The Survival and Price Impact of Irrational Traders

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Abstract

We examine the long-run survival of irrational traders who use persistently wrong beliefs to make their portfolio choices. Using a parsimonious model, we show that the partial equilibrium arguments to support the long-run survival of irrational traders are flawed. The impact irrational traders have on equilibrium prices is important in determining their long-run fortunes; in particular, we find that irrational traders with very little wealth and CRRA preferences may still have a large impact on stock prices. However, absent intermediate consumption and rational traders with logarithmic preferences, irrational traders with beliefs mildly different from the true probabilities can survive in the long run. In the presence of intermediate consumption, we show that under fairly realistic conditions irrational traders do not survive or have a long term price impact, but that these results are sensitive to assumptions about preferences and the aggregate endowment.

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1 Introduction

Classical asset pricing models rely on the assumption that market participants (traders) are rational in the sense that they behave in ways that are consistent with the objective probabilities of the states of the economy (e.g., Radner (1971) and Lucas (1978)). In particular, they maximize expected utilities using the true probabilities of uncertain economic events. Such an assumption is not based on the observed behavior of average traders in the market. It is often based on two types of arguments. The first argument is that asset prices are not determined by average traders, but rather by informed traders with easy access to capital, namely, the large traders whose behavior is (close to) rational. The second argument (see, e.g., Friedman (1953)), which further reinforces the first line of reasoning, is that irrational traders who use wrong probabilities do not survive in a competitive market. Trading under the wrong beliefs makes them lose money to the rational traders and eventually lose their wealth. It is the rational traders who survive and ultimately dominate in the market.

However, whether or not irrational traders with persistently wrong beliefs can survive in a competitive market remains an open question. For example, De Long, Shleifer, Summers and Waldmann (1991) (DSSW, thereafter) use a partial equilibrium model to argue that in the absence of intermediate consumption, traders with wrong beliefs may survive in the long run because they may hold portfolios with higher growth rate and therefore can eventually outgrow the rational traders. In contrast, in a stationary general equilibrium setting with intermediate consumption, Sandroni (2000) and Blume and Easley (2001) show that irrational traders do not survive in the long run. DSSW also point out the negative impact intermediate consumption has on the long-run survival of irrational traders in their partial equilibrium framework. Reconciling these seemingly contradictory results in a unified framework has remained an open and challenging problem. In particular, how robust is the intuition based on partial equilibrium arguments? Can irrational traders survive in equilibrium in the absence of intermediate consumption? And finally, a question central to the debate about the role of irrational traders in financial markets, does there exists a conceptual distinction between the long-run survival and the long-run price impact of irrational traders?

In this paper we address these questions using a parsimonious general equilibrium model analogous to the partial equilibrium framework of DSSW. By deriving an explicit solution to the model, we show that the partial equilibrium arguments of DSSW are not reliable, demonstrating that the impact of the irrational traders on asset prices is important to their long-run fortunes and should not be overlooked. However, we find that in the absence of intermediate consumption, the irrational traders can survive in the long-run, unless the rational traders have logarithmic utility. The intuition is well known (see, e.g., Hakansson (1971)). With non-logarithmic utility, traders in general hold portfolios that have lower growth than the maximum growth portfolio. Wrong beliefs can drive the irrational traders to hold portfolios closer to the maximum growth portfolio than those of the rational traders. As a result, their wealth sustains a higher growth rate in the long-run and can eventually dominate the allocation of wealth in the economy. However, we find that the long-run survival of the irrational traders only occurs when the deviations in their beliefs from the true probability measure is mild.

Another question of interest to us is how critical the survival of irrational traders, in terms of their relative wealth, is to their ability to impact prices. This is directly related to the first argument for the rationality assumption mentioned above. Sandroni (2000) gives an affirmative answer, showing "the intuitive result that agents whose wealth converges to zero eventually have no influence on prices." However, in our framework we demonstrate that irrational traders do not need a significant share of wealth to affect the behavior of asset prices. It is possible for a group of irrational traders with small share of wealth to exert significant influence on prices. The price impact of non-surviving irrational traders dies much more slowly than their share of wealth. Figlewski (1979) has argued that it may take the irrational traders a very long time to lose their share of wealth. Our analysis shows that it may take an order of magnitude longer for them to lose their influence on prices.

Even when the irrational traders' wealth share is small, they affect the dynamics of instantaneous moments of stock returns. Over long periods of time, the wealth distribution in the economy may change drastically, inducing changes in the investment opportunity set. For example, if the irrational traders are pessimistic and relatively small in the economy, they will hold proportionately less of their wealth in the stock. If the stock price were to decline significantly due to a negative shock to the aggregate state (the expectations of the terminal dividend), the relative wealth of the irrational traders would rise and the stock price moments would be affected accordingly. Such a potential change in the investment opportunity set induces a non-trivial hedging demand on behalf of the rational traders, which in turn can have a significant impact on the level of prices and the moments of returns. Thus, the price impact of the relatively small traders is *indirect* and relies critically on the dynamic nature of the financial markets; it comes through the hedging demand of the larger traders rather than directly through the demand of the smaller traders.

We extend our analysis of the survival and price impact of irrational traders by considering economies with time-varying irrational beliefs and economies with intermediate consumption. Several papers, including De Long, Shleifer, Summers, and Waldman (1990), show that under certain conditions irrational traders can earn higher expected returns than rational traders. This result is not sufficient for survival because it ignores the relative volatility of the portfolios of rational and irrational traders. We demonstrate in our framework that when irrational traders have a deterministically fluctuating bias in their beliefs, they do not survive and do not affect prices in the long run. Because the beliefs of irrational traders change over time, their portfolio holdings fluctuate as well. The irrational traders become extinct not necessarily because they "buy high and sell low", but also because their portfolio exhibits a higher average volatility than for the rational traders.

Our second extension is to consider economies with intermediate consumption. In such economies the irrational traders must make both portfolio and consumption choices, both of which are affected by their beliefs. We find that under fairly plausible conditions, the irrational trader does not survive and has no price impact, but that this is not a completely general conclusion. In contrast to earlier results, e.g., Sandroni (2000), who claims that the extinction of irrational traders is a generic evolutionary outcome, we show that when the aggregate endowment is allowed to be unbounded the irrational traders may survive.

The rest of the paper is organized as follows. Section 2 describes a canonical, pure exchange economy similar to that of Black and Scholes (1973), but in presence of irrational traders who have persistently wrong beliefs about the economy. Section 3 describes the general equilibrium of the economy. In Section 4, we first consider the partial equilibrium argument of DSSW using our setting to analyze the conditions under which the irrational traders may survive in the long run. We then present the general equilibrium results on the survival of irrational traders. In Section 5, we examine in detail the shortcomings of the partial equilibrium analysis. We also study the portfolio policies, the wealth dynamics, and the price impact of the irrational traders. We then extend our basic model with time-varying irrational beliefs. Section 6 describes and evaluates the economy with intermediate consumption. Section 7 concludes.

2 The Model

Since our objective is to analyze the economic mechanisms that determine the survival of irrational traders and their impact on prices, we use a simple model for parsimony. Generalizations are discussed in Section 7.

Information structure

We consider a continuous-time, finite-horizon economy. The uncertainty of the economy is described by a one-dimensional, standard Brownian motion B_t for $0 \leq t \leq T$, defined on a complete probability space (Ω, F, P), where F is the augmented filtration generated by B_t.

Financial markets

There is a single share of a risky asset in the economy, the stock, which is a claim on a dividend payment D_T at time T. D_T is the value of a geometric Brownian motion D_t at time T, where $D_0 = 1$ and

$$dD_t = D_t \left(\mu dt + \sigma dB_t \right), \quad \sigma > 0. \tag{1}$$

There is also a risk-free bond, available in zero net supply. Each unit of the risk-free bond makes a payment of one at time T. We use the risk-free bond as the numeraire and denote the price of the stock at time t by S_t.

Endowments

There are two competitive traders in the economy, each endowed with half a share of the stock and zero unit of the bond at time zero.

Trading strategies

Financial markets are frictionless, and there are no constraints on lending and borrowing. Traders' trading strategies satisfy the standard integrability condition

$$\int_0^T \theta_t^2 \, d\langle S \rangle_t < \infty \tag{2}$$

where θ_t is the number of shares of the stock held in the portfolio at time t and $\langle S \rangle_t$ is the quadratic variation process of S_t.

Preferences and beliefs

Both traders have constant relative risk aversion utility, defined over their consumption at time T:

$$\frac{1}{1-\gamma}C_T^{1-\gamma}, \quad \gamma \geq 1.$$

For simplicity in exposition, we only consider the cases when γ is no less than one. The cases when $0 < \gamma < 1$ can be analyzed similarly and the results are similar in spirit.

Standard aggregation results imply that each trader in our model can actually represent a collection of traders with the same preferences. This provides a justification for our competitive assumption for each of the traders.

The first, rational trader, knows the true probability measure P and maximizes expected utility

$$\mathrm{E}_0^P\left[\frac{1}{1-\gamma}C_{r,T}^{1-\gamma}\right] \tag{3}$$

where the subscript r denotes quantities associated with the rational trader.

The second, irrational trader, believes incorrectly that the probability measure is Q, under which

$$dB_t = (\sigma\eta)dt + dB_t^Q \tag{4}$$

and hence

$$dD_t = D_t\left[\left(\mu + \sigma^2\eta\right)dt + \sigma dB_t^Q\right] \tag{5}$$

where B_t^Q is the standard Brownian motion under the measure Q and η is a constant, parameterizing the degree of irrationality of the irrational trader. When η is positive, the irrational trader is optimistic about the prospects of the economy, overestimating the rate of growth of the aggregate endowment. Similarly, negative η corresponds to a pessimistic irrational trader. The irrational trader maximizes expected utility using belief Q:

$$\mathrm{E}_0^Q\left[\frac{1}{1-\gamma}C_{n,T}^{1-\gamma}\right] \tag{6}$$

where the subscript n denotes quantities associated with the irrational trader.

Because η is assumed to be constant, the probability measure of the irrational trader Q is absolutely continuous with respect to the objective measure P, i.e., both traders agree on zero-probability events. Let $\xi_t \equiv (dQ/dP)_t$ denote the Radon-Nikodym derivative of the probability measure Q with respect to P. Then

$$\xi_t = e^{-\frac{1}{2}\eta^2\sigma^2 t + \eta\sigma B_t} \tag{7}$$

and the irrational trader maximizes

$$\mathrm{E}_0^Q\left[\frac{1}{1-\gamma}C_{n,T}^{1-\gamma}\right] = \mathrm{E}_0^P\left[\xi_T\frac{1}{1-\gamma}C_{n,T}^{1-\gamma}\right]. \tag{8}$$

Thus, the objective of the irrational trader can be equivalently expressed as the expected value of a state-dependent utility function, $\xi_T\frac{1}{1-\gamma}C_{n,T}^{1-\gamma}$, under the true probability measure P.

In what follows, we adopt this equivalent expression of the irrational trader's objective and use the true probability measure P without further clarification.

3 The Equilibrium

The competitive equilibrium of the economy defined above can be solved analytically. Since there is only one source of uncertainty in the economy, the financial markets are dynamically complete, as long as the volatility of stock returns remains non-zero almost surely. In fact, in equilibrium, the instantaneous volatility of stock returns is bounded below by σ. Consequently, the equilibrium allocation is efficient and can be characterized as the solution to a central planner's problem:

$$\max \quad \left[\frac{1}{1-\gamma}C_{r,T}^{1-\gamma} + b\,\xi_T\frac{1}{1-\gamma}C_{n,T}^{1-\gamma}\right] \tag{9a}$$

$$\text{s.t.} \quad C_{r,T} + C_{n,T} = D_T \tag{9b}$$

where b is the ratio of utility weights. The equilibrium allocation is characterized in the following proposition.

Proposition 1 *For the economy defined in Section 2, the equilibrium allocation between the* *two traders is*

$$C_{r,T} = \frac{1}{1+(b\,\xi_T)^{1/\gamma}} D_T \tag{10a}$$

$$C_{n,T} = \frac{(b\,\xi_T)^{1/\gamma}}{1+(b\,\xi_T)^{1/\gamma}} D_T \tag{10b}$$

where

$$b = e^{(\gamma-1)\eta\sigma^2 T}. \tag{11}$$

The price of a financial security with the terminal payoff Z_T is given by

$$P_t = \frac{\mathrm{E_t}\left[\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma} Z_T\right]}{\mathrm{E_t}\left[\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}\right]}. \tag{12}$$

For the stock, $Z_T = D_T$ and its volatility is bounded between σ and $\sigma(1+|\eta|)$.

4 The Survival of Irrational Traders

In this section, we examine the survival of each type of traders in the long-run when T becomes large. We start by defining long-run survival and extinction.

Definition 1 *The irrational trader is said to experience relative extinction in the long-run if*

$$\lim_{T\to\infty} \frac{C_{n,T}}{C_{r,T}} = 0 \quad a.s. \tag{13}$$

The relative extinction of the rational trader can be defined symmetrically. A trader is said to survive in the long-run if relative extinction does not occur.

In our model, the final wealth of each trader equals the terminal consumption. Thus, the above definition of survival and extinction is equivalent to a similar definition using wealth.

4.1 Heuristic Partial Equilibrium Analysis

De Long, Shleifer, Summers and Waldmann (1991) have examined the survival of irrational traders, using partial equilibrium arguments. In this section, we follow their argument and derive 'conditions' under which the irrational trader may survive in the long run despite the wrong belief. We start with two limiting cases in which one of the two traders controls most of the aggregate wealth in the economy while the other trader is infinitesimal. We then assume that the infinitesimal agent has no impact on market prices and compute the implied growth rate of both traders' wealth. If the wealth of the infinitesimally small trader has a higher growth rate, and therefore the share of wealth of such a trader is growing over time, then DSSW conclude that such a trader would be able to successfully "invade" the economy and hence must survive in the long run, "in the sense that their wealth share does not drop toward zero in the long run with probability one."

Assume first that the rational trader dominates the economy and the prices of financial assets are as if there are no irrational traders in the economy. Let μ_S and σ_S denote the drift and volatility of the stock price process:

$$dS_t = S_t \left(\mu_S dt + \sigma_S dB_t\right).$$

It is easy to show that

$$\mu_S = \gamma\sigma^2, \quad \sigma_S = \sigma.$$

The rational trader invests only in the stock. The rate of growth of the rational trader's portfolio, defined as the drift of the logarithm of the trader's wealth, is given by

$$\mu_S - \frac{1}{2}\sigma_S^2 = \frac{1}{2}(2\gamma - 1)\sigma^2.$$

Under the beliefs of the irrational trader (under the measure Q), the drift of the stock price process is

$$\widehat{\mu}_S = \mu_S + \sigma^2\eta$$

and the volatility remains σ. The irrational trader invests a fraction $w_n = \widehat{\mu}_S/\left(\gamma\sigma^2\right) = 1+\eta/\gamma$

of the wealth in the stock. Thus, the growth rate of the irrational trader's portfolio is

$$\mu_S - \frac{1}{2}\sigma^2 + \frac{1}{2}\frac{\sigma^2}{\gamma^2}\eta\left(\gamma\eta^\star - \eta\right)$$

where we define

$$\eta^\star \equiv 2(\gamma - 1). \tag{14}$$

The growth rate of wealth of the "invading" irrational trader is higher than that of the dominant rational trader if and only if $0 < \eta < \gamma\eta^\star$.

Next, assume that the irrational trader dominates the economy. Repeating the steps of the previous analysis, the volatility of the stock price remains at σ and the drift becomes

$$\mu_S = \gamma\sigma^2 - \eta\sigma^2.$$

For the irrational trader, the growth rate of the portfolio is

$$\mu_S - \frac{1}{2}\sigma^2.$$

while for the rational trader it is

$$\mu_S - \frac{1}{2}\sigma^2 + \frac{1}{2}\frac{\sigma^2}{\gamma^2}(2\gamma - 1)\eta\left(\eta - \frac{\gamma}{2\gamma - 1}\eta^\star\right).$$

The rational trader's portfolio grows faster than the irrational trader's portfolio if and only if $\eta < 0$ or $\eta > \frac{\gamma}{2\gamma-1}\eta^\star$.

The partial equilibrium analysis thus appears to provide sufficient conditions for long-run survival of both types of traders. In particular,

$$\begin{array}{lcl} 0 < \eta < \frac{\gamma}{2\gamma-1}\eta^\star & \Rightarrow & \text{Irrational trader survives} \\ \frac{\gamma}{2\gamma-1}\eta^\star < \eta < \gamma\eta^\star & \Rightarrow & \text{Both traders survive} \\ \eta < 0 \text{ or } \eta > \gamma\eta^\star & \Rightarrow & \text{Rational trader survives} \end{array} \tag{15}$$

For $\gamma = 1$, only the rational trader survives regardless of the value of η. These results can be summarized in the following phase diagram in the parameter space. Note that $\gamma/(2\gamma-1) \le 1$ for $\gamma \ge 1$, therefore $\eta^\star$ belongs to the second region defined by (15).

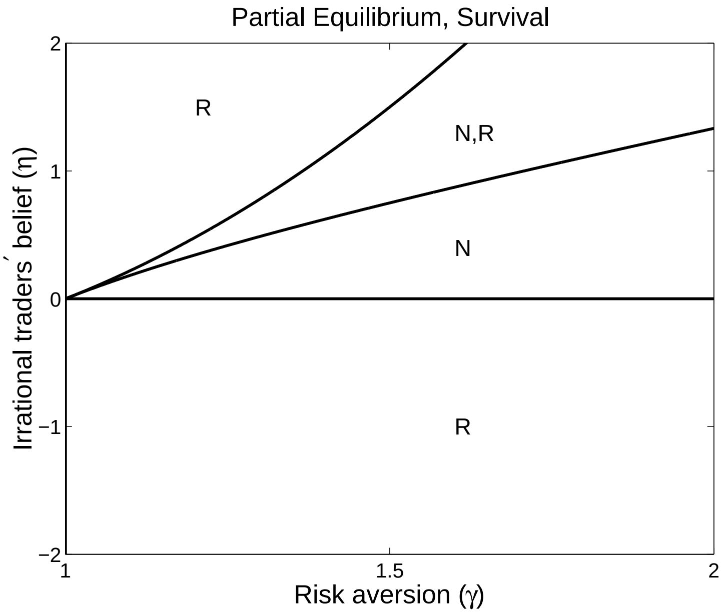

Figure 1: The survival of rational and irrational traders for different values of η and γ in partial equilibrium. For each region in the parameter space, we document which of the agents survives in the long run. "R" means that survival of the rational trader is guaranteed inside the region, "N" corresponds to the irrational trader, "N,R" means that both traders survive.

4.2 General Equilibrium Analysis

The partial equilibrium analysis seems straightforward and intuitive. However, it relies on the assumption that the stock price is unaffected by the diminishing trader, whose wealth approaches zero in the long run. As we will demonstrate, this assumption does not always hold. Thus, the results from the partial equilibrium analysis on survival and extinction can be incorrect. Given the competitive equilibrium derived in Section 3, we have the following result:

Proposition 2 *Suppose* $\eta \neq 0$*.*

(i) For $\gamma = 1$*, the irrational trader never survives.*

(ii) For $\gamma > 1$*, only one of the traders survives in the long run. In particular, defining*

$\eta^\star = 2(\gamma - 1)$,

$$
\begin{aligned}
&\text{Pessimistic irrational trader: } \eta < 0 && \Rightarrow \text{ Rational trader survives} \\
&\text{Moderately optimistic irrational trader: } 0 < \eta < \eta^\star && \Rightarrow \text{ Irrational trader survives} \quad (16)\\
&\text{Strongly optimistic irrational trader: } \eta > \eta^\star && \Rightarrow \text{ Rational trader survives}
\end{aligned}
$$

For $\eta = \eta^\star$, both rational and irrational traders survive.

The result for $\gamma = 1$ is intuitive. For $\gamma = 1$, the rational trader holds the portfolio with maximum growth. Any deviation in the belief from the true probability causes the irrational trader to move away from the maximum growth portfolio, which leads to long-run relative extinction.

For $\gamma > 1$, Proposition 2 identifies three distinct regions in the parameter space, which is shown in Figure 2. For $\eta < 0$, the irrational trader is pessimistic and does not survive in the long-run. For $0 < \eta < \eta^\star$, the irrational trader is moderately optimistic and survives in the long-run while the rational trader does not. For $\eta > \eta^\star$, the irrational trader is strongly optimistic and does not survive.

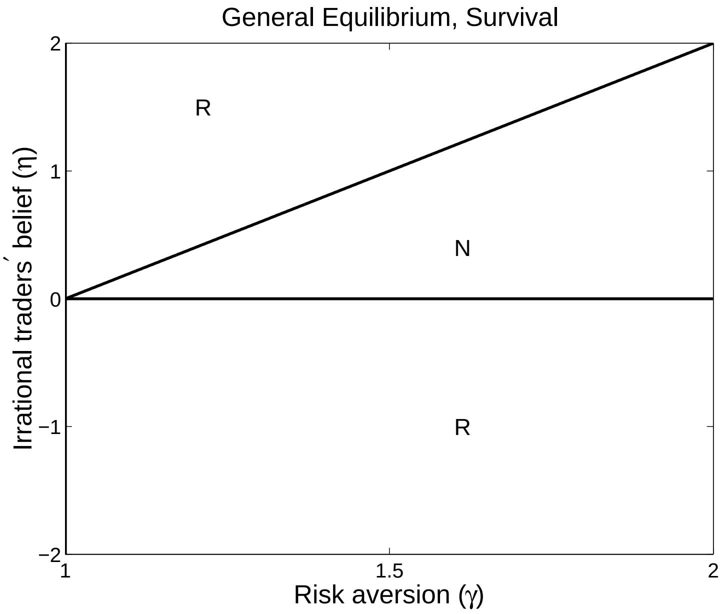

Figure 2: The survival of rational and irrational traders for different values of η and γ in general equilibrium. For each region in the parameter space, we document which of the agents survives in the long run. "R" means that survival of the rational trader is guaranteed inside the region, "N" corresponds to the irrational trader.

It is interesting to compare Figure 2, the phase diagram in general equilibrium, with Figure 1, the phase diagram from partial equilibrium arguments. Other than the knife-edge case ($\eta = \eta^{\star}$), only one of the traders can survive in general equilibrium. Moreover, for certain parameter values, the partial equilibrium argument of DSSW leads to incorrect predictions about the survival of irrational traders. We discuss the reasons for the failure of the partial equilibrium argument in the next section. It is also worth noting that in the current setting, it is impossible for both traders to survive. In fact, the surviving trader eventually dominates the economy and own most of the wealth. In general, however, survival and dominance may not be equivalent.

In the above discussion, we used the notion of extinction in the relative sense, when the relative wealth of a trader goes to zero in the long-run. We can also consider the notion of extinction in the absolute sense.

Definition 2 *The irrational trader is said to experience long-run extinction in the absolute sense if*

$$\lim_{T\to\infty} C_{n,T} = 0 \quad a.s. \tag{17}$$

The absolute extinction of the rational trader can be defined symmetrically.

Our equilibrium analysis gives the following result:

Proposition 3 *Assume that the aggregate endowment is growing, i.e.,* $\mu > \frac{1}{2}\sigma^2$.

(i) The irrational trader experiences long-run absolute extinction if

$$\eta < (\gamma - 1) - \sqrt{(\gamma - 1)^2 + \gamma\,(2\mu/\sigma^2 - 1)}$$

or

$$\eta > (\gamma - 1) + \sqrt{(\gamma - 1)^2 + \gamma\,(2\mu/\sigma^2 - 1)}.$$

(ii) The rational trader experience long-run absolute extinction if $(\gamma-1)^2 - \gamma\,(2\mu/\sigma^2 - 1) > 0$ *and*

$$(\gamma - 1) - \sqrt{(\gamma - 1)^2 - \gamma\,(2\mu/\sigma^2 - 1)} < \eta < (\gamma - 1) + \sqrt{(\gamma - 1)^2 - \gamma\,(2\mu/\sigma^2 - 1)}.$$

Figure 3 shows the ranges of η, given γ, for the relative and absolute extinction of the two traders.

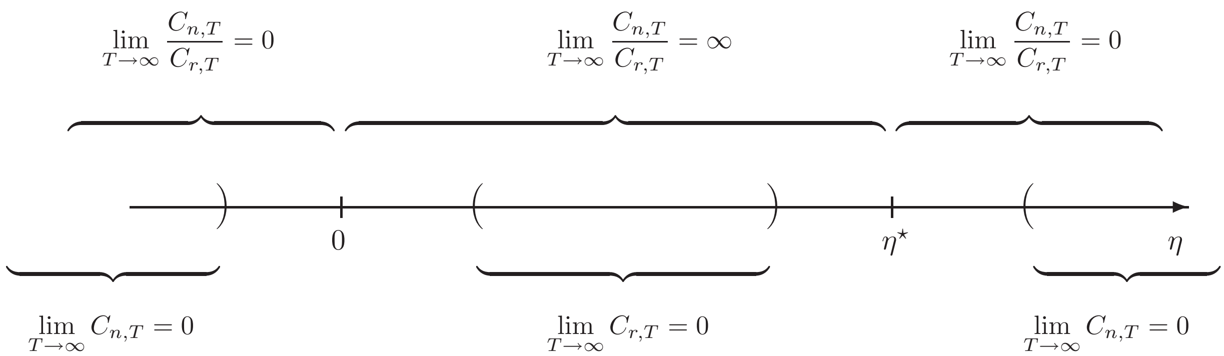

Figure 3: Asymptotic consumption behavior.

In order to gain more intuition about what determines the survival of the irrational trader, we examine the terminal wealth profile of the rational and irrational traders. Figure 4 shows the two traders' terminal wealth profile and the state price density when the irrational trader is pessimistic. The left panel shows the terminal wealth distributions of the rational trader (solid line) and the irrational trader (dashed line) for three values of T (5, 25 and 125). Obviously, the rational trader ends up with more wealth in good states of the economy (when the dividend is high) while the irrational trader, being pessimistic, ends up with more wealth in the bad states of the economy. In the right panel, we show the probability distribution of the terminal state with the dashed line and the state price density with the solid line. Apparently, the rational trader has more wealth (consumption) in cheap states, those states with low state prices adjusted by the true probabilities. The irrational trader, however, has more wealth in expensive states. (Of course, adjusted by his own belief, these states look cheap.) Since the more likely states tend to be the cheap states, the rational trader is more likely to end up with more wealth. The probability for the rational trader to win increases as the horizon increases.

When the irrational trader is mildly optimistic, the situation is different. His impact on the prices make the bad states (i.e., the low dividend states) cheaper than the good states. This induces the rational trader to accumulate more wealth in the bad states by giving up wealth in the good states, including those with high probabilities. As a result, the irrational trader is more likely to end up with more wealth. When strongly optimistic, the irrational

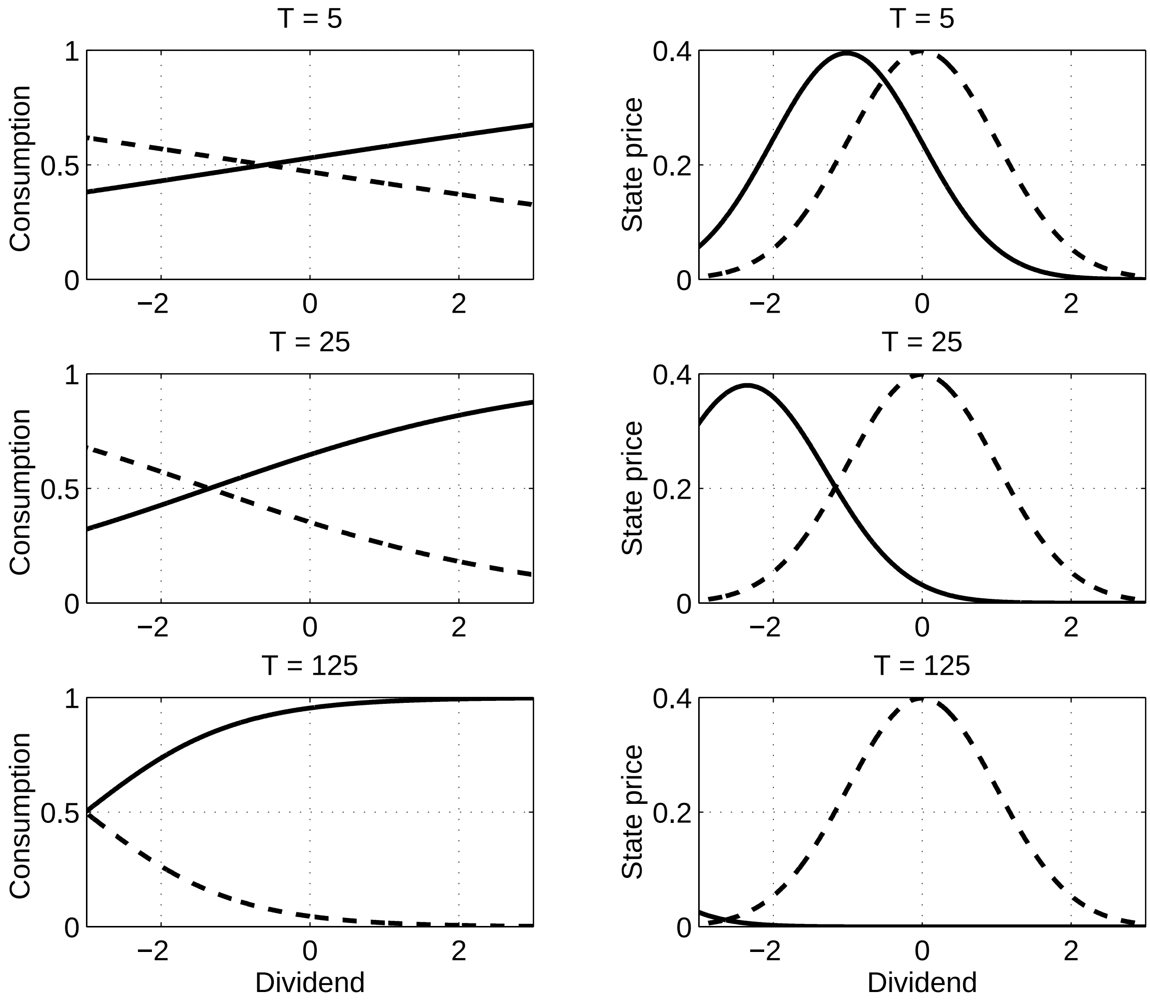

Figure 4: The terminal consumption of rational and irrational traders and the state prices for different values of the economy horizon T. We consider three values of the economy horizon, $T = 5, 25, 125$. We set the model parameters to $\mu = 0.12$, $\sigma = 0.18$, and $\gamma = 2$. We assume that the irrational trader is pessimistic and set $\eta = -0.5\eta^{\star}$. The horizontal axis in all panels is the normalized value of the terminal dividend, i.e., $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$. The three panels on the left show the terminal consumptions of the rational trader (solid line) and the irrational trader (dashed line) as a fraction of the aggregate endowment, i.e., $C_{r,T}/D_T$ and $C_{n,T}/D_T$. The three panels on the right show the probability distribution of the normalized dividend (dashed line), which is a standard normal random variable, and the state price density with respect to the Lebesque measure over $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$ (solid line).

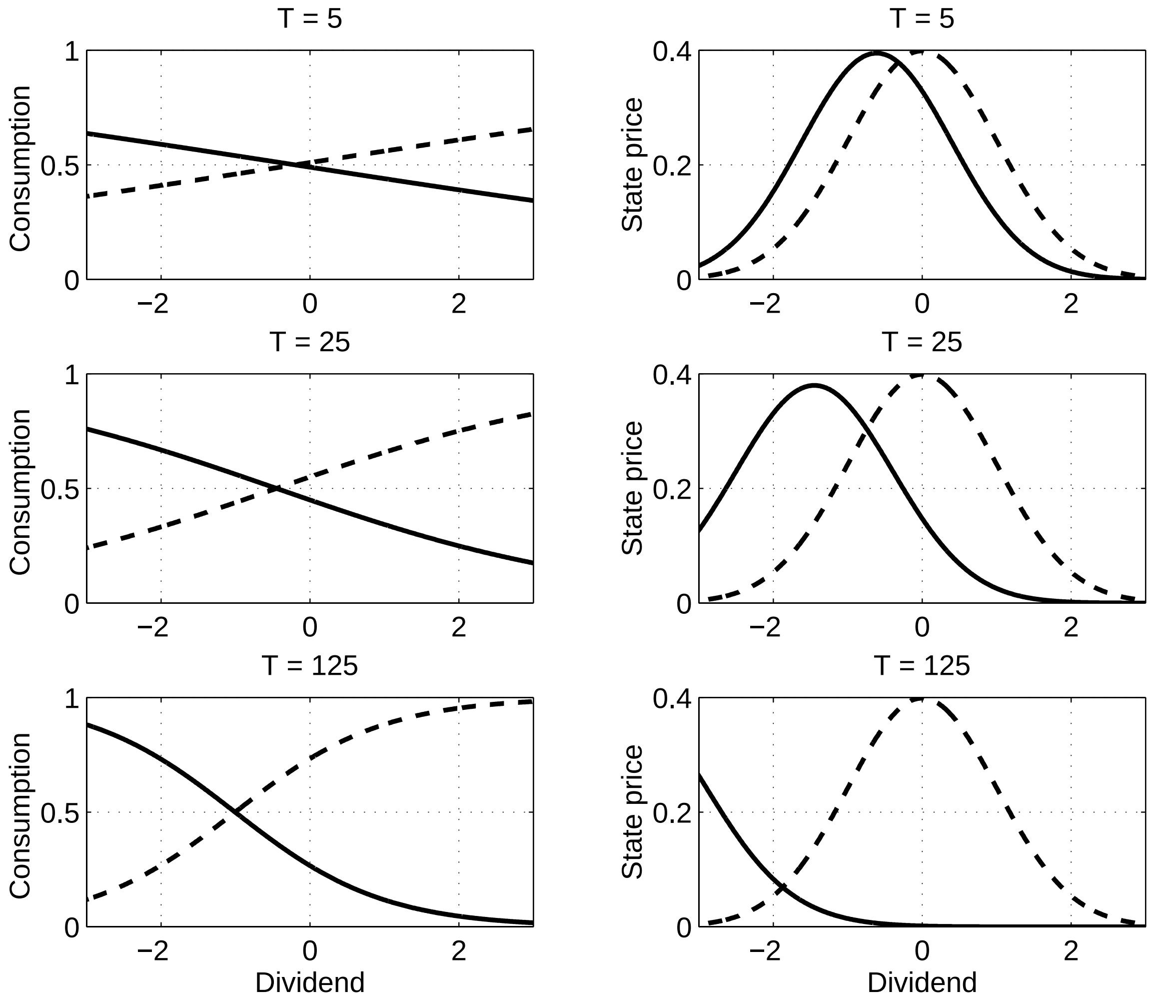

Figure 5: The terminal consumption of rational and irrational traders and the state prices for different values of the economy horizon T. We consider three values of the economy horizon, $T = 5, 25, 125$. We set the model parameters to $\mu = 0.12$, $\sigma = 0.18$, and $\gamma = 2$. We assume that the irrational trader is moderately optimistic and set $\eta = 0.5\eta^{\star}$. The horizontal axis in all panels is the normalized value of the terminal dividend, i.e., $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$. The three panels on the left show the terminal consumptions of the rational trader (solid line) and the irrational trader (dashed line) as a fraction of the aggregate endowment, i.e., $C_{r,T}/D_T$ and $C_{n,T}/D_T$. The three panels on the right show the probability distribution of the normalized dividend (dashed line), which is a standard normal random variable, and the state price density with respect to the Lebesque measure over $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$ (solid line).

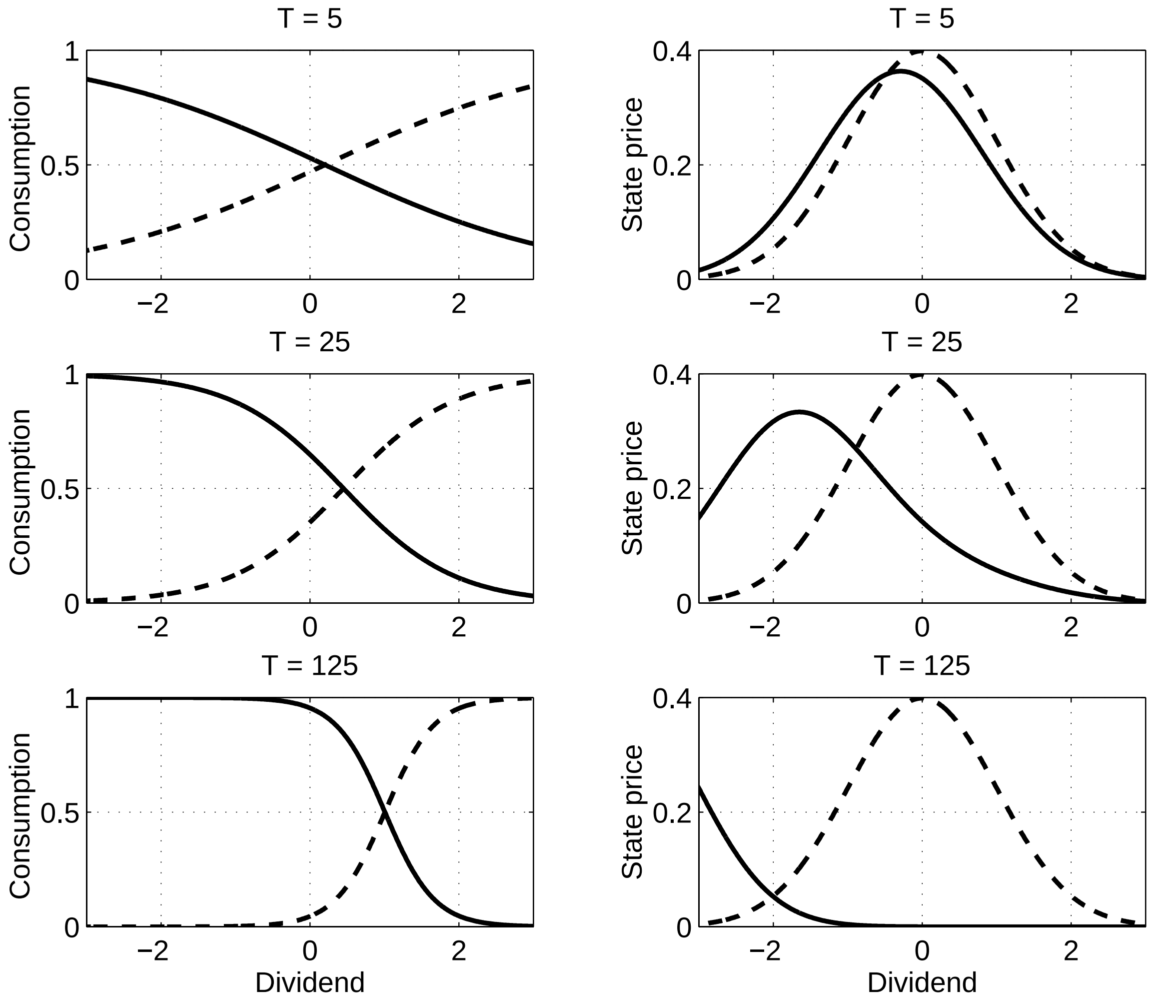

Figure 6: The terminal consumption of rational and irrational traders and the state prices for different values of the economy horizon T. We consider three values of the economy horizon, $T =$ 5, 25, 125. We set the model parameters to $\mu = 0.12$, $\sigma = 0.18$, and $\gamma = 2$. We assume that the irrational trader is strongly optimistic and set $\eta = 1.5\eta^\star$. The horizontal axis in all panels is the normalized value of the terminal dividend, i.e., $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$. The three panels on the left show the terminal consumptions of the rational trader (solid line) and the irrational trader (dashed line) as a fraction of the aggregate endowment, i.e., $C_{r,T}/D_T$ and $C_{n,T}/D_T$. The three panels on the right show the probability distribution of the normalized dividend (dashed line), which is a standard normal random variable, and the state price density with respect to the Lebesque measure over $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$ (solid line).

trader ends up accumulating wealth in very unlikely, good states by giving up wealth in most other states, which leads to his extinction in the long-run.

5 Stock Prices, Wealth Dynamics and Portfolio Policies

The application of the partial equilibrium "invasion" argument to the general equilibrium economy, as described by De Long *et al.*, relies on the following three assumptions. If one of the traders becomes infinitesimal, i.e., controls only a negligible fraction of the total wealth in the economy, then (i) such an trader has no impact on prices, (ii) the portfolio policies of both traders are the same as if the prices are exclusively set by the dominant trader, and (iii) as long as under these conditions the infinitesimal trader's wealth is growing as a fraction of the total, one can conclude that such an trader must survive in the long run. The third statement appears very intuitive and one may be tempted to conclude that it is an immediate implication of the first two assumptions, rather than an independent assumption. As we argue below, this is not the case. Having derived the exact solution to the general equilibrium economy, we investigate the validity of each of these assumptions and characterize the precise combinations of the model parameters under which they fail to hold.

The general expressions for the equilibrium asset prices and individual policies involve conditional expectations which cannot always be computed in closed form. To derive an explicit characterization of the objects of interest, we resort to asymptotic analysis. In this section and in the rest of the paper, we call two stochastic processes asymptotically equivalent if for large values of T, their ratio converges to one. Formally, we have

Definition 3 (Asymptotic Equivalence) *Two stochastic processes, X_t and Y_t are said to be asymptotically equivalent if*

$$\lim_{T\to\infty} \frac{X_T}{Y_T} = 1$$

denoted by $X_T \sim Y_T$.

As we consider a sequence of economies with increasing horizon, the time of observation is chosen to be $t = \lambda T$, $0 < \lambda \leq 1$. Thus, as the horizon of the economy approaches infinity, the "current" time t increases as well, but remains at a constant fraction of the horizon of the economy. Moreover, the time remaining until the final date of the economy is increasing proportionally to T. Since the properties of the equilibrium prices and quantities depend on how much time is remaining till the final date, they depend on λ. We introduce the following three constants, which help us identify the points of change in the limiting behavior:

$$\lambda_S \equiv \frac{2}{2\gamma - \eta}, \quad \lambda_r \equiv \frac{\eta}{(\gamma-1)(2\gamma-\eta)}, \quad \lambda_n \equiv \frac{\eta}{\eta(\gamma+1) - 2\gamma(\gamma-1)}. \tag{18}$$

It is easy to verify that for $\eta < \eta^\star$, $0 < \lambda_S \leq 1$; for $0 < \eta \leq \eta^\star$, $0 < \lambda_r \leq 1$; and for $\eta < 0$ or $\eta > \eta^\star$, $0 < \lambda_n \leq 1$.

5.1 Stock Prices

We first characterize the limiting behavior of the stock price process.

Proposition 4 (Stock Price) *At $t = \lambda T$, the stock price behaves as follows when $T \to \infty$.*

Case 1. Pessimistic Irrational Trader ($\eta < 0$):

$$S_t \sim \begin{cases} e^{\left(\mu/\sigma^2 - \gamma + \eta\right)\sigma^2 T + \frac{1}{2}\left[(2\gamma-1) - 2\gamma\eta + \eta^2\right]\sigma^2 t + (1-\eta)\sigma B_t}, & 0 < \lambda < \lambda_S \\ e^{\left(\mu/\sigma^2 - \gamma\right)\sigma^2 T + \frac{1}{2}(2\gamma-1)\sigma^2 t + \sigma B_t}, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 2. Moderately Optimistic Irrational Trader ($0 < \eta < \eta^\star$):

$$S_t \sim \begin{cases} e^{\left(\mu/\sigma^2 - \gamma\right)\sigma^2 T + \frac{1}{2}\left[(2\gamma-1) + 2(\gamma-1)\eta - \eta^2\right]\sigma^2 t + (1+\eta)\sigma B_t}, & 0 < \lambda < \lambda_S \\ e^{\left(\mu/\sigma^2 - \gamma + \eta\right)\sigma^2 T + \frac{1}{2}\left[(2\gamma-1) - 2\eta\right]\sigma^2 t + \sigma B_t}, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 3. Strongly Optimistic Irrational Trader ($\eta^\star < \eta$):

$$S_t \sim e^{\left(\mu/\sigma^2 - \gamma\right)\sigma^2 T + \frac{1}{2}(2\gamma-1)\sigma^2 t + \sigma B_t}$$

The limiting values of the instantaneous moments of stock returns are equal to the moments of the corresponding asymptotic expressions above.

Observe that in the first two cases the stock price process does not immediately converge to its value in the economy populated exclusively by the trader who survives in the long run. Instead, over long periods of time, i.e., for t between 0 and $\lambda_S T$, the stock price process is affected by the presence of both traders. We illustrate this property of the stock price in Figure 7, by showing which of the traders has a finite asymptotic impact on the stock price for different combinations of model parameters. We pick a particular value of λ, 2/3, to illustrate that both traders can potentially affect the stock price even if one of them becomes infinitesimally small asymptotically. Eventually, when t reaches $\lambda_S T$, convergence of the stock price to its value in a single-trader economy does occur. As we show in the next proposition, in each of the cases considered in Proposition 4, for any non-zero value of λ, the proportion of aggregate wealth controlled by the trader who does not survive in the long run becomes arbitrarily small already at time $t = \lambda T$.

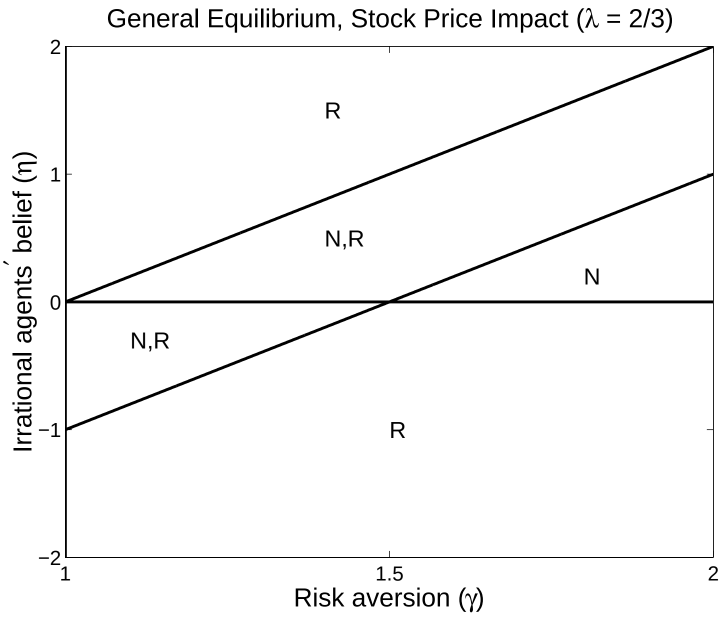

Figure 7: The price impact of rational and irrational traders for different values of η and γ in general equilibrium. The time is fixed at $t = (2/3)T$. For each region in the parameter space, we document which of the traders has an impact on the stock price in the long run. "R" means that the rational trader has an impact inside the region, "N" means that the irrational trader has an impact, and "N,R" means that both traders have an impact.

The first assumption of the partial equilibrium "invasion" argument appears incorrect. As the horizon of the economy expands, a trader can control an asymptotically infinitesimal fraction of the total wealth and yet exert a non-negligible effect on the price. This happens despite the fact that for any given economy, with fixed horizon T and the observation period t, the first partial equilibrium assumption applies, as shown in Proposition 5 below.

Proposition 5 (Stock Price, Fixed T)

For any fixed values of T and t, the instantaneous moments of stock returns, μ_{St} and σ_{St}, behave as follows.

Case 1. Pessimistic Irrational Trader ($\eta < 0$)*:*

$$B_t \to \infty \quad \Rightarrow \quad \frac{W_r}{W_n} \to \infty, \qquad \mu_{St} \to \gamma\sigma^2, \qquad \sigma_{St} \to \sigma$$

$$B_t \to -\infty \quad \Rightarrow \quad \frac{W_r}{W_n} \to 0, \qquad \mu_{St} \to \gamma\sigma^2 - \eta\sigma^2, \qquad \sigma_{St} \to \sigma$$

Case 2. Optimistic Irrational Trader ($\eta < 0$)*:*

$$B_t \to \infty \quad \Rightarrow \quad \frac{W_r}{W_n} \to \infty, \qquad \mu_{St} \to \gamma\sigma^2 - \eta\sigma^2, \qquad \sigma_{St} \to \sigma$$

$$B_t \to -\infty \quad \Rightarrow \quad \frac{W_r}{W_n} \to 0, \qquad \mu_{St} \to \gamma\sigma^2, \qquad \sigma_{St} \to \sigma$$

As the relative wealth of any one of the traders becomes "sufficiently small", the price impact of such trader becomes small as well. What constitutes "sufficiently small" in this context depends on the horizon of the economy and the observation period. Therefore, it must be that as one allows the horizon of the economy to expand, the critical level of relative wealth below which the price impact becomes "small" declines as well. Conclusions of Propositions 4 and 5 can thus be reconciled: for a range of values of λ and $t = \lambda T$, as the horizon of the economy increases, the relative wealth of one of the traders converges to zero, but not rapidly enough for the result of Proposition 5 to apply. Therefore, one can observe finite price impact while the relative wealth of the trader approaches zero as $T \to \infty$.

5.2 Wealth Distribution

In Section 4, we have discussed the limiting distribution of wealth between the rational and irrational traders at the final date T, as T goes to infinity. We now characterize the dynamics of the wealth distribution in the economy.

Proposition 6 (Individual Wealth) *At $t = \lambda T$, the individual wealth processes behave as follows when $T \to \infty$.*

Case 1. Pessimistic Irrational Trader ($\eta < 0$)*:*

$$\frac{W_{r,t}}{W_{n,t}} \sim \begin{cases} e^{\frac{1}{2}[\eta-2(\gamma-1)]\eta\sigma^2 t-\eta\sigma B_t}, & 0<\lambda<\lambda_n \\ e^{\frac{1}{2}(\eta^2/\gamma^2)(\gamma-1)\sigma^2 T+\frac{1}{2}\left[(\eta^2/\gamma^2)-2\eta(\gamma-1)/\gamma\right]\sigma^2 t-(\eta/\gamma)\sigma B_t}, & \lambda_n<\lambda\leq 1 \end{cases}$$

$$W_{r,t} \sim S_t$$

Case 2, Moderately Optimistic Irrational Trader ($0 < \eta < \eta^\star$)*:*

$$\frac{W_{r,t}}{W_{n,t}} \sim \begin{cases} e^{\frac{1}{2}[\eta-2(\gamma-1)]\eta\sigma^2 t-\eta\sigma B_t}, & 0<\lambda<\lambda_r \\ e^{-\frac{1}{2}(\eta^2/\gamma^2)(\gamma-1)\sigma^2 T+\frac{1}{2}\left[(\eta^2/\gamma^2)(2\gamma-1)-2\eta(\gamma-1)/\gamma\right]\sigma^2 t-(\eta/\gamma)\sigma B_t}, & \lambda_r<\lambda\leq 1 \end{cases}$$

$$W_{n,t} \sim S_t$$

Case 3. Strongly Optimistic Irrational Trader ($\eta^\star < \eta$)*:*

$$\frac{W_{r,t}}{W_{n,t}} \sim \begin{cases} e^{\frac{1}{2}[\eta-2(\gamma-1)]\eta\sigma^2 t-\eta\sigma B_t}, & 0<\lambda<\lambda_n \\ e^{\frac{1}{2}(\eta^2/\gamma^2)(\gamma-1)\sigma^2 T+\frac{1}{2}\left[(\eta^2/\gamma^2)-2\eta(\gamma-1)/\gamma\right]\sigma^2 t-(\eta/\gamma)\sigma B_t}, & \lambda_n<\lambda\leq 1 \end{cases}$$

$$W_{r,t} \sim S_t$$

The third case provides a stark example of the failure of the second assumption behind the partial equilibrium analysis. In this case only the rational trader survives in the long run. Moreover, according to Proposition 4, convergence of the stock price process at $t = \lambda T$ occurs for any non-zero value of λ.[1] Nevertheless, the growth rate of the irrational trader's wealth converges to the value suggested by the partial-equilibrium arguments only for $\lambda > \lambda_n$. This implies that even if the price process and the moments of returns do converge to their partial-equilibrium values, the individual portfolio policies may not. We make this explicit in the following discussion.

5.3 Portfolio Policies

Expressions for portfolio policies are not available in closed form. However, using the same arguments as in the proof of the bounds on the volatility of stock returns, Proposition 1,

[1]We establish convergence of the price level, as well as convergence of the drift and diffusion coefficients of the return process in the appendix.

it is easy to show that individual portfolio holdings are bounded in absolute value, $|w| \leq 1 + |\eta|(\gamma + 1)/\gamma$. This implies that our asymptotic survival results do not rely on the traders being able to use progressively higher levels of leverage and our solution for the equilibrium would remain valid even if agents were constrained in their portfolio choice, as long as the constraint was sufficiently loose to allow for $w = \pm(1 + |\eta|(\gamma + 1)/\gamma)$. The traders fail to survive asymptotically either because they take short positions in a stock with positive expected excess return, or because they adopt portfolio policies with excessively high volatility of returns.

To analyze the traders' portfolio policies further, we decompose a trader's stock demand into two components, the myopic component and the hedging component. The sum of the two gives the trader's total stock demand.

Proposition 7 (Portfolio Policies) *At $t = \lambda T$, the individual stock holdings behave as follows when $T \to \infty$.*

Case 1. Pessimistic Irrational Trader ($\eta < 0$):[2]

$$w_{r,t} \sim \begin{cases} \frac{\gamma-\eta}{\gamma(1-\eta)} & - & \frac{(\gamma-1)\eta}{\gamma(1-\eta)} & = & 1, & 0 < \lambda < \lambda_S \\ \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ 1 & + & 0 & = & 1, & \lambda_S < \lambda \leq 1 \end{cases}$$

$$w_{n,t} \sim \begin{cases} \frac{1}{1-\eta} & + & 0 & = & \frac{1}{1-\eta}, & 0 < \lambda < \min(\lambda_n, \lambda_S) \\ \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ 1 + \frac{\eta}{\gamma} & + & 0 & = & 1 + \frac{\eta}{\gamma}, & \max(\lambda_n, \lambda_S) < \lambda \leq 1 \end{cases}$$

[2]The limit of the portfolio policy for values of $\lambda \in [\min(\lambda_n, \lambda_S), \max(\lambda_n, \lambda_S)]$ can be characterized explicitly as well, but the results depend on the ordering between λ_n and λ_S, which in turn is determined by the values of model parameters. We chose to omit these results to simplify the exposition.

Case 2. Moderately Optimistic Irrational Trader $(0 < \eta < \eta^\star)$:

$$w_{r,t} \sim \begin{cases} \frac{1}{1+\eta} & + & 0 & = & \frac{1}{1+\eta}, & 0 < \lambda < \lambda_r \\ \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ \frac{1}{1+\eta} & + & \frac{\eta(\gamma-1)}{\gamma(1+\eta)} & = & 1 - \frac{\eta}{\gamma(1+\eta)}, & \lambda_r < \lambda < \lambda_S \\ 1 - \frac{\eta}{\gamma} & + & 0 & = & 1 - \frac{\eta}{\gamma}, & \lambda_S < \lambda \leq 1 \end{cases}$$

$$w_{n,t} \sim \begin{cases} \frac{\gamma+\eta}{\gamma(1+\eta)} & + & \frac{\eta(\gamma-1)}{\gamma(1+\eta)} & = & 1, & 0 < \lambda < \lambda_S \\ \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ 1 & + & 0 & = & 1, & \lambda_S < \lambda \leq 1 \end{cases}$$

Case 3. Strongly Optimistic Irrational Trader, $(\eta^\star < \eta)$:

$$w_{r,t} \sim 1 + 0 = 1, \quad 0 < \lambda \leq 1$$

$$w_{n,t} \sim \begin{cases} 1 + \frac{\eta}{\gamma} & + & \frac{\eta(\gamma-1)}{\gamma} & = & 1 + \eta, & 0 < \lambda < \lambda_n \\ \text{(myopic)} & & \text{(hedging)} & & \text{(total)} & \\ 1 + \frac{\eta}{\gamma} & + & 0 & = & 1 + \frac{\eta}{\gamma}, & \lambda_n < \lambda \leq 1 \end{cases}$$

Although the moments of stock returns are asymptotically state-independent, the portfolio policy is not myopic, i.e., it does not converge to the corresponding values in the economy where the moments of returns are exactly *equal* to their limiting values. The reason for this lack of convergence of the individual portfolio policies is due to the presence of non-trivial hedging demand in our economy, in addition to the myopic demand. By definition, the myopic component of portfolio holdings is a function of the instantaneous moments of stock returns and hence it converges to its partial equilibrium value as long as the instantaneous moments of stock returns converge. It may seem surprising, however, that even when the moments of stock returns are asymptotically state-independent, the hedging component of portfolio holdings does not necessarily converge to zero, as can be seen, for example, from the first case in Proposition 7. This is because for finite values of T, investment opportunities do not remain constant. In fact, the instantaneous Sharpe ratio of returns, which completely summarizes the instantaneous investment opportunities, can vary drastically, as illustrated in Figure 8. The Sharpe ratio is a function of the distribution of wealth between the two traders. Figure 8 illustrates the properties of the economy with pessimistic irrational traders. For the chosen set of parameter values, $\lambda_S = 0.29$, hence the panels on the left correspond to "small" values of t, $t < \lambda_S T$, while panels on the right correspond to "large" values of t, $t > \lambda_S T$. It is straightforward to show that for very low values of D_t, holding time t fixed, the economy is dominated by the irrational traders (see the bottom row of panels) and the instantaneous Sharpe ratio of returns converges to its value in a homogeneous economy populated by irrational traders only, $\sigma(\gamma - \eta)$. Portfolio holdings of both agents converge to their values in the homogeneous economy as well. For very large values of D_t, the Sharpe ratio is the same as in an economy populated exclusively by the rational traders, $\sigma\gamma$, and equals the asymptotic value for $t > \lambda_S T$.[3] These properties of the Sharpe ratio are shown in the top two panels of Figure 8. Note that for small values of t (left panels), with high probability the Sharpe ratio is close to $\sigma(\gamma - \eta)$, while for large values of t (right panels), it is close to $\sigma\gamma$. This behavior is captured by the asymptotic result in Proposition 4. However, there always exists a possibility of a significant change in the Sharpe ratio, provided a sufficiently large movement in the state variable. Such a possibility is reflected in the indirect utility function. Since traders in our model have constant relative risk aversion, their indirect utility function can be expressed in the form

$$V(t, W_t, D_t) = \frac{1}{1-\gamma} e^{h(t,D_t)} W_t^{1-\gamma} \tag{19}$$

State dependence of the indirect utility function is captured by the function $h(t, D_t)$. As shown in the second row of plots, function h is non-constant over a wide range of values of D_t. In particular, for small values of t, the indirect utility function exhibits significant state dependence even when the contemporaneous Sharpe ratio of returns is approximately constant. This captures the effect of possible future changes in the Sharpe ratio. On the other hand, as he remaining time horizon $T - t$ declines, the indirect utility function reflects more closely the behavior of the contemporaneous Sharpe ratio, as shown in the panel on the right. State dependence in the indirect utility function induces hedging demand. The third row of panels shows hedging demand of the rational trader. For small values of t, over a wide range of values of D_t, hedging demand is close to its asymptotic value $(\gamma - 1)\eta/(\gamma(1 - \eta))$ (see Proposition 7), which equals 0.375 for the chosen values of parameters. For large values of t, hedging demand is close to zero with high probability, although it remains positive over

[3]When the irrational trader is optimistic, for large values of D_t the Sharpe ratio is the same as in a homogeneous economy populated by the irrational traders, while for small values of D_t the Sharpe ratio is the same as in a homogeneous economy populated by the rational traders.

the range of values of D_t for which the Sharpe ratio of returns is variable.

Finally, we illustrate precisely how the third assumption of the partial equilibrium argument can fail as well. Consider a combination of model parameters satisfying $\eta^\star < \eta < \gamma\eta^\star$. Even though for sufficiently large values of t the stock prices, the individual portfolio policies, and the growth rate of individual wealth eventually do converge to their partial equilibrium values, the irrational trader still becomes extinct in the long run. The reason for this can be seen from Proposition 6 and is further illustrated in Figure 9. Even though for $t = \lambda T$, $\lambda > \lambda_n$ the relative wealth of the irrational trader is growing, it does so only following a period of decline, i.e., $0 < \lambda < \lambda_n$. This happens because the first two assumptions of the partial equilibrium analysis do not hold for $0 < \lambda < \lambda_n$. After all, these assumptions do not explicitly state how small the fraction of wealth of the irrational (or rational) trader must become before one can safely ignore the trader's impact on the market prices. As the horizon of the economy increases, the relative wealth threshold below which the trader has no impact on the equilibrium prices and policies declines. The net result is the ultimate relative extinction of the irrational trader.

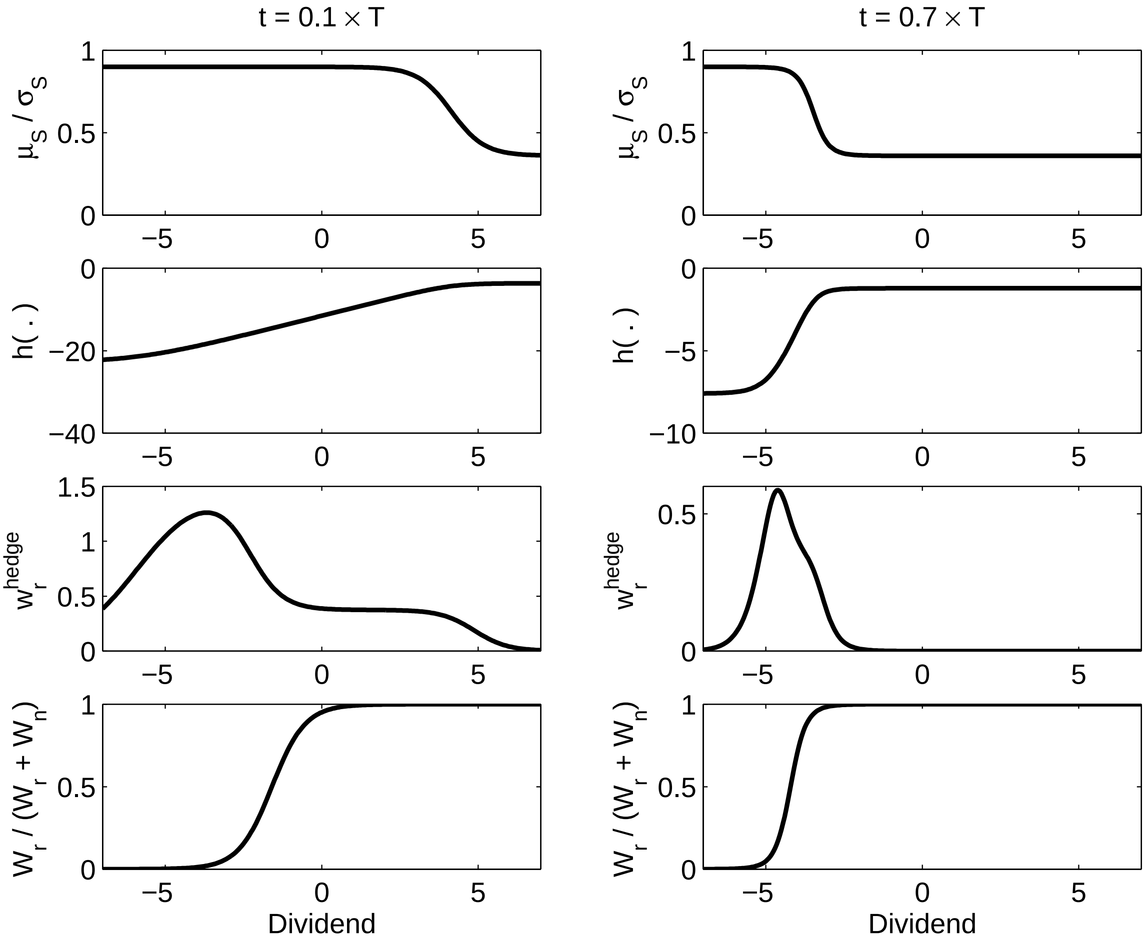

Figure 8: The figure illustrates the properties of the economy for the following set of parameter values: $\mu = 0.12$, $\sigma = 0.18$, $\gamma = 2$, $T = 125$. We assume that the irrational trader is pessimistic and set $\eta = -1.5\eta^\star = -3$. We consider two value of t, $0.1 \times T$ (left panels) and $0.7 \times T$ (right panels). The horizontal axis in all panels is the normalized value of the terminal dividend, i.e., $\left[\ln D_T - (\mu - \sigma^2/2)T\right]/(\sigma\sqrt{T})$. The four rows of panels show (i) the instantaneous Sharpe ratio of returns, μ_S/σ_S; (ii) the state dependence of the indirect value function, as captured by the function $h(t, D_t)$ in (19); (iii) the portion of the portfolio strategy of the rational trader attributable to hedging demand, defined as $w_r^{\text{hedge}} = w_r - \mu_S/(\gamma\sigma_S^2)$; (iv) the fraction of the aggregate wealth controlled by the rational agent, $W_r/(W_r + W_n)$.

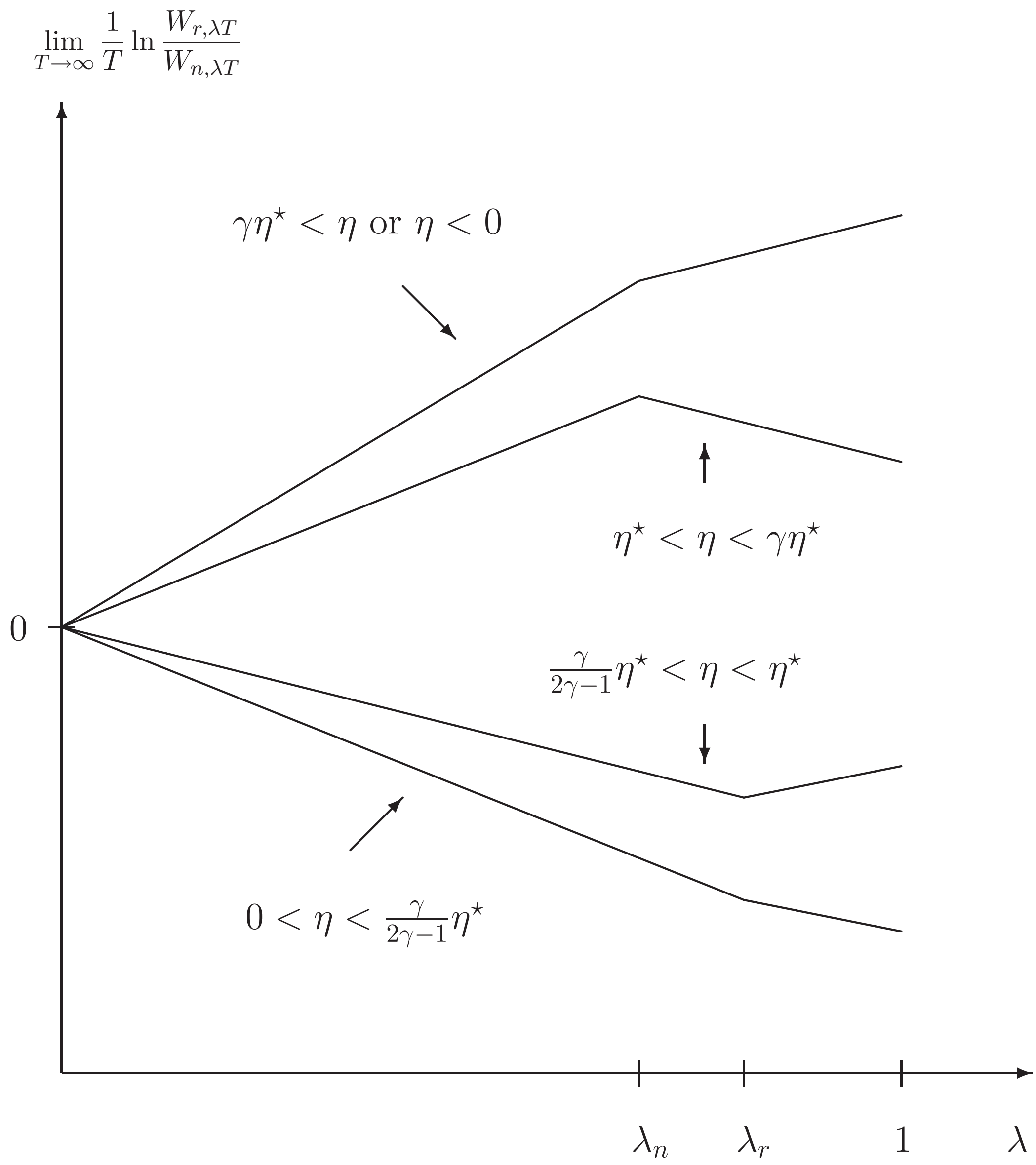

Figure 9: Asymptotic Wealth Behavior.

5.4 Utility Loss of Irrational Beliefs

As we have established in Section 5.1, the irrational trader can have a significant impact on the behavior of stock returns. In this subsection we show that the impact on prices is associated with utility loss by irrational traders. By revealed preference, the rational trader always benefits from the presence of irrational traders, since the autarchic solution remains feasible in our equilibrium economy. We quantify this benefit and show that it is asymptotically independent of the exact belief of the irrational traders, i.e., independent of the long run survival of the rational trader.

Assuming $\gamma > 1$ and $\eta \neq 0$, the certainty equivalent of the terminal consumption of the rational trader in the equilibrium economy we are analyzing is given by

$$\text{C.E.}\left(C_{r,T}\right) \equiv \left(\text{E}_0\left[C_{r,T}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \sim e^{\left(\mu-\frac{1}{2}\sigma^2\gamma\right)T}.$$

The above expression is the same as the certainty equivalent of consuming the entire aggregate dividend at time T. Thus, we find that, in the limit of economy horizon approaching infinity, the rational trader realizes the largest possible benefit of risk sharing, and this conclusion is independent of the exact belief bias η exhibited by the irrational trader. This remains the case even if the rational trader does not survive in the long run.

Next, we compute the certainty equivalent of the consumption of the irrational trader. We are using the objective probability distribution in our calculation, i.e., we are finding the certainty equivalent the the rational trader would assign to the terminal consumption profile of the irrational trader. As before, we assume $\gamma > 1$ and $\eta \neq 0$. We find

$$\text{C.E.}\left(C_{n,T}\right) \equiv \left(\text{E}_0\left[C_{n,T}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \sim e^{\left(\mu-\frac{1}{2}\sigma^2\gamma-\frac{1}{2}\eta^2\sigma^2(2\gamma-1)/\gamma^2\right)T} \tag{20}$$

We know from our previous analysis that the irrational traders can have long-run impact on the asset prices in equilibrium. To quantify this impact, we consider that certainty equivalent of the consumption chosen by the irrational trader facing the prices free of the impact of the irrational traders, i.e., the prices in the economy populated exclusively by the rational traders. Assuming that the endowment of the irrational trader is one half of the aggregate dividend, a straightforward calculation yields the certainty equivalent of

$$\frac{1}{2}e^{\left(\mu-\frac{1}{2}\sigma^2\gamma-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}\right)T}.$$

Since $\gamma > 1$, the above expression has lower growth rate in T than (20), i.e., asymptotically the irrational traders are hurt by their own price impact. As a quantitative measure of the resulting utility loss one can use the fraction of the aggregate dividend with which the irrational agent must be endowed in an economy without price impact to achieve the same certainty equivalent consumption as in our general equilibrium economy. We find that this fraction converges to zero asymptotically.

5.5 Time-Varying Beliefs

So far we have focused on the case when irrational traders are persistently wrong in their beliefs, they are either constantly optimistic or pessimistic. We now consider an extension of our basic model by allowing the beliefs of the irrational trader to be time-varying. Specifically, we let η be a deterministic function of time, $\eta_t = \eta_0 \sin(\kappa t)$. Then

$$\xi_t = \exp\left(-\frac{\eta_0^2 \sigma^2}{4\kappa}\left(\kappa t - \frac{1}{2}\sin(2\kappa t)\right) + \int_0^t \sigma \eta_0 \sin(\kappa t)\, dB_t\right)$$

Thus, "on average", the irrational trader holds correct beliefs, i.e., the time-average value of η_t is zero. However, at any point in time they the irrational trader is either optimistic or pessimistic, with beliefs varying periodically. Following the proof of Proposition 1, one can show that the equilibrium allocations are given by (10) with

$$b = \exp\left(\frac{\eta_0 \sigma^2(\gamma - 1)}{\kappa}(1 - \cos(\kappa T))\right)$$

Since b is bounded as a function of T, it follows that the irrational trader fails to survive in the long run, as long as $\eta_0 \neq 0$. Moreover, the wealth of the irrational trader decays exponentially as a fraction of the total.

Asymptotically, the irrational trader has no price impact in this economy, i.e.,

$$S_t \sim e^{\left(\mu/\sigma^2 - \gamma\right)\sigma^2 T + \frac{1}{2}(2\gamma - 1)\sigma^2 t + \sigma B_t}$$

for $t = \lambda T$, $\lambda > 0$. In the limit of $T \to \infty$, the portfolio policies of the two agents take the form

$$w_{r,t} \sim 1, \qquad 0 < \lambda \leq 1$$

$$w_{n,t} \sim \begin{cases} 1 + \eta_0 \sin(\kappa t), & 0 < \lambda < \frac{1}{\gamma+1} \\ 1 + \frac{1}{\gamma}\eta_0 \sin(\kappa t), & \frac{1}{\gamma+1} < \lambda \leq 1 \end{cases}$$

Intuitively, one would expect the lack of survival of the irrational trader to be attributable to the price impact exerted due to changing belief bias, η_t. In particular, when η_t is positive, one would expect the irrational traders to increase their stock holdings, temporarily raising the stock price. The opposite would happen when η_t is negative. Thus, irrational traders would "buy high and sell low." Our analysis shows that this is not the only mechanism by which irrational traders can disappear in the long run. Asymptotically, the irrational trader has no price impact at $t = \lambda T$, for any positive λ. Nevertheless, the relative wealth of the irrational trader continues decaying exponentially. The reason lies in fluctuations in the trader's portfolio policy. While the average return on the irrational trader's portfolio is the same as for the rational trader, the former exhibits a higher average variance of returns, which results in a lower average growth rate of wealth.

6 Intermediate Consumption

Our previous discussion relies on a model with only terminal consumption. A natural question is how intermediate consumption may affect our results. With intermediate consumption, each trader has two decisions to make: how much to consume today and how to invest savings for the future. These two decisions are interdependent, so that a trader who perceives better investment opportunities may well choose to save less. Thus, in our analysis, an irrational trader's wrong beliefs may well affect his consumption/saving decision and significantly change his chance for long-run survival. In this section we consider an extension of the model in Section 2 to allow intermediate consumption. We then examine how intermediate consumption may affect irrational traders' survival.

6.1 The Model with Intermediate Consumption

We start with the setting in Section 2 and make the appropriate adjustments to introduce intermediate consumption. With intermediate consumption, we can extend the horizon of the economy to infinity and use the current price of the consumption good as the numeraire. The uncertainty of the economy is defined by a one-dimensional, standard Brownian motion B_t for $t \geq 0$.

Two financial securities are traded, a bond and a stock. The bond pays an instantaneous, risk-free interest rate of r_t. The stock pays a stream of dividend at an instantaneous rate of D_t, where $D_0 = 1$ and

$$dD_t = D_t \left(\mu dt + \sigma dB_t\right), \quad \sigma > 0. \tag{21}$$

It is worth emphasizing that even though D_t here follows the same process as in Section 2, its meaning is different. In the current context, it stands for the *rate* of dividends paid at each point in time while previously it stands for the *level* of dividends paid on the terminal date. Furthermore, we use current consumption rather than the bond as the numeraire.

We still consider two competitive traders, each endowed with a fraction share of the stock. Each chooses a consumption strategy C_t, which denotes the rate of consumption at time t, and a trading strategy θ_t.[4] The consumption and trading strategies satisfy the following integrability conditions:

$$\forall \;\; t > 0: \qquad \int_0^t |C_s| ds < \infty, \qquad \int_0^t \theta_s^2 d\langle S_s \rangle < \infty \tag{22}$$

where S_t denotes the stock price as before.

The rational trader knows the correct probability measure P. The irrational trader believes the probability measure to be Q, under which $dB_t = (\sigma\eta)dt + dB_t^Q$ and

$$dD_t = D_t \left[\left(\mu + \sigma^2 \eta\right) dt + \sigma dB_t^Q \right]$$

where, as before, B_t^Q is the standard Brownian motion under Q and η is a constant.

Both traders have a constant coefficient of relative risk aversion, and each maximizes expected utility over his lifetime consumption. For the rational trader, the expected utility function is

$$\mathrm{E}_0^P \left[\frac{1}{1-\gamma} \int_0^\infty e^{-\rho t} C_{r,t}^{1-\gamma} dt \right] \tag{23}$$

[4] As with dividends, C_t is an abuse of notation. With intermediate consumption, it stands for the rate of consumption at each point in time, while with terminal consumption it stands for the level of consumption at the terminal date.

where $\rho > 0$ is the time-discount coefficient and E^P denotes the expectation under probability measure P. For the irrational trader, the expected utility function is

$$\mathrm{E}_0^Q \left[\frac{1}{1-\gamma} \int_0^\infty e^{-\rho t} C_{n,t}^{1-\gamma} dt \right] = \mathrm{E}_0^P \left[\frac{1}{1-\gamma} \int_0^\infty e^{-\rho t} C_{n,t}^{1-\gamma} \xi_t dt \right] \tag{24}$$

where $\xi_t \equiv (dQ/dP)_t$ is given in equation (7).

In addition, we will assume

$$\rho > \left(\mu + \eta\sigma^2 - \frac{1}{2}\gamma\sigma^2 \right) (1-\gamma) \tag{25}$$

$$\rho > \left(\mu - \frac{1}{2}\gamma\sigma^2 \right) (1-\gamma) \tag{26}$$

so that the expected utilities of both agents are guaranteed to be finite.[5]

6.2 The Survival and Price Impact of Irrational Traders

Since the financial markets are dynamically complete, the solution to the equilibrium in the case of intermediate consumption is similar to the case of terminal consumption. In fact, we obtain the same sharing rule for the traders' optimal consumption, except the social utility weight b now depends on the initial wealth distribution, as opposed to the horizon of the economy. We use the same criteria to examine the survival of irrational traders. That is, if his relative share of aggregate consumption diminishes eventually, the irrational trader is said to experience relative extinction. We have the following result.

Proposition 8 *In the economy defined in Section 6.1, the irrational trader does not survive, as long as* $\eta \neq 0$.

This result is stronger than the result for the economy with only terminal consumption. Apparently, the irrational trader's consumption decision, which is based on his incorrect belief about his future wealth growth, is a decisive factor in his fate. The following proposition illustrates this intuition for integer values of γ, showing that both optimistic and strongly

[5] The first condition ensures that if the irrational trader consumes his initial endowment then resulting expected utility is finite. This provides a lower bound on the expected utility in equilibrium: $1/(1-\gamma)\mathrm{E}_0 \left[\int_0^\infty (D_t/2)^{1-\gamma} \xi_t dt \right] = 2^{\gamma-1}/(1-\gamma) \int_0^\infty e^{\left[-\rho + \left(\mu + \eta\sigma^2 - \frac{1}{2}\gamma\sigma^2\right)(1-\gamma)\right]t} dt$. Equation (25) ensures that this integral is finite. Condition (26) is derived similarly for the rational trader.

pessimistic irrational traders eventually consume more of their wealth than rational traders. The proposition also shows that, in the long run, the growth rate of the traders' wealth and consumption is the same, therefore our results on the long-run relative consumption shares apply to the agents' wealth as well.

Proposition 9 *For integer values of $\gamma \geq 1$,*

$$\lim_{t\to\infty} \frac{C_{r,t}/W_{r,t}}{C_{n,t}/W_{n,t}} = k$$

where

$$k = \frac{\int_0^\infty e^{-\rho t} \mathrm{E}_0 \left[D_t^{1-\gamma} \xi_t^{1/\gamma} \right] dt}{\int_0^\infty e^{-\rho t} \mathrm{E}_0 \left[D_t^{1-\gamma} \right] dt}$$

and $k > 1$ if and only if $-2\gamma < \eta < 0$.

We also find (again, for integer γ), that in the long run irrational traders have no impact on neither the stock price nor the interest rate. This stands in contrast to the result for the terminal-consumption economies, where long-run extinction did not necessarily imply the absence of long-run price impact.

Proposition 10 *For integer $\gamma \geq 1$,*

$$\lim_{t\to\infty} \frac{S_t}{D_t} = \frac{1}{\rho + (\gamma - 1)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)}$$

and

$$\lim_{t\to\infty} r_t = \rho + \gamma\mu - \frac{1}{2}\gamma(\gamma + 1)\sigma^2$$

6.3 Further Discussion

Proposition 8 establishes the extinction of irrational traders in the setting specified in Section 6.1. To examine the robustness of this result, we now relax our assumptions on the traders' preferences and the aggregate endowment process.

Proposition 11 *Suppose that rational and irrational traders have the same utility function with the relative risk aversion γ bounded from above, and that the equilibrium consumption* *allocation is Pareto optimal and corresponds to an interior solution, i.e.,* $u_r'/u_n' = b\,\xi_t$*. If the bias of the irrational trader is sufficiently large, so that* $\xi_t \to 0$*, then* $C_{n,t}/C_{r,t} \to 0$*.*

Proposition 11 shows that as long as both traders have the same utility function with relative risk aversion bounded from above, irrational traders who do not learn the true probability distribution sufficiently quickly cannot survive. The exact specification of the endowment process is irrelevant.

Our results so far have involved conditions on three aspects of the economy. The first condition is on the degree of irrationality, which can be measured by the asymptotic behavior of ξ_t. In terms of survival, $\xi_t \to 0$ implies significant degree of irrationality. The second condition is on the traders' preferences. An upper bound on relative risk aversion is non-trivial assumption which has important exceptions. The third condition is on the aggregate endowment process.

Sandroni (2000) and Blume and Easley (2001) use a setting similar to ours with intermediate consumption. They show that "agents making inaccurate predictions are driven out of the market" and claim this to be a general "evolutionary" result, independent of the traders' preferences. However, they assume that the level of dividends (i.e. the aggregate endowment) is bounded. Other than being unrealistic, a bounded dividend process is crucial to their result, as highlighted by the following counter-example.

Example 1 *Assume that irrational and rational agents have the same utility function with absolute risk aversion bounded away from zero by a positive constant. Assume that the equilibrium consumption allocation is Pareto optimal and corresponds to an interior solution, i.e.,* $u_r'/u_n' = b\,\xi_t$*. If the aggregate endowment is growing fast enough relative to the bias in beliefs of the irrational trader, so that* $(\ln \xi_t)/D_t \to 0$*, then the irrational trader will survive and* $C_{n,t}/C_{r,t} \to 1$*.*

This example, together with the results in this paper and in Sandroni (2000) and Blume and Easley (2001), illustrates that in general the survival of irrational traders depends on their preferences and the properties of the endowment process.

7 Conclusion

In this paper, we have examined the long-run survival of irrational traders who use persistently wrong beliefs to make their portfolio choices. Using a parsimonious model, we have shown that the partial equilibrium arguments to support the long-run survival of irrational traders are flawed. The impact irrational traders have on equilibrium prices is important in determining their long-run fortunes; in particular, we have found that an irrational trader with a very small wealth and CRRA preferences may still have a large impact on the stock price. However, in absence of intermediate consumption and rational traders with logarithmic preferences, irrational traders with beliefs mildly different from the true probabilities can survive in the long run. In the presence of intermediate consumption, we have shown that under fairly realistic conditions irrational traders do not survive or have a long term price impact, but that these results are sensitive to assumptions about preferences and aggregate consumption.

For our analysis, we have adopted a continuous-time model. Formulating the model in continuous time allows us to obtain explicit analytic expressions for equilibrium portfolio policies of agents and the moments of asset returns. Our main results do not rely on the continuous-time assumption and remain valid as long as the financial markets are complete. We restricted ourselves to preferences with constant relative risk aversion primarily for tractability. Extensions of our analysis to more general preferences are possible. We have also assumed that the rational and irrational traders differ only in their beliefs but not in their preferences. This allows us to focus on the impact of irrational beliefs on survival and prices. Of course, differences in time and risk preferences can have their own implications for long-run survival.

A Appendix

Proof of Proposition 1

The optimality conditions of the maximization problem in (9a) require that

$$C_{r,T} = C_{n,T}\,(b\,\xi_T)^{1/\gamma}\,.$$

Combined with the market clearing condition (9b), this implies (10a) and (10b).

The state price density must be proportional to the traders' marginal utilities. Since we set the interest rate equal to zero, the state price density conditional on the information available at time t is given by

$$\frac{\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}}{\mathrm{E_t}\left[\left(1+(b\,\xi_T)^{1/\gamma}\right)^{\gamma} D_T^{-\gamma}\right]}.$$

The price of any payoff Z_T is therefore given by (12).

The individual budget constraint in a dynamically complete market is equivalent to the static constraint that the initial wealth of an trader is equal to the present value of the trader's consumption (e.g., Duffie (1996, Sec. 9.E)). Since the two traders in our model have identical endowments at time $t = 0$, their budget constraints imply

$$W_{r,0} = \frac{\mathrm{E}_0\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_0\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]} = \frac{\mathrm{E}_0\left[D_T^{1-\gamma}\,(b\,\xi_T)^{\frac{1}{\gamma}}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_0\left[D_T^{-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]} = W_{n,0}. \quad (27)$$

We now verify that $b = e^{\eta\sigma^2(\gamma-1)T}$ satisfies (27). Note that

$$D_T^{1-\gamma} = e^{[(1-\gamma)(\mu-\frac{\sigma^2}{2})+\frac{1}{2}(1-\gamma)^2\sigma^2]T}\,e^{-\frac{1}{2}(1-\gamma)^2\sigma^2 T+(1-\gamma)\sigma B_T}$$

where the second term is an exponential martingale. Define a new measure Q, such that

$$\left(\frac{dQ}{dP}\right)_t = e^{-\frac{1}{2}(1-\gamma)^2\sigma^2 t+(1-\gamma)\sigma B_t}$$

where P is the original probability measure. By Girsanov's theorem, the process

$$B_t^Q = B_t - (1-\gamma)\sigma t$$

is a Brownian motion under Q. Thus, Girsanov's theorem implies that the equality

$$\mathrm{E}_0\left[D_T^{1-\gamma}\,(b\,\xi_T)^{\frac{1}{\gamma}}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]=\mathrm{E}_0\left[D_T^{1-\gamma}\left(1+(b\,\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]$$

is equivalent to

$$\mathrm{E}_0^Q\left[(\xi_T^Q)^{\frac{1}{\gamma}}\left(1+(\xi_T^Q)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]=\mathrm{E}_0^Q\left[\left(1+(\xi_T^Q)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]$$

where $\xi_t^Q=\exp(-\frac{1}{2}\sigma^2\eta^2 t+\sigma\eta B_t^Q)$. Since the process B_t^Q is equivalent in distribution to B_t, we can restate the last equality equivalently as

$$\mathrm{E}_0\left[\xi_T^{\frac{1}{\gamma}}\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]=\mathrm{E}_0\left[\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right].$$

To verify that the above equality holds, consider a function $F(z)$ defined as

$$F(z)=\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}zT}+e^{-\frac{1}{2\gamma}zT}\xi_T^{\frac{1}{\gamma}}\right)^{\gamma}\right]. \tag{28}$$

Changing the order of differentiation and expectation operators, (see Billingsley 1995, Th. 16.8),

$$F'(z)\big|_{z=0}=\mathrm{E}\left[\frac{1}{2}\left(1-\xi_T^{\frac{1}{\gamma}}\right)\left(1+\xi_T^{\frac{1}{\gamma}}\right)^{\gamma-1}\right].$$

Thus it suffices to prove that $F'(z)\big|_{z=0}=0$.

Since

$$\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}zT}+e^{-\frac{1}{2\gamma}zT}\xi_T^{\frac{1}{\gamma}}\right)^{\gamma}\right]=\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}\left(zT-\frac{1}{2}\eta^2\sigma^2T+\eta\sigma B_T\right)}+e^{-\frac{1}{2\gamma}\left(zT-\frac{1}{2}\eta^2\sigma^2T+\eta\sigma B_T\right)}\right)^{\gamma}\xi_T^{\frac{1}{2}}\right] \tag{29}$$

if we define a new measure Q so that

$$\left(\frac{dQ}{dP}\right)_t=e^{-\frac{1}{8}\eta^2\sigma^2 t+\frac{1}{2}\eta\sigma B_t}$$

and use Girsanov's theorem in a manner similar to its earlier application in this proof, we find that (29) equals

$$\mathrm{E}_0\left[\left(e^{\frac{1}{2\gamma}(zT+\eta\sigma B_T)}+e^{-\frac{1}{2\gamma}(zT+\eta\sigma B_T)}\right)^{\gamma}\right]e^{-\frac{1}{8}\eta^2\sigma^2 T}.$$

The symmetry of the distribution of the normal random variable B_T implies that $F(z) = F(-z)$, therefore $F'(z)|_{z=0} = 0$. This verifies that $b = e^{\eta\sigma^2(\gamma-1)T}$.

The stock price is given by

$$S_t = \frac{\mathrm{E_t}\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{1/\gamma}\right)^\gamma\right]}{\mathrm{E_t}\left[D_T^{-\gamma}\left(1+(b\xi_T)^{1/\gamma}\right)^\gamma\right]}$$

Define

$$A = e^{\left(\frac{-\eta\sigma^2}{\gamma}\right)(T-t)}$$

$$g = e^{-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}T+\sigma^2\eta(\gamma-1)\frac{1}{\gamma}t+\frac{\eta\sigma}{\gamma}B_T}$$

then, as we have shown above, the stock price can be equivalently expressed as

$$S_t = e^{\left(\mu-\sigma^2\gamma\right)T+\left(-\frac{1}{2}\sigma^2(1-2\gamma)\right)t}e^{\sigma B_t}\frac{\mathrm{E_t}\left[(1+g)^\gamma\right]}{\mathrm{E_t}\left[(1+gA)^\gamma\right]}$$

and therefore, by Ito's lemma, its volatility σ_{St} is given by

$$\sigma_{St} = \frac{\partial \ln S_t}{\partial B_t} = \sigma + \eta\sigma\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^\gamma\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^\gamma\right]}\right) \tag{30}$$

To establish the bounds on volatility, we prove that

$$\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^\gamma\right]} - \frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^\gamma\right]} \geq 0 \tag{31}$$

for $A \leq 1$ with the opposite inequality for $A \geq 1$. Note that for any twice-differentiable function $F(A,\gamma)$,

$$\frac{\partial}{\partial\gamma}\frac{\partial}{\partial A}\ln\left(F(A,\gamma)\right) \geq 0 \Rightarrow \frac{\partial}{\partial A}\ln\left(F(A,\gamma-1)\right) - \frac{\partial}{\partial A}\ln\left(F(A,\gamma)\right) \leq 0 \Rightarrow \frac{\partial}{\partial A}\frac{F(A,\gamma-1)}{F(A,\gamma)} \leq 0$$

Thus, to prove (31), it suffices to show that $\partial^2 \ln\left(\mathrm{E_t}\left[(1+gA)^\gamma\right]\right)/\partial A\partial\gamma \geq 0$. The function $(1+gA)^\gamma$ is log-supermodular in A, g, and γ, since it is positive and it's cross-partial derivatives in all arguments are positive. Thus, according to the additivity property of log-supermodular functions (see Athey (2002)), $\mathrm{E_t}\left[(1+gA)^\gamma\right]$ is log-supermodular in A and γ, i.e., $\partial^2 \ln\left(\mathrm{E_t}\left[(1+gA)^\gamma\right]\right)/\partial A\partial\gamma \geq 0$.

Because $A > 1$ if and only if $\eta < 0$, we have shown that

$$\eta\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]}-\frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}\right)\geq 0$$

and hence $\sigma_{St} \geq \sigma$.

Because $\left(\frac{\mathrm{E_t}\left[(1+gA)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+gA)^{\gamma}\right]}-\frac{\mathrm{E_t}\left[(1+g)^{\gamma-1}\right]}{\mathrm{E_t}\left[(1+g)^{\gamma}\right]}\right)$ is bounded between -1 and 0 for $\eta < 0$ and between 0 and 1 for $\eta > 0$, we obtain the stated upper bound from (30): $\sigma_{St} \leq \sigma\left(1+|\eta|\right)$. ■

Proof of Proposition 2

According to (10a) and (10b),

$$\frac{C_{n,T}}{C_{r,T}} = \left(b\,\xi_T\right)^{1/\gamma} = \exp\left[\frac{1}{\gamma}\left(-\frac{1}{2}\sigma^2\eta^2+\eta\sigma^2\left(\gamma-1\right)\right)T+\frac{1}{\gamma}\eta\sigma B_T\right].$$

Using the strong Law of Large Numbers for Brownian motion (see Karatzas and Shreve (1991, Sec. 2.9.A)), for any value of σ,

$$\lim_{T\to\infty} e^{a\,T+\sigma\,B_T} = \begin{cases} 0, & a<0 \\ \infty, & a>0 \end{cases} \tag{32}$$

where convergence takes place almost surely. The statement of the proposition then follows. ■

Proof of Proposition 3

Equations (10a) and (10b) state that

$$\begin{aligned} C_{r,T} &= \frac{1}{1+\left(b\,\xi_T\right)^{1/\gamma}}D_T \\ C_{n,T} &= \frac{\left(b\,\xi_T\right)^{1/\gamma}}{1+\left(b\,\xi_T\right)^{1/\gamma}}D_T \end{aligned}$$

If $\mu > \frac{1}{2}\sigma^2$, then the irrational trader's consumption converges to zero if and only if $\left(b\,\xi_T\right)^{1/\gamma} D_T$ converges to zero:

$$\left(b\,\xi_T\right)^{1/\gamma} D_T = \exp\left[\left(\eta\sigma^2\frac{1}{\gamma}\left(\gamma-1\right)-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+\mu-\frac{1}{2}\sigma^2\right)T+\left(\sigma\eta\frac{1}{\gamma}+\sigma\right)B_T\right] \tag{33}$$

Similarly, the rational trader's consumption converges to zero if and only if $\frac{D_T}{(b\,\xi_T)^{1/\gamma}}$ converges to zero:

$$\frac{D_T}{(b\,\xi_T)^{1/\gamma}} = \exp\left[\left(-\eta\sigma^2\frac{1}{\gamma}(\gamma-1)+\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+\mu-\frac{1}{2}\sigma^2\right)T+\left(-\sigma\eta\frac{1}{\gamma}+\sigma\right)B_T\right] \quad (34)$$

According to (32), (33) and (34) imply that

$$C_{n,T}\to 0 \quad\Leftrightarrow\quad \gamma\left(2\frac{\mu}{\sigma^2}-1\right) < -2\eta(\gamma-1)+\eta^2 \quad (35a)$$

$$C_{r,T}\to 0 \quad\Leftrightarrow\quad \gamma\left(2\frac{\mu}{\sigma^2}-1\right) < 2\eta(\gamma-1)-\eta^2 \quad (35b)$$

The right hand side of (35a) is convex in η, thus $C_{n,T}\to 0$ for $\eta<(\gamma-1)-\sqrt{(\gamma-1)^2+\gamma\left(2\frac{\mu}{\sigma^2}-1\right)}$ and $\eta>(\gamma-1)+\sqrt{(\gamma-1)^2+\gamma\left(2\frac{\mu}{\sigma^2}-1\right)}$.

Similarly, $C_{n,T}\to 0$ for $\eta>(\gamma-1)-\sqrt{(\gamma-1)^2-\gamma\left(2\frac{\mu}{\sigma^2}-1\right)}$ and $\eta<(\gamma-1)+\sqrt{(\gamma-1)^2-\gamma\left(2\frac{\mu}{\sigma^2}-1\right)}$. Such a value of η exists as long as $(\gamma-1)^2-\gamma\left(2\frac{\mu}{\sigma^2}-1\right)>0$. ■

Proof of Propositions 4 – 6

Our analysis will make use of the following technical result.

Lemma 1 *Consider a stochastic process $X_t=e^{c\,t+v\,B_t}$ and a constant $a\geq 0$. Assume that $ac+\frac{1}{2}v^2a^2(1-\lambda)\neq 0$, $0\leq\lambda<1$. Then the limit $\lim_{T\to\infty}\mathrm{E}_t[X_T^a]$ is equal to either zero or infinity almost surely, where we set $t=\lambda T$. The following convergence results hold:*

(i) (Point-wise convergence)

$$\lim_{T\to\infty}\frac{\mathrm{E}_t\left[(1+X_T)^a\right]}{1+\mathrm{E}_t\left[X_T^a\right]}=1. \quad (36)$$

(ii) (Convergence of moments)

$$\lim_{T\to\infty}\frac{\mathrm{mean_t}\ \mathrm{E}_t\left[(1+X_T)^a\right]}{\mathrm{mean_t}\ \left(1+\mathrm{E}_t\left[X_T^a\right]\right)}=1, \quad (37)$$

$$\lim_{T\to\infty}\frac{\mathrm{vol_t}\ \mathrm{E}_t\left[(1+X_T)^a\right]}{\mathrm{vol_t}\ \left(1+\mathrm{E}_t\left[X_T^a\right]\right)}=1, \quad (38)$$

where $\mathrm{mean_t f_t}$ *and* $\mathrm{vol_t f_t}$ *denote the instantaneous mean and standard deviation of the process* $\ln f_t$ *respectively.*

Proof of Lemma 1

(i) Consider the conditional expectation

$$\mathrm{E_t}\left[X_T^a\right] = \exp\left[acT + \frac{1}{2}v^2a^2(1-\lambda)T + avB_t\right] \tag{39}$$

The limit of $\mathrm{E}_t[X_T^a]$ is equal to zero if $ac+\frac{1}{2}v^2a^2(1-\lambda) < 0$ and equal to infinity if the opposite inequality holds (according to the strong Law of Large Numbers for Brownian motion, see Karatzas and Shreve, 1991, Sec. 2.9.A).

Because the function $acT + \frac{1}{2}v^2a^2(1-\lambda)T$ is convex in a and equal to zero when $a = 0$, we find that for $a \geq 1$

$$\mathrm{E}_t\left[X_T^a\right] \to \infty \;\Rightarrow\; \frac{\mathrm{E}_t\left[X_T^z\right]}{\mathrm{E}_t\left[X_T^a\right]} \to 0, \quad \forall z \in (0,a) \tag{40}$$

$$\mathrm{E}_t\left[X_T^a\right] \to 0 \;\Rightarrow\; \mathrm{E}_t\left[X_T^z\right] \to 0, \quad \forall z \in (0,a) \tag{41}$$

We prove the result of the lemma separately for six regions that cover the entire parameter space.

Case 1: $0 \leq a \leq 1, \mathrm{E}_t\left[X_T^a\right] \to \infty$.

If $X_T \leq 1$, $(X_T+1)^a \leq 2^a$, while if $X_T \geq 1 \Rightarrow (X_T+1)^a - X_T^a \leq aX_T^{a-1} \leq a$ since $(X_T+1)^a$ is concave and $a-1 \leq 0$. Therefore, $X_T^a \leq (1+X_T)^a \leq X_T^a + 2^a + a$, and hence $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right]/\mathrm{E_t}\left[X_T^a\right] = 1$, which implies $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right]/(1+\mathrm{E_t}\left[X_T^a\right]) = 1$.

Case 2: $1 \leq a \leq 2, \mathrm{E}_t\left[X_T^a\right] \to \infty$.

By the mean value theorem, $(1+X_T)^a = X_T^a + a\,(w+X_T)^{a-1}$ for some $w \in [0,1]$. Using the analysis of case 1, $(w+X_T)^{a-1} \leq (1+X_T)^{a-1} \leq X_T^{a-1} + 2^{a-1} + a - 1$, which, combined with (40), implies that $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right]/\mathrm{E_t}\left[X_T^a\right] = 1$ and the main result follows.

Case 3: $2 \leq a, \mathrm{E_t}\left[X_T^a\right] \to \infty$.

By the mean value theorem, $(1+X_T)^a = X_T^a + a\,(w+X_T)^{a-1}$ for some $w \in [0,1]$. By

Jensen's inequality, $((1+X_T)/2)^{a-1} \le (1+X_T^{a-1})/2$. Thus,

$$0 \le (w+X_T)^{a-1} \le (1+X_T)^{a-1} \le 2^{a-2} + 2^{a-2}X_T^{a-1}$$

which, combined with (40) implies that $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^a\right]/\mathrm{E}_t\left[X_T^a\right] = 1$ and the main result follows.

Case 4: $0 \le a \le 1, \mathrm{E_t}\left[X_T^a\right] \to 0$:

If $X_T \le 1$, $(1+X_T)^a \le 1+X_T \le 1+X_T^a$, while if $X_T \ge 1$, $(1+X_T)^a \le X_T^a + a \le 1+X_T^a$ since $(1+X_T)^a$ is concave. Thus, $1 \le (1+X_T)^a \le 1+X_T^a$ and therefore $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right] = 1$, which implies the main result.

Case 5: $1 \le a \le 2, \mathrm{E_t}\left[X_T^a\right] \to 0$.

By the mean value theorem, $(1+X_T)^a = 1 + aX_T(1+wX_T)^{a-1}$ for some $w \in [0,1]$. Further, $X_T(1+wX_T)^{a-1} \le X_T(1+X_T)^{a-1} \le X_T\left(X_T^{a-1} + 2^{a-1} + a - 1\right)$, using the same argument as in case 1. Since $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^a\right] = 0$, according to (41), $\lim_{T\to\infty} \mathrm{E}_t\left[X_T\right] = 0$ and hence $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right] = 1$.

Case 6: $2 \le a, \mathrm{E_t}\left[X_T^a\right] \to 0$.

By the mean value theorem, $(1+X_T)^a = 1 + aX_T(1+wX_T)^{a-1}$ for some $w \in [0,1]$. Further, $X_T(1+wX_T)^{a-1} \le X_T(1+X_T)^{a-1} \le 2^{a-2}X_T + 2^{a-2}X_T^a$ by Jensen's inequality. Since $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^a\right] = 0$, according to (41), and $\lim_{T\to\infty} \mathrm{E}_t\left[X_T\right] = 0$ and hence $\lim_{T\to\infty} \mathrm{E_t}\left[(1+X_T)^a\right] = 1$.

(ii) Since the conditional expectations $\mathrm{E}_t\left[(1+X_T)^a\right]$ and $\mathrm{E}_t\left[1+X_T^a\right]$ are martingales, they have zero drift for all values of T and t. By Ito's lemma, convergence of the first moments of the natural logarithms of the same processes follows from convergence of the second moments.

We now establish convergence of volatility of the process $\mathrm{E_t}\left[(1+X_T)^a\right]$. According to Ito's lemma, one must show that

$$\lim_{T\to\infty} \frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t}{\partial \ln(1+\mathrm{E}_t\left[X_T^a\right])/\partial B_t} = 1, \quad \forall a \ge 0$$

Given (39), it suffices to prove that $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = 0$ if $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^a\right] = 0$ and $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = av$ if $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^a\right] = \infty$.

First, changing the order of differentiation and expectation operators (see Billingsley

1995, Th. 16.8),

$$\frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]}{\partial B_t} = av\frac{\mathrm{E}_t\left[X_T\left(1+X_T\right)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]} = av\left(1-\frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]}\right).$$

Furthermore, according to part (i),

$$\frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{\mathrm{E}_t\left[(1+X_T)^a\right]} \sim \frac{\mathrm{E}_t\left[(1+X_T)^{a-1}\right]}{1+\mathrm{E}_\mathrm{t}\left[X_T^a\right]}. \tag{42}$$

Assume $a \geq 1$. As we have shown in case 1 of the proof of part (i), $X_T^{a-1} \leq (1+X_T)^{a-1} \leq X_T^{a-1} + 2^{a-1} + a - 1$. If $\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to \infty$, according to (40), $\mathrm{E}_\mathrm{t}\left[X_T^{a-1}\right]/\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to 0$, which yields

$$\lim_{T\to\infty} \frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]}{\partial B_t} = av.$$

Similarly, if $\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to 0$, then, according to (41), $\lim_{T\to\infty} \mathrm{E}_t\left[X_T^{a-1}\right] = 0$, which, according to part (i), implies that $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^{a-1}\right] = 1$ and

$$\lim_{T\to\infty} \frac{\partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]}{\partial B_t} = 0.$$

Next, consider the case of $0 < a < 1$. If $\mathrm{E}_\mathrm{t}\left[X_T^a\right] \to \infty$, because $\mathrm{E}_\mathrm{t}\left[(1+X_T)^{a-1}\right] \leq 1$, (42) implies $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = a\,v$.

Suppose that $\lim_{T\to\infty} \mathrm{E}_\mathrm{t}\left[X_T^a\right] = 0$. By Markov's inequality, $\mathrm{P}_t\left[X_T > \epsilon\right] \leq \mathrm{E}_t\left[X_T^a\right]/\epsilon^a \to 0$, for any $\epsilon > 0$. Similarly, $P_t\left[X_T < \epsilon\right] \leq \mathrm{E}_t\left[(1+X_T)^{a-1}\right]/(1+\epsilon)^{a-1}$. Thus, $1 \geq \mathrm{E}_t\left[(1+X_T)^{a-1}\right] \geq P_t\left[X_T < \epsilon\right](1+\epsilon)^{a-1}$, and therefore $\liminf_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^{a-1}\right] \geq (1+\epsilon)^{a-1}$ for any $\epsilon > 0$. This implies that $\lim_{T\to\infty} \mathrm{E}_t\left[(1+X_T)^{a-1}\right] = 1$, and therefore $\lim_{T\to\infty} \partial \ln \mathrm{E}_t\left[(1+X_T)^a\right]/\partial B_t = 0$. ■

We establish the long-run behavior of the quantities S_t and $W_{r,t}/W_{n,t}$ for the case when $\gamma > 1$ and $0 < \eta < \eta^\star = 2(\gamma - 1)$. The results for all other regions in the parameter space can be obtained similarly.

The equilibrium stock price and the ratio of the individual wealth processes are given by

$$S_t = \frac{\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^\gamma\right]}{\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^\gamma\right]}.$$

$$\frac{W_{r,t}}{W_{n,t}} = \frac{\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}{\mathrm{E}_t\left[D_T^{1-\gamma}(b\xi_T)^{\frac{1}{\gamma}}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]}.$$

We therefore need to characterize the long-run behavior of the following four quantities:

$$\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]$$

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right]$$

$$\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]$$

$$\mathrm{E}_t\left[D_T^{1-\gamma}(b\xi_T)^{\frac{1}{\gamma}}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma-1}\right].$$

We will establish the asymptotic behavior of the stock price process in detail, the corresponding results for the wealth ratio process in Proposition 6 are obtained in a very similar fashion.

Consider the first expression,

$$\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] = D_t^{1-\gamma}\mathrm{E}_t\left[\left(\frac{D_T}{D_t}\right)^{1-\gamma}\left(1+\left(b\xi_t\frac{\xi_T}{\xi_t}\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right].$$

Given the aggregate dividend process,

$$\left(\frac{D_T}{D_t}\right)^{1-\gamma} = e^{(T-t)\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)}e^{-\frac{1}{2}(1-\gamma)^2\sigma^2(T-t)+(1-\gamma)\sigma(B_T-B_t)}.$$

As in the proof of Proposition 1, we introduce a new measure Q with the Radon-Nikodym derivative

$$\left(\frac{dQ}{dP}\right)_t = e^{-\frac{1}{2}(1-\gamma)^2\sigma^2(T-t)+(1-\gamma)\sigma(B_T-B_t)}.$$

By Girsanov's theorem, $B_T - B_t = B_T^Q - B_t^Q - (1-\gamma)\sigma(T-t)$, where B_t^Q is a Brownian motion under the measure Q. Using the expression for b from Proposition 1, $b = e^{T(\gamma-1)\sigma^2\eta}$, we find

$$\begin{aligned}\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] &= e^{T\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)+t\left(-\frac{1}{2}\sigma^2(1-\gamma)^2\right)+B_t(\sigma(1-\gamma))} \\ &\quad\times \mathrm{E}_t^Q\left[\left(1+e^{\left(-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}\right)T+\left(\frac{1}{\gamma}(\gamma-1)\sigma^2\eta\right)t+\frac{\eta\sigma}{\gamma}B_T^Q}\right)^{\gamma}\right]. \qquad (43)\end{aligned}$$

We will omit the superscript Q in (43), since the distribution of B_t^Q under the measure Q is the same as the distribution of B_t under the original measure P.

Using the assumption that $t = \lambda T$, define

$$X_T = e^{\left(-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+(1-\lambda)\frac{1}{\gamma}(\gamma-1)\sigma^2\eta\right)T+\frac{\eta\sigma}{\gamma}B_T}.$$

We now apply the result of lemma 1, with

$$\begin{aligned} c &= -\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+(1-\lambda)\frac{1}{\gamma}(\gamma-1)\,\sigma^2\eta, \\ v &= \frac{\eta\sigma}{\gamma}, \\ a &= \gamma. \end{aligned}$$

Since we are assuming $\gamma > 1$ and $0 < \eta < 2(\gamma - 1)$, $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$.

According to lemma 1,

$$\mathrm{E}_t\left[D_T^{1-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{\left(\mu(1-\gamma)-\frac{1}{2}\sigma^2(1-\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2(\eta+1-\gamma)^2\right)t+\sigma(\eta+1-\gamma)B_t}.$$

We next examine

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right].$$

Using a similar change of measure, we find

$$\begin{aligned} \mathrm{E}_t\left[D_T^{-\gamma}\left(1+(b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] &= e^{\left(-\mu\gamma+\frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T+\left(-\frac{1}{2}\sigma^2\gamma^2\right)t+(-\sigma\gamma)B_t} \\ &\quad \times \mathrm{E}_t\left[\left(1+e^{\left(-\sigma^2\eta\frac{1}{\gamma}-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}\right)T+\sigma^2\eta t+\frac{\eta\sigma}{\gamma}B_T}\right)^{\gamma}\right]. \end{aligned}$$

We apply lemma 1, setting $X_T = e^{c\,T+v\,B_T}$ and

$$\begin{aligned} c &= -\sigma^2\eta\frac{1}{\gamma}-\frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma}+(1-\lambda)\sigma^2\eta, \\ v &= \frac{\eta\sigma}{\gamma}, \\ a &= \gamma. \end{aligned}$$

The value of $\lim_{T\to\infty} \mathrm{E}_t[X_T^a]$ depends on the exact combination of the model parameters. In particular,

$$\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \begin{cases} \infty, & -2\eta + \lambda(2\gamma\eta - \eta^2) \;>\; 0, \\ 0, & -2\eta + \lambda(2\gamma\eta - \eta^2) \;<\; 0, \end{cases}$$

(see the proof of lemma 1, part (i)). Define

$$\lambda_S = \frac{2}{2\gamma - \eta}.$$

Note that, because $\gamma > 1$ and $0 < \eta < 2(\gamma - 1)$, $0 < \lambda_S < 1$. Then, $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$ if $\lambda > \lambda_S$ and $2\gamma\eta - \eta^2 > 0$ or if $\lambda < \lambda_S$ and $2\gamma\eta - \eta^2 < 0$, and the limit is equal to zero otherwise.

By lemma 1, if $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = \infty$,

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1 + (b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{T\left(-\mu\gamma + \frac{1}{2}\sigma^2(1+\gamma)\gamma - \sigma^2\eta\right)T + \left(-\frac{1}{2}\sigma^2(\eta-\gamma)^2\right)t + \sigma(\eta-\gamma)B_t}$$

while if $\lim_{T\to\infty} \mathrm{E}_t[X_T^a] = 0$, then

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1 + (b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \sim e^{\left(-\mu\gamma + \frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T + \left(-\frac{1}{2}\sigma^2\gamma^2\right)t + (-\sigma\gamma)B_t}.$$

Using our definition of λ_S, we re-state these results as

$$\mathrm{E}_t\left[D_T^{-\gamma}\left(1 + (b\xi_T)^{\frac{1}{\gamma}}\right)^{\gamma}\right] \;\sim\; \begin{cases} e^{\left(-\mu\gamma + \frac{1}{2}\sigma^2(1+\gamma)\gamma\right)T + \left(-\frac{1}{2}\sigma^2\gamma^2\right)t + (-\sigma\gamma)B_t}, & 0 \le \lambda < \lambda_S \\ e^{\left(-\mu\gamma + \frac{1}{2}\sigma^2(1+\gamma)\gamma - \sigma^2\eta\right)T + \left(-\frac{1}{2}\sigma^2(\eta-\gamma)^2\right)t + \sigma(\eta-\gamma)B_t}, & \lambda_S < \lambda \le 1 \end{cases}$$

Having established the behavior of both the numerator and the denominator of the expression for the stock price, we have proven the limiting result for the stock price itself. According to part (ii) of lemma 1, not only the stock price, but also the mean and volatility of returns behave according to the asymptotic expressions of Proposition 4 in the limit of the economy horizon T approaching infinity. The same is true for the ratio of individual wealth and the volatility of that ratio in Proposition 6. ■

Proof of Proposition 7

When the financial markets are dynamically complete and there is a single source of uncertainty (driven by a Brownian motion), the fraction of the agent's wealth invested in stock can be computed as a ratio of the instantaneous volatility of the agent's wealth to the instantaneous volatility of the cumulative stock return process. Propositions 4 and 6 provide expression for the long-run behavior of the volatility of stock returns and individual wealth processes, from which the expression for portfolio holdings follow immediately. To decompose the portfolio holdings of the rational trader into as a sum of the myopic and hedging demands, we compute the hedging demand as $\mu_S/(\gamma\sigma_S^2)$, where μ_S and σ_S are the drift and the diffusion coefficients of the stock return process. The difference between the total portfolio holdings and the myopic component define the agent's hedging demand. For the irrational trader, the calculations are analogous, except the myopic demand is given by $\hat{\mu}_S/(\gamma\sigma_S^2) = (\mu_S + \eta\sigma\sigma_S)/(\gamma\sigma_S^2)$, where $\hat{\mu}_S$ is the expected stock return as perceived by the irrational trader. ■

Proof of Propositions 8 and 11.

Because Proposition 8 is a special case of Proposition 11, we prove the latter directly. Let

$$\frac{\gamma(c)}{c} \equiv -\frac{u''(c)}{u'(c)} = -\frac{d}{dc}\ln\left(u'(c)\right)$$

where $\gamma(c)$ is the coefficient of relative risk aversion. After integrating, Pareto optimality implies

$$\exp\left[-\int_{C_{n,t}}^{C_{r,t}} \frac{\gamma(s)}{s} ds\right] = \frac{u'\left(C_{r,t}\right)}{u'\left(C_{n,t}\right)} = b\xi_t \to 0.$$

For every path on which $\xi_t \to 0$, there exists a t' such that for all $t > t'$, $C_{r,t} > C_{n,t}$. Since $\gamma(c) \le M$, on each of these paths when $t > t'$ we have

$$M\left(\ln C_{r,t} - \ln C_{n,t}\right) \ge \int_{C_{n,t}}^{C_{r,t}} \frac{\gamma(s)}{s} ds \to \infty$$

and so $C_{n,t}/C_{r,t} \to 0$. ■

Proof of Proposition 9

Our specification of the economy implies that for $\gamma \ge 1$ and an integer,

$$\frac{W_{n,t}}{W_{r,t}} = \frac{\mathrm{E_t}\left[\int_t^\infty e^{\rho(t-s)} D_s^{1-\gamma} (b\,\xi_s)^{1/\gamma} \left(1 + (b\,\xi_s)^{1/\gamma}\right)^{\gamma-1} ds\right]}{\mathrm{E_t}\left[\int_t^\infty e^{\rho(t-s)} D_s^{1-\gamma} \left(1 + (b\,\xi_s)^{1/\gamma}\right)^{\gamma-1} ds\right]}$$

$$= \frac{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} \int_t^\infty \mathrm{E_t}\left[e^{\rho(t-s)} D_s^{1-\gamma} (b\,\xi_s)^{\frac{i+1}{\gamma}}\right] ds}{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} \int_t^\infty \mathrm{E_t}\left[e^{\rho(t-s)} D_s^{1-\gamma} (b\,\xi_s)^{\frac{i}{\gamma}}\right] ds}.$$

where we changed the order of integration by Fubini's Theorem for positive processes (see, for example, Halmos (1950, pg. 147)). Notice that

$$\int_t^\infty \mathrm{E_t}\left[e^{\rho(t-s)} D_s^{1-\gamma} (b\,\xi_s)^{\frac{j}{\gamma}}\right] ds = D_t^{1-\gamma} (b\,\xi_t)^{\frac{j}{\gamma}} m_j$$

where m_j is a constant equal to

$$m_j \equiv \int_t^\infty e^{\rho(t-s)} \mathrm{E_t}\left[\left(\frac{D_s}{D_t}\right)^{1-\gamma} \left(\frac{\xi_s}{\xi_t}\right)^{j/\gamma}\right] ds = \int_0^\infty e^{-\rho s} \mathrm{E_0}\left[D_s^{1-\gamma} \xi_s^{j/\gamma}\right] ds.$$

Equation (25) and $0 \leq j \leq \gamma - 1$ imply the m_j are finite. The wealth ratio becomes

$$\frac{W_{n,t}}{W_{r,t}} = \frac{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} D_t^{1-\gamma} (b\,\xi_t)^{\frac{j+1}{\gamma}} m_{j+1}}{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} D_t^{1-\gamma} (b\,\xi_t)^{\frac{j}{\gamma}} m_j}$$

Recall that $\frac{C_{n,t}}{C_{r,t}} = (b\,\xi_t)^{\frac{1}{\gamma}}$, so we can write

$$\frac{W_{n,t}/W_{r,t}}{C_{n,t}/C_{r,t}} = \frac{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} (b\,\xi_t)^{\frac{j}{\gamma}} m_{j+1}}{\sum_{i=0}^{i=\gamma-1} \binom{\gamma-1}{i} (b\,\xi_t)^{\frac{j}{\gamma}} m_j}$$

Because $\xi_t \to 0$, both the numerator and denominator have a dominant term, and so

$$\lim_{t \to 0} \frac{W_{n,t}/W_{r,t}}{C_{n,t}/C_{r,t}} = \frac{m_1}{m_0}$$

Define

$$k \equiv \frac{m_1}{m_0} = \frac{\rho + (\gamma - 1)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)}{\rho + (\gamma - 1)\left(\mu - \frac{1}{2}\gamma\sigma^2 + \eta\sigma^2\frac{1}{\gamma} + \frac{1}{2}\eta^2\sigma^2\frac{1}{\gamma^2}\right)}$$

and the result follows. ■

Proof of Proposition 10

This proof follows very closely the proof of Proposition 9, and so we omit some intermediate steps. Our specification of the economy implies that for $\gamma \geq 1$ and an integer,

$$S_t = \frac{\mathrm{E_t}\left[\int_t^\infty e^{\rho(t-s)} D_s^{1-\gamma}\left(1+(b\xi_s)^{1/\gamma}\right)^\gamma ds\right]}{D_t^{-\gamma}\left(1+(b\xi_t)^{1/\gamma}\right)^\gamma}$$

$$= \frac{\sum_{i=0}^{i=\gamma}\binom{\gamma}{i}\int_t^\infty \mathrm{E_t}\left[e^{\rho(t-s)} D_s^{1-\gamma}(b\xi_s)^{\frac{i}{\gamma}}\right] ds}{\sum_{i=0}^{i=\gamma}\binom{\gamma}{i} D_t^{-\gamma}(b\xi_t)^{\frac{i}{\gamma}}}$$

Notice that

$$\int_t^\infty \mathrm{E_t}\left[e^{\rho(t-s)} D_s^{1-\gamma}(b\xi_s)^{\frac{j}{\gamma}}\right] ds = D_t^{1-\gamma}(b\xi_t)^{\frac{j}{\gamma}} m_j$$

where m_j is defined in Proposition 9. Because $\xi_t \to 0$, both the numerator and denominator have a dominant term, and so we write the limit of the price-dividend ratio as

$$\lim_{t\to\infty}\frac{S_t}{D_t} = m_0 = \frac{1}{\rho + (\gamma-1)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)}$$

To evaluate the interest rate, consider the marginal utility of the rational trader,

$$\phi_t = e^{-\rho t} D_t^{-\gamma}\left(1+(b\xi_t)^{1/\gamma}\right)^\gamma.$$

The instantaneous interest rate is given by

$$r_t = -\frac{E_t[d\phi_t]}{\phi_t dt} = \rho + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2 + \frac{\gamma-1}{2\gamma}\eta^2\sigma^2\frac{(b\xi_t)^{1/\gamma}}{\left(1+(b\xi_t)^{1/\gamma}\right)^2} + \gamma\eta\sigma^2\frac{(b\xi_t)^{1/\gamma}}{1+(b\xi_t)^{1/\gamma}}$$

and therefore

$$\lim_{t\to\infty} r_t = \rho + \gamma\mu - \frac{1}{2}\gamma(\gamma+1)\sigma^2$$

■

Proof of Example 1

We write $f = -\ln(u')$ so that by Pareto optimality,

$$f(C_{r,t}) - f(C_{n,t}) = -\ln(b\,\xi_t).$$

Dividing by D_t, we have

$$\frac{f(C_{r,t}) - f(C_{n,t})}{D_t} = -\frac{\ln(b\xi_t)}{D_t} \to 0.$$

Since $a(c) = -\frac{u''(c)}{u'(c)} = f'(c) \geq M$,

$$\frac{C_{r,t} - C_{n,t}}{f(C_{r,t}) - f(C_{n,t})} = \frac{C_{r,t} - C_{n,t}}{\int_{C_{n,t}}^{C_{r,t}} f'(s)ds} \leq \frac{C_{r,t} - C_{n,t}}{\int_{C_{n,t}}^{C_{r,t}} M ds} = \frac{1}{M}.$$

We have shown the expressions for $C_{r,t} > C_{n,t}$. Those for $C_{r,t} < C_{n,t}$ are analogous, and we ignore the case when $C_{r,t} = C_{n,t}$ because we have $\frac{C_{n,t}}{C_{r,t}} = 1$ directly. Furthermore, $\frac{C_{r,t} - C_{n,t}}{f(C_{r,t}) - f(C_{n,t})} > 0$ for $C_{r,t} \neq C_{n,t}$ because f is increasing. Continuing,

$$0 < \frac{C_{r,t} - C_{n,t}}{f(C_{r,t}) - f(C_{n,t})} \leq \frac{1}{M}$$

so that

$$\frac{C_{r,t} - C_{n,t}}{D_t} = \frac{C_{r,t} - C_{n,t}}{f(C_{r,t}) - f(C_{n,t})} \frac{f(C_{r,t}) - f(C_{n,t})}{D_t}$$

The first term on the right-hand side is bounded, and the second term converges to zero. Thus, we have

$$\frac{C_{r,t} - C_{n,t}}{D_t} \to 0$$

and so $C_{r,t} + C_{n,t} = D_t$ implies $\frac{C_{n,t}}{C_{r,t}} \to 1$. ■

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