated numerically.

We then compute the duration-contingent unemployment rates by solving a system of two ordinary differential equations with time-varying coefficients:

$$\dot{u}_r(t) = -u_r(t)(\lambda + \underline{h}_r(t) + \hat{h}_r(t)) \text{ and } \dot{u}_s(t) = -u_s(t)\alpha + u_r(t)(\lambda + \underline{h}_r(t)) \tag{39}$$

for all $t > 0$. The number of workers in rest unemployment falls as markets shut down, as they exit the market for search unemployment, and as they reenter employment. In the first two events, they become search unemployed, while search unemployment falls at rate α as these workers find jobs. To solve these differential equations, we require two boundary conditions; however, to compute the share of rest unemployed in the unemployed population with duration t, $\frac{u_r(t)}{u_r(t)+u_s(t)}$, we need only a single boundary condition,

$$\frac{\int_0^\infty u_r(t)dt}{\int_0^\infty u_s(t)dt} = \frac{U_r}{U_s}, \tag{40}$$

where U_r and U_s are given in equations (33) and (35).

The hazard rate is particularly easy to characterize both at short and long durations. When t is small, we find that $\hat{h}_r(t) \approx \frac{1}{2t}$. Intuitively, consider a market with $\omega = \hat{\omega}$. After a short time interval—short enough that the variance of the Brownian motion dominates the drift—there is a $\frac{1}{2}$ probability that ω has increased, so the worker is reemployed, and a $\frac{1}{2}$ chance it has fallen. But a one-half probability over any horizon t implies a hazard rate $1/2t$.

When t is large, the first term of the partial sum in equation (38) dominates,

$$\lim_{t\to\infty} \hat{h}_r(t) = \frac{\psi_1}{1 + e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}} \text{ and } \lim_{t\to\infty} \underline{h}_r(t) = \frac{\psi_1 e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}{1 + e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}}.$$

In addition, if $\alpha > \lambda + \psi_1$,

$$\lim_{t\to\infty} \frac{u_r(t)}{u_s(t)} = \frac{(\alpha - \psi_1 - \lambda)\left(1 + e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}\right)}{\lambda + (\lambda + \psi_1)e^{-\frac{\mu(\hat{\omega}-\underline{\omega})}{\sigma^2}}},$$

while otherwise the limiting ratio is zero. Together this implies $\lim_{t\to\infty} h(t) = \min\{\psi_1 + \lambda, \alpha\}$, a function only of the slow exit rate. In our baseline calibration, $\alpha = 2.5$, $\lambda = 0$, and $\psi_1 = 1.98$, so the latter governs the exit rate from long-term unemployment. The efficiency of search affects the hazard of exiting long-term unemployment only indirectly, through its influence on the distance between the rest unemployment boundaries $\hat{\omega} - \bar{\omega}$.

Figure 5 shows the annual hazard rate of finding a job in our baseline calibration, including

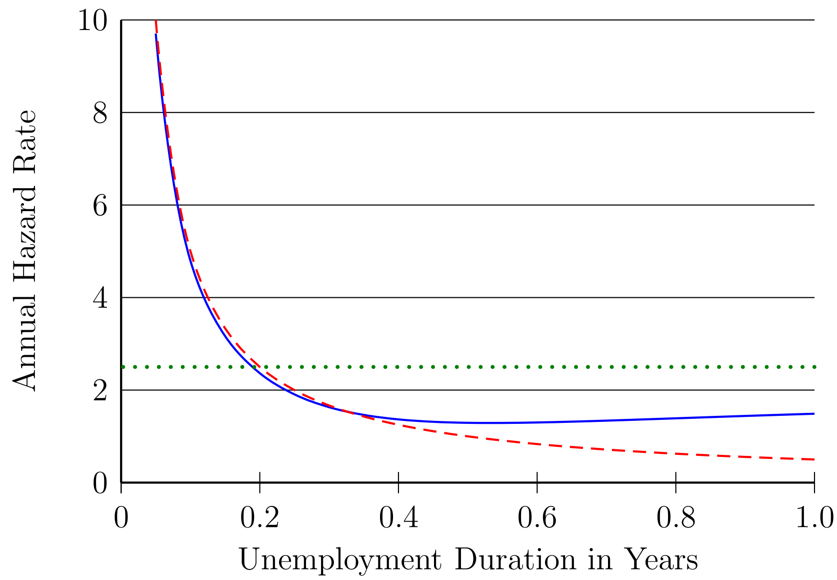

Figure 5: The solid blue line shows the hazard rate of finding a job as a function of unemployment duration. The dashed red line shows $1/2t$. The dotted green line shows $\alpha = 2.5$. The parameter values are in the text.

a 2.8% rest unemployment rate. The overall hazard rate generally mimics the behavior of $\hat{h}_r(t)$, especially at short unemployment durations, when most unemployed workers are in rest unemployment. At longer durations, $\hat{h}_r(t)$ continues to fall, asymptoting to about 0.91. The overall hazard rate actually starts to increase after about half a year of unemployment, as the share of workers in search unemployment, which is negligible for the first two months, starts to grow (Figure 6), eventually reaching 67% at long unemployment durations.

In the model, the mean duration of an in progress unemployment spell is 0.55 years while the median duration is 0.35 years. Both these numbers are significantly larger than in the data, but they do reflect some of the skewness in that we observe in the unemployment duration distribution. With a constant hazard, a mean of 0.55 years would imply $\alpha = 1.83$, while a median of 0.35 years would imply $\alpha = 1.97$.

Our finding of a constant hazard rate for workers in search unemployment and a decreasing hazard rate for workers in rest unemployment is consistent with Katz and Meyer's (1990) observation that the empirical decline in the job finding hazard rate is concentrated among workers on temporary layoff. Moreover, those authors find that workers who expect to be recalled to a past employer and are not—in the parlance of our model, workers who end a spell of rest unemployment by searching for a new labor market, at hazard $\underline{h}_r(t)$—experience longer unemployment duration than observationally equivalent workers who immediately entered search unemployment. In our model, this last group might correspond to workers

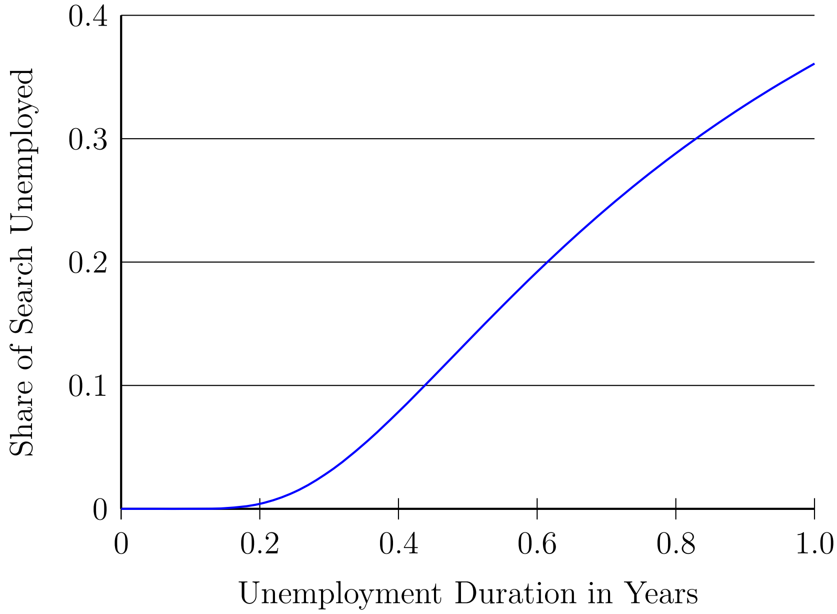

Figure 6: The solid blue line shows $\frac{u_s(t)}{u_r(t)+u_s(t)}$. The parameter values are in the text.

experiencing a λ shock.

6 Discussion

We end this paper by answering the two questions posed in the introduction: Can we distinguish between economies with different amounts of search and rest unemployment? And is the life of a worker in search unemployment substantially different from that of one in rest unemployment?

Our analysis in Section 5 provides a partial answer to the first question. Rest unemployment helps explain why wages are so persistent yet some workers cycle frequently between jobs. It helps explain why job creation and destruction are such concave functions of elapsed time. And it helps explain why the hazard of exiting unemployment declines with unemployment duration. In each case, this is a consequence of workers' ability to cycle costlessly between rest unemployment and work. A worker who just switched between employment and rest unemployment is at the margin between the two states. A small shock will move him back. But the longer a worker remains unemployed, the more likely her labor market has suffered a series of adverse shocks, reducing the hazard of finding a job. The low hazard rate of exiting long-term unemployment may be important for understanding the coexistence of many workers who move easily between jobs and a relatively small number of workers who suffer extended unemployment spells (Juhn, Murphy, and Topel, 1991).

One way to test for the importance of rest unemployment is to examine the fraction of unemployment spells that end with a worker switching industries.[14] Using data from the Panel Study of Income Dynamics (PSID), Loungani and Rogerson (1989) classify workers into two groups, "switchers" and "stayers." Switchers are employed in a base year t, move to a new industry or to unemployment in year $t + 1$, and have not returned to their industry by year $t + 3$. They find that workers who switch two digit industries account for about a quarter of all unemployment spells and a quarter of all weeks of unemployment, while "stayers" account for the remaining three-quarters.

Starr-McCluer (1993) also looks at PSID data, but classifies workers as stayers if they report they returned to the same employer or took a similar job with a new employer. Unfortunately, she does not have any information for workers who are still unemployed at the interview date. She finds that stayers account for half of all unemployment spells but a quarter of all weeks of unemployment, i.e. stayers experience short unemployment spells, 10.1 weeks on average. Switchers account for a quarter of all spells and a quarter of all weeks of unemployment, so their average unemployment duration is about twice as long, 20.9 weeks (or 4.8 months, which implies $\alpha = 0.21$). The remaining quarter of all spells and half of all weeks of unemployment are accounted for by censored spells.

Although Starr-McCluer (1993) guesses that most of the censored spells are switchers, our model's declining hazard rate for the rest-unemployed suggests that many of them would ultimately be classified as stayers. This may be important for reconciling Starr-McCluer's (1993) finding that stayers have a short unemployment duration with Loungani and Rogerson's (1989) that unemployment durations are equal for stayers and switchers.

In short, rest unemployment may be important for understanding the coexistence of a large number of very short unemployment spells with a small number of workers who stay unemployed for years. For example, one would not want to infer that unemployment is nearly costless from the fact that so many unemployment spells are short. In our model, rest unemployment is a manifestation of the type of labor market risk that all workers, employed and unemployed face.

Finally, we are in a position to talk with a worker in rest unemployment. She might tell us that she routinely moves in and out of jobs, and perhaps is classified as working part time.[15] When we ask her why she does not search for a job in a different industry, she explains all

[14] One should be cautious using this type of evidence, however, since there are important idiosyncratic reasons why workers switch labor markets that are absent from our model, such as learning (Jovanovic, 1979) and career matching (Neal, 1999).

[15] From 1994 to 2007, the U.S. unemployment rate averaged 5.1 percent. Including discouraged workers and other marginally attached workers raises this to 6.1 percent, while also adding those who work "part time for economic reasons" raises the average to 8.9 percent. See `http://www.bls.gov/webapps/legacy/cpsatab12.htm`.

the costs of doing so, including gathering the necessary information, retraining, and moving to a new city. Things really are not that bad and might get a lot better.

When we catch up with our worker a year later, she tells us that she has not worked since we last met. Again we ask her why she doesn't look for a better job. She tells us that she is thinking about moving, but in fact many of her former coworkers have already left town. If labor market conditions should improve, she could easily find a job paying a high wage. We ask her, with the benefit of hindsight, if she should have looked for a new job after our previous conversation. She says of course, but stresses that there is no way she could have known how badly things would turn out. When we try to find her after another year has passed, her neighbor tells us she has left town and is now working at a higher wage in a new job. In the end, he says, it only took her five months to find the job.

Appendix

A Density of Productivity x

The density f_x solves a Kolmogorov forward equation:

$$\lambda f_x(\tilde{x}; x_0) = -\mu_x f_x'(\tilde{x}; x_0) + \frac{\sigma_x^2}{2} f_x''(\tilde{x}; x_0)$$

at all $\tilde{x} \neq \tilde{x}_0 \equiv \log x_0$. The solution to this equation takes the form

$$f_x(\tilde{x}; x_0) = \begin{cases} D_1^1(x_0)e^{\tilde{\eta}_1 \tilde{x}} + D_2^1(x_0)e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} < \log x_0 \\ D_1^2(x_0)e^{\tilde{\eta}_1 \tilde{x}} + D_2^2(x_0)e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} > \log x_0, \end{cases}$$

For this to be a well-defined density, integrating to 1 on $(-\infty, \infty)$, we require that $D_1^1(x_0) = D_2^2(x_0) = 0$. To pin down the remaining constants, we use two more conditions: the density is continuous at $\tilde{x} = \log x_0$; and it is a proper density and so must integrate to 1. Imposing these boundary conditions delivers equation (7).

B Steady State Density f

B.1 Proof of Proposition 2

Appendix B.2 characterizes the density in a discrete time, discrete state-space analog of our model. In the limit as we move to a continuous time diffusion, the stationary distribution solves a Kolmogorov forward equation:

$$\lambda f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2} f''(\omega). \tag{41}$$

Solving this differential equation requires two additional constraints. First is a boundary condition at the lower reflecting threshold $\underline{\omega}$:

$$\left(\mu + \frac{\theta\sigma^2}{2}\right) f(\underline{\omega}) = \frac{\sigma^2}{2} f'(\underline{\omega}). \tag{42}$$

Second is the requirement that f is a density:

$$\int_{\underline{\omega}}^{\bar{\omega}} f(\omega) d\omega = 1.$$

Simple algebra then delivers equation (28).

To find L, we use the boundary condition at the upper reflecting threshold $\bar{\omega}$:

$$L = \frac{2\lambda L_0}{\sigma^2 f'(\bar{\omega}) - (2\mu + \theta\sigma^2) f(\bar{\omega})}, \tag{43}$$

which we also derive from the discrete time, discrete state-space model in Appendix B.2. Here L_0 is the number of workers in an average new labor market. A market with productivity x_0 has a log full-employment wage $\bar{\omega}$ when there are $Yu'(Y)^\theta (Ax_0)^{\theta-1} e^{-\theta\bar{\omega}}$ workers in it, so

$$L_0 \equiv \int_0^\infty Yu'(Y)^\theta (Ax_0)^{\theta-1} e^{-\theta\bar{\omega}} dF(x_0) = Yu'(Y)^\theta (AX_0)^{\theta-1} e^{-\theta\bar{\omega}}.$$

With some algebra, we find that

$$\sigma^2 f'(\bar{\omega}) - (2\mu + \theta\sigma^2) f(\bar{\omega}) = \frac{-\sigma^2(\theta+\eta_1)(\theta+\eta_2)\left(e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})}\right)}{\sum_{i=1}^2 |\theta+\eta_i| \frac{e^{\eta_i(\bar{\omega}-\underline{\omega})}-1}{\eta_i}}.$$

Equation (29) then follows by substituting these expressions into equation (43).

B.2 Discrete Approximation

We start by using a discrete time, discrete state space model to obtain the Kolmogorov forward equations and boundary conditions for the density f. Divide $[\underline{\omega}, \bar{\omega}]$ into n intervals of length $\Delta\omega = (\bar{\omega} - \underline{\omega})/n$. Let the time period be $\Delta t = (\Delta\omega/\sigma)^2$ and assume that ω increases by $\Delta\omega$ with probability $(1-\lambda\Delta t)\frac{1}{2}(1+\Delta p)$ for $\Delta p = \mu\Delta\omega/\sigma^2$, decreases by $\Delta\omega$ with probability $(1-\lambda\Delta t)\frac{1}{2}(1-\Delta p)$, and returns to $\bar{\omega}$ with probability $\lambda\Delta t$. Also let $\Delta\tilde{l} = \theta\Delta\omega$ denote the percent decrease in employment needed to keep ω constant following an adverse shock. In the limit as $\Delta\omega \to 0$, the expected value of $\omega(t+\Delta t)$ conditional on $\omega(t)$ is $\omega(t) + \mu\Delta t$ and the conditional standard deviation is $\sigma\sqrt{\Delta t}$.

Let $f_\Delta(\omega)$ denote the discrete time, discrete state version of $f(\omega)$; in the limit as $\Delta\omega \to 0$, the densities are related via $\frac{f_\Delta(\omega)}{\Delta\omega} \to f(\omega)$. Also let $f_\Delta(\omega, t)$ denote a time-varying counterpart of $f_\Delta(\omega)$. For $\omega \in [\underline{\omega} + \Delta\omega, \bar{\omega} - \Delta\omega]$, we can write a backward-looking equation for f_Δ:

$$f_\Delta(\omega, t+\Delta t) = (1-\lambda\Delta t)\tfrac{1}{2}\Big((1+\Delta p) f_\Delta(\omega - \Delta\omega, t) + (1-\Delta p) f_\Delta(\omega + \Delta\omega, t)\Big). \tag{44}$$

The workers at ω at $t+\Delta t$ either were at $\omega - \Delta\omega$ at t and had a positive shock or at $\omega + \Delta\omega$ at t and had a negative shock. Now impose stationarity on f_Δ, dropping the second argument. Take a second order approximation to $f_\Delta(\omega + \Delta\omega)$ and $f_\Delta(\omega - \Delta\omega)$ around ω, substituting

Δt and Δp by the expressions above:

$$f_\Delta(\omega) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(f_\Delta(\omega) - \mu\frac{\Delta\omega^2}{\sigma^2}f'_\Delta(\omega) + \frac{\Delta\omega^2}{2}f''_\Delta(\omega)\right)$$
$$\Rightarrow \lambda f_\Delta(\omega) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(-\mu f'_\Delta(\omega) + \frac{\sigma^2}{2}f''_\Delta(\omega)\right)$$

Taking the limit as $\Delta\omega$ converges to zero, $\frac{f_\Delta(\omega)}{\Delta\omega} \to f(\omega)$ solving equation (41).

Now consider the behavior of f_Δ at the lower threshold $\underline{\omega}$. A similar logic implies

$$f_\Delta(\underline{\omega}, t + \Delta t) = (1 - \lambda\Delta t)\tfrac{1}{2}(1 - \Delta p)\big(f_\Delta(\underline{\omega} + \Delta\omega, t) + f_\Delta(\underline{\omega}, t)(1 - \Delta\tilde{l})\big).$$

The workers at $\underline{\omega}$ at $t + \Delta t$ either were at $\underline{\omega} + \Delta\omega$ or at $\underline{\omega}$ at t; in both cases, they had a negative shock. Moreover, in the latter case, a fraction $\Delta\tilde{l}$ of the workers exited the market. Again impose stationarity but now take a first order approximation to $f_\Delta(\underline{\omega} + \Delta\omega)$ at $\underline{\omega}$; the higher order terms will drop out later in any case. Replacing Δt, Δp, and $\Delta\tilde{l}$ with the expressions described above gives

$$f_\Delta(\underline{\omega}) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1 - \frac{\mu\Delta\omega}{\sigma^2}\right)\left(f_\Delta(\underline{\omega})\left(1 - \frac{\theta\Delta\omega}{2}\right) + \frac{\Delta\omega}{2}f'_\Delta(\underline{\omega})\right)$$

Again eliminating terms in $f_\Delta(\underline{\omega})$ and taking the limit as $\Delta\omega \to 0$, we obtain $\frac{f_\Delta(\underline{\omega})}{\Delta\omega} \to f(\underline{\omega})$ solving equation (42).

Finally consider the behavior of f_Δ at the upper threshold $\bar{\omega}$:

$$f_\Delta(\bar{\omega}, t + \Delta t) = (1 - \lambda\Delta t)\tfrac{1}{2}(1 + \Delta p)\big(f_\Delta(\bar{\omega} - \Delta\omega, t) + f_\Delta(\bar{\omega}, t)(1 + \Delta\tilde{l})\big) + \lambda\Delta t L_0/L.$$

Compared to the equation at the lower threshold, the only significant change is the last term, which reflects the fact that on average a fraction L_0/L workers enter at the upper threshold when a newmarket is created. Recall also that in a period of length Δt, then λ markets are destroyed and hence also λ market are created. Thus, $L_0/L\Delta t\lambda$ is the fraction of workers added to the upper thresholds due to newly created markets. Impose stationarity and take limits to get

$$f_\Delta(\bar{\omega}) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1 + \frac{\mu\Delta\omega}{\sigma^2}\right)\left(f_\Delta(\bar{\omega})\left(1 + \frac{\theta\Delta\omega}{2}\right) - \frac{\Delta\omega}{2}f'_\Delta(\bar{\omega})\right) + \lambda\frac{\Delta\omega^2}{\sigma^2}L_0/L$$

Eliminate terms in $f_\Delta(\bar{\omega})$ and take the limit as $\Delta\omega \to 0$ to obtain $\frac{f_\Delta(\bar{\omega})}{\Delta\omega} \to f(\bar{\omega})$ solving equation (43).

C Exit Rates from Labor Markets

A worker exits her labor market if the log full-employment wage is $\underline{\omega}$ and the market is hit by an adverse shock or if the labor market closes. In the discrete time, discrete state space model, the first event hits a fraction $\frac{1}{2}\Delta\tilde{l}(1-\Delta p)$ of the workers who are in a surviving labor market with $\omega = \underline{\omega}$:

$$N_s \Delta t \equiv (1-\lambda\Delta t)\tfrac{1}{2}(1-\Delta p)\Delta\tilde{l} f_\Delta(\underline{\omega})L + \lambda\Delta t L$$

Reexpress $\Delta\omega$, $\Delta\tilde{l}$, and Δp in terms of Δt, take the limit as $\Delta t \to 0$, and use $\frac{f_\Delta(\omega)}{\Delta\omega} \to f(\omega)$, to get equation (34).

D Job Destruction: Approximation

We propose to approximate the job destruction rate by ignoring the fact that ω may hit the boundaries $\underline{\omega}$ or $\bar{\omega}$. We take a job with initial ω_0 at time 0 and compute $\omega(t)$ at time t, which we assume is Normally distributed with mean $\mu t + \omega_0$ and variance $\sigma^2 t$. We then use this to compute the resulting change in employment. We integrate over the initial density of ω_0 across employed workers to get the job destruction rate. We assume throughout that $\mu = -\frac{1}{2}\theta\sigma^2$ and $\lambda \to 0$.

We break the problem into two pieces.

Market with $\omega_0 \leq \hat{\omega}$. Take a market with $\omega_0 \in (\underline{\omega}, \hat{\omega})$. Let $x \equiv \omega(t) - \omega_0$ denote the change in ω. If $x < 0$, the job destruction rate is $1 - e^{\theta x}$, while otherwise it is zero. Thus the expected job destruction rate is

$$\int_{-\infty}^{0} (1-e^{\theta x})\phi_t(x)dx,$$

where $\phi_t(x)$ is the density of the $N(\mu t, \sigma^2 t)$. Using $\mu = -\frac{1}{2}\theta\sigma^2$, this integral evaluates to

$$2\Phi_t(0) - 1,$$

where $\Phi_t(0)$ is the cumulative distribution of a $N(\mu t, \sigma^2 t)$.

Since the number of workers employed at $\omega \in [\underline{\omega}, \hat{\omega}]$ is $\frac{e^{\theta(\omega-\hat{\omega})}}{\bar{\omega}-\underline{\omega}}$, the density of ω multiplied by the employment rate, the total number of workers employed in such markets is

$$\frac{1}{\bar{\omega}-\underline{\omega}}\int_{\underline{\omega}}^{\hat{\omega}} e^{\theta(\omega-\hat{\omega})}d\omega = \frac{1-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta(\bar{\omega}-\underline{\omega})}.$$

Hence the job destruction coming from such markets is

$$\frac{1-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta(\bar{\omega}-\underline{\omega})}\left(2\Phi_t(0)-1\right).$$

Market with $\omega_0 > \hat{\omega}$. Now take a market with $\omega_0 > \hat{\omega}$ and again define define $x_0 \equiv \hat{\omega} - \omega_0$. Now we require $x \equiv \omega(t) - \omega_0 < x_0$ in order to have any job destruction. If $x < x_0$, the job destruction rate is $1 - e^{\theta(x-x_0)}$, while otherwise it is zero. Hence for a market that starts with $x_0 < 0$, the job destruction rate is

$$\int_{-\infty}^{x_0}\left(1-e^{\theta(x-x_0)}\right)\phi_t(x)dx.$$

Again we can simplify this integral using $\mu = -\frac{1}{2}\theta\sigma^2$ to get that it is equal to

$$\Phi_t(x_0) - e^{-\theta x_0}(1-\Phi_t(-x_0)).$$

The total number of workers employed at $\omega \in [\hat{\omega}, \bar{\omega}]$ is $\frac{1}{\bar{\omega}-\underline{\omega}}$. To compute the total job destruction coming from such markets, we need to integrate this over $x_0 \in [\hat{\omega}-\bar{\omega}, 0]$. In practice, however, the job destruction rate is nearly zero for x_0 much smaller than zero, and so we find it more convenient to integrate over $(-\infty, 0)$:

$$\int_{\hat{\omega}-\bar{\omega}}^{0}\frac{\Phi_t(x_0)-e^{-\theta x_0}(1-\Phi_t(-x_0))}{\bar{\omega}-\underline{\omega}}dx_0 \approx \int_{-\infty}^{0}\frac{\Phi_t(x_0)-e^{-\theta x_0}(1-\Phi_t(-x_0))}{\bar{\omega}-\underline{\omega}}dx_0$$

We can solve this integral to get

$$\frac{1}{\bar{\omega}-\underline{\omega}}\left(\frac{1}{\theta}+\frac{e^{-\frac{1}{8}\theta^2\sigma^2 t}\sigma}{\sqrt{2\pi}}\sqrt{t}+\left(\frac{\theta\sigma^2}{2}t-\frac{2}{\theta}\right)\Phi_t(0)\right)$$

Summing the two pieces. Summing the two approximations, we get that total job destruction per elapsed time t is approximately

$$\frac{1}{\bar{\omega}-\underline{\omega}}\left(\frac{e^{-\frac{1}{8}\theta^2\sigma^2 t}\sigma\sqrt{t}}{\sqrt{2\pi}}+\frac{\theta\sigma^2\Phi_t(0)t}{2}-\frac{e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta}\left(2\Phi_t(0)-1\right)\right)$$

We must divide this by employment, which is $\frac{\bar{\omega}-\hat{\omega}+\frac{1-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta}}{\bar{\omega}-\underline{\omega}}$. This gives the job destruction rate,

$$\tilde{J}D(t) = \frac{\frac{e^{-\frac{1}{8}\theta^2\sigma^2 t}\sigma\sqrt{t}}{\sqrt{2\pi}}+\frac{\theta\sigma^2\Phi_t(0)t}{2}-\frac{e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta}\left(2\Phi_t(0)-1\right)}{\bar{\omega}-\hat{\omega}+\frac{1-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta}}.$$

Now define $\tilde{J}D_2(t) = \tilde{J}D(t^2)$ for all t. To a second order, we can approximate $\tilde{J}D_2(t)$ as

$$\tilde{J}D_2(t) \approx \frac{\frac{(1-e^{-\theta(\hat{\omega}-\underline{\omega})})\sigma t}{\sqrt{2\pi}} + \frac{\theta\sigma^2 t^2}{4}}{\bar{\omega} - \hat{\omega} + \frac{1-e^{-\theta(\hat{\omega}-\underline{\omega})}}{\theta}}.$$

This implies $\tilde{J}D(t) \approx JD(t)$ in equation (37).

E Hazard Rates

Consider a Brownian motion with initial $\omega \in [\underline{\omega}, \bar{\omega}]$. Let $G(t;\cdot;\cdot)$ and $\underline{G}(t;\cdot;\cdot)$ denote the cumulative distribution function for the times until each of the barriers is hit, conditional on the initial value of ω:

$$\begin{aligned}\hat{G}(t;\hat{\omega},\underline{\omega};\omega) &= \Pr\{t \le T_{\hat{\omega}}, T_{\hat{\omega}} < T_{\underline{\omega}} \,|\omega(0) = \omega\} \\ \underline{G}(t;\hat{\omega},\underline{\omega};\omega) &= \Pr\{t \le T_{\underline{\omega}}, T_{\underline{\omega}} < T_{\hat{\omega}} \,|\omega(0) = \omega\},\end{aligned}$$

with associated densities $\hat{g}$ and $\underline{g}$. Kolkiewicz (2002, pp. 17-17) proves

$$\begin{aligned}\hat{g}(t;\hat{\omega},\underline{\omega};\omega) &= \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\omega-\underline{\omega})}{\bar{\omega}-\underline{\omega}}\right) e^{\frac{\mu(2(\hat{\omega}-\omega)-\mu t)}{2\sigma^2} - \frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}} \\ \underline{g}(t;\hat{\omega},\underline{\omega};\omega) &= \frac{\pi\sigma^2}{(\hat{\omega}-\underline{\omega})^2}\sum_{n=1}^{\infty} n(-1)^{n-1}\sin\left(\frac{\pi n(\bar{\omega}-\omega)}{\bar{\omega}-\underline{\omega}}\right) e^{\frac{-\mu(2(\omega-\underline{\omega})-\mu t)}{2\sigma^2} - \frac{\pi^2 n^2\sigma^2 t}{2(\hat{\omega}-\underline{\omega})^2}}.\end{aligned}$$

The hazard rate of the first hitting time, conditional on a rest unemployment spell starting at time 0, i.e conditional on $\omega = \hat{\omega}$, is

$$\hat{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\hat{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)} \text{ and } \underline{h}_r(t) \equiv \lim_{\omega\uparrow\hat{\omega}} \frac{\underline{g}(t;\hat{\omega},\underline{\omega},\omega)}{1-\hat{G}(t;\hat{\omega},\underline{\omega},\omega)-\underline{G}(t;\hat{\omega},\underline{\omega},\omega)}.$$

Equation (38) follows using L'Hopital's rule.

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Additional Appendixes not for Publication

F Derivation Hamilton-Jacobi-Bellman

This appendix proves that if $v(\omega_0)$ is given by:

$$v(\omega_0) = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_\omega(\omega;\omega_0)d\omega \tag{45}$$

for an arbitrary continuous function $R(\cdot)$ and where the local time function $\Pi_\omega(\cdot)$ is given as in Stokey (2006) Proposition 10.4:

$$\Pi_\omega(\omega;\omega_0) = \begin{cases} \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\bar{\omega}} - \zeta_1 e^{\zeta_1\bar{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\underline{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\underline{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \underline{\omega} \le \omega < \omega_0 \\[2ex] \dfrac{\left(\zeta_2 e^{\zeta_1\omega_0+\zeta_2\underline{\omega}} - \zeta_1 e^{\zeta_1\underline{\omega}+\zeta_2\omega_0}\right)\left(\zeta_2 e^{\zeta_2(\bar{\omega}-\omega)} - \zeta_1 e^{\zeta_1(\bar{\omega}-\omega)}\right)}{(\rho+\lambda)(\zeta_2-\zeta_1)\left(e^{\zeta_1\underline{\omega}+\zeta_2\bar{\omega}} - e^{\zeta_1\bar{\omega}+\zeta_2\underline{\omega}}\right)} & \text{if } \omega_0 \le \omega \le \bar{\omega}, \end{cases} \tag{46}$$

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation $\rho + \lambda = \mu\zeta + \frac{\sigma^2}{2}\zeta^2$, then

$$(\rho+\lambda)v(\omega_0) = R(\omega_0) + \mu v'(\omega_0) + \frac{\sigma^2}{2}v''(\omega_0).$$

Proof. Differentiating v with respect to ω_0 we get

$$\begin{aligned} v'(\omega_0) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_{\omega\omega_0}(\omega;\omega_0)d\omega \\ v''(\omega_0) &= \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)d\omega + R(\omega_0)\big(\lim_{\omega\uparrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0) - \lim_{\omega\downarrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0)\big) \end{aligned}$$

where we use that Π_ω is continuous but $\Pi_{\omega\omega_0}$ has a jump at $\omega = \omega_0$. Then

$$\begin{aligned} &(\rho+\lambda)v(\omega_0) - \mu v'(\omega_0) - \frac{\sigma^2}{2}v''(\omega_0) \\ &\qquad = \int_{\underline{\omega}}^{\bar{\omega}} R(\omega)\left((\rho+\lambda)\Pi_\omega(\omega;\omega_0) - \mu\Pi_{\omega\omega_0}(\omega;\omega_0) - \frac{\sigma^2}{2}\Pi_{\omega\omega_0\omega_0}(\omega;\omega_0)\right)d\omega \\ &\qquad\qquad - \frac{\sigma^2}{2}R(\omega_0)\left(\lim_{\omega\uparrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0) - \lim_{\omega\downarrow\omega_0}\Pi_{\omega\omega_0}(\omega;\omega_0)\right). \end{aligned}$$

Using the functional form of Π_ω we have, for $\omega < \omega_0$:

$$\Pi_\omega(\omega; \omega_0) = e^{\zeta_1 \omega_0} \tilde{h}_1(\omega) - e^{\zeta_2 \omega_0} \tilde{h}_2(\omega)$$

where

$$\begin{aligned} \tilde{h}_1(\omega) &= \frac{\zeta_2 e^{\zeta_2 \bar{\omega}} \left(\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega)} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega)} \right)}{(\rho + \lambda)(\zeta_2 - \zeta_1) \left(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}} \right)} \\ \text{and } \tilde{h}_2(\omega) &= \frac{\zeta_1 e^{\zeta_1 \bar{\omega}} \left(\zeta_2 e^{\zeta_2 (\underline{\omega} - \omega)} - \zeta_1 e^{\zeta_1 (\underline{\omega} - \omega)} \right)}{(\rho + \lambda)(\zeta_2 - \zeta_1) \left(e^{\zeta_1 \underline{\omega} + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \underline{\omega}} \right)}. \end{aligned}$$

Thus for all $\omega < \omega_0$:

$$\begin{aligned} &(\rho + \lambda)\Pi_\omega(\omega; \omega_0) - \mu \Pi_{\omega\omega_0}(\omega; \omega_0) - \frac{\sigma^2}{2} \Pi_{\omega\omega_0\omega_0}(\omega; \omega_0) \\ &\quad = \left[(\rho + \lambda) - \zeta_1 \mu - (\zeta_1)^2 \frac{\sigma^2}{2} \right] e^{\zeta_1 \omega_0} \tilde{h}_1(\omega) - \left[(\rho + \lambda) - \zeta_2 \mu - (\zeta_2)^2 \frac{\sigma^2}{2} \right] e^{\zeta_2 \omega_0} \tilde{h}_2(\omega) = 0 \end{aligned}$$

where the last equality follow from the definition of the roots ζ_i . Hence

$$\int_{\underline{\omega}}^{\omega_0} R(\omega) \left((\rho + \lambda)\Pi_\omega(\omega; \omega_0) - \mu \Pi_{\omega\omega_0}(\omega; \omega_0) - \frac{\sigma^2}{2} \Pi_{\omega\omega_0\omega_0}(\omega; \omega_0) \right) d\omega = 0.$$

Using a symmetric calculation for $\omega > \omega_0$ we have:

$$\int_{\omega_0}^{\bar{\omega}} R(\omega) \left((\rho + \lambda)\Pi_\omega(\omega; \omega_0) - \mu \Pi_{\omega\omega_0}(\omega; \omega_0) - \frac{\sigma^2}{2} \Pi_{\omega\omega_0\omega_0}(\omega; \omega_0) \right) d\omega = 0.$$

Next, differentiating $\Pi_\omega(\omega; \omega_0)$ when $\omega < \omega_0$ and when $\omega > \omega_0$ and let $\omega \to \omega_0$ from below and from above, tedious—but straightforward—algebra, gives:

$$\lim_{\omega \uparrow \omega_0} \Pi_{\omega\omega_0}(\omega; \omega_0) - \lim_{\omega \downarrow \omega_0} \Pi_{\omega\omega_0}(\omega; \omega_0) = -\frac{\zeta_1 \zeta_2}{\rho + \lambda}.$$

Then use the expression for the roots: $\zeta_1 \zeta_2 = -(\rho + \lambda)/(\sigma^2/2)$. Putting this together proves the result. $\square$

G Joint Distribution of ω and $\log l$

G.1 Solution to the Partial Differential Equation

This section characterizes the joint distribution of ω and $\tilde{l} = \log l$. We start with the easier problem of characterizing the ergodic marginal density of ω across islands, $\tilde{f}(\omega)$. Appendix G.2 characterizes the density in a discrete time, discrete state-space analog of our model. In the limit as we move to a continuous time diffusion, the stationary distribution solves a Kolmogorov forward equation:

$$\lambda \tilde{f}(\omega) = -\mu \tilde{f}'(\omega) + \frac{\sigma^2}{2} \tilde{f}''(\omega). \tag{47}$$

Solving this differential equation requires two additional constraints. First is a boundary condition at the lower reflecting threshold $\underline{\omega}$:

$$\mu \tilde{f}(\underline{\omega}) = \frac{\sigma^2}{2} \tilde{f}'(\underline{\omega}). \tag{48}$$

Note that there is no corresponding boundary condition at $\bar{\omega}$ since new labor markets enter at this point in the distribution. Second is the requirement that $\tilde{f}$ is a density:

$$\int_{\underline{\omega}}^{\bar{\omega}} \tilde{f}(\omega) d\omega = 1.$$

Using these, it is straightforward to prove that the ergodic marginal density of the full-employment wage is

$$\tilde{f}(\omega) = \frac{\eta_2 e^{\eta_2(\omega - \underline{\omega})} - \eta_1 e^{\eta_1(\omega - \underline{\omega})}}{e^{\eta_2(\bar{\omega} - \underline{\omega})} - e^{\eta_1(\bar{\omega} - \underline{\omega})}}. \tag{49}$$

Next consider the joint density of ω and log employment $\tilde{l}$, $\hat{f}(\omega, \tilde{l}; x_0, Y)$. Again following the discrete time, discrete state approximation in Appendix G.2 gives a Kolmogorov forward equation when $\omega \in (\underline{\omega}, \bar{\omega})$ and hence the labor force is constant:

$$\lambda \hat{f}(\omega, \tilde{l}; x_0, Y) = -\mu \hat{f}_\omega(\omega, \tilde{l}; x_0, Y) + \frac{\sigma^2}{2} \hat{f}_{\omega\omega}(\omega, \tilde{l}; x_0, Y). \tag{50}$$

The general solution takes the form

$$\hat{f}(\omega, \tilde{l}; x_0, Y) = D_1(\tilde{l}; x_0, Y) e^{\eta_1 \omega} + D_2(\tilde{l}; x_0, Y) e^{\eta_2 \omega}. \tag{51}$$

Note that we now need to solve for two functions $D_i(\tilde{l}; x_0, Y)$, rather than simply two numbers.

To find these functions, we first use a pair of boundary conditions obtained in Appendix G.2:

$$\mu\hat{f}(\underline{\omega},\tilde{l};x_0,Y)=\frac{\sigma^2}{2}\big(\hat{f}_\omega(\underline{\omega},\tilde{l};x_0,Y)+\theta\hat{f}_{\tilde{l}}(\underline{\omega},\tilde{l};x_0,Y)\big)\text{ for all }\tilde{l} \tag{52}$$

$$\mu\hat{f}(\bar{\omega},\tilde{l};x_0,Y)=\frac{\sigma^2}{2}\big(\hat{f}_\omega(\bar{\omega},\tilde{l};x_0,Y)+\theta\hat{f}_{\tilde{l}}(\bar{\omega},\tilde{l};x_0,Y)\big)\text{ for all }\tilde{l}\neq l_0(x_0,Y), \tag{53}$$

where

$$\tilde{l}_0(x_0,Y)\equiv\log Y+\theta\log u'(Y)+(\theta-1)\log(Ax_0)-\theta\bar{\omega}$$

is log-employment in a new market with productivity x_0. The second equation does not necessarily hold at $\tilde{l}_0(x_0,Y)$, since there is an additional inflow at this point when new new markets are created.

Substituting these restrictions into equation (51), we obtain a pair of ordinary differential equations for the functions D_i:

$$\left(\mu-\frac{\sigma^2}{2}\eta_i\right)D_i(\tilde{l};x_0,Y)=\theta\frac{\sigma^2}{2}D_i'(\tilde{l};x_0,Y)$$

for all $\tilde{l}\neq\tilde{l}_0(x_0,Y)$. Noting that $\eta_1+\eta_2=\frac{2\mu}{\sigma^2}$, we can rewrite this more compactly as

$$\frac{D_1'(\tilde{l};x_0,Y)}{D_1(\tilde{l};x_0,Y)}=\frac{\eta_2}{\theta}\text{ and }\frac{D_2'(\tilde{l};x_0,Y)}{D_2(\tilde{l};x_0,Y)}=\frac{\eta_1}{\theta}. \tag{54}$$

For each $i=1,2$, this is a pair of differential equations, above and below $\tilde{l}_0(x_0,Y)$. To find the two constants of integration, note that for each x_0, the long-run distribution of ω is given by $\tilde{f}$:

$$\tilde{f}(\omega)=\int_{-\infty}^{\infty}\hat{f}(\omega,\tilde{l};x_0,Y)d\tilde{l}. \tag{55}$$

Solving this integral, we find that

$$D_1(\tilde{l};x_0,Y)=\begin{cases}\dfrac{2\lambda}{\theta\sigma^2}\dfrac{e^{\frac{\eta_2}{\theta}(\tilde{l}-\tilde{l}_0(x_0,Y))-\eta_1\underline{\omega}}}{e^{\eta_2(\bar{\omega}-\underline{\omega})}-e^{\eta_1(\bar{\omega}-\underline{\omega})}} & \text{if }\tilde{l}<\tilde{l}_0(x_0,Y)\\ 0 & \text{if }\tilde{l}>\tilde{l}_0(x_0,Y)\end{cases} \tag{56}$$

$$D_2(\tilde{l};x_0,Y)=\begin{cases}0 & \text{if }\tilde{l}<\tilde{l}_0(x_0,Y)\\ \dfrac{2\lambda}{\theta\sigma^2}\dfrac{e^{\frac{\eta_1}{\theta}(\tilde{l}-\tilde{l}_0(x_0,Y))-\eta_2\underline{\omega}}}{e^{\eta_2(\bar{\omega}-\underline{\omega})}-e^{\eta_1(\bar{\omega}-\underline{\omega})}} & \text{if }\tilde{l}>\tilde{l}_0(x_0,Y)\end{cases}$$

Now substitute into equation (51) to get

$$\hat{f}(\omega, \tilde{l}; x_0, Y) = \begin{cases} \dfrac{2\lambda}{\theta\sigma^2} \dfrac{e^{\frac{\eta_2}{\theta}(\tilde{l}-\tilde{l}_0(x_0,Y))+\eta_1(\omega-\underline{\omega})}}{e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})}} & \text{if } \tilde{l} < \tilde{l}_0(x_0, Y) \\ \dfrac{2\lambda}{\theta\sigma^2} \dfrac{e^{\frac{\eta_1}{\theta}(\tilde{l}-\tilde{l}_0(x_0,Y))+\eta_2(\omega-\underline{\omega})}}{e^{\eta_2(\bar{\omega}-\underline{\omega})} - e^{\eta_1(\bar{\omega}-\underline{\omega})}} & \text{if } \tilde{l} > \tilde{l}_0(x_0, Y). \end{cases} \tag{57}$$

This is the joint distribution of ω and $\tilde{l}$ in a market that started with productivity x_0. □

G.2 Discrete Approximation

In this section, we use a discrete time, discrete state space model to obtain the Kolmogorov forward equations and boundary conditions for the density $\tilde{f}$ in Appendix G.1. We start with $\tilde{f}$, the marginal distribution of ω across labor markets. Divide $[\underline{\omega}, \bar{\omega}]$ into n intervals of length $\Delta\omega = (\bar{\omega} - \underline{\omega})/n$. Let the time period be $\Delta t = (\Delta\omega/\sigma)^2$ and assume that ω increases by $\Delta\omega$ with probability $(1 - \lambda\Delta t)\frac{1}{2}(1 + \Delta p)$ for $\Delta p = \mu\Delta\omega/\sigma^2$, decreases by $\Delta\omega$ with probability $(1-\lambda\Delta t)\frac{1}{2}(1-\Delta p)$, and returns to $\bar{\omega}$ with probability $\lambda\Delta t$. It limit as $\Delta\omega \to 0$, the expected value of $\omega(t + \Delta t)$ conditional on $\omega(t)$ is $\omega(t) + \mu\Delta t$ and the conditional standard deviation is $\sigma\sqrt{\Delta t}$.

Let $\tilde{f}_\Delta(\omega)$ denote the discrete time, discrete state version of $\tilde{f}(\omega)$; in the limit as $\Delta\omega \to 0$, the densities are related via $\frac{\tilde{f}_\Delta(\omega)}{\Delta\omega} \to \tilde{f}(\omega)$. Also let $\tilde{f}_\Delta(\omega, t)$ denote a time-varying counterpart of denote a time-varying counterpart of $\tilde{f}_\Delta(\omega)$. For $\underline{\omega} < \omega < \bar{\omega}$, we can write a backward-looking equation for $\tilde{f}_\Delta$:

$$\tilde{f}_\Delta(\omega, t + \Delta t) = (1 - \lambda\Delta t)\tfrac{1}{2}\big((1 + \Delta p)\tilde{f}_\Delta(\omega - \Delta\omega, t) + (1 - \Delta p)\tilde{f}_\Delta(\omega + \Delta\omega, t)\big). \tag{58}$$

To obtain a value ω at $t + \Delta t$, either we must have been at $\omega - \Delta\omega$ at t and had a positive shock or been at $\omega + \Delta\omega$ at t and had a negative shock. Now stationarity of $\tilde{f}_\Delta$, dropping the second argument. Take a second order approximation to $\tilde{f}_\Delta(\omega + \Delta\omega)$ and $\tilde{f}_\Delta(\omega - \Delta\omega)$ around ω, substituting Δt and Δp by the expressions above:

$$\tilde{f}_\Delta(\omega) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(\tilde{f}_\Delta(\omega) - \mu\frac{\Delta\omega^2}{\sigma^2}\tilde{f}'_\Delta(\omega) + \frac{\Delta\omega^2}{2}\tilde{f}''_\Delta(\omega)\right)$$
$$\Rightarrow \lambda\tilde{f}_\Delta(\omega) = \left(1 - \lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(-\mu\tilde{f}'_\Delta(\omega) + \frac{\sigma^2}{2}\tilde{f}''_\Delta(\omega)\right)$$

Taking the limit as $\Delta\omega$ converges to zero, $\frac{\tilde{f}_\Delta(\omega)}{\Delta\omega} \to \tilde{f}(\omega)$ solving equation (47).

Now consider the behavior of $\tilde{f}_\Delta$ at the lower threshold $\underline{\omega}$. A similar logic implies

$$\tilde{f}_\Delta(\underline{\omega}, t+\Delta t) = (1-\lambda\Delta t)\tfrac{1}{2}(1-\Delta p)\big(\tilde{f}_\Delta(\underline{\omega}+\Delta\omega, t) + \tilde{f}_\Delta(\underline{\omega}, t)\big), \tag{59}$$

since it is only possible to reach $\underline{\omega}$ following an adverse shock. Again impose stationarity but now take a first order approximation to $\tilde{f}_\Delta(\underline{\omega}+\Delta\omega)$ $\underline{\omega}$; the higher order terms will drop out later in any case. Replacing Δt and Δp with the expressions described above gives

$$\tilde{f}_\Delta(\underline{\omega}) = \left(1-\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1-\frac{\mu\Delta\omega}{\sigma^2}\right)\left(\tilde{f}_\Delta(\underline{\omega}) + \frac{\Delta\omega}{2}\tilde{f}'_\Delta(\underline{\omega})\right)$$

Again eliminating terms in $\tilde{f}_\Delta(\underline{\omega})$ and taking the limit as $\Delta\omega \to 0$, we obtain $\frac{\tilde{f}_\Delta(\underline{\omega})}{\Delta\omega} \to \tilde{f}(\underline{\omega})$ solving equation (42).

We turn our attention now to the discrete time, discrete state density $\tilde{f}_\Delta(\omega, \tilde{l}; x_0, Y)$. Let $\Delta\tilde{l} \equiv \theta\Delta\omega$. In the limit as $\Delta\omega$ to zero, $\frac{\tilde{f}_\Delta(\omega,\tilde{l};x_0,Y)}{\Delta\omega\,\Delta\tilde{l}} \to \tilde{f}(\omega, \tilde{l}; x_0, Y)$. When $\underline{\omega} < \omega < \bar{\omega}$, $\tilde{l}$ is constant and so the logic for the Kolmogorov forward equation for $\tilde{f}$ is unchanged. In other words, there is no surprise that equations (47) and (50) have the same form. At the lower boundary, the discrete time, discrete state space model implies

$$\tilde{f}_\Delta(\underline{\omega}, \tilde{l}, t+\Delta t; x_0) = (1-\lambda\Delta t)\tfrac{1}{2}(1-\Delta p)\tilde{f}_\Delta(\underline{\omega}+\Delta\omega, \tilde{l}, t; x_0) + \tilde{f}_\Delta(\underline{\omega}, \tilde{l}+\Delta\tilde{l}, t; x_0)\big).$$

To get to the state $(\underline{\omega}, \tilde{l})$ at $t+\Delta t$, we must have either been at $(\underline{\omega}+\Delta\omega, \tilde{l})$ the previous period and suffered an adverse productivity shock, reducing ω but leaving $\tilde{l}$ unchanged; or we were at $(\underline{\omega}, \tilde{l}+\Delta\tilde{l})$ the previous period and suffered an adverse productivity shock. In the latter case, ω cannot fall, and so instead $\tilde{l}$ must decline by enough to leave ω unchanged.

As usual, we impose impose stationarity of $\tilde{f}_\Delta$ and take a first order approximation around $(\underline{\omega}, \tilde{l})$; higher order terms would drop out later. Replacing Δt, Δp, and $\Delta\tilde{l}$ gives

$$\tilde{f}_\Delta(\underline{\omega}, \tilde{l}; x_0) = \left(1-\lambda\frac{\Delta\omega^2}{\sigma^2}\right)\left(1-\frac{\mu\Delta\omega}{\sigma^2}\right)\left(\tilde{f}_\Delta(\underline{\omega}, \tilde{l}; x_0) + \frac{\Delta\omega}{2}\frac{\partial\tilde{f}_\Delta(\underline{\omega}, \tilde{l}; x_0)}{\partial\omega} + \frac{\theta\Delta\omega}{2}\frac{\partial\tilde{f}_\Delta(\underline{\omega}, \tilde{l}; x_0)}{\partial\tilde{l}}\right).$$

Now eliminating terms in $\tilde{f}_\Delta(\underline{\omega}, \tilde{l}; x_0)$ and taking the limit as $\Delta\omega$ converges to zero, we obtain $\frac{\tilde{f}_\Delta(\underline{\omega},\tilde{l};x_0)}{\Delta\omega\Delta\tilde{l}} \to \tilde{f}(\underline{\omega}, \tilde{l}; x_0)$ in equation (52). A symmetric logic yields equation (53) when $\omega = \bar{\omega}$, except at the point where new labor markets enter, except at the point where new labor markets enter, $\tilde{l}_0(x_0, Y)$.

H Market Social Planner's Problem

In this section we introduce a dynamic programming problem whose solution gives the equilibrium value for the thresholds $\underline{\omega}, \bar{\omega}$. This problem has the interpretation of a fictitious social planner located in a given market who maximizes net consumer surplus by deciding how many of the agents currently located in the market work and how many rest and whether to adjust the number of workers in the market. The equivalence of the solution of this problem with the equilibrium value of the labor market participant has the following implications. First, it establishes that our market decentralization is rich enough to attain an efficient equilibrium, despite the presence of search frictions. Second, it gives an alternative argument to establish the uniqueness of the equilibrium values for the thresholds $\underline{\omega}$ and $\bar{\omega}$. Third, it connects our results with the decision theoretic literature analyzing investment and labor demand model with costly reversibility.

The market planner maximizes the net surplus from the production of the final good in a market with current log productivity $\tilde{x}$ and l workers, taking as given aggregate consumption C and aggregate output Y. The choices for this planner are to increase the number of workers located in this market (hire), paying $\bar{v}$ to the households for each or them, or to decrease the number of workers located at the market number (fire), receiving a payment $\underline{v}$ for each. Increases and decreases are non-negative, and the prices associated with them have the dimension of an asset value, as opposed to a rental. We let $M(\tilde{x}, l)$ be the value function of this planner, hence:

$$M(\tilde{x}, l) = \max_{l_h(t), l_f(t)} \mathbb{E}\left(\int_0^\infty e^{-(\rho+\lambda)t} \big(S(\tilde{x}(t), l(t)) - \bar{v} dl_h(t) + \underline{v} dl_f(t) \big) dt \,\middle|\, \tilde{x}(0) = \tilde{x}, l(0) = l \right)$$
$$\text{subject to } dl(t) = dl_h(t) - dl_f(t) \text{ and } d\tilde{x} = \mu_x dt + \sigma_x dz. \tag{60}$$

The $l_h(t)$ and $l_f(t)$ are increasing processes describing the cumulative amount of "hiring" and "firing" and hence $dl_h(t)$ and $dl_f(t)$ intuitively have the interpretation of hiring and firing during period t. The planner discounts at rate $\rho + \lambda$, accounting both for the discount rate of households and for the rate at which her labor market disappears.

The function $S(\tilde{x}, l)$ denotes the return function of the market social planner per unit of time and is given by

$$S(\tilde{x}, l) = \max_{E \in [0,l]} u'(C) \int_0^{EAe^{\tilde{x}}} \left(\frac{Y}{q} \right)^{\frac{1}{\theta}} dq + b_r(l - E) + \lambda l \underline{v}.$$

The first term is the consumer's surplus associated with the particular good, obtained by the output produced by E workers with log productivity $\tilde{x}$. The second term is value of the workers that the planner chooses to send back to the household, receiving $\underline{v}$ for each. The third term is the value of the "sale" of all the workers if the market shuts down. Setting $\lambda = b_r = 0$ our problem is formally equivalent to Bentolila and Bertola's (1990) model of a firm deciding employment subject to a hiring and firing cost and to Abel and Eberly's (1996) model of optimal investment subject to costly irreversibility, i.e. a different buying and selling price for capital.

Using the envelope theorem, we find that the marginal value of an additional worker is:

$$
\begin{aligned}
S_l(\tilde{x}, l) &= \max\left\{u'(C)\left(\frac{Y(Ae^{\tilde{x}})^{\theta-1}}{l}\right)^{\frac{1}{\theta}}, b_r\right\} + \lambda\underline{v} \qquad (61)\\
&\equiv s\left(\frac{(\theta-1)(\tilde{x}+\log A)+\log Y-\log l}{\theta}+\log u'(C)\right)
\end{aligned}
$$

where the function $s(\cdot)$ is given by $s(\omega) = \max\{e^{\omega}, b_r\} + \lambda\underline{v}$ and is identical to the expression for the per-period value of a labor market participant in our equilibrium. This is critical to the equivalence between the two problems.

To prove this equivalence, we write the market social planner's Hamilton-Jacobi-Bellman equation. For each $\tilde{x}$, there are two thresholds, $\underline{l}(\tilde{x})$ and $\bar{l}(\tilde{x})$ defining the range of inaction. The value function $M(\cdot)$ and thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve the Hamilton-Jacobi-Bellman equation if the following two conditions are met:

1. For all $\tilde{x}$, and $l \in (\underline{l}(\tilde{x}), \bar{l}(\tilde{x}))$ employment stays constant and hence the value function M solves
$$(\rho+\lambda)M(\tilde{x}, l) = S(\tilde{x}, l) + \mu_x M_{\tilde{x}}(\tilde{x}, l) + \frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}}(\tilde{x}, l). \qquad (62)$$

2. For all $(\tilde{x}, l)$ outside the interior of the range of inaction,
$$(\rho+\lambda)M(\tilde{x}, l) - \mu_x M_{\tilde{x}}(\tilde{x}, l) - \frac{\sigma_x^2}{2}M_{\tilde{x}\tilde{x}}(\tilde{x}, l) \leq S(\tilde{x}, l), \qquad (63)$$
$$\underline{v} = M_l(\tilde{x}, l)\ \forall l \geq \bar{l}(\tilde{x}), \text{ and } \bar{v} = M_l(\tilde{x}, l)\ \forall l \leq \underline{l}(\tilde{x}) \qquad (64)$$

Equation (64) is also referred to as *smooth pasting*. Since $M(\tilde{x}, \cdot)$ is linear outside the range of inaction, a twice-continuously differentiable solution implies *super-contact*, or that for all $\tilde{x}$:

$$0 = M_{ll}(\tilde{x}, \bar{l}(\tilde{x})) = M_{ll}(\tilde{x}, \underline{l}(\tilde{x})). \qquad (65)$$

According to Verification Theorem 4.1, Section VIII in Fleming and Soner (1993), a twice-continuously differentiable function $M(\tilde{x}, l)$ satisfying equations (62), (64), and (65) solves the market social planner's problem.

If M is sufficiently smooth, finding the optimal thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ can be stated as a boundary problem in terms of the function $M_l(\tilde{x}, l)$ and its derivatives. To see this, differentiate both sides of equation (62) with respect to l and replace S_l using equation (61):

$$(\rho + \lambda) M_l(\tilde{x}, l) = s\left(\frac{(\theta - 1)(\tilde{x} + \log A) + \log Y - \log l}{\theta} + \log u'(C)\right) \\ + \mu_x M_{\tilde{x}l}(\tilde{x}, l) + \frac{\sigma_x^2}{2} M_{\tilde{x}\tilde{x}l}(\tilde{x}, l). \quad (66)$$

If the required partial derivatives exist, any solution to the market social planner's problem must solve equations (64)–(66). Moreover, there is a clear relationship between the value function $v(\omega)$ in the decentralized problem and the marginal value of a worker M_l in the market social planner's problem:

Lemma 2. Assume that $\theta \neq 1$ and that the functions $M_l(\cdot)$ and $v(\cdot)$ satisfy

$$M_l(\tilde{x}, l) = v(\omega), \text{ where } \omega = \frac{\log Y + (\theta - 1)(\log A + \tilde{x}) - \log l}{\theta} + \log u'(C) \quad (67)$$

and that thresholds functions $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ and the thresholds levels $\{\underline{\omega}, \bar{\omega}\}$ satisfy

$$\log \bar{l}(\tilde{x}) = \log Y + (\theta - 1)(\tilde{x} + \log A) - \theta(\underline{\omega} - \log u'(C)) \quad (68)$$

$$\log \underline{l}(\tilde{x}) = \log Y + (\theta - 1)(\tilde{x} + \log A) - \theta(\bar{\omega} - \log u'(C)). \quad (69)$$

Then, $M_l(\cdot)$ and $\{\underline{l}(\cdot), \bar{l}(\cdot)\}$ solve equations (64)–(66) for all $\tilde{x}$ and $l \in [\underline{l}(\tilde{x}), \bar{l}(\tilde{x})]$ if and only if $v(\cdot)$ and $\{\underline{\omega}, \bar{\omega}\}$ solve equations (21).

Proof. Differentiate equation (67) with respect to $\tilde{x}$ to get

$$M_{l\tilde{x}}(\tilde{x}, l) = v'(\omega)\, \frac{\theta - 1}{\theta} \text{ and } M_{l\tilde{x}\tilde{x}}(\tilde{x}, l) = v''(\omega) \left(\frac{\theta - 1}{\theta}\right)^2.$$

Recall that a solution of equation (21) is equivalent to a solution to equations (26), (27), and $v(\bar{\omega}) = \bar{v}$ and $v(\underline{\omega}) = \underline{v}$. The equivalence between equation (21) and equations (64)–(66) is immediate. $\square$

This lemma has important implications. First, it establishes, not surprisingly, that the equilibrium allocation is Pareto Optimal. Second, since the market social planner's problem is a maximization problem, the solution is easy to characterize. For instance, since the problem is convex, it has at most one solution and hence the equilibrium value of a labor market participant is uniquely defined, for given $u'(C)$ and Y. The fact that v is increasing is then equivalent to the concavity of $S(\tilde{x}, \cdot)$. Finally, notice that Proposition 1 in Section 4.3 we establish existence and uniqueness of the solution to equation (21) only under mild conditions on $s(\cdot)$, i.e. that it was weakly increasing and bounded below. Proposition 1 can be used to extend the uniqueness and existence results of the literature of costly irreversible investment to a wider class of production functions. Currently the literature uses that the production function is the of the form $x^{a_x}\, l^{a_l}$ for some constants a_x and a_l, with $0 < a_l < 1$, as in Abel and Eberly (1996). Proposition 1 shows that the only assumption required is that the production function be concave in l, and that the marginal productivity of the factor l can be written as a function of the ratio of the quantity of the input l to (a power of) the productivity shock x.