HOMEOWNERSHIP: VOLATILE HOUSING PRICES, LOW LABOR MOBILITY AND HIGH INCOME DISPERSION

François Ortalo-Magné
London School of Economics & University of Wisconsin

Sven Rady
University of Munich

July 17, 2002
Preliminary and incomplete

Abstract

We develop a dynamic stochastic equilibrium model of a city with two locations where heterogeneous households make joint location and tenure mode decisions. To investigate the effect of homeownership on equilibrium prices and allocations, we compare the response of this model economy to a population shock with that of a rental-only version. This comparison yields three results. First, homeownership adds to the volatility of the housing market. Second, homeownership enables more relatively poor households to remain in the desirable location at the expense of newcomers. Third, homeownership may amplify the dispersion of household income within a location. These results are driven by the fact that homeownership makes the ranking of households according to wealth differ from the ranking of households according to income. Next, we introduce securities in the model that eliminate the market incompleteness inherent in the ownership economy. We show that such securities can lead to a more volatile housing market.

1 Introduction

This paper studies how homeownership affects local housing markets and the distribution of households across residential properties. Housing differs from other durable consumption goods insofar as a housing purchase is typically tied to the purchase of land. This makes housing a desirable asset to hold in one's portfolio as a hedge against shocks to the local economy, which provides an insurance motive for homeownership. Recent research has explored the significance of this motive at the household level and reports evidence that differences between the financial risks of renting and of owning are significant drivers of housing tenure choices.[1] The novelty of the present paper is to embed key features of existing models of household tenure choice into a dynamic stochastic framework where heterogenous households choose both housing tenure and consumption.[2]

This framework enables us to analyze formally the type of issues raised in the following story. Let us consider the case of taxicab drivers in central London who do not expect their income to grow as fast as local housing costs. If they want to live in central London and remain there for years, they have a strong incentive to buy their home; buying their home allows them to fix their housing expenditures. If the London economy booms, attracting new economic activity and labor, the resulting pressure on the housing market will force housing costs upward. Thanks to homeownership, the taxicab drivers' wealth increases with the opportunity cost of remaining in their current home. Some move out of central London but fewer than if they had all been renters. As such, homeownership adds upward pressure on housing costs. Consequently, only the richer newcomers choose to locate in central London, the only ones who are willing to compete for housing with the enriched taxicab drivers. The positive effect of homeownership on housing prices therefore leads to somewhat surprising levels of income heterogeneity within neighborhoods where we observe taxicab drivers living next to investment bankers.

This story captures the main insights gathered from our model as far as the effects of homeownership are concerned. Without homeownership, the cost of housing in the prime location does not rise as much in response to a positive shock because more households move out of the prime location given one and the same price rise. The

[1] E.g., Davidoff (2001), Hilber (2002), Sinai and Souleles (2001).

[2] A review of this literature is provided in Ortalo-Magné and Rady (forthcoming). This paper proposes a model of household behavior closely related to the one on which we build the equilibrium model analyzed here.

lower rise in housing cost incites a broader cross-section of newcomers to move into the prime location. Without ownership, therefore, there is less income disparity between the newcomers and the initial households who remain in the prime location after the boom.

Does homeownership generate inefficiencies? This paper argues that the main consequence of allowing people to own their home is that aggregate shocks have the potential to yield an equilibrium distribution of wealth disconnected from the distribution of human capital (or income). Although the incoming teachers may have the higher level of human capital, they find themselves living in worse accommodation than the taxicab drivers who benefited from getting on the property ladder ahead of the boom. The taxicab drivers enjoy capital gains, they win. The newcomers face higher housing costs, they lose.[3] Homeownership thus raises first and foremost distributional issues.

As such, our model provides a new explanation for a long-standing puzzle in the literature concerned with the formation and composition of local communities: standard models predict too much stratification of households according to income into overly homogeneous communities.[4] This is particularly the case for static models concerned with mobility among local jurisdictions which assume a single dimension of household heterogeneity.[5] Bénabou (1993) showed that stratification also arises in a dynamic model with local complementarities in human capital investment and endogenously determined local land rents. To obtain some degree of income heterogeneity within communities, Epple and Platt (1998) as well as Epple and Sieg (1999) extend the local jurisdiction model by assuming that households differ not only in terms of income but also in terms of preferences. Here we propose an alternative explanation for imperfect stratification based on the fact that the wealth distribution may differ from the income distribution due to the heterogeneity of households' housing market experiences.

If homeownership does not generate any inefficiencies in and of itself, it still remains that homeownership may amplify existing inefficiencies. This is the case for example if efficiency in the labor market requires a spatial allocation of workers driven by human capital and not wealth. Oswald (1996, 1997) sparked a heated debated when reporting evidence of a positive correlation between homeownership and unemployment. Nickell and Layard (1998) and Green and Hendershott (2001a, 2001b) provide further evidence

[3]Public policy in both the US and the UK has been concerned for years with a particular group of losers: new teachers, nurses, police officers and other so-called key workers.

[4]See for example Epple and Platt (1998), Ioannides (2001) and Ioannides and Seslen (2001).

[5]See Epple, Filimon, and Romer (1984, 1993) and Goodspeed (1989) for examples of such models with a housing market.

in support of this correlation. Researchers engaged with this debate have failed to point out convincingly any market failure that could lead to inefficiencies arising from the interaction between the labor and housing markets. In this paper, we focus on the inability of households to vary their exposure to housing returns independently of their consumption of housing services. This is not a problem of market failure but one of market incompleteness.

Our model therefore provides a natural framework to study the introduction of housing price derivatives that eliminate this market incompleteness.[6] Rent futures or other forms of price derivatives would allow households to disconnect their housing investment from their housing consumption. Our model demonstrates that introducing such securities can have a dramatic impact on the dynamics of the housing market. In particular, resulting changes in the pattern of housing consumption may lead to a more volatile housing market, a reversal of the results we would obtain in a model without endogenously chosen state-contingent housing consumption plans. Incorporating such a feature in a dynamic stochastic equilibrium model is a key contribution of this paper.

This paper abstracts from a further dimension of market incompleteness: credit constraints. There is ample empirical evidence that credit constraints may override other concerns in the tenure decision for a subset of households. Linneman and Wachter (1989), for example, find that wealth-poor households are less likely to own. At the aggregate level, Ortalo-Magné and Rady (1999) argue that credit constraints affect primarily the tenure decisions of the young in equilibrium. Chiuri and Jappelli (forthcoming) report cross-county evidence in support of this theoretical prediction. While we acknowledge the determinant role credit constraints play in the tenure decision of a subset of households, we focus here on the equilibrium interaction of households sufficiently wealthy that credit constraints are not likely to affect their tenure choice.

In the following section, we consider the decision problem of an individual household. Section 3 studies the equilibrium allocation and prices in an economy made up of a continuum of such households who differ in their earnings. Section 4 analyzes a rental-only version of the model. This version of the model serves as a benchmark to evaluate the effects of homeownership on prices and the allocation of properties. Section 5 studies the effect of completing markets by means of housing rent futures. Section 6 offers some concluding remarks.

[6] Arguments in favor of the introduction of such securities have been put forward for example by Case, Shiller and Weiss (1993) and Caplin et al. (1997).

2 Tenure Choice and Housing Consumption

We consider a household living in a two-period economy with two locations, labelled 1 and 0, and three commodities: homes in location 1, homes in location 0, and a numeraire consumption good. This good can be saved or borrowed from period 1 to period 2 at the exogenously given interest rate r.

The household derives additively separable utility from the consumption of housing and the numeraire. Housing is enjoyed at the end of periods 1 and 2, the numeraire good only at the end of period 2. There is no discounting of housing utility across periods. For concreteness, we assume that location 1 is more desirable than location 0: housing utility derived from a home in location 0 is normalized to zero, whereas a home in location 1 yields an additive utility premium of $\mu > 0$ per period, independent of tenure. The non-housing utility derived from consumption of c units of the numeraire good is described by the constant absolute risk aversion function $U(c) = -e^{-ac}$ where $a > 0$ is the coefficient of absolute risk aversion.

Over its lifetime, the household receives a stream of endowments of the numeraire good. The capitalized value of these endowments in period 2 is denoted by W. Trade takes place in the middle of each period, after the household receives its endowment and before it gets to enjoy its home. The household may either buy or rent a home at the start of each period. The initial owners of all homes are risk-neutral competitive landlords.

The household faces uncertainty about the cost of housing in the second period. With probability π, state H occurs, and the second-period rental cost of housing in location 1 is R_H; with the complementary probability $1 - \pi$, state L occurs, and the second-period rental cost of housing in location 1 is $R_L < R_H$. By contrast, rental costs of a home in location 0 are independent of the state of the economy and constant over time. For ease of exposition, we normalize them to zero.

The state of the economy in period 2 is revealed before trading takes place. As period 2 is the last period of the economy, renting a home in period 2 is equivalent to buying it, so the price of a home in period 2 coincides with the rental cost of that home in period 2. Moreover, arbitrage on the part of the landlords ensures that the price of a home in period 1 equals the first-period rent plus discounted expected second-period rent. For a home in location 1, this means that the first-period price is

$$p_1 = R_1 + \frac{\bar{R}_2}{1+r} \tag{1}$$

where R_1 denotes the first-period rent and $\bar{R}_2 = \pi R_H + (1-\pi) R_L$ the expected second-period rent.

Equation (1) highlights what ownership means in this model: by purchasing a home in the first period, the household effectively signs a two-period rental contract, locking in the second-period rent at its expected level. If the household plans to stay in location 1 in the second period, purchasing the home in the first period provides insurance against rent fluctuations. If the household plans to stay in location 1 in the second period, it is insured against rent fluctuations. If it plans to sell and move to location 0, it is exposed to potential capital gains or losses. Whether buying in location 1 instead of renting makes housing in itself more or less risky therefore depends on the household's desired housing consumption in the second period.

For a home in location 0, the normalization of rents to zero implies that the first-period price is zero as well by the analogue of equation (1). This means in particular that it makes no difference for a household wishing to live in location 0 in the first period whether it rents or buys its home.

Taking the three rent levels R_1, R_H, R_L and the price p_1 as given, the household must solve for housing and numeraire consumption in each possible state of the world in period 2, and for housing consumption and tenure choice in period 1. The housing consumption plan is denoted by the triple (h_1, h_H, h_L), where h_1, h_H and h_L take the value of 1 for location 1, and 0 otherwise. To indicate the tenure choice in case $h_1 = 1$, we denote the combined location and tenure choice by $(1_B, h_H, h_L)$ if the household buys a home, and $(1_R, h_H, h_L)$ if it rents one.

The household has to choose among eight housing location combinations, two alternatives for each of period 1, period 2 state H, and period 2 state L. In addition, for the four combinations that involve living in location 1 in period 1, the household must decide whether to buy or rent. That is a total of twelve housing consumption and tenure options.

Ex post, given that the economy is in state $s \in \{H, L\}$, given the corresponding rental price R_s, and given the household's housing and tenure choice, the budget constraint determines the consumption of the numeraire good in period 2. Hence, the household's ex-post utility is

$$U\Big(W - h_s R_s\Big) + h_s \mu \tag{2}$$

if it lived in location 0 in the first period,

$$U\Big(W - (1+r)R_1 - h_s R_s\Big) + (1+h_s)\mu \tag{3}$$

if it rented a home in location 1 in the first period, and

$$\begin{aligned} & U\Big(W - (1+r)p_1 + (1-h_s)R_s\Big) + (1+h_s)\mu \\ = \;& U\Big(W - (1+r)R_1 - \bar{R}_2 + (1-h_s)R_s\Big) + (1+h_s)\mu \end{aligned} \tag{4}$$

if it bought a home in location 1 in the first period. Under this last scenario, the household pays period 1 plus expected period 2 rent and gets back the realized period 2 rent if it moves out of location 1 after the first period.

These equations highlight what is at issue with regards to the tenure choice: the stochastic properties of numeraire consumption. For a given housing location choice, the utility derived from housing consumption is independent of tenure choice, by assumption. What the tenure determines is how shocks to housing costs translate into fluctuations in non-housing consumption.[7] In this regard, equation (1) implies that *expected* non-housing consumption is independent of tenure choice, as can easily be seen from computing the expectation of the consumption levels figuring in (3) and (4), respectively. So the tenure choice reduces to choosing the option that induces the smallest absolute difference between the non-housing consumption levels in the two states of the economy.

As the tenure mode in location 0 is of no consequence, we can restrict our analysis of tenure choice to the four housing consumption plans that involve living in location 1 in the first period: $(1,1,1)$, $(1,1,0)$, $(1,0,1)$, and $(1,0,0)$. It turns out that the household prefers to own if its housing consumption plan is $(1,1,1)$ or $(1,1,0)$, and prefers to rent if its plan is $(1,0,1)$ or $(1,0,0)$. The crucial difference for tenure is therefore whether or not the household plans to move out of the more desirable home in case the cost of occupying such a home turns out to be high. This is fairly obvious

[7] In the present framework, fluctuations in non-housing consumption are entirely driven by shocks to housing costs in location 1. If the household' earnings were stochastic, the tenure choice would depend on the extent to which rents and income co-vary, not just on the stochastic properties of the housing rent alone. Furthermore, if housing costs in location 0 were stochastic, the covariance between housing costs in the two locations would matter for tenure choice. These effects are analyzed in Ortalo-Magné and Rady (forthcoming).

for the plans with a deterministic horizon in the type 1 home, $(1, 1, 1)$ and $(1, 0, 0)$, since in these cases one of the tenure modes provides full insurance whereas the other does not. Thus, ownership dominates rental for a household who has a deterministic two-period horizon in location 1, while the opposite holds true for a household with a deterministic one-period horizon in location 1.

Under the plans $(1, 1, 0)$ and $(1, 0, 1)$, by contrast, either tenure mode imposes some risk on the household. Under $(1, 1, 0)$, the household's non-housing consumption is necessarily higher in state L. If it rents in the first period, non-housing consumption in state L is higher by R_H since the second-period rent is paid precisely when it is high. If the household buys the home in the first period, on the other hand, non-housing consumption in state L is higher by R_L since the household's non-housing consumption is boosted by the revenue from the sale of the home precisely when this revenue is low. Under $(1, 1, 0)$, buying is thus less risky. Under $(1, 0, 1)$, this logic is reversed. The household's non-housing consumption is now necessarily higher in state H. If it rents in the first period, non-housing consumption in state H is higher by R_L since the second-period rent is paid when it is low. If the household buys the home in the first period, on the other hand, non-housing consumption in state H is higher by R_H since the household's non-housing consumption is boosted by the revenue from the sale of the home when this revenue is high. So buying is more risky under $(1, 0, 1)$.[8]

From a total of 66 pairwise comparisons of potential housing choices (twelve alternatives), therefore, only 28 comparisons (eight alternatives) remain because four options are dominated by their alternative tenure mode; i.e., the household never chooses $(1_R, 1, 1)$, $(1_R, 1, 0)$, $(1_B, 0, 1)$ or $(1_B, 0, 0)$. Next, using the CARA specification of non-housing utility, it is easy to verify that the preference ranking of the plans $(1_R, 0, 0)$ and $(0, 1, 1)$ does not depend on the household's endowment W. In fact, the household weakly prefers $(1_R, 0, 0)$ over $(0, 1, 1)$ if and only if

$$e^{a(1+r)R_1} \leq \pi e^{aR_H} + (1 - \pi) e^{aR_L}, \tag{5}$$

with a strict preference if the inequality is strict. So there remain at most seven alternatives to be considered, and 21 comparisons to be carried out.

[8]Note that these results on tenure choice rely merely on risk aversion, and not on the CARA specification of utility. For a treatment of tenure choice with arbitrary risk averse utility of numeraire consumption, see Ortalo-Magné and Rady (forthcoming). In the present paper, the CARA specification is adopted to make the household's choice of housing consumption plan and the computation of equilibrium tractable.

Each possible housing consumption plan determines a curve in the plane with coordinates W (the household endowment) and U (the expected utility level). For any given W, the optimal plan is the one which yields the highest level of utility. Determining the optimal plan for every W amounts to characterizing the upper envelope of the utility curves for the seven housing consumption alternatives that remain.

For every pair of such utility curves, we first ask whether they are likely to intersect at a point on the upper envelope, that is, whether their intersection is likely to be useful in characterizing the household's choice across housing alternatives. If not, we dismiss the pair. Using the fact that housing is a normal good, we can dismiss pairs involving total amounts of housing consumption in location 1 sufficiently distinct in the sense that there exists a housing alternative with a total amount of housing consumption in between the two.[9] For example, the plans $(1_B, 1, 1)$ and $(1_R, 0, 0)$ will be separated by $(1_B, 1, 0)$ or $(1_R, 0, 1)$. This strategy enables us to dismiss eleven pairs.

We are thus left with ten pairwise comparisons to carry out. Table 1 shows these comparisons for the case where inequality (5) holds, so that the plan $(0, 1, 1)$ is weakly dominated.[10] For each pair of housing consumption plans being compared, the table presents the endowment level that would make the household just indifferent between the two.

Note that whether π, the probability of state H, is greater than $\frac{1}{2}$ or not affects which of the options $(., 1, 0)$ and $(., 0, 1)$ delivers a larger promise of housing consumption in location 1 ex ante. It therefore affects the ranking of housing plans. We choose to focus on the case $\pi < \frac{1}{2}$.

Relying primarily on the monotonicity and the convexity of the exponential function, it is possible to rank the critical endowment levels in Table 1 analytically. In particular, it is straightforward (but somewhat tedious) to see that $(1_B, 1, 0)$ and $(0, 1, 0)$ are never optimal.

We can summarize our findings so far as follows.

Lemma 1 *If* $\pi < \frac{1}{2}$*, the household chooses a housing consumption plan within the following subset of alternatives:* $(1_B, 1, 1)$*,* $(1_R, 0, 1)$*,* $(1_R, 0, 0)$*,* $(0, 1, 1)$*,* $(0, 0, 1)$ *and*

[9] The amount of housing consumption in location 1 can be 2, $2 - \pi$, $1 + \pi$, 1, $1 - \pi$, π or 0.

[10] This will turn out to be the relevant case for the equilibrium analysis that we shall carry out in the following section.

Plans compared		First plan preferred if μe^{aW} exceeds
$(1_B,1,1)$	$(1_R,0,1)$	$e^{a(1+r)R_1}\left[e^{a\bar{R}_2}-1+\frac{1-\pi}{\pi}\left(e^{a\bar{R}_2}-e^{aR_L}\right)\right]$
	$(1_B,1,0)$	$e^{a(1+r)R_1+\bar{R}_2}\left(1-e^{-aR_L}\right)$
$(1_R,0,1)$	$(1_B,1,0)$	$e^{a(1+r)R_1}\left[\frac{1-\pi}{1-2\pi}\left(e^{aR_L}-e^{a[\bar{R}_2-R_L]}\right)-\frac{\pi}{1-2\pi}\left(e^{a\bar{R}_2}-1\right)\right]$
	$(1_R,0,0)$	$e^{a(1+r)R_1}\left(e^{aR_L}-1\right)$
$(1_B,1,0)$	$(1_R,0,0)$	$e^{a(1+r)R_1}\left[e^{a\bar{R}_2}-1+\frac{1-\pi}{\pi}\left(e^{a[\bar{R}_2-R_L]}-1\right)\right]$
$(1_R,0,0)$	$(0,0,1)$	$e^{a(1+r)R_1}-1+\frac{1-\pi}{\pi}\left(e^{a(1+r)R_1}-e^{aR_L}\right)$
	$(0,1,0)$	$e^{a(1+r)R_1}-1+\frac{\pi}{1-\pi}\left(e^{a(1+r)R_1}-e^{aR_H}\right)$
$(0,0,1)$	$(0,1,0)$	$\frac{1-\pi}{1-2\pi}\left(e^{aR_L}-1\right)-\frac{\pi}{1-2\pi}\left(e^{aR_H}-1\right)$
	$(0,0,0)$	$e^{aR_L}-1$
$(0,1,0)$	$(0,0,0)$	$e^{aR_H}-1$

Table 1: Pairwise housing and tenure plan comparisons

$(0,0,0)$. *The preference ranking of the plans* $(1_R,0,0)$ *and* $(0,1,1)$ *does not depend on the household's endowment. More precisely, the plan* $(0,1,1)$ *is weakly dominated if and only if inequality* (5) *holds.*

Under condition (5), we do not need to consider the plan $(0,1,1)$ when we compute the critical endowments that make the household indifferent between any two of the housing options in Lemma 1. Given that housing is a normal good, four critical endowment levels then fully characterize the decision of the household:

• for indifference between $(1_B,1,1)$ and $(1_R,0,1)$, the critical endowment level is W^1 with

$$\mu e^{aW^1}=e^{a(1+r)R_1}\left[e^{a\bar{R}_2}-1+\frac{1-\pi}{\pi}\left(e^{a\bar{R}_2}-e^{aR_L}\right)\right]; \tag{6}$$

• for indifference between $(1_R,0,1)$ and $(1_R,0,0)$, the critical endowment level is W^2 with

$$\mu e^{aW^2}=e^{a(1+r)R_1}\left(e^{aR_L}-1\right); \tag{7}$$

• for indifference between $(1_R, 0, 0)$ and $(0, 0, 1)$, the critical endowment level is W^3 with

$$\mu e^{aW^3} = e^{a(1+r)R_1} - 1 + \frac{1-\pi}{\pi}\left(e^{a(1+r)R_1} - e^{aR_L}\right); \tag{8}$$

• for indifference between $(0, 0, 1)$ and $(0, 0, 0)$, the critical endowment level is W^4 with

$$\mu e^{aW^4} = e^{aR_L} - 1. \tag{9}$$

It is straightforward to see that $W^1 > W^2$ whenever $R_H > R_L$, and $W^3 > W^4$ whenever $(1+r)R_1 > R_L$. Finally, $W^2 > W^3$ if and only if

$$\left(e^{a(1+r)R_1} - 1\right)\left(e^{aR_L} - 1\right) > \frac{1}{\pi}\left(e^{a(1+r)R_1} - e^{aR_L}\right). \tag{10}$$

This concludes our analysis of a single household's decision problem. We now turn to the analysis of equilibrium in an economy populated by a large number of such households with heterogeneous endowments.

3 A Housing Market with Heterogeneous Households

3.1 Economic Environment

We consider a two-period model of a city with two locations, 1 and 0. Absentee landlords own a measure S of homes in location 1. No more homes can be built in location 1. The landlords are risk neutral and do not derive any consumption benefit from owning properties. Their sole investment alternative is a savings technology available to all. All agents can save or borrow between periods 1 and 2 at the exogenously given interest rate r. The alternative to living in location 1 is to live in location 0 where there is an unlimited supply of housing whose price is normalized to zero.[11]

The city is populated by a measure one of households of the type considered in the previous section. These native households are distributed uniformly over the unit interval, each being identified by an index $i \in [0, 1]$. The total endowment household i receives, $W(i)$, is an increasing function of i.

[11] For the sake of simplicity and without loss of generality, we assume extreme differences in the supply of housing between the two locations: perfectly inelastic in location 1, perfectly elastic in location 0. The key feature here is that the supply of housing in 1 is less elastic than in 0.

The city is potentially subject to a population shock in period 2. With probability $\pi < \frac{1}{2}$, a measure ν of newcomer households arrives in period 2 (state H); with the complementary probability, no shock occurs (state L). Like native households, newcomers are distributed uniformly over the unit interval; they are characterized by the index $n \in [0, 1]$. Their endowment is defined by the function $\phi W(n)$ with $\phi < 1$ to capture the idea that these are younger households. They have the same utility function as natives adjusted for the fact that they cannot obtain any utility from housing in period 1. The only decision they face is whether to live in location 1 or 0 in the second period and how much of the numeraire good to consume.

3.2 Households' Behavior

The difference between the current setup and that in the previous section is that housing rents and prices are now endogenous. The population shock will generate the ranking of rents assumed above. Arbitrage by the landlords will generate relationship (1) between first-period price and rents.

The four critical endowment levels that characterized housing consumption and tenure choice in the previous section translate here into four critical indices i^1, i^2, i^3, i^4 that separate households according to their housing choice: $W(i^k) = W^k$, $k = 1, \ldots, 4$.

Newcomers face a deterministic one-period problem since they only enter if state H occurs. To decide where to live, a newcomer household compares the cost of living in location 1 in terms of lower utility from numeraire consumption with the utility premium derived from housing. The newcomer household with index n thus prefers location 1 to location 0 if and only if

$$U\Big(\phi W(n) - R_H\Big) + \mu > U\Big(\phi W(n)\Big). \tag{11}$$

Given the monotonicity of the endowment function, we obtain a critical index n^1 such that only newcomers with index $n > n^1$ strictly prefer location 1. This index is implicitly defined by the equation

$$\mu e^{a\phi W(n^1)} = e^{aR_H} - 1. \tag{12}$$

3.3 Equilibrium

We need to solve for the rental costs (R_1, R_H, R_L) of housing in location 1 for period 1, period 2 state H and period 2 state L, respectively. The supply of housing in location 1 is S. Aggregate demand is computed by adding up the measures of agents in location 1.

Market clearing together with Lemma 1 rules out equilibria where some (and therefore all) native households strictly prefer $(0,1,1)$ to $(1_R,0,0)$. This immediately implies

Lemma 2 *In equilibrium, the rental costs* (R_1, R_H, R_L) *of housing in location 1 satisfy inequality* (5)*. The plan* $(0,1,1)$ *can arise as a native household's equilibrium choice only if* (5) *holds as an equality, i.e., if this plan yields precisely the same utility as* $(1_R,0,0)$.

Now, if $1 > i^1 > i^2 > i^3 > i^4 > 0$ and (5) holds as a strict inequality, the market clearing conditions for housing in location 1 in period 1, period 2 state H and period 2 state L are

$$1 - i^3 = S, \tag{13}$$

$$1 - i^1 + (1-n^1)\nu = S, \tag{14}$$

$$1 - i^2 + i^3 - i^4 = S. \tag{15}$$

This is a system of three non-linear equations for the three unknowns R_1, R_H and R_L. Note that the two equations (13) and (15) involve R_1 and R_L only. Given these two rents, R_H can then be determined from equation (14). We have the following result.

Proposition 1 *If* $W(0)$ *is sufficiently small,* $W(1)$ *sufficiently large, and*

$$\left(\mu e^{aW(1-S)} + 1\right) e^{aW(1-S)} < \exp\left\{a\, W\left(1 - S + \nu + \nu W^{-1}(W(1-S)/\phi)\right)\right\}, \tag{16}$$

then there is a unique equilibrium. The equilibrium rental prices (R_1^o, R_H^o, R_L^o) *are the unique solution to the market clearing conditions* (13)–(15) *and satisfy*

$$R_H^o > (1+r)R_1^o > R_L^o > 0$$

as well as

$$e^{a(1+r)R_1^o} < \pi e^{aR_H^o} + (1-\pi)e^{aR_L^o}.$$

In equilibrium, each native household chooses one of the following housing consumption and tenure plans: $(1_B,1,1)$, $(1_R,0,1)$, $(1_R,0,0)$, $(0,0,1)$ *and* $(0,0,0)$.

The proof of this result, which can be found in the appendix, states precisely what it means for the endowment levels $W(0)$ and $W(1)$ to be sufficiently small and large, respectively. Note that condition (16) holds if the difference in wealth between native households $i = 1-S$ and $i = 1-S+\nu+\nu W^{-1}(W(1-S)/\phi)$ is sufficiently large. Given the other primitives of the model, therefore, the conditions in Proposition 1 essentially require that the endowment profile be sufficiently steep, both over the entire range and between the two households mentioned in the previous sentence.

If $1 > i^1 > i^2 > i^3 > i^4 > 0$ and (5) holds as an equality, only two of the three rental prices R_1, R_H and R_L remain to be determined, but we need to solve for an additional variable: the proportion ρ of native households who choose the plan $(1_R, 0, 0)$ over the (equally attractive) alternative $(0, 1, 1)$. The market clearing conditions in this case are

$$1 - i^2 + (i^2 - i^3)\rho = S, \tag{17}$$

$$1 - i^1 + (i^2 - i^3)(1-\rho) + (1 - n^1)\nu = S, \tag{18}$$

$$1 - i^2 + i^3 - i^4 + (i^2 - i^3)(1-\rho) = S. \tag{19}$$

Solving for ρ in (17) and substituting into (18) and (19), we find that

$$i^1 + i^3 + \nu n^1 = 2(1-S) + \nu, \tag{20}$$

$$i^2 + i^4 = 2(1-S). \tag{21}$$

As $e^{a(1+r)R_1} = \pi e^{aR_H} + (1-\pi)e^{aR_L}$, this is effectively a system of two equations in two unknowns. We do not pursue its solution any further here since it does not add to the insights that can be gathered from the previous, simpler case. These insights concern the comparison of a mixed ownership/rental economy with a pure rental one, and with a complete-market economy where certain futures contracts can be traded. It is to the first of these comparisons that we now turn.

4 A Rental Economy

We wish to contrast the results of the previous section with the equilibrium in an economy where households are not given the opportunity of owning their accommodation. So all homes have to be rented from the absentee landlords at the relevant rental price R_1, R_H or R_L.

Maintaining the assumption $\pi < \frac{1}{2}$, we can replicate the arguments of the previous section and verify that native households choose housing consumption plans within the following subset of alternatives: $(1_R, 1, 1)$, $(1_R, 0, 1)$, $(1_R, 0, 0)$, $(0, 1, 1)$, $(0, 0, 1)$ and $(0, 0, 0)$. Again, $(0, 1, 1)$ can arise as a native household's equilibrium choice only if (5) holds as an equality, i.e., if this plan yields precisely the same utility as $(1_R, 0, 0)$. So there are again four critical indices that characterize marginal households. We find the same critical indices i^2, i^3, i^4 as in the previous section. For indifference between $(1_R, 1, 1)$ and $(1_R, 0, 1)$, however, the critical index is now j^1 with

$$\mu e^{W(j^1)} = e^{a(1+r)R_1} \left(e^{aR_H} - 1 \right). \tag{22}$$

Note that $j^1 > i^2$ whenever $R_H > R_L$. As to the ranking of the other indices, our earlier results carry over.

If (5) holds as a strict inequality and $1 > j^1 > i^2 > i^3 > i^4 > 0$, the market clearing conditions for housing in location 1 in period 1, period 2 state H and period 2 state L are

$$\begin{aligned} 1 - i^3 &= S, & (23) \\ 1 - j^1 + (1 - n^1)\nu &= S, & (24) \\ 1 - i^2 + i^3 - i^4 &= S. & (25) \end{aligned}$$

Note that the two equations (23) and (25), which involve R_1 and R_L only, are the same as equations (13) and (15) in the economy with ownership. So we know that they determine unique values of R_1 and R_L, and these are the same as in the analogous configuration of the economy with ownership. Given these two rents, R_H can be determined from equation (24). We have the following result which is again proved in the appendix.

Proposition 2 *Under the conditions of Proposition 1, there is a unique equilibrium in the pure-rental economy. The equilibrium rental prices* (R_1^r, R_H^r, R_L^r) *are the unique solution to the market clearing conditions* (23)–(25) *and satisfy*

$$R_1^r = R_1^o, \quad R_H^r < R_H^o, \quad R_L^r = R_L^o.$$

In equilibrium, each native household chooses one of the following housing consumption and tenure plans: $(1_R, 1, 1)$, $(1_R, 0, 1)$, $(1_R, 0, 0)$, $(0, 0, 1)$ *and* $(0, 0, 0)$.

As Proposition 2 shows, allowing households to own their home affects both price dynamics and the response of the distribution of households across locations. First, allowing households to own their home increases the volatility of housing prices. This is an immediate consequence of the fact that R_1 and R_L are independent of the availability of the ownership option, whereas R_H is strictly larger in the case where households are allowed to own their homes.

Second, allowing households to own their home increases the number of native households that choose to remain in location 1 in state H, and reduces the number of newcomer households moving in. This follows directly from the fact that R_H is higher in the economy with ownership. As a consequence, the newcomers' critical index n^1 is higher in the ownership economy while the native households' critical index i^1 is smaller than j^1.

Third, allowing households to own their home increases the income dispersion in location 1 in state H as long as newcomers are not too rich, as a group, relative to natives. This is the case if ϕ is sufficiently small so that n^1 in the rental equilibrium is greater than j^1. Then, as i^1 is lower than j^1 and n^1 under rental is lower than under ownership, poorer native households stay put in location 1 in state H under ownership, while the income distribution of the newcomers who move in location 1 is truncated at a higher level. Thanks to the capital gains they enjoy on their home, a group of native households remains in location 1 for whom it would have been optimal to leave their home to a newcomer had they not enjoyed these capital gains, as in the rental economy.

We summarize these findings in the following corollary.

Corollary 1 *Under the conditions of Proposition 1, allowing households to own their home increases the volatility of housing prices and reduces the number of newcomer households moving to location 1 when state H occurs. If newcomers are not too wealthy relative to natives, allowing households to own their home also increases the income dispersion in location 1 in state H.*

5 Completing the Markets with Rent Futures

We defined homeownership as a long-term rent contract which enables households to lock in future rents at their expected value. As such, homeownership allows households to insure themselves against future rent fluctuations. The major drawbacks of this insurance are its indivisibility and the fact that it is tied to housing consumption. In this section, we investigate the consequences of eliminating this market incompleteness by allowing households to trade rent futures: in period 1, they can buy or sell any amount of a contract that pays realized second-period rent on a location 1 home, that is, either R_H or R_L. This contract is supplied by the risk neutral landlords. In a competitive market, the first-period price of this contract equals the discounted expected second period rent. A household purchasing rent futures pays $\bar{R}_2$ units of second-period numeraire consumption per contract; a household selling rent futures receives $\bar{R}_2$ units of second-period numeraire consumption per contract.[12]

Such future contracts allow the risk averse households to insure themselves against fluctuations in non-housing consumption. As this insurance is provided at the actuarily fair premium, households optimally choose to insure themselves fully by taking whatever position on the futures market is necessary to eliminate any fluctuations in non-housing consumption. As buying a home in the first period is equivalent to renting the home in that period and buying one futures contract, we can assume without loss of generality that all households rent in the first period. Under the plans $(0,1,1)$ and $(1,1,1)$, the household fully insures itself by purchasing one futures contract. Under the plans $(0,0,1)$ and $(1,0,1)$, the household sells $R_L/(R_H - R_L)$ contracts. Under the plans $(0,1,0)$ and $(1,1,0)$, the household buys $R_H/(R_H - R_L)$ contracts. Under the plans $(0,0,0)$ and $(1,0,0)$, the household is already fully insured when renting and so finds it optimal to stay out of the futures market.[13]

As a result, to consume housing in location 1 in period 2 for sure, the household has to pay the expected rent, thus giving up $\bar{R}_2$ units of second-period numeraire consumption. To consume housing in location 1 in state L only, the household gives up $(1-\pi)R_L$ units of second-period numeraire consumption, and πR_H units to consume housing in location 1 in state H only.

With optimal trades on the futures market, the overall expected utility obtained from a given housing consumption plan is

$$\begin{aligned}
&- e^{-a(W-(1+r)R_1h_1-\bar{R}_2)} + (h_1+1)\mu && \text{for} \quad (h_1,1,1);\\
&- e^{-a(W-(1+r)R_1h_1-(1-\pi)R_L)} + (h_1+1-\pi)\mu && \text{for} \quad (h_1,0,1);\\
&- e^{-a(W-(1+r)R_1h_1-\pi R_H)} + (h_1+\pi)\mu && \text{for} \quad (h_1,1,0);\\
&- e^{-a(W-(1+r)R_1h_1)} + h_1\mu && \text{for} \quad (h_1,0,0).
\end{aligned}$$

[12] Alternatively, we can think of these securities as forward contracts with forward price $\bar{R}_2$.

[13] Note that these demands for futures would be different if the households were subject to earnings uncertainty.

Maintaining the assumption $\pi < \frac{1}{2}$, we can again rank housing consumption plans in decreasing order according to the expected amount of housing consumption they promise to deliver: $(1,1,1)$, $(1,0,1)$, $(1,1,0)$ $(1,0,0)$, $(0,1,1)$, $(0,0,1)$, $(0,1,0)$ and $(0,0,0)$. The ordering is strict except for the two plans $(1,0,0)$ and $(0,1,1)$ which deliver the same amount of housing consumption. It is obvious that the preference ranking of the plans $(1,0,0)$ and $(0,1,1)$ does not depend on the household's endowment. In fact, the household weakly prefers $(1,0,0)$ to $(0,1,1)$ if and only if

$$(1+r)R_1 \leq \bar{R}_2, \tag{26}$$

with a strict preference if the inequality is strict. Otherwise, we have again a set of critical endowment levels at which a household would be indifferent between two given plans. These endowment levels are implicitly defined in Table 2.

Plans compared		First plan preferred if μe^{aW} exceeds
$(h_1,1,1)$	$(h_1,0,1)$	$e^{a(1+r)R_1h_1}\ e^{a(1-\pi)R_L}\ \frac{1}{\pi}\left(e^{a\pi R_H}-1\right)$
	$(h_1,1,0)$	$e^{a(1+r)R_1h_1}\ e^{a\pi R_H}\ \frac{1}{1-\pi}\left(e^{a(1-\pi)R_L}-1\right)$
$(h_1,0,1)$	$(h_1,1,0)$	$e^{a(1+r)R_1h_1}\ \frac{1}{1-2\pi}\left(e^{a(1-\pi)R_L}-e^{a\pi R_H}\right)$
	$(h_1,0,0)$	$e^{a(1+r)R_1h_1}\ \frac{1}{1-\pi}\left(e^{a(1-\pi)R_L}-1\right)$
$(h_1,1,0)$	$(h_1,0,0)$	$e^{a(1+r)R_1h_1}\ \frac{1}{\pi}\left(e^{a\pi R_H}-1\right)$
$(1,0,0)$	$(0,0,1)$	$\frac{1}{\pi}\left(e^{a(1+r)R_1}-e^{a(1-\pi)R_L}\right)$
	$(0,1,0)$	$\frac{1}{1-\pi}\left(e^{a(1+r)R_1}-e^{a\pi R_H}\right)$

Table 2: Pairwise housing plan comparisons in the complete-market economy

Comparing the critical endowment for $(h_1,0,1)$ versus $(h_1,1,0)$ with that for $(h_1,1,0)$ versus $(h_1,0,0)$, we see that the former is below the latter if and only if

$$\frac{e^{a(1-\pi)R_L}-1}{e^{a\pi R_H}-1} \leq \frac{1-\pi}{\pi}. \tag{27}$$

Under this condition, therefore, the plans $(1,1,0)$ and $(0,1,0)$ are dominated for all households.[14]

[14]Note that as π approaches $\frac{1}{2}$, this condition reduces to $R_L \leq R_H$, which will always be the case given the influx of newcomers in state H. For smaller π, however, condition (27) can well be violated in equilibrium, as we shall see below.

On the other hand, if we suppose that in equilibrium some households choose $(1,1,1)$ and some $(0,0,0)$, then market clearing requires that some households choose the plans $(1,0,1)$ and some $(0,0,1)$.[15] If we further suppose that both (26) and (27) hold as strict inequalities, the same five housing consumption plans arise here as in the previous two setups, i.e., $(1,1,1)$, $(1,0,1)$, $(1,0,0)$, $(0,0,1)$ and $(0,0,0)$. In this case, it is easy to compare the complete-market equilibrium with the equilibrium of the ownership economy constructed in Section 3. We write R_s^c $(s = 1, H, L)$ for the equilibrium rents in the complete-market economy. First, we observe that for $R_1^c = R_1^o$ and

$$R_L^c = \frac{1}{a(1-\pi)} \ln \left[(1-\pi) e^{aR_L^o} + \pi \right], \tag{28}$$

the three cutoff indices i_2, i_3, and i_4 in the complete-market economy coincide with their counterparts in the ownership equilibrium, which means that these rental prices R_1^c and R_c^L clear the housing markets in period 1 and period 2 state L. As $R_L^o = 0$ implies $R_L^c = 0$ and the derivative of R_L^c with respect to R_L^o exceeds 1 for $R_L^o > 0$, we see that $R_L^c > R_L^o$.

Next, we consider the housing market in period 2 state H and the newcomers' location decision. If expected second-period rents satisfied $\bar{R}_2^c = \bar{R}_2^o$, then i_1 in the complete-market economy would be the same as its analogue in the ownership equilibrium. However, given that the rent in state L is higher with complete markets, the rent in state H would have to be lower. A lower rent in turn would imply a greater number of newcomers choosing location 1, hence excess demand for housing in location 1. So we must have $\bar{R}_2^c > \bar{R}_2^o$. On the other hand, if we had $R_H^c = R_H^o$, then $\bar{R}_2^c > \bar{R}_2^o$ since $R_L^c > R_L^o$. Then i^1 would be higher in the complete-market economy than in the ownership equilibrium. However, we would have the same number of newcomers choosing location 1, hence excess supply of housing in location 1 in state H. Therefore, the rent R_H^c is somewhere in between the value that would keep the expected second-period rent at the same level as in the ownership economy, and the equilibrium rent in the ownership economy. We have the following result.

Proposition 3 *Under conditions strictly stronger than those of Proposition 1, the rental prices in the unique equilibrium of the complete-market economy satisfy*

$$R_1^c = R_1^o, \quad R_H^c < R_H^o, \quad R_L^c > R_L^o,$$

[15] Note the similarity of the expression for the relevant cutoffs around $(1,0,1)$ and around $(0,0,1)$. If one of the two plans is chosen in equilibrium, so is the other one.

and each native household chooses one of the following housing consumption plans: $(1,1,1)$*,* $(1,0,1)$*,* $(1,0,0)$*,* $(0,0,1)$ *and* $(0,0,0)$*.*

This yields the following corollary.

Corollary 2 *Under the conditions of Proposition 3, the introduction of rent futures decreases the volatility of housing prices and increases the number of newcomer households moving to location 1 when state* H *occurs. If newcomers are not too wealthy relative to natives, introducing rent futures decreases the income dispersion in location 1 in state* H*.*

However, the conditions specified for the equilibrium under ownership to display the configuration we have been working with does not guarantee the same equilibrium configuration when futures are introduced. Actually, it is easy to show by example that when households have access to futures contracts, some may choose to consume $(1,1,0)$ or $(0,1,0)$. In other words, (27) can fail when the conditons of Proposition 1 hold.

What does this mean for the equilibrium outcome? To understand the consequences of this change in the structure of the housing allocation, let us start with the same reasoning as above and consider the prices we would obtain assuming no change in the structure of the allocation. We discussed how this leads to less volatile housing markets, less sluggish labor and (if newcomers are not too rich) less income dispersion. But we are now saying that a second effect kicks in: some agents previously consuming $(h_1,0,1)$ and some agents previously consuming $(h_1,0,0)$ now choose to consume $(h_1,1,0)$, where $h_1 \in \{1,0\}$. This implies a decrease in demand for location 1 homes in state L and an increase in demand for location 1 homes in state H. Hence prices need to adjust; in particular, R_H must increase. Numerical simulations show that this second effect on R_H can outweigh the first effect described above and so reverse the prediction as to the consequences of the introduction of securities. Thus we have the following result.

A high R_H leading to fewer newcomers moving into location 1 homes in state H leads to an allocation with a greater response of housing prices to a positive shock, more sluggish labor and greater income dispersion.

Proposition 4 *Under the conditions of Proposition 1, the introduction of rent futures can increase the volatility of housing prices and decrease the number of newcomer* *households moving to location 1 when state H occurs. If newcomers are not too wealthy relative to natives, introducing rent futures can increase the income dispersion in location 1 in state H.*

The proof of this result is simply by example.

6 Concluding Remarks

Ongoing work considers the welfare implications of the introduction of securities on various groups of households.

Numerical experiments confirm the robustness of our analytical findings when we relax the parameter assumptions required for the derivations above. Adding income shocks does not change the nature of the results. This is can be confirmed analytically at the expense of clarity of exposition. Note that the households who do not expect their income to follow housing costs yet want to remain in the desirable location have the strongest incentive to own. Under homeownership, more of them choose to stay put, so the insights gathered above remain.

The model we propose here is a natural candidate within which one could study the effect of homeownership incentives and other policy measures that distort housing consumption choices. In future research we intend to explore not only the consequences of such measures for equilibrium prices and allocations but also the distribution of welfare gains and losses.

References

Bénabou, Roland (1993): "Workings of a City: Location, Education, and Production," *Quarterly Journal of Economics*, 108:612–52.

Chiuri, Maria C., and Tullio Jappelli (forthcoming): "Financial Market Imperfections and Home Ownership: A Comparative Study," *European Econonmic Review.*

Davidoff, Thomas (2001): "Labor Income, Housing Prices and Homeownership," mimeo, Dept. of Economics, MIT.

Epple, Dennis, Radu Filimon, and Thomas Romer (1984): "Equilibrium among Local Jurisdictions: Toward an Integrated Approach of Voting and Residential Choice," *Journal of Public Economics*, 281–308.

Epple, Dennis, Radu Filimon, and Thomas Romer (1993): "Existence of Voting and Housing Equilibrium in a System of Communities with Property Taxes," *Regional Science and Urban Economics*, 23:585-610.

Epple, Dennis, and Glenn J. Platt (1998): "Equilibrium and Local Redistribution in an Urban Economy when Households Differ in Both Preferences and Incomes," *Journal of Urban Economics*, 43, 23-51.

Epple, Dennis, and Holger Sieg (1999): "Estimating Equilibrium Models of Local Jurisdictions," *Journal of Political Economy*, 107, 645-681.

Green, Richard K., and Patric H. Hendershott (2001a): "Homeownership and Unemployment in the U.S.," *Urban Studies*, 38, 1509–1520.

Green, Richard K., and Patric H. Hendershott (2001b): "Home Ownership and the Duration of Unemployment: A Test of the Oswald Hypothesis," mimeo, Dept. of Real Estate and Urban Land Economics, The School of Business, University of Wisconsin – Madison.

Hilber, Christian A.L. (2002): "Neighborhood Externality Risk and the Homeownership Status of Properties," mimeo, The Wharton School, University of Pennsylvania.

Ioannides, Yannis M. (2001): "Neighborhood Income Distributions," Working Paper 2001-03, Dept. of Economics, Tufts University.

Ioannides, Yannis M., and Tracey N. Seslen (2001): "Neighborhood Wealth Distributions," Working Paper 2001-16, Dept. of Economics, Tufts University.

Linneman, P., and Susan M. Wachter (1989): "The Impacts of Borrowing Constraints on Homeownership," *Journal of the American Real Estate Urban Economics Association*, 17:389-402.

Nickell, Stephen, and Richard Layard (1998): "Labour Market Institutions and Economic Performance," Centre for Economic Performance Discussion Paper No. 407, London School of Economics (also in: O. Ashenfelter and D. Card, eds., *Handbook of Labor Economics*, Amsterdam: North Holland).

Ortalo-Magné, François, and Sven Rady (forthcoming): "Tenure Choice and the Riskiness of Non-Housing Consumption," to appear in *Journal of Housing Economics*.

Ortalo-Magné, François, and Sven Rady (1999): "Boom In, Bust Out: Young Households and the Housing Price Cycle," *European Economics Review* (43):755–66.

Oswald, Andrew J. (1996): "A Conjecture on the Explanation for High Unemployment in the Industrialized Nations: Part I," Working Paper No. 475, University of Warwick.

Oswald, Andrew. J. (1997): "Thoughts on NAIRU," *Journal of Economic Perspectives*, 11: 227-228.

Sinai, Todd and Nicholas Souleles (2001): "Owner-Occupied Housing as a Hedge Against Rent Risk," mimeo, The Wharton School, University of Pennsylvania.

Appendix

To ease notation, define the variables

$$x_1 = e^{a(1+r)R_1} - 1, \quad x_H = e^{aR_H} - 1, \quad x_L = e^{aR_L} - 1, \tag{A.1}$$

$$\bar{x} = e^{a\bar{R}_2} - 1 = (x_H + 1)^{\pi}(x_L + 1)^{1-\pi} - 1 \tag{A.2}$$

and the constants

$$y = \mu e^{aW(1-S)}, \quad v = \mu \exp\big\{ a\, W\big(1 - S + \nu + \nu W^{-1}(W(1-S)/\phi)\big)\big\}\,. \tag{A.3}$$

Proof of Proposition 1: The market clearing condition (13) implies $i^3 = 1 - S$. By the definition of the critical index i^3, this yields

$$x_1 = \pi y + (1-\pi)x_L, \tag{A.4}$$

which implicitly expresses R_1 as an increasing function of R_L. The market clearing conditions (13) and (15) imply $i^2 + i^4 = 2(1-S)$. Together with (A.4), the definitions of i^2 and i^4 thus yield the following equation for x_L (and hence R_L):

$$\begin{aligned} 2(1-S) \quad &= \quad W^{-1}\Big(\left[\ln(\pi y + (1-\pi)x_L + 1) + \ln x_L - \ln \mu\right]/a\Big) \\ &\quad + W^{-1}\Big(\left[\ln x_L - \ln \mu\right]/a\Big)\,. \end{aligned} \tag{A.5}$$

Monotonicity and continuity of the endowment function W imply the same properties for its inverse, so there is at most one solution x_L^o to (A.5). That a strictly positive solution exists follows from the fact that for $x_L \to 0$, the right-hand side of (A.5) tends to 0, while it tends to 2 for $x_L \to \infty$. In fact, the right-hand side of (A.5) already exceeds $2(1-S)$ at $x_L = y$, so we can conclude that the solution satisfies $x_L^o < y$. This in turn implies $x_L^o < x_1^o < y$ by (A.4). To get a lower bound on x_L^o, define z to be the unique positive solution of the quadratic equation $y = [\pi y + (1-\pi)z + 1]\, z$. It is trivial to check that $z < y$. Evaluated at $x_L = z$, the first term on the right-hand side of (A.5) equals $1 - S$ whereas the second term is strictly smaller than $1 - S$. So $x_L^o > z$ and $x_1^o > \pi y + (1-\pi)z$.

We have thus determined unique rents R_1^o and R_L^o. As $x_1^o > x_L^o$, we have $(1+r)R_1^o > R_L^o$ and hence $i^3 > i^4$ as assumed at the outset. Moreover, $x_L^o > z$ implies $(x_1^o + 1)x_L^o = [\pi y + (1-\pi)x_L^o + 1]\, x_L^o > y$ and hence $i^2 > 1 - S = i^3$ by (7) and (A.3), again in accordance with the assumed configuration.

Next, the market clearing condition (14) and the definition of the critical indices i^1 and n^1 yield

$$\begin{aligned} 1 + \nu - S \quad &= \quad W^{-1}\Big(\left[\ln(x_1 + 1) + \ln(\bar{x} + (1-\pi)(\bar{x} - x_L)/\pi) - \ln \mu\right]/a\Big) \\ &\quad + \nu\, W^{-1}\Big(\left[\ln x_H - \ln \mu\right]/(a\phi)\Big) \end{aligned} \tag{A.6}$$

where $\bar{x}$ is monotonically increasing in x_H. For fixed x_1 and x_L, continuity and monotonicity of the inverse function W^{-1} imply that there is at most one solution x_H of (A.6). To show existence of such a solution (and its positivity), it suffices to note that the right-hand side of (A.6) tends to 0 and $1+\nu$ as x_H tends to 0 and ∞, respectively. We thus have unique values R_1^o, R_H^o and R_L^o for the three rents.

Next, we need to show that $R_H^o > R_L^o$ (so $i^1 > i^2$) and that condition (5) holds as a strict inequality. The latter requirement translates into $x_1^o < \pi x_H^o + (1-\pi)x_L^o$, which by (A.4) amounts to the inequality $x_H^o > y$. As $x_L^o < y$, therefore, we only need to show that the left-hand side of (A.6) is larger than the right-hand side evaluated at $x_H = y$. After some simple manipulations, this turns out to be equivalent to the inequality

$$(x_1^o + 1)\left(\frac{1}{\pi}\left[(y+1)^{\pi}(x_L^o + 1)^{1-\pi} - 1\right] - \frac{1-\pi}{\pi}\, x_L^o\right) < v. \tag{A.7}$$

As $x_1^o < y$ and the second factor on the left-hand side of (A.7) is strictly increasing in x_L for $x_L < y$, a sufficient condition for $x_H^o > y$ in terms of the primitives of the model is the inequality $(y+1)y < v$. This is precisely condition (16).

To complete the proof, we spell out conditions on the primitives of the model that guarantee $i^1 < 1$, $i^4 > 0$ and $0 < n^1 < 1$. First, $i^1 < 1$ if and only if

$$\mu e^{aW(1)} > (x_1^o + 1)\left(\frac{1}{\pi}\left[(y+1)^{\pi}(x_L^o+1)^{1-\pi} - 1\right] - \frac{1-\pi}{\pi}\,x_L^o\right). \tag{A.8}$$

In view of (A.7), a sufficient condition for $i^1 < 1$ is therefore $\mu e^{aW(1)} > v$. Second, $i^4 > 0$ if and only if $\mu e^{aW(0)} < x_L^o$. As $x_L^o > z$, a sufficient condition for $i^4 > 0$ is therefore $\mu e^{aW(0)} < z$. Since $z < x_H^o$, this condition also implies $\mu e^{a\phi W(0)} < x_H^o$ and hence $n^1 > 0$. Finally, $n^1 < 1$ if and only if $\mu e^{a\phi W(1)} > x_H^o$. We can use (A.6) to derive an upper bound on x_H^o in terms of the primitives of the model, and then use this bound to formulate a sufficient condition for $n^1 < 1$. As this is straightforward, we omit the details. ■

Proof of Proposition 2: We have already seen that the market clearing conditions (23) and (25) imply $R_1^r = R_1^0$ and $R_L^r = R_L^o$. To ease the notational burden, we write x_1 for $x_1^r = x_1^o$ and similarly, x_L for $x_L^r = x_L^o$.

As to R_H, the market clearing condition (24) and the definition of the critical indices j^1 and n^1 yield

$$\begin{aligned} 1 + \nu - S \;&=\; W^{-1}\Big(\left[\ln(x_1+1) + \ln x_H - \ln\mu\right]/a\Big) \\ &\quad + \nu\, W^{-1}\Big(\left[\ln x_H - \ln\mu\right]/(a\phi)\Big), \end{aligned} \tag{A.9}$$

which clearly has a unique solution $x_H^r > 0$. To compare x_H^r with x_H^o, we note that strict convexity of the function e^{ax} implies $\pi e^{aR_H} + (1-\pi)e^{aR_L} > e^{a\bar{R}_2}$ unless $R_H = R_L$, so $\pi x_H + (1-\pi)x_L > \bar{x}$ and $x_H > \bar{x} + (1-\pi)(\bar{x} - x_L)/\pi$ unless $x_H = x_L$. For each $x_H \neq x_L$, therefore, the right-hand side of (A.9) is strictly larger than the right-hand side of (A.6), which implies that either $x_H^r < x_H^o$ or $x_H^r = x_H^o = x_L$. As $x_L < y$, the latter case can be ruled out if $x_H^r > y$, which is equivalent to $(x_1+1)y < v$ by (A.9). As $x_1 < y$, the inequality $(y+1)y < v$, that is, (16), is a sufficient condition for $x_H^r > y$ as well.

The rest of the proof proceeds exactly as the proof of Proposition 1. ■

Proof of Proposition 3: We need to formulate conditions under which the configuration considered in the run-up to Proposition 3 is indeed an equilibrium, i.e., conditions ensuring that $i^1 > i^2$ and that (26) and (27) hold as strict inequalities.

First, market clearing in period 1 implies $x_1^c = \pi y + (x_L^c + 1)^{1-\pi} - 1$. Now, if $x_H^c > y$, then

$$\bar{x}^c > (y+1)^{\pi}(x_L^c+1)^{1-\pi} - 1 > \pi y + (1-\pi)x_L^c > \pi y + (x_L^c+1)^{1-\pi} - 1 = x_1^c,$$

so (26) holds as a strict inequality. In turn, market clearing in period 2 state H implies that $x_H^c > y$ if and only if

$$(x_1^c + 1)\frac{1}{\pi}\left[(y+1)^{\pi} - 1\right](x_L^c+1)^{1-\pi} < v. \tag{A.10}$$

As $x_1^c < y$ and $(x_L^c+1)^{1-\pi} = (1-\pi)x_L^o + 1 < (1-\pi)y + 1$, sufficient conditions for (A.10) are

$$(y+1)\frac{1}{\pi}\left[(y+1)^{\pi} - 1\right]\left[(1-\pi)y + 1\right] \quad < \quad v \tag{A.11}$$

and

$$(y+1)y\left[(1-\pi)y + 1\right] \quad < \quad v, \tag{A.12}$$

with the latter obviously implying (16). Finally, $i^1 > i^2$ if and only if

$$\bar{x}^c > \frac{1}{1-\pi}\left[(x_L^c+1)^{1-\pi} - 1\right] = x_L^o. \tag{A.13}$$

As $\bar{x}^c > \bar{x}^o$ and $\bar{x}^o > x_L^o$ under (16), condition (A.12) is therefore sufficient for $i^1 > i^2$ as well.

Since we have already formulated conditions that are stronger than those of Proposition 1, we omit the additional constraint that would ensure (27). ■