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Equilibrium Directed Search with Multiple Applications*

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1 Introduction

In this paper, we construct an equilibrium model of directed search. Unemployed workers, observing the wages posted at all vacancies, direct their applications towards the vacancies they find most attractive. At the same time, (the owners of) vacancies post their wages taking into account that the wages they post will influence the number of applicants they attract. Burdett, Shi, and Wright (2001) (hereafter BSW) develop a model of this type, albeit in a product market setting. In the labor market version of BSW, each unemployed worker makes a single application. They show that there is a unique symmetric equilibrium in which all vacancies post a wage between zero (the monopsony wage) and one (the competitive wage). The value of this common posted wage depends on the number of unemployed, u, and the number of vacancies, v, in the market.

In our model, each unemployed worker makes a fixed number of applications, a, where $a \in \{1, 2, ..., v\}$. BSW is thus the special case of $a = 1$. In addition to being more realistic – job seekers in fact typically apply to more than one vacancy at a time –, the model with multiple applications is of interest because its results differ qualitatively from those of the single application model. In particular, when $a \in \{2, .., v\}$, all vacancies post the monopsony wage in the unique symmetric equilibrium. However, when workers apply for two or more jobs at the same time, there is a possibility that more than one vacancy will want to hire the same worker. In this case, we assume that the vacancies in question can compete for this worker's services. The end result is that some workers (those who receive no offers) remain unemployed, some workers (those who receive exactly one offer) are employed at the monopsony wage, and some workers (those who receive two or more offers) are employed at the competitive wage.

We make two contributions by considering the case of $a \in \{2, ..., v\}$. First, we provide a new microfoundation for the matching function,[1] an essential ingredient in much of the search literature (Pissarides 2000). The standard microfoundation for the matching function is the urn-ball model. In that model, each worker makes a single application, and there is a coordination problem among applicants because some vacancies can receive applications from more than one worker, while others can receive none. With multiple applications, there is a second coordination problem, this time among vacancies. When workers apply for more than one job at a time, some workers

[1] Our derivation of the matching function is taken from Albrecht, Gautier and Vroman (2002). Relative to that paper, our contribution here is to derive our matching function in an equilibrium setting.

can receive offers from more than one vacancy, while others receive none. Ultimately, a worker can only take one job, and the vacancies that "lose the race" for a worker will have wasted time and effort while considering his or her application. Our matching function incorporates both the urn-ball and the multiple application coordination frictions.

The second contribution of our multiple application model is that it generates equilibrium wage dispersion with directed search, even though workers and vacancies are homogeneous. Some workers are employed at the monopsony wage and others at the competitive wage. Postel-Vinay and Robin (2000) have a similar result in an undirected, random search framework. In their model, as in Burdett and Mortensen (1998), wage offers arrive at Poisson rates to both the unemployed and the employed. If a worker who is already employed receives another offer, then that worker's current employer and prospective new employer engage in Bertrand competition for his or her services.

In addition to looking at positive issues – the matching function and the equilibirum wage distribution –, we also examine a normative question, namely, whether vacancy creation in a labor market with wage posting and directed search is constrained efficient. That is, is the equilibrium, free-entry level of labor market tightness the same as the level that a social planner would choose? In competitive search equilibrium, as u and v become arbitrarily large, the results of Moen (1997) suggest an affirmative answer to this question. To look at this issue, we investigate a limiting version of our model. We let $u, v \to \infty$ with $v/u = \theta$ and a fixed. We verify that the standard efficiency result holds when $a = 1$, but for each fixed $a > 1$, we show that equilibrium is inefficient; specifically, there is excess vacancy creation. Interestingly, however, if we let $a = v$, which corresponds to the model of Julien, Kennes and King (2002), then equilibrium is always efficient, both in the finite (u, v) case and in the limiting version of the model.

We now turn in the next section to our basic model. Then, in Section 3, we consider efficiency. In the final section, we discuss additional issues that we plan to address in future versions of this paper.

2 The Basic Model

We consider a game played by u homogeneous unemployed workers and (the owners of) v homogeneous vacancies, where u and v are given. This game has several stages:

1. Each vacancy posts a wage.

2. Each unemployed worker observes all posted wages and then submits a applications with no more than one application going to any one vacancy.

3. Each vacancy that receives at least 1 application randomly selects one to process. Any excess applications are returned as rejections.

4. A vacancy with a processed application offers the applicant the posted wage. If more than one vacancy makes an offer to a particular worker, then those vacancies can bid against one another for that worker's services.

5. A worker with one offer can accept or reject that offer. A worker with more than one offer can accept one of the offers or reject all of them.

Workers who fail to match with a vacancy and vacancies that fail to match with a worker receive payoffs of zero. The payoff for a worker who matches with a vacancy is w, where w is the wage that he or she is paid. A vacancy that hires a worker at a wage of w receives a payoff of $1 - w$.

Before we analyze this game, some comments on the underlying assumptions are in order. First, this is a model of directed search in the sense that workers observe all wage postings and send their applications to vacancies with attractive wages and/or where relatively little competition is expected. Second, we are treating a as a parameter of the search technology; that is, the number of applications is taken as given. In general, $a \in \{1, 2, ..., v\}$. The case of $a = 1$ corresponds to BSW. Third, we assume that it takes a period for a vacancy to process an application. This is why vacancies return excess applications as rejections. This processing time assumption is important for our results. It captures the idea that when workers apply for several jobs at the same time, firms can waste time and effort pursuing applicants who ultimately go elsewhere. Finally, we assume that 2 or more vacancies that are competing for the same worker can engage in *ex post* Bertrand competition for that worker. This means that workers who receive more than one offer will have their wages bid up to $w = 1$, the competitive wage. There are, of course, other possible "tie-breaking" assumptions. For example, one might assume that vacancies hold to their posted wages, that is, refuse to engage in *ex post* bidding. This, however, would not be in the individual interest of vacancies.

We consider symmetric equilibria in which all vacancies post the same wage and all workers use the same strategy to direct their applications.

We will show that for each (u, v, a) combination there is a unique symmetric equilibrium, and we will derive the corresponding equilibrium matching function and posted wage. Assuming (for the moment) the existence of a symmetric equilibrium, we begin with the matching function. The following result is from Albrecht et.al. (2002).

Proposition 1 *The expected number of matches in symmetric equilibrium is*

$$M(u, v; a) = u(1 - (1 - \frac{v}{au}(1 - (1 - \frac{a}{v})^u))^a). \tag{1}$$

Proof: Let q be the probability that any one application leads to a job offer. This equals the number of vacancies with applications divided by the total number of applications, that is, $q = pv/au$, where p is the probability that a particular vacancy will receive at least one application. If all vacancies post the same wage, then the optimal mixed strategy for each unemployed (given that all other unemployed follow the same strategy) is to send applications to randomly selected vacancies. The number of applications received by any one vacancy is then a binomial random variable with parameters u and a/v, so $p = 1 - (1 - \frac{a}{v})^u$ and $q = \frac{1 - (1 - \frac{a}{v})^u}{au/v}$. The probability that at least one of a worker's applications leads to a job offer is $1 - (1 - q)^a$; so, the total number of matches is $u[1 - (1 - q)^a]$. Substitution gives equation (1). *QED.*

For $a = 1$, this result is analogous to the one given in Proposition 2 of BSW. That is, with $a = 1$ (and with the notational change of $m = v$ and $n = u$) our results exactly match those of BSW. For $a \in \{1, 2, .., v\}$, $M(u, v; a)$ is increasing at a decreasing rate in both u and v. In addition, $M(u, v; a)$ exhibits decreasing returns to scale in (u, v) for each fixed a. (Proofs available on request.) The basic point of these results is that the qualitative properties of the matching process in BSW hold for general values of a, i.e., not just for the special case of $a = 1$.

The properties of $M(u, v; a)$ as a function of a are of more interest. With $a = 1$, the familiar urn-ball friction operates in the labor market. Some vacancies receive more than one application, while others receive none, so the expected number of matches is less than the minimum of u and v. When workers submit more than one application, the urn-ball friction is reduced in the sense that the probability that any particular vacancy receives no applications decreases, but with $a > 1$, a new friction is introduced by the multiple applications. A worker who gets multiple offers can only accept one job. A vacancy that has processed a particular application may find at the end of the period that the worker whose application it processed takes a job elsewhere. The urn-ball friction results from a lack of coordination among job seekers; the multiple-application friction is due to a lack of coordination among vacancies. The urn-ball friction decreases with a; the multiple-application friction increases with a. The result of this interplay between the two frictions is that the expected number of matches first increases but then decreases as a function of a.

The special case of $a = v$ is of particular interest. A matching process in which workers apply to all vacancies and each vacancy then randomly selects one applicant is an urn-ball process with the role of urn played by workers and that of ball played by vacancies. That is, the case of $a = v$ is essentially the same as that of $a = 1$, except that the roles of workers and vacancies are reversed. The symmetry between the cases of $a = 1$ and $a = v$ can be seen in $M(u,v;1) = v(1-(1-\frac{1}{v})^u)$ and $M(u,v;v) = u(1-(1-\frac{1}{u})^v)$. The case of $a = v$ is the one considered in Julien, Kennes, and King (2002), a paper that we discuss below.

Proposition 1 and its implications are only interesting if a symmetric equilibrium exists. We now turn to the existence question.

Proposition 2 *There exists a unique symmetric equilibrium. When $a = 1$, all vacancies post a wage of*

$$w(u,v;1) = \frac{u(v-1)\left(\frac{v-1}{v}\right)^u}{v(v-1) - \left(\frac{v-1}{v}\right)^u (u + v(v-1))}. \qquad (2)$$

When $a \in \{2, ..., v\}$, all firms post the monopsony wage of $w = 0$. As in Proposition 1, let $q = \dfrac{1-(1-\frac{a}{v})^u}{au/v}$. A fraction $aq(1-q)^{a-1}$ of all job seekers receive exactly one offer. These workers are paid $w = 0$. A fraction $1-(1-q)^a - aq(1-q)^{a-1}$ of job seekers receive two or more offers. These workers are paid $w = 1$.

Proof: To prove the existence of a symmetric equilibrium, we need to show that there is a wage $w \geq 0$ such that if all vacancies, with the possible exception of a "potential deviant" (D), post w, then it is also in the interest of D to post w. If all vacancies post the same wage, then the unemployed can do no better than to send their applications to a vacancies selected at random.

To proceed, we need some notation. First, we let D denote the potential deviant, posting a wage of w^D, and N the nondeviant vacancies; that is, those posting the common, putative equilibrium wage, which we call w^N. Second, let k be the probability that any individual applies to D. In symmetric equilibrium, k must be the same for all workers. Finally, let q^D be the probability that a worker is offered the D job, conditional on applying for that job, and let q^N be the probability that a worker is offered any particular N job, conditional on applying for that job.

The expected profit of D as a function of w^D, taking w^N as fixed, is

$$\pi(w^D; w^N) = (1 - w^D)(1 - (1-k)^u)(1 - q^N)^{a-1}. \tag{3}$$

When D posts a wage of w^D, there are 3 possible outcomes. One is that no one applies to this vacancy. This occurs with probability $(1-k)^u$. In this case, D's profit is zero. With probability $1-(1-k)^u$, D receives at least one application. With probability $1-(1-q^N)^{a-1}$, the applicant to whom D offers its job has at least one other offer. In this case, Bertrand competition bids the wage up to $w = 1$, and D's profit is again zero. With probability $(1-q^N)^{a-1}$, however, the applicant has no other offers. In this case, the applicant accepts D's offer of w^D, leading to a profit of $1 - w^D$. Using this notation, a symmetric equilibrium wage is thus a w such that

$$w \text{ solves } \max_{w^D \geq 0} \pi(w^D; w).$$

We next develop explicit expressions for q^D and q^N and an implict expression for k. The derivation of q^D is as follows. The probability that a particular worker is offered the D job is kq^D. At the same time, given that all workers choose the same value of k, each worker has an equal chance of being offered the D job, so the probability that the worker is offered this job equals the probability that this vacancy has at least one applicant divided by u. That is, $kq^D = \dfrac{1-(1-k)^u}{u}$ or

$$q^D = \frac{1-(1-k)^u}{ku}. \tag{4}$$

To derive q^N, we reason in a similar fashion. There are $v-1$ N vacancies. Each worker sends $a-1$ applications to the N vacancies; a worker sends his or her a^{th} application to an N vacancy with probability $1-k$. The probability that a worker applies to any particular N vacancy is thus $\dfrac{(a-1)+(1-k)}{v-1} = \dfrac{a-k}{v-1}$, so the probability that an N vacancy has at least one applicant is $1-(1-\dfrac{a-k}{v-1})^u$. The probabililty that a worker gets a particular N job is $\left(\frac{a-k}{v-1}\right) q^N = \frac{1-(1-\frac{a-k}{v-1})^u}{u}$; thus,

$$q^N = \frac{1-(1-\frac{a-k}{v-1})^u}{\left(\frac{a-k}{v-1}\right) u}. \quad (5)$$

For future reference, we note that

$$\frac{\partial q^D}{\partial k} = \frac{ku(1-k)^{u-1} - (1-(1-k)^u)}{k^2 u} \quad (6)$$

and

$$\frac{\partial q^N}{\partial k} = \left(\frac{v-1}{u}\right) \frac{-u\left(1-\frac{a-k}{v-1}\right)^{u-1}\left(\frac{a-k}{v-1}\right) + 1 - \left(1-\frac{a-k}{v-1}\right)^u}{(a-k)^2}. \quad (7)$$

To derive an implicit expression for k, we begin with the fact that each worker has two possible application strategies:

1. Send $a-1$ applications to randomly selected N vacancies and also apply to D;

2. Send all a applications to randomly selected N vacancies.

Note that if $a = v$, only the first strategy is possible and $k = 1$.

Given w^D and w^N, we can compute the expected payoffs to the two strategies. The expected payoff to the first strategy is

$$\begin{aligned} &q^D(1-q^N)^{a-1}w^D + q^D(1-(1-q^N)^{a-1}) \\ &+(1-q^D)(a-1)q^N(1-q^N)^{a-2}w^N \\ &+(1-q^D)(1-(1-q^N)^{a-1} - (a-1)q^N(1-q^N)^{a-2}). \end{aligned}$$

The first term in this expression reflects the fact that a worker who follows the first strategy is offered only the D job with probability $q^D(1-q^N)^{a-1}$; in this case, a payoff of w^D is realized. With probability $q^D(1-(1-q^N)^{a-1})$, the worker's application to D is accepted along with at least one of his or her applications to the N vacancies; in this case the worker's payoff is 1. With probability $(1-q^D)(a-1)q^N(1-q^N)^{a-2}$, the worker is rejected at D and accepted at exactly one of the N vacancies; the resulting payoff is

w^N. With probability $(1-q^D)(1-(1-q^N)^{a-1}-(a-1)q^N(1-q^N)^{a-2})$, the worker is rejected at D but gets 2 or more offers at w^N; in this case, a payoff of 1 is realized. The only other possibility is that all of the worker's applications are rejected, implying a payoff of zero.

The expected payoff to the second strategy is

$$aq^N(1-q^N)^{a-1}w^N + [1-(1-q^N)^a - aq^N(1-q^N)^{a-1}].$$

The first term reflects the fact that the probability of being offered only one N job is $aq^N(1-q^N)^{a-1}$. In this case, the worker receives w^N. The second term gives the probability that the worker is offered more than one job in which case the worker receives 1. For $k \in (0,1)$, the expected payoffs from the two strategies must be equal giving the indifference condition,

$$(1-q^N)\left(q^D w^D - q^N + aq^N(1-w^N)\right) - (1-q^D)(a-1)q^N(1-w^N) = 0. \quad (8)$$

Holding w^N fixed, we can differentiate (8) with respect to w^D. Solving for $\frac{\partial k}{\partial w^D}$ gives

$$\frac{\partial k}{\partial w^D} = \frac{-q^D(1-q^N)}{M}, \quad (9)$$

where

$$\begin{aligned} M &= \frac{\partial q^D}{\partial k}[w^D(1-q^N) + (a-1)q^N(1-w^N)] \\ &\quad + \frac{\partial q^N}{\partial k}[-(1-2q^N)(1-a(1-w^N)) - q^D w^D - (1-q^D)(a-1)(1-w^N)] \end{aligned}$$

Now we can return to D's choice of w^D. The derivative of D's expected profit (equation (3)) with respect to w^D is

$$\frac{\partial \pi(w^D; w^N)}{\partial w^D} = -(1-(1-k)^u)(1-q^N)^{a-1} + (1-w^D)\frac{\partial k}{\partial w^D} \quad (10)$$

$$\left(\begin{array}{c} u(1-k)^{u-1}(1-q^N)^{a-1} - \\ (a-1)(1-q^N)^{a-2}\dfrac{\partial q^N}{\partial k}(1-(1-k)^u) \end{array} \right) \quad (11)$$

We now show that (i) for $a = 1$, there is a unique $w \in (0,1)$, namely, the wage given by equation (2), such that $\frac{\partial \pi(w;w)}{\partial w^D} = 0$ but (ii) for $a \in \{2, ..., v\}$, $\frac{\partial \pi(w;w)}{\partial w^D} < 0$ for all $w \in [0,1]$.

To establish these claims, we need to evaluate $\frac{\partial \pi(w^D; w^N)}{\partial w^D}$ at $w^D = w^N = w$. When D posts the same wage as the other vacancies, we have

$$k = \frac{a-k}{v-1} = a/v$$

$$q^D = q^N = \frac{1-(1-k)^u}{ku} \equiv q$$

$$\frac{\partial q^N}{\partial k} = -\frac{v}{a(v-1)}((1-k)^{u-1} - q)$$

$$\frac{\partial q^D}{\partial k} = -(v-1)\frac{\partial q^N}{\partial k}$$

$$\frac{\partial k}{\partial w^D} = \frac{-a(v-1)q(1-q)}{v^2((1-k)^{u-1} - q)(w(1-aq) + (a-1)q)}.$$

Note that (i) $(1-k)^{u-1} - q < 0$[2] and (ii) $w(1-aq)+(a-1)q > 0 \; \forall w \in [0,1]$[3]. These inequalities, which we use below, imply $\frac{\partial q^N}{\partial k} > 0$, $\frac{\partial q^D}{\partial k} < 0$, and $\frac{\partial k}{\partial w^D} > 0$, as expected.

When $a = 1$,

$$\frac{\partial \pi(w;w)}{\partial w^D} = -(1-(1-k)^u) - \frac{1-w}{w}\left(u(1-k)^{u-1}\frac{(v-1)q}{v^2((1-k)^{u-1} - q)}\right). \tag{12}$$

Setting this expression equal to zero and substituting for k and q gives the wage in equation (2).

The situation when $a \in \{2, .., v\}$ is more complicated. In this case, we have

$$\begin{aligned} \frac{\partial \pi(w;w)}{\partial w^D} &= -(1-(1-k)^u)(1-q)^{a-1} \\ &\quad + (1-w)\frac{\partial k}{\partial w^D}\begin{pmatrix} u(1-k)^{u-1}(1-q)^{a-1} - \\ (a-1)(1-q)^{a-2}\frac{\partial q^N}{\partial k}(1-(1-k)^u) \end{pmatrix} \end{aligned}$$

[2] **Proof**: Let X be a binomial random variable with parameters $u > 1$ and $k \in (0,1)$. Then $(1-k)^{u-1} - q = \frac{ku(1-k)^{u-1} - (1-(1-k)^u)}{ku} = \frac{P[X=0] + P[X=1] - 1}{ku} < 0$.

[3] **Proof**: The inequality holds at $w = 0$ and at $w = 1$, and the expression is linear in w.

Using $1-(1-k)^u = kuq$ and substituting for $\frac{\partial k}{\partial w^D}$ and $\frac{\partial q^N}{\partial k}$ we have

$$\frac{\partial \pi(w;w)}{\partial w^D} = kuq\left(-1 + \frac{1-w}{v(w(1-aq)+(a-1)q)}\left(\frac{(v-1)(1-q)(1-k)^{u-1}}{q-(1-k)^{u-1}} - (a-1)q\right)\right) \tag{13}$$

Since $w(1-aq)+(a-1)q > 0 \ \forall w \in [0,1]$, the inequality we want to prove reduces to

$$v(w(1-aq)+(a-1)q) > (1-w)\left(\frac{(v-1)(1-q)(1-k)^{u-1}}{q-(1-k)^{u-1}} - (a-1)q\right) \tag{14}$$

Note that this inequality is (i) true for $w = 1$ and (ii) linear in w. Thus, if we can prove that this inequality holds at $w = 0$, we will have our result. That is, we need to show

$$v(a-1)q > \frac{(v-1)(1-q)(1-k)^{u-1}}{q-(1-k)^{u-1}} - (a-1)q. \tag{15}$$

Since $q-(1-k)^{u-1} > 0$, the inequality we want to show can be expressed as

$$(v+1)(a-1)q(q-(1-k)^{u-1}) - (v-1)(1-q)(1-k)^{u-1} > 0.$$

If this inequality holds for $a = 2$, then it holds for $a \in \{3, .., v\}$, so we set $a = 2$ and show

$$(v+1)q(q-(1-k)^{u-1}) - (v-1)(1-q)(1-k)^{u-1} > 0.$$

To do this, we rewrite the inequality as

$$v(q^2-(1-k)^{u-1}) + (1-q)(1-k)^{u-1} + q(q-(1-k)^{u-1}) > 0.$$

Since the second and third terms on the left-hand side of this inequality are positive, it suffices to show $q^2-(1-k)^{u-1} \geq 0$, i.e.,

$$(1-(1-k)^u)^2 - k^2u^2(1-k)^{u-1} \geq 0. \tag{16}$$

The proof of this final inequality is as follows.[4] First,

$$1-(1-k)^u = k(1+(1-k)+\ldots+(1-k)^{u-1}),$$

[4] We are extremely grateful to Harald Lang for this proof.

so (16) can be expressed as

$$k^2\left((1+(1-k)+...+(1-k)^{u-1})^2-u^2(1-k)^{u-1}\right).$$

Next, since

$$\frac{1}{u}(1+(1-k)+...+(1-k)^{u-1})\geq(1\times(1-k)\times...\times(1-k)^{u-1})^{\frac{1}{u}},$$

i.e., the arithmetic mean is at least as large as the geometric mean, and

$$1\times(1-k)\times...\times(1-k)^{u-1}=(1-k)^{\frac{u(u-1)}{2}}$$

we have

$$(1+(1-k)+...+(1-k)^{u-1})^2-u^2(1-k)^{u-1}\geq 0.$$

QED.

The equilibrium wage for the case of $a = 1$ is equal to one minus the price given in Proposition 2 in BSW – again with the appropriate notational change. The tradeoff that leads to a well-behaved equilibrium wage, $w \in (0, 1)$, when $a = 1$ is the standard one in equilibrium search theory. As any particular vacancy increases its posted wage, holding the wages posted by other vacancies constant, the probability that it will attract at least one applicant also increases. At the same time, however, the profit that this vacancy will generate conditional on attracting an applicant decreases. This tradeoff varies smoothly with u and v; so the equilibrium wage varies smoothly between zero and one as v increases and/or u decreases. Thus, as emphasized in BSW (p. 1069), there is a sense in which frictions "smooth" the operation of the labor market.

When $a \in \{2, ..., v\}$, matters are radically different. No matter what the values of u and v, so long as workers make more than one application, the posted wage collapses to the Diamond (1971) monopsony level. The intuition for this result is based on the change in the tradeoff underlying equilibrium wage determination. It is still the case that as any particular vacancy increases its posted wage, holding all other posted wages constant, the probability that at least one applicant will be attracted also increases.[5] However, the profit that a vacancy generates conditional on attracting an

[5] As a increases, the rate of increase in this probability decreases. In the limiting case of $a = v$, an increase in the posted wage cannot increase the probability of attracting an applicant since that probability is necessarily already one.

applicant now decreases for *two* reasons when the posted wage is increased. First, if the vacancy manages to employ the worker at the posted wage, then an increase in that wage obviously decreases profit. This is the same factor that limits increases in the posted wage when $a = 1$. Second, and this is the new factor, the probability that the applicant will have other offers increases. The reason is that in symmetric equilibrium, all workers respond to an increase in the wage posted by one vacancy by increasing the probability of applying for that job. Equivalently, the probability that other workers will apply to the same vacancies as the ones applied to by the applicant selected by the vacancy that increased its wage decreases. The probability that the selected applicant will get multiple offers and so generate zero profit thus increases.

Despite the fact that the posted equilibrium wage is zero, there is still a sense in which "the wage" varies smoothly with u and v. The *ex post* fraction of wages equal to one, $\gamma = \dfrac{1-(1-q)^a - aq(1-q)^{a-1}}{1-(1-q)^a}$, increases with v and decreases with u, and in the limit, as $v \to \infty$ holding u fixed (as $u \to \infty$ holding v fixed), $\gamma \to 1$ ($\gamma \to 0$). Note that since the wage is either 0 (the posted wage) or **1** (the Bertrand wage), that γ is the expected wage (paid, as opposed to posted).

3 Efficiency

We now turn to the question of constrained efficiency. Since this issue is usually discussed in the context of "large" labor markets, we first look at the limiting properties of the labor market described in the preceding section. To do this, we let the labor market get large in the standard way, namely, we let the number of unemployed and vacancies increase without limit, but we do so in such a way that the ratio of vacancies to unemployed, i.e., labor market tightness, is held fixed. In Proposition 3, we carry out this limiting exercise holding the number of applications per worker fixed. To make this clear, we use the notation $a \in \{1, ..., A\}$, where A is an integer greater than 1 and does not change as u and v go to infinity.

Proposition 3 *Let $u, v \to \infty$ with $v/u = \theta$ and $a \in \{1, ..., A\}$ fixed. The number of matches increases without limit, but the probability that any one worker finds a job converges to*

$$m(\theta; a) = 1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a. \tag{17}$$

In the case of $a = 1$, the wage converges to

$$w(\theta; 1) = \frac{(1/\theta)\exp(-1/\theta)}{1 - \exp(-1/\theta)}. \tag{18}$$

In the case of $a \in \{2, ..., A\}$, the fraction of wages paid that equals one converges to

$$\gamma(\theta; a) = \frac{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a - \theta(1 - e^{-a/\theta})(1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}}{1 - (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^a}$$

Proof: The probability that an unemployed worker finds a job is

$$\frac{M(u, v; a)}{u} = 1 - (1 - \frac{v}{au}(1 - (1 - \frac{a}{v})^u))^a.$$

Taking the limit as $u, v \to \infty$ with $v/u = \theta$ and a fixed gives

$$m(\theta; a) = 1 - (1 - \frac{\theta}{a}(1 - \lim_{u\to\infty}(1 - \frac{a}{\theta u})^u))^a = 1 - (1 - \frac{\theta}{a}(1 - \exp(-\frac{a}{\theta})))^a.$$

Similarly,

$$\begin{aligned} w(\theta; 1) &= \lim_{\substack{u, v \to \infty \\ v/u = \theta}} \frac{u(v-1)\left(\frac{v-1}{v}\right)^u}{v(v-1) - \left(\frac{v-1}{v}\right)^u (u + v(v-1))} \\ &= \lim_{u,v\to\infty} \frac{\frac{1}{\theta}\frac{v-1}{v}\left(\frac{v-1}{v}\right)^u}{\frac{v-1}{v} - \left(\frac{v-1}{v}\right)^u (\frac{1}{v\theta} + \frac{v-1}{v})} \\ &= \frac{(1/\theta)\exp(-1/\theta)}{1 - \exp(-1/\theta)}. \end{aligned}$$

The expression for $\gamma(\theta; a)$ is derived using $\lim_{u,v\to\infty} q = \frac{\theta}{a}(1 - e^{-a/\theta})$. *QED.*

With $a = 1$, it is easy to verify that our limiting matching function and the limiting wage for the case of $a = 1$ match the corresponding entities in a labor-market version of BSW. In general, i.e., for $a \in \{1, 2, ..., A\}$, the limiting matching probability has the following properties:

(i) $m(\theta; a)$ is increasing and concave in θ and $\frac{m(\theta; a)}{\theta}$ is decreasing and convex in θ,

(ii) For large values of u and v, the matching function exhibits approximate constant returns to scale for each fixed a in the sense that in the limit, the matching probability depends only on the ratio of v to u,
(iii) $m(\theta; a)$ first increases and then decreases in a.

For the case of $a \in \{2, ..., A\}$, we also need to investigate the properties of $\gamma(\theta; a)$. In a series of plots (not shown), we can see that the expected wage is increasing and concave in both θ and a. The result for θ is exactly as one would expect – as the labor market gets tighter, the chance that an individual worker gets multiple offers increases. To understand $\gamma_a(\theta; a) > 0$, especially in light of the fact that $m(\theta; a)$ first increases but then decreases with a, it is important to remember that $\gamma(\theta; a)$ is the expected wage for those workers who match with a vacancy; in particular, those workers who fail to match are not treated as receiving a wage of zero.

Proposition 3 describes the limiting properties of the labor market taking a as a given constant. Alternatively, we could let the number of applications per worker become arbitrarily large as well. The most natural way to do this is to let $a = v$, and to then let u and v go to infinity in the standard way. This allows us to consider a limiting version of the Julien, Kennes, and King (2002) model.

Proposition 4 *Let $a = v$, and let $u, v \to \infty$ with $v/u = \theta$. Then the probability that any one worker finds a job converges to*

$$m(\theta) = 1 - e^{-\theta}$$

and the fraction of wages paid equal to one converges to

$$\gamma(\theta) = \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - e^{-\theta}}.$$

We omit the proof, as it is straightforward.

We now verify that the standard result on the efficiency of competitive search equilibrium holds in our setting when $a = 1$; however, when $a \in \{2, ..., A\}$, this result breaks down. As in Proposition 3, we let $u, v \to \infty$ with $v/u = \theta$ and a fixed. We now imagine that vacancies are set up at the beginning of the period and that each vacancy is created at cost c. The efficient level of labor market tightness[6] is determined as the solution to

$$\max_{\theta}\{-c\theta + m(\theta; a)\}$$

[6] In a finite labor market with u given, the social planner chooses v to maximize $-cv + M(u, v; a)$; i.e., expected output (equal to the expected number of matches since each match produces an output of 1) minus the vacancy creation costs. Dividing the maximand by u and letting $u, v \to \infty$ gives the maximand in the text.

i.e.,

$$-c + m_\theta(\theta^*; a) = 0. \tag{19}$$

The equilibrium level of labor market tightness is determined by free entry. When $a = 1$, this means

$$-c + \frac{m(\theta^{**}; 1)}{\theta^{**}}(1 - w(\theta^{**}; 1)) = 0, \tag{20}$$

whereas for $a \in \{2, ..., A\}$, the condition is

$$-c + \frac{m(\theta^{**}; a)}{\theta^{**}}(1 - \gamma(\theta^{**}; a)) = 0. \tag{21}$$

Equations (21) and (22) reflect the condition that entry (vacancy creation) occurs up to the point that the cost of vacancy creation is just offset by the value of owning a vacancy. This value equals the probability of hiring a worker times the expected surplus generated by a hire – equal to 1 minus the wage when $a = 1$ and to 1 minus the expected wage when $a \in \{2, ..., A\}$.

Here, we are using θ^* to denote the constrained Pareto efficient level of labor market tightness and and θ^{**} to denote the equilibrium level of labor market tightness. At issue is the relationship between θ^* and θ^{**}.

Proposition 5 *Let $u, v \to \infty$ with $v/u = \theta$ and $a \in \{1, ..., A\}$ fixed. For $a = 1$, $\theta^* = \theta^{**}$. For $a \in \{2, ..., A\}$, $\theta^{**} > \theta^*$.*

Proof: When $a = 1$, differentiating equation (17) gives

$$m_\theta(\theta; 1) = 1 - e^{-1/\theta} - \frac{1}{\theta}e^{-1/\theta};$$

and equations (17) and (18) give

$$\frac{m(\theta; 1)}{\theta}(1 - w(\theta; 1)) = 1 - e^{-1/\theta} - \frac{1}{\theta}e^{-1/\theta}.$$

Thus, equations (19) and (20) imply $\theta^* = \theta^{**}$.
Similarly, we find that when $a \in \{2, ..., A\}$, θ^* solves

$$c = (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta} - \frac{a}{\theta}e^{-a/\theta}), \tag{22}$$

whereas θ^{**} solves

$$c = (1 - \frac{\theta}{a}(1 - e^{-a/\theta}))^{a-1}(1 - e^{-a/\theta}). \tag{23}$$

The right-hand sides of both (22) and (23) are decreasing in θ. Since the right-hand side of (23) is greater than that of (22) for all $\theta > 0$, it follows that $\theta^{**} > \theta^{*}$. *QED.*

Posting a vacancy has the standard congestion and thick-market effects in our model – adding one more vacancy makes it more difficult for the incumbent vacancies to find workers but makes it easier for the unemployed to generate offers. A striking result of the competitive search equilibrium literature is that adding one more vacancy causes the wage to adjust in such a way as to balance these external effects correctly. Equivalently, one can say that competition leads to a wage that equals the one that would be dictated by the Hosios (1990) condition in a Nash bargaining model. The first part of Proposition 5 shows that this result continues to hold when one uses the urn-ball ($a = 1$) microfoundation for the matching function. However, when workers make multiple applications, the result that $\theta^{**} > \theta^{*}$ indicates that the equilibrium level of vacancy creation is too high. Equivalently, the equilibrium expected wage is too low, i.e., below the level that would be indicated by the Hosios condition. The effects of the marginal vacancy are more complicated with multiple applications than in the urn-ball model. Adding one more vacancy makes it less likely that each incumbent vacancy will attract any applicants but, conditional on attracting an applicant, makes it more likely for the incumbent vacancy to "win the race" for that applicant. Adding another vacancy to the market puts upward pressure on the (expected) wage but not to the extent required to achieve the efficient level of entry.

Proposition 5 lets the labor market get large holding a fixed. We now consider the question of efficiency with $a = v$ as the labor market gets arbitrarily large.

Proposition 6 *Let $a = v$, and let $u, v \to \infty$ with $v/u = \theta$. Then $\theta^{**} = \theta^{*}$; i.e., the equilibrium level of labor market tightness is constrained efficient.*

The proof follows directly from Julien, Kennes, and King (2002); in fact, they show that this efficiency result holds even for finite u and v. Alternatively, the result can be shown by mimicking the proof of Proposition 5.

The intuition for why we find constrained efficiency with $a = 1$ and $a = v$ but not with a fixed number of multiple applications has to do with the fact that with $a = 1$ and $a = v$, only the urn-ball coordination problem affects the operation of the labor market, whereas with a fixed $a \in \{2, ..., A\}$, the urn-ball and the multiple applications coordination problems operate simultaneously. Adjusting the wage can only solve one coordination problem at a time.

4 Concluding Remarks

Thus far, we have constructed a one-period model with homogeneous workers and vacancies, and we have used our model to examine equilibria in which all vacancies post the same wage. The preceding sentence suggests three possible extensions.

First, it would be useful to extend our analysis to the steady-state. Steady-state analysis would offer two advantages relative to the single-period framework. First, both u and v can be endogenized in the steady state – by a steady-state (Beveridge curve) condition and by free entry, respectively. Second, the steady-state framework allows those unemployed who fail to find an acceptable job in the current period to apply again in the next period; that is, the unemployed will have a positive reservation wage. The ability of the unemployed to hold out for a situation in which firms engage in Bertrand competition for their services, albeit at the cost of delay, will in turn have an impact on the wage-posting problem faced by vacancies.

Second, it would be useful to examine the implications of multiple applications in a model with heterogeneous workers and/or vacancies, i.e., to incorporate multiple applications into a model along the lines of Shimer (2001). When an unemployed worker can send out more than one application, a plausible strategy might be to risk some applications on "long-shot" vacancies while reserving others for relatively "safe" vacancies. It would be very interesting to know whether a strategy of this type makes sense in equilibrium and, if so, what the implications for equilibrium matching and wage setting might be.

Third, one could think about equilibria in which not all vacancies post the same wage. Our argument in the multiple application case that it is in the interest of each vacancy to post the monopsony wage is made conditional on the provisional equilibrium assumption that all other vacancies post that wage. There is nothing in our argument, however, that rules out a situation in which some vacancies post the monopsony wage while others post a wage between 0 and 1. (Obviously, there may be other arguments that preclude this outcome.) That is, there may be scope for a directed search version of a result along the lines of Burdett and Judd (1983).

While these extensions are potentially quite interesting, we feel that examining the role of multiple applications in a homogeneous worker and homogeneous vacancy, one-period symmetric equilibrium directed search model has been very fruitful. Multiple applications (i) introduce a new coordination friction into the matching process, (ii) generate equilibrium wage dispersion, and (iii) imply excessive vacancy creation in large labor markets.

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