24/7 Competitive Innovation

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May 2002

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ABSTRACT

Intellectual property (IP) rights differ from ordinary property rights. Historically, societies have tolerated monopolistic inefficiency from IP protection to incentivize intellectual asset creation. This paper considers how competitive markets can optimally allocate resources, bypassing that monopolistic inefficiency. It departs from earlier related work in three ways: First, it allows economic actions undertaken progressively rapidly as technology advances. Second, it weakens property rights yet further, allowing both consumers and asset holders to make and sell copies. Third, it distinguishes nonrivalry from infinite reproduction. The first departure restores the traditional view that competitive markets fail. The second and third, surprisingly, have competitive markets achieve social efficiency.

Keywords: bootlegging, continuous time, finitely expansibility, infinitely expansible, intellectual asset, intellectual property, Internet, IPR, knowledge product, MP3, Open Source Software, rival

JEL Classification: D90, O30, O41

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1 Introduction

Since at least Arrow (1962) knowledge has been recognized to have peculiar properties in economic analysis. Knowledge and thus innovations and ideas are, in a useful idealization, nonrival and aspatial.

A good is *nonrival* when its use by one agent does not degrade its usefulness to a yet different agent. Thus, ideas, mathematical theorems, videogames, engineering blueprints, computer software, cookery recipes, the decimal expansion of π, gene sequences, and so on are nonrival. By contrast, food is distinctly rival: consumption renders it immediately no longer existent.

A good is *aspatial* when its extent is not localized to a physical spatial neighborhood. Thus, all the examples of nonrivalry mentioned previously, including perhaps more vividly rich media filestreams—sounds and images—on an Internet server, are all also aspatial.[1]

For compactness I will refer to all such products as *intellectual assets* or *knowledge products*—even if, for instance, a Spice Girls MP3 file might typically be viewed as neither knowledge nor intellectual. Nonrival, aspatial knowledge products are important for at

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[1] It is difficult to think of interesting nonrival economic goods that are not at the same time aspatial. Emphasizing both features, however, serves to remind why "increasing returns" is not necessarily the most useful way to model nonrivalry. Quah (2002b) describes how these properties describe a subclass of cultural goods as well.

least three reasons: First, in the endogenous growth formulations of Aghion and Howitt (1998), Grossman and Helpman (1991), Helpman (1993), Romer (1990), and others, knowledge advance is the driver of economic growth, and intellectual property rights (IPRs) protect the incentives for continued innovation.

Second, as more and more everyday economic activity becomes the creating and disseminating of knowledge products, the associated incentive mechanisms become correspondingly more important. These can no longer be relegated to historically- and haphazardly-determined patent and copyright law.[2]

Third, the aspatial nature of knowledge products creates powerful forces that will redraw the economic landscape across realworld geographies. Economic analyses and policy formulations that rely on the sanctity of national boundaries or on transportation costs across physical distance will, a fortiori, need to be re-examined. (See, e.g., Quah (2000, 2001a).)

Issues raised by these developments are large and complex, and are not usefully treated in a single article. I mention them here because they outline the economic importance of mechanisms for creating and disseminating knowledge and other knowledge-like nonrival goods.

Economists have long recognized that efficient allocations for in-

[2] Take, as just one example, Microsoft Corp.: If this company and its actions are as central to the modern economy's operation as both plaintiff and defendant in high-profile antitrust suits through the 1990s have made them out to be, then certainly the economics surrounding Linux and the Open Source Software movement matters importantly. And that economics is, in essence, that of the creation and dissemination of knowledge products. So too, over the same period, is the concern over ownership of knowledge on the human genome, provision of pharmaceuticals cheaply to developing economies, and proliferation of music on the Internet, among others. Internet development then, at its rapid rate of technical progress, amplifies the importance of appropriate institutions for managing IPRs. See, among others, Boldrin and Levine (2002a), Romer (2002), and section 2.3 below.

tellectual assets fail to obtain in competitive markets to decentralize trading after the usual assignment of property rights. When an asset has zero marginal costs of reproduction, the stream of rents it commands under perfect competition is also zero. But if so then a costly first instantiation will never occur, even if social efficiency dictates it should.

Recently, Boldrin and Levine (2002b) have challenged the view that competitive markets fail for nonrival intellectual assets. They allow perfectly competitive markets in the asset and in the consumption flows derived from the asset. In their analysis, for intellectual assets, as for all ordinary economic commodities, competitive markets produce a socially efficient outcome. (To be clear, it is only nonrivalry that is of concern below and in Boldrin and Levine (2002b), not aspatiality.)

This paper uses the Boldrin-Levine framework to address three issues. First, what if market participants can act 24/7, i.e., more and more frequently as technology advances? Second, what if IPRs are relaxed even further, so that consumers who use an intellectual asset—through buying only its consumption service flows—can legally make copies and compete with the original owner, without having to purchase the asset itself? Third, the paper distinguishes nonrivalry—in the sense described above—from infinite reproduction; what if goods are nonrival but allow only finite reproduction in finite time? Allowing 24/7 actions restores the conventional wisdom, that markets fail. However, keeping to the discrete-time convention, relaxing IPRs as described maintains the viability of competitive markets in achieving social efficiency. Nonrivalry with finite reproduction also has competitive markets attain social efficiency. In a special case (log utility) even the 24/7 version of the weakened-IPRs, nonrival economy continues to have markets retain optimal allocation properties.

Comparing the workings of competitive markets with IPR institutions, the latter create monopolies that reduce social efficiency. The effect is two-fold. First, IPR institutions that create monopolies restrict consumption and dissemination, away from socially optimal levels. Second, they increase the rents accorded an intellectual asset and therefore incentivize too much innovation.

The organizing principle used throughout the paper is dynamic equilibrium pricing of intellectual assets. In this, the current paper follows Boldrin and Levine (2002b) and differs from the approach used in evaluating innovation taken by, among others, Jones and Williams (2000).

The paper is organized as follows. Section 2 briefly describes the traditional economics of intellectual assets in particular and nonrival goods more generally. Subsection 2.3 segues this discussion into what is done in the current paper, to emphasize the progression in this line of ideas.

Section 3 examines the Boldrin-Levine analysis, expositing and confirming their findings. In Section 4 I allow demanders and suppliers to take actions progressively frequently as technology improves. This turns out to restore the conventional wisdom on the need for institutions like IPR protection. This does not say that IPR regimes are socially optimal, only that the search continues for reasonable institutions that support efficiency in knowledge-intensive economies.

Throughout sections 3–4 I follow Boldrin and Levine (2002b) in distinguishing an intellectual asset from the use or consumption flows that that asset generates. Thus, for instance, an MP3 music track is distinct from its consumption. The assumption in the model thus far is that the former can generate further copies; the latter, cannot. In reality, however, this distinction is blurred for many knowledge products. Indeed, because it is impossible to use many intellectual assets without making a copy—if only temporarily in computer memory—several legal cases have argued that existing copyright law is overly restrictive, varying with "fair use" interpretations. Moreover, nonrivalry as described above differs from infinite reproduction.

Section 5 takes up both these observations and considers markets where the good is nonrival but only gradually reproduced and where ordinary users can legally make copies for resale. The section shows that, under the same timing conventions as used in section 3, perfectly competitive markets achieve social efficiency. The section establishes this optimality result also in the Boldrin-Levine specification, where nonrivalry and infinite expansibility are not distinguished, when users, in addition to asset holders, are allowed to make and sell copies.

Thus, section 5 confirms the optimality of competitive markets, going even further than Boldrin and Levine (2002b) in dispensing with IPR protection. Section 6 shows, however, that the continuous-time version of the economy in section 5 suffers from the same problem as in section 4—markets generally fail if agents can act sufficiently frequently. A special case—unavailable in sections 3–4—forms a counter-example to the general proposition, although it seems more a curiosity than substantive. Section 7 briefly concludes.

2 Issues

Since at least Arrow (1962), Machlup (1962), and Nordhaus (1969) the creation and allocation of intellectual assets—nonrival goods—have been relegated to institutions other than competitive markets. The view was that because of zero marginal costs of reproduction, the present value of an intellectual asset would, under perfect competition, turn out to be zero. But then the costly first instantation would not be undertaken, and what would be a socially improving innovation would fail to occur.

2.1 The traditional view — Institutions

One potential solution to this market failure is the set of institutions that protect IPRs. Patent and copyright laws are examples. Yet other institutions have been observed in economic history and have efficiency properties that have been studied by economists (e.g., Dasgupta, 1988; David, 1992, 1993; Scotchmer, 1991; Wright, 1983). David (1992) divides all such schemes into what he calls 3P's: property, patronage, and procurement. Under property, institutions sacrifice ex post social efficiency by sanctioning a legal monopolist who controls the ownership and use of the intellectual asset in exchange for society's providing enough incentives for costly first instantiation

of that intellectual asset.[3] The alternatives of patronage and procurement involve yet other inefficiencies, so that comparing the different schemes leads often to ambiguous conclusions—see, e.g., Klemperer (1990), Scotchmer (1991), and Wright (1983).

Managing intellectual assets—their creation and dissemination—matters importantly not just in individual-level efficiency but also in macroeconomic issues surrounding economic growth. The creation of scientific and engineering knowledge—nonrival and only partially excludable—is central in endogenous growth, a point made explicit in Romer (1990). (See also Keely (2001) and O'Donoghue and Zweimüller (1998).)

2.2 Competitive markets and finite expansibility

The Boldrin-Levine analysis is an important and profound development. It seeks to overturn nearly half a century of formal economic thinking on intellectual property, suggesting instead that perfectly competitive markets in intellectual assets function in the usual Arrow-Debreu way and therefore lead to socially efficient outcomes.

Following Boldrin and Levine (2002b) this paper will consider not just nonrivalry in the extreme but instead varying degrees of nonrivalry. For this, it helps to have a positive terminology for nonrivalry. Taking a lead from David (1992) I will refer to nonrivalry as *infinite expansibility* for some of the discussion to follow. (See also Thomas Jefferson's 13 August 1813 letter to I. McPherson in Koch and Peden,

[3] There is a further important economic distinction between patents and copyright but that will not be central in this paper. Copyright protects only the implementation of an idea, not the idea itself. The latter is deemed part of nature and so cannot have been the result of creative activity and thereby awarded monopolistic protection. Patents, on the other hand, provide a stronger form of intellectual property protection. See, among others, the discussion in David (1993), Quah (2001b), and Waterson and Ireland (1998). In practice, copyright protection has strengthened to where in many instances it is indistinguishable from patent protection.

eds (1944).) Goods that are near nonrival are goods with high but finite expansibility.

Boldrin and Levine (2002b) take nonrivalry as the limit of finitely expansibility. In their model, nonrivalry means the ability to make an infinite number of copies instantly. Finite expansibility, conversely, means a bounded rate of reproduction, although the possibility remains that an infinite number of copies can be made, eventually. Boldrin and Levine (2002b) allow untrammeled competition in the sale and resale of near-nonrival assets. Thus they imagine a world where property rights over intellectual assets are exactly the same as property rights over any other kind of asset. This contrasts with current patent and copyright law where both ownership and use of intellectual assets are severely controlled.[4]

Boldrin and Levine (2002b) show that perfectly competitive markets achieve social efficiency and incentivize positive innovation, without the need for additional IPR protection beyond that implicit in Arrow-Debreu general competitive analysis. No one disputes that conclusion in economies with goods that have high but finite expansibility, i.e., are only nearly nonrival. Remarkable in the Boldrin-Levine analysis is that the same conclusion holds even in the limit as expansibility increases without bound. In their view, therefore, economies where nonrival goods are important can be usefully approximated by economies requiring only perfectly competitive markets, not IPR-type institutions like patent and copyright laws. Such second-best institutions are *not* required to achieve socially optimal outcomes. Conventional markets do the job perfectly well.

[4] To be clear, Boldrin and Levine (2002b) do not permit *illegal* reproduction of intellectual assets for sale. Illegal copying is distinct from legally purchasing a copy of an intellectual asset and then setting up a reproduction service that competes with the original owner—the latter is allowed in their discussion, the former is not. However, with appropriate adjustment of prices this distinction turns out to be inessential—see section 5.2 below.

2.3 24/7 and Open Source Software

This paper takes up below two alternative formulations of the framework. In Boldrin and Levine (2002b), the decision interval is fixed; the model operates in discrete time. So, first, I allow consumers and firms to operate instead at time intervals that grow progressively finer, i.e., in continuous time in the limit. Popular descriptions of "working 24/7 in Internet time" reflect, after all, that emerging possibility. I find this overturns the Boldrin-Levine conclusion, restoring the conventional wisdom that nonrival goods require something other than just competitive markets.

Second, notice that nonrivalry means not just that an infinite number of copies can be made. In the definition from the Introduction, nonrivalry means simply that one use of the knowledge product does not detract from yet another use. This says nothing about the magnitude of copying rates. I consider therefore an economy where copying rates are bounded but nonrivalry holds in this alternative meaning. In this new model, property rights are even weaker than those in Boldrin and Levine (2002b) as the model no longer distinguishes purchases of the services of the asset from purchases of the asset itself. Those who have bought only the consumption flow from the asset can capture it into the stock of the asset itself, and thereby make yet further copies. Surprisingly, competitive markets achieve social efficiency here too, under the same timing conventions as in Boldrin and Levine (2002b).

Third, I return to the structure in Boldrin and Levine (2002b), now allowing consumers as well as asset-holders to copy and sell the intellectual asset. What happens then is that the equilibrium prices in Boldrin and Levine (2002b) need to be adjusted, but competitive markets again achieve social efficiency.[5]

One interpretation of this paper and Boldrin and Levine (2002b) is

[5] These descriptions appear in a section called **Issues**, a section that proceeds chronologically, for two reasons. First, they seem to me the natural next questions to ask of intellectual assets in general, in light of the analysis in Boldrin and Levine (2002b). Second, these last variations in the model, in my view, correspond to Open Source Soft-

that each considers different approximations to nonrival goods. The Boldrin-Levine approximation leads to where perfectly competitive markets work even in the limit. One of the approximations in this paper leads to where they don't; yet others, to where they do. Why take one approximation more seriously than another?

Technically the description of the previous paragraph is correct. However, it hides substantive differences in what economic agents in the models do in the different approximations. In my continuous-time scheme agents are free to select when they trade—they could, for example, choose to act only at discrete time intervals if that is what they wished. In a discrete-time model, by contrast, agents are prohibited from taking action within an otherwise-arbitrary time period. In many economic models, this restriction is irrelevant. The contention in this paper is that here, it matters. The incentive to act more and more often increases when the losses from being prevented to do so are rising in technological improvements. Those improvements on the one hand mirror real-world developments in computer, media, and Internet technology; and on the other simply constitute a mathematical device to approximate, better and better, the concept of nonrivalry more generally. This part of the analysis says that institutions like IPRs are needed to overcome the traditional market failure in intellectual assets.

The other approximation—where nonrivalry is unrelated to reproduction rates—confirms the Boldrin-Levine insight. Perfectly competitive markets function optimally, here under even weaker restrictions on trading than in Boldrin and Levine (2002b). The socially efficient outcome has intellectual assets disseminated as widely as possible. In equilibrium the price system supports that outcome; it

ware, a set of ideas and institutions recently enabled by Internet-based worldwide collaboration (e.g., `http://www.opensource.org/`). My emphasis on pricing intellectual assets—i.e., the output or software part of Open Source Software—distinguishes the current work from other Open Source economics research such as Lerner and Tirole (2001) that focus more on the input side in studying individual incentives and corporate strategies.

does not restrict dissemination and consumption below what is technically feasible.

However, even with the Boldrin-Levine specification on technology, competitive markets function optimally without restricting users' rights to make copies and redistribute them further. The kind of market competition envisioned here, without intellectual property protection and with all asset exchange and consumption flows restricted only by the competitive price system, comes close—in my view—to the kind of economic organization envisioned by the Open Source Software community. This part of the analysis below, therefore, supports the proposition that strong intellectual property rights are not just unnecessary, but impede social efficiency.

3 The model

The key assertion in Boldrin and Levine (2002b) is that many putatively nonrival knowledge products are not literally infinitely expansible. Copying rates are finite. Reproduction to an infinite number of copies from the first instance of a knowledge product might well occur, eventually, but cannot happen instantaneously. What happens in the first few periods after instantiation then becomes critical for determining economic outcomes.

This section goes over the Boldrin-Levine analysis. It treats the decentralized markets allocation explicitly, beginning from a two-period case and extending to the Boldrin-Levine model. This serves principally to exposit how the infinite horizon socially efficient outcome, already compactly characterized in Boldrin and Levine (2002b), is achieved by firms and consumers trading in markets.

3.1 Technology

Time extends from 0 to infinity. I use t subscripts to denote integer time when the decision interval is discrete, and (t) parenthesized when time is taken as continuous. First, consider the fixed time-interval model. I will turn to the continuous time model only in Section 4 below.

Let s_t denote the stock of the intellectual asset or knowledge product at the beginning of time t, for $t = 0, 1, 2, \ldots$. Begin the analysis at time 0, and assume that the (costly) act of instantiating the knowledge product occurs just before then.

The copying rate is given, exogenously, by $\gamma > 1$ when s is not otherwise being used. (The mnemonic is γ for either *copying* or *growth*.) Let $c_t \in [0, s_t]$ denote the flow of consumption services; when consumption occurs, the copying rate from the knowledge product deteriorates to $\widehat{\gamma} \in [0, \gamma)$. Call $\widehat{\gamma}$ the rate of *degraded copying*. After time 0 the knowledge product evolves only in volume and through copying,

$$\forall \text{ integer } t \geq 0: \; s_{t+1} = (s_t - c_t) \cdot \gamma + c_t \cdot \widehat{\gamma} = \gamma s_t - (\gamma - \widehat{\gamma}) c_t.$$

To interpret the specification, consider a hypothetical copying technology for videogames. After the first few instances of the game exist, all necessary resources can be devoted to creating further copies of it. Copying proceeds at rate γ. Alternatively, the firm can set aside some instances of the videogame for employee or customer play. But then the bit-manipulating copying machines that are working off those copies slow down to the degraded copying rate $\widehat{\gamma} < \gamma$. This might happen either because the magnetic heads reading the hard disk medium have to allow for conflicting demands, or because the firm's employees are distracted and cannot operate at maximum efficiency.

As one extreme, it might be that no copies can be made at all when s is used to generate consumption services, so that degradation is complete, $\widehat{\gamma} = 0$. This special case be used to develop intuition but what matters for the discussion below is only that $\widehat{\gamma} < \gamma$.

When γ is finite, an infinite number of copies of s cannot be made instantaneously. Then, provided that consumer preferences show nonsatiation and impatience, the market will clear in the first period only at a positive price. The higher is γ, the faster will copies propagate and the faster will equilibrium price, one suspects, converge to zero. At the same time, however, the more will asset holders value s, as

more and more copies can be made, to be sold at putatively positive prices. Equilibrium below will reflect both effects.

It will turn out that for nondegenerate equilibrium the copying rate γ must be sufficiently high. How high exactly depends on the kind of equilibrium to be studied—whether the economy exists only for two periods or infinitely; whether decisions occur only at discrete time intervals or continuously.

3.2 Preferences

The economy has an infinitely-lived representative consumer with preferences

$$W = \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad \beta \in (0, 1), \tag{1}$$

where U is strictly increasing, concave, at least twice continuously differentiable, and has

$$\lim_{c \to 0} U'(c) = \infty. \tag{2}$$

This Inada condition, as well as its failure, will be important for parts of the discussion below.

3.3 Allocation

Given initial stock $s_0 > 0$, an *allocation* is a sequence of quantities in consumption flows and asset stocks:

$$\{ c_t, s_{t+1} : t = 0, 1, \ldots \}.$$

An allocation is *feasible* when it satisfies the technology constraints:

$$\begin{aligned} c_t &\in [0, s_t] \\ s_{t+1} &= \gamma s_t - (\gamma - \widehat{\gamma}) c_t. \end{aligned} \tag{3}$$

Given $s_0 > 0$ denote $\mathsf{C}(s_0)$ to be the collection of all consumption sequences $c = \{ c_t : \; t = 0, 1, 2, \ldots \}$ each part of a feasible allocation.

Iterating on (3) gives

$$s_t = (\gamma - \widehat{\gamma})\gamma^{-1} \sum_{j=0}^{\infty} \gamma^{-j} c_{t+j} + \lim_{j \to \infty} \gamma^{-j} s_{t+j}.$$

Thus, provided that $\gamma^{-j} s_{t+j} \to 0$ as $j \to \infty$, we have

$$\forall c \in \mathsf{C}(s_0): \qquad \sum_{t=0}^{\infty} \gamma^{-t} \frac{dc_t}{ds_0} = (\gamma - \widehat{\gamma})^{-1}\gamma. \tag{4}$$

Definition 3.1 (Social efficiency) *At initial stock $s_0 > 0$, an allocation is socially efficient (SE) when it is feasible and maximizes the representative consumer's preferences W in* (1)*. Define the value function $V : \mathbb{R}_+ \to \mathbb{R}$*

$$V(s_0) \stackrel{\text{def}}{=} \sup_{\mathsf{C}(s_0)} \sum_{t=0}^{\infty} \beta^t U(c_t).$$

∎

Evaluating W in (1) at the consumption component of an SE allocation gives the value function V.

3.4 Competitive Equilibrium

Denote by q_t the share price at the beginning of time period t. Let p_t be the period t price of a unit flow of consumption services.

Consumers own shares in the representative firm. They take prices as given and choose consumption c_t and asset holdings s_{t+1} to maximize preferences (1) subject to the period budget constraint

$$p_t c_t + q_{t+1} s_{t+1} \le q_t s_t. \tag{5}$$

Correspondingly, the representative firm takes prices as given and begins time 0 with stock $s_0 > 0$. At each time period t the firm supplies to the market consumption services c_t and asset holdings s_{t+1} subject to technology (3) so that it maximizes value

$$q_t s_t \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} p_{t+j} c_{t+j}. \tag{6}$$

Since copying technology (3) is linear in s_t, production of (c_t, s_{t+1}) displays constant returns to scale. At each time period the maximum number of copies that can be made is the same whether a single firm owns all of s or that amount of s is divided across multiple firms.

Definition 3.2 (Competitive Equilibrium) *At initial stock $s_0 > 0$, a competitive equilibrium (CE) is a sequence of prices and quantities*

$$\left\{ p_t^*, q_t^*, c_t^*, s_{t+1}^* : \ t = 0, 1, 2, \ldots, \right\}$$

such that when consumers and firms take CE prices as given, then markets clear at CE quantities, with consumers maximizing preferences (1) *subject to constraints* (5) *and firms maximizing value* (6) *subject to constraints* (3). ∎

The definition allows that, at time t, any agent can purchase some quantity of the stock s_t at price q_t, and set up a copying service to compete with the original owner. That agent can, moreover, further sell on that asset to yet other downstream resellers. Hence, as in standard competitive equilibrium, legal rights are assigned only over asset ownership, not over asset use and resale—dissemination practices that current IPRs typically prohibit. Note, however, that an agent's purchasing copies of the asset for potential resale differs from that agent claiming to buy only consumption service flows c_t and then, illegally, making copies off those flows for re-distribution. In the first case the agent pays q per copy of the asset; the second, p per unit of consumption flow. In the legal system envisioned in the model of this section and the next—but not in sections 5 and 6—the asset stock s and its flow of consumption services c are two distinct economic goods. In equilibrium their prices q and p differ. We return to this in section 5 below.

By constant returns to scale in (3), competitive equilibrium will be independent of the ownership of s_t. The central issue then is, What is the value of the firm $q_0 s_0$ in CE? Instantiation of the first copy of s will occur provided that the costs of such creation don't exceed q_0. In this no-IPR world, whether innovation, artistic endeavor, music recording, and so on take place hinges on the competitive equilibrium value of q_0. At time 0 intellectual assets with costs of creation up to q_0 will be instantiated.

3.5 Two-period equilibrium

Suppose that consumers live only for two periods, $t = 0$ and 1. This will turn out to have all the features of the infinite horizon case below, with some interesting but inessential differences. The two-period case is useful to describe explicitly, however, to develop intuition for the more intricate situations later. Also, as noted in footnote 9 below, the two-period outcome—with its finite endpoint—provides an alternative confirmation that the positive asset prices calculated for the infinite-horizon cases are not simply so-called "rational bubbles" or Ponzi-game outcomes.

Assume:

Condition $\mathcal{G}$: *Copying rate γ is bounded from below,*

$$\gamma > \widehat{\gamma} + \beta^{-1} \times \sup_{s>0} \frac{U'(s)}{U'(\widehat{\gamma}s)}.$$

Some special cases are useful to provide intuition for this restriction.[6] It always holds when U satisfies the Inada condition (2) and degradation is complete, $\widehat{\gamma} = 0$, as the denominator on the right side is then infinite. More generally, since U is concave the larger is $\widehat{\gamma}$ the larger will be $\sup_{s>0} U'(s)/U'(\widehat{\gamma}s)$, so that the more restrictive becomes the inequality in condition $\mathcal{G}$, and the larger must be the difference $\gamma - \widehat{\gamma}$. When $\widehat{\gamma} < 1$ it suffices that the difference $\gamma - \widehat{\gamma}$ be at least $\beta^{-1} > 1$. For given $\widehat{\gamma}$, the higher is the subjective discount rate (the smaller is β), the higher must be γ.

[6] Boldrin and Levine (2002b) leave implicit conditions $\mathcal{G}$, $\mathcal{G}'$, and $\mathcal{G}''$ (the latter two to follow below). Sections 5–6 below will do away with these altogether.

If preferences (1) imply constant intertemporal elasticity of substitution or, equivalently, $U(c)$ shows constant relative risk aversion:

$$U(c) = \frac{c^{1-R} - 1}{1 - R}, \qquad R > 0 \tag{7}$$

then the inequality in condition $\mathcal{G}$ becomes

$$\gamma > \widehat{\gamma} + \beta^{-1}\widehat{\gamma}^R.$$

In words, copying must be faster than degraded copying by $\beta^{-1}\widehat{\gamma}^R$, a quantity increasing in both the rates of degraded copying $\widehat{\gamma}$ and subjective discount.

I characterize competitive equilibrium in two steps. First, obtain the SE outcome. Second, find market-clearing prices such that consumers and firms optimally demand and supply those SE quantities.

Proposition 3.3 *Under condition $\mathcal{G}$ the SE allocation (c_0^*, s_1^*, c_1^*) satisfies*

$$U'(c_0^*) = \beta U'(c_1^*) \times (\gamma - \widehat{\gamma})$$
$$c_1^* = s_1^* = \gamma s_0 - (\gamma - \widehat{\gamma})c_0^*.$$

(All proofs appear in section 8 below.) The first of the equations in Prop. 3.3 gives the relation between marginal utilities in consumption across the two time periods. At the SE allocation those marginal utilities have slope in time that depends on $\gamma - \widehat{\gamma}$, the difference between the rates of copying and degraded copying. The higher is this difference, the more consumption is tilted towards the future. This is efficient because undegraded copying is then sufficiently better than degraded copying that, other things equal, welfare is improved by refraining from consumption today. The prediction on U' is not, however, always unambiguous. When γ changes, so does the opportunity set described in (3): general equilibrium wealth effects then affect the outcome as well.

Turn now to competitive equilibrium. The firm's problem is

$$\max_{\{c_0,s_1,c_1,\}} q_0 s_0 = p_0 c_0 + p_1 c_1$$
$$\text{s.t. } c_t \le s_t, \quad t = 0,1$$
$$s_1 = \gamma s_0 - (\gamma - \widehat{\gamma}) c_0,$$

while the consumer's problem is

$$\max_{\{c_0,s_1,c_1\}} U(c_0) + \beta U(c_1)$$
$$\text{s.t. } p_0 c_0 + q_1 s_1 \le q_0 s_0$$
$$p_1 c_1 \le q_1 s_1.$$

Knowing the SE allocation allows calculating market-clearing prices. Boldrin and Levine (2002b) use equivalence between a recursive Bellman equation description, social efficiency, and competitive equilibrium in their elegant and efficient discussion. however, since part of the analysis below develops conditions under which the perfectly competitive markets mechanism fails, I have chosen to undertake all the reasoning here explicitly from first principles.

Proposition 3.4 *Assume condition $\mathcal{G}$. For any positive constant μ, the SE allocation*

$$(c_0^*, c_1^*, s_1^*)$$

gives market-clearing quantities at prices

$$p_0^* = U'(c_0^*) \times \mu$$
$$p_1^* = (\gamma - \widehat{\gamma})^{-1} p_0^*$$
$$q_0^* = (\gamma - \widehat{\gamma})^{-1} \gamma \times p_0^* = (\gamma - \widehat{\gamma})^{-1} \gamma \times U'(c_0^*) \times \mu$$
$$q_1^* = (\gamma - \widehat{\gamma})^{-1} p_0^* = \gamma^{-1} q_0^* = p_1^*.$$

Set $s_2^* = 0$. The resulting $\left\{ (p_t^*, q_t^*, c_t^*, s_{t+1}^*) : \ t = 0,1 \right\}$ is a CE. ∎

The idea in Prop. 3.4 is straightforward and familiar from general equilibrium analysis. Knowing the SE allocation, seek prices such that consumers optimally select that allocation. Since Prop. 3.3 characterizes social efficiency as a relation between marginal utilities, such decentralizing prices are readily available—just take scaled versions of the marginal utilities evaluated at the SE allocation. Moreover, social efficiency respects feasibility. Those marginal utilities then must also align with the boundaries of the technologically feasible set, thereby maximizing firm value. The proof given in the Technical Appendix below simply formalizes this intuition. These same ideas are used again in Props. 3.7 and 4.4 below.

Useful to notice explicitly is a further relation between q and p, which will reappear below.

Proposition 3.5 *In CE we have $q_0^* = s_0^{-1} \times (p_0^* c_0^* + p_1^* c_1^*)$. Setting $\mu = 1$ in Prop. 3.4, we also have $q_0^* = V'(s_0)$.* ∎

In words, the competitive equilibrium price on the asset is the present value of the stream of consumption flows that that asset will generate over its lifetime. Further, that asset price turns out also to equal the social value of a marginal increment in the initial asset s_0.

Prop. 3.4 indicates that in competitive equilibrium, as copying proceeds, the price of consumption flows declines at rate $(\gamma - \widehat{\gamma})$. Provided that early on, capacity remains finite, Prop. 3.5 says that those initial positive prices ensure a positive valuation to q_0 in CE.

In Prop. 3.4 the rate of decline in price p reflects both the copying rate γ, i.e., the rate at which copies are flooding the market, and $\widehat{\gamma}$, the degraded copying rate, at the end of the economy's lifetime, when all s is used to generate consumption flows. With an infinite horizon this second, endpoint effect vanishes and the rate of price decline on c will be simply γ^{-1}. Notice that the rate of decline in p is, in Prop. 3.4, also the rate of decline on the asset price q. Asset and consumption prices in infinite horizon economies, below, will again show equal rates of decline.

Without loss, the proportionality constant μ in Prop. 3.4 can be normalized to 1 so that $p_0^* = U'(c_0^*)$. This normalization reflects only choice of numeraire; all subsequent price accounting is in units of period 0 marginal utility. Then

$$q_0^* = (\gamma - \widehat{\gamma})^{-1}\gamma \times U'(c_0^*) \geq U'(c_0^*) > 0,$$

with the inequality strict whenever $\widehat{\gamma} > 0$. Private agents will therefore undertake, prior to period 0, any innovation or knowledge production costing no more than q_0^*. Provided then that q_0^* remains bounded away from zero, innovation occurs even with only competitive markets and no IPRs.

Is such innovation socially efficient? What "socially efficient innovation" means can be subtle—partly because innovation involves new goods, partly because an earlier partial-equilibrium literature has made up ad hoc welfare functions in defining social efficiency for analyzing these issues. Here, because the analysis is general equilibrium with a representative agent, social efficiency can be transparently and unambiguously defined relative to that agent's utility, and, in turn, the economy's value function. Because this two-period case is intended only as an example to confirm intuition, it will be efficient to postpone complete discussion until after Prop. 3.8 below.

3.6 Discrete-time infinite horizon equilibrium

Equilibrium in an infinite-horizon economy has all the same essential features described for the two-period one.

Replace condition $\mathcal{G}$ with something weaker:

Condition $\mathcal{G}'$: *Copying rate γ is bounded from below,*

$$\gamma > \beta^{-1} \times \sup_{s>0} \frac{U'(s)}{U'(\widehat{\gamma}s)}.$$

∎

The inequality in $\mathcal{G}'$ is satisfied whenever that in $\mathcal{G}$ holds. Recall that $\mathcal{G}$ rules out the boundary outcome when all the period 0 asset is used to generate consumption flows. When consumers live beyond two periods, the marginal value to having more s is correspondingly higher. Therefore, the less likely will consumers wish to have as great a consumption in the initial period, the less likely will c_0 be driven to its upper boundary at s_0.

The analysis parallels that for the two-period model.

Proposition 3.6 *Assume $\mathcal{G}'$ and fix $s_0 > 0$. The SE allocation $\left\{ c_t^*, s_{t+1}^* \right\}_{t=0}^{\infty}$ satisfies*

$$U'(c_t^*) = \beta U'(c_{t+1}^*) \times \gamma, \quad t = 0, 1, 2, \ldots \tag{8}$$

$$s_0 = (\gamma - \widehat{\gamma})\gamma^{-1} \sum_{j=0}^{\infty} \gamma^{-j} c_j^* \tag{9}$$

$$s_t^* = (\gamma - \widehat{\gamma})\gamma^{-1} \sum_{j=0}^{\infty} \gamma^{-j} c_{t+j}^*, \quad t = 1, 2, 3, \ldots. \tag{10}$$

∎

Condition (8) differs from its two-period model counterpart in Prop. 3.3 in its slope coefficient being γ, not $\gamma - \widehat{\gamma}$. This is because the infinite-horizon setting shows no endpoint effects, as already earlier described. In condition (9) the initial stock s_0 is given—the relation confirms feasibility in the sequence $\{ c_t^* \}_{t=0}^{\infty}$, which can be solved for using (8)–(9). Condition (10) gives the s_t^* implied by the sequence $\{ c_t^* \}_{t=0}^{\infty}$. Together, conditions (9) and (10) rule out a Ponzi game with $\left\{ \gamma^{-t} s_t \right\}$ remaining bounded away from zero. The proof to Prop. 3.6 shows a Ponzi game would be socially inefficient.

The equivalence earlier established for the two-period model in Prop. 3.4 is available here as well. Taking prices (p, q) as given, the representative firm solves for each t,

$$\begin{aligned} \sup_{\{ c_{t+j},\ s_{t+1+j} \}_{j=0}^{\infty}} \quad & q_t s_t = \sum_{j=0}^{\infty} p_{t+j} c_{t+j} \\ \text{s.t. } & c_{t+j} \leq s_{t+j} \\ & s_{t+j+1} = \gamma s_{t+j} - (\gamma - \widehat{\gamma}) c_{t+j}, \quad j = 0, 1, 2, \ldots, \\ & \text{given } s_t. \end{aligned}$$

Similarly, consumers take prices (p, q) as given, and solve for each t,

$$\sup_{\{c_{t+j},\ s_{t+1+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j U(c_{t+j})$$
$$\text{s.t. } p_{t+j}c_{t+j} + q_{t+j+1}s_{t+j+1} = q_{t+j}s_{t+j}, \quad j = 0, 1, 2, \ldots,$$
$$\text{given } s_t.$$

Market clearing then gives the competitive equilibrium

$$\{\, p_t^*, q_t^*, c_t^*, s_{t+1}^* \,\}_{t=0}^{\infty}.$$

Proposition 3.7 *Assume $\mathcal{G}'$ and fix $s_0 > 0$ from Prop. 3.6. For any positive constant μ, the SE allocation*

$$\{\, (c_t^*, s_{t+1}^*),\ t = 0, 1, 2, \ldots \,\}$$

gives market-clearing quantities at prices

$$\begin{aligned} p_0^* &= U'(c_0^*) \times \mu \\ p_t^* &= \gamma^{-1} p_{t-1}^*, \quad t = 1, 2, 3, \ldots \\ q_t^* &= (\gamma - \widehat{\gamma})^{-1}\gamma \times p_t^* \\ &= (\gamma - \widehat{\gamma})^{-1}\gamma \times \gamma^{-t} U'(c_0^*) \times \mu, \quad t = 0, 1, 2, \ldots . \end{aligned}$$

The resulting $\{\, p_t^*, q_t^*, c_t^*, s_{t+1}^* \,\}_{t=0}^{\infty}$ is a CE. ∎

Although more intricate, the argument to establish Prop. 3.7 is, in concept, the same as that for the two-period Prop. 3.4. Now, however, both consumption and asset prices decline at rate γ^{-1}. While, again, Prop. 3.7 gives asset price q as a simple contemporaneous relation to the price of consumption flows p, the former is more insightfully viewed as the present value of all current and future revenues on supplying c to the market:

Proposition 3.8 *In CE we have $q_0^* = s_0^{-1} \times \sum_{t=0}^{\infty} p_t^* c_t^*$. Setting $\mu = 1$ in Prop. 3.7, we also have $q_0^* = V'(s_0)$.* ∎

The period 0 asset price q_0^* is the cost threshold below which socially useful but costly innovation occurs. If q_0^* is zero then no innovation occurs, even if positively valued by society according to Prop. 3.6. Since the relation $q_0^* = V'(s_0)$ also occurs in the two-period model of Prop. 3.5 and all the cases considered below—Props. 4.5, 5.4, 5.6, and 6.2—that asset price is, in many different circumstances, the social value of a marginal change in the initial stock s_0. As long as this holds—exceptions where $q^* = 0$ are described in footnote 11 below—when considering an innovation project that costs $\psi \leq q_0^*$ and that takes the initial stock from 0 to s_0, activating that project maximizes consumer surplus and thus social welfare. As in the usual demand/supply-curve analysis, this marginal condition captures within it maximization of the entire integral under the demand curve. Given s_0, any lower-cost innovation benefits society even more; any higher-cost innovation will reduce social surplus from the maximum. Undertaking the marginal innovation at $\psi = q_0^*$ is, therefore, socially efficient.[7]

[7] To be clear, from $q_0^* = V'(s_0)$ being the derivative with respect to s, at $\psi = q_0^*$ the social surplus maximization is over all projects equally costly but with different instantiation quantities in the asset stock s. Because this model explicitly treats only one intellectual asset (as in Boldrin and Levine (2002b)), this is a sensible definition for social efficiency. An alternative definition for socially efficient innovation—with multiple projects and multiple intellectual assets—is to maximize consumer welfare net of costs across *different* projects creating different kinds of intellectual assets. This takes into account new goods and their associated Dupuit triangles—see, e.g., Quah (2002a) and Romer (1994). The optimal policy then is to accept all projects for which $V(s_0) - V(0) \geq \psi$, or at the margin to drive $V(s_0) - V(0) = \psi$. This obviously differs from $V'(s_0) = \psi$. Quah (2002b) uses that alternative to evaluate the social efficiency of IPR regimes. There, market-based outcomes can display too much or too little innovation relative to the social optimum. This occurs even without the externalities induced by patent races and creative destruction from Aghion and Howitt (1998), Dasgupta (1988), and Jones and

By choice of numeraire, we can again set μ in Prop. 3.7 to unity so that

$$q_0^* = (\gamma - \widehat{\gamma})^{-1}\gamma \times U'(c_0^*).$$

Two factors enter the determination of q_0^*. The first is the multiplier $(\gamma - \widehat{\gamma})^{-1}\gamma$; the second is period 0 marginal utility $U'(c_0^*)$. Under almost all reasonable assumptions on utility U marginal utility $U'(c_0^*)$ at the SE allocation is bounded away from zero.[8] Notice then that when γ grows without bound, q_0^* always remains positive.

This then is the key Boldrin-Levine insight. Even when technology advances to allow near-nonrival reproduction, perfectly competitive markets survive and positive rents allow ongoing innovation. Consumption and asset prices might decline to zero (at rate $\gamma \uparrow \infty$), but in period 0 the positive price $U'(c_0^*)$ guarantees positive q_0^* in competitive equilibrium.[9]

This analysis implies that to a reasonable approximation nonrival goods present no difficulties to the workings of perfectly competitive markets. This result from Boldrin and Levine (2002b) is a profound and remarkable finding that overturns decades of economic thinking on intellectual property rights. The result relies crucially, however, on what happens in period 0. Regardless of how high γ gets, actions by consumers and competing firms can only occur once every fixed time period. A positive price in period 0 prevails for all of period 0. This need not be reasonable. Agents will, in general, wish to adjust demand and supply decisions more and more frequently, and prices change more and more rapidly—should this be what maintains equilibrium with ever higher γ. The right approximation then is one that allows actions by economic agents in continuous time.

Williams (2000).

[8] Marginal utility might even diverge to infinity, but ignoring that case loses no insight and is not central to the discussion. See the example in the Technical Appendix following the proof of Prop. 4.5.

[9] It should be apparent that such an asset price is based on fundamentals, namely the stream of revenues that the asset generates. But if any worry lingers that this price might be simply a so-called "rational bubble" or Ponzi-game outcome, the two-period model of Section 3.5 fixes that. The finite endpoint after $t = 1$ would not support a rational bubble, but clearly the prices in Prop. 3.7 are simply the infinite-horizon counterparts to those in Prop. 3.4.

4 Continuous Time

In continuous time while consumption price $p = U'$ can be bounded away from zero at $t = 0$, the length of time that it remains positive can be fleetingly brief. The integral form used to asset price q (the continuous time counterpart to Prop. 3.8) then evaluates to zero even when consumption flow price remains positive. But then no innovation occurs even if social efficiency dictates the opposite. This continuous-time analysis, therefore, restores the conventional wisdom that perfectly competitive markets fail with nonrival (or near nonrival) goods.

Rewrite consumer preferences (1) and technology constraints (3) as

$$W = \int_{t=0}^{\infty} e^{-\rho t} U(c(t))\, dt, \qquad \rho > 0 \tag{1'}$$

and

$$\begin{aligned} &c(t) \in [0, s(t)] \\ &\dot{s}(t) = \gamma s(t) - (\gamma - \widehat{\gamma})c(t), \end{aligned} \tag{3'}$$

with $\gamma > \widehat{\gamma} \geq 0$. Similarly, the consumer's budget constraint (5) and the firm's value (6) become:

$$\frac{d}{dt}[qs] = -pc \tag{5'}$$

and

$$q(t)s(t) \stackrel{\text{def}}{=} \int_0^{\infty} p(t+r)c(t+r)\, dr. \tag{6'}$$

For completeness explicitly give definitions for social efficiency and competitive equilibrium in continuous time. Given initial stock $s(0) > 0$, an allocation is a time profile of quantities

$$\{\, c(t), s(t) : t \geq 0 \,\} .$$

An allocation is feasible when it satisfies ($3'$). As before, call $\mathsf{C}(s(0))$ the collection of every consumption profile $c = \{\, c_t : \; t \geq 0 \,\}$ that is part of a feasible allocation. Provided that $\lim_{t\to\infty} e^{-\gamma t} s(t) = 0$, integrating ($5'$) gives

$$s(0) = (\gamma - \widehat{\gamma}) \; \int_{t=0}^{\infty} e^{-\gamma t} c(t) \, dt,$$

so that for all $c \in \mathsf{C}(s(0))$ we have

$$\int_{t=0}^{\infty} e^{-\gamma t} \frac{dc(t)}{ds(0)} \, dt = (\gamma - \widehat{\gamma})^{-1}. \qquad (11)$$

Definition 4.1 (SE – Continuous Time) *At initial stock $s(0) > 0$, an allocation is socially efficient (SE) when it is feasible and maximizes the representative consumer's preferences W in ($1'$). Define the value function $V : \mathbb{R}_+ \to \mathbb{R}$*

$$V(s(0)) \stackrel{\text{def}}{=} \sup_{\mathsf{C}(s(0))} \; \int_{t=0}^{\infty} e^{-\rho t} U(c(t)) \, dt. \quad ▮$$

As before, we will compare the SE allocation with competitive equilibrium.

Definition 4.2 (CE – Continuous Time) *At initial stock $s(0) > 0$, a competitive equilibrium (CE) is a time profile of prices and quantities*

$$\{\, p^*(t), q^*(t), c^*(t), s^*(t) : \; t \geq 0 \,\}$$

with $s^*(0) = s(0)$, such that when consumers and firms take CE prices as given, then markets clear at CE quantities, with consumers maximizing preferences ($1'$) subject to ($5'$) and firms maximizing value ($6'$) subject to ($3'$). ▮

Next, we need the continuous-time counterpart to conditions $\mathcal{G}$ and $\mathcal{G}'$. Define

$$R(c) = -cU''(c)/U'(c) > 0.$$

This would be the coefficient of relative risk aversion in a model of choice under uncertainty. Here, equivalently, it is also the reciprocal of the intertemporal elasticity of substitution. Assume:

Assumption $\mathcal{G}''$: *Copying rate γ is bounded from below,*

$$\gamma > \rho + \widehat{\gamma} \times \sup_{s>0} R(s). \quad ▮$$

Condition $\mathcal{G}''$ might at first appear to contradict the usual $\gamma < \rho$ restriction familiar from the theory of economic growth. However, the model's steady-state growth rate is not γ but a convex combination of γ and $\widehat{\gamma}$—see this by using, e.g., that c is a constant fraction of s from the Example in the Technical Appendix.

I can now state the continuous-time versions of the social efficiency and competitive equilibrium equivalence propositions.

Proposition 4.3 *Under condition $\mathcal{G}''$ the SE allocation*

$$\{\, (c^*(t), s^*(t)), \; t \geq 0 \,\}$$

satisfies for some non-negative $\{\, \lambda(t) : \; t \geq 0 \,\}$ the set of equations

$$\begin{aligned} U'(c^*(t)) &= (\gamma - \widehat{\gamma})\lambda(t) \\ \dot{\lambda}/\lambda &= \rho - \gamma < 0 \\ s^*(0) = s(0) &= (\gamma - \widehat{\gamma}) \int_{t=0}^{\infty} e^{-\gamma t} c^*(t) \, dt \\ s^*(t) &= (\gamma - \widehat{\gamma}) \int_{r=0}^{\infty} e^{-\gamma r} c^*(t + r) \, dr, \quad t > 0. \end{aligned} \quad ▮$$

Interpretation of the equations characterizing SE above follows exactly that in their discrete-time counterparts from Prop. 3.6.

Proposition 4.4 *Assume $\mathcal{G}''$. For any positive constant μ, the SE allocation*

$$\{\,(c^*(t), s^*(t)),\ t \geq 0\,\}$$

gives market-clearing quantities at the prices

$$\begin{aligned} p^*(0) &= U'(c^*(0)) \times \mu \\ \dot{p^*}/p^* &= -\gamma < 0 \quad \text{or} \quad p^*(t) = e^{-\gamma t} p^*(0) \\ q^*(t) &= (\gamma - \widehat{\gamma})^{-1} \times p^*(t) \\ &= (\gamma - \widehat{\gamma})^{-1} \times e^{-\gamma t} U'(c^*(0)) \times \mu. \end{aligned}$$

Set $s^*(0) = s(0)$. The resulting $\{\,(p^*(t), q^*(t), c^*(t), s^*(t)) :\ t \geq 0\,\}$ is a CE. ∎

As before, while asset price bears a simple contemporaneous relation with consumption price, it is also the present value of the revenue stream from current and future sales of consumption services.

Proposition 4.5 *In CE we have*

$$q^*(0) = s(0)^{-1} \times \int_{t=0}^{\infty} p^*(t) c^*(t)\, dt.$$

Setting $\mu = 1$ in Prop. 4.4 we also have $q^*(0) = V'(s(0))$. ∎

Prop. 4.5 gives the same present value interpretation to $q^*(0)$ as in the discrete-time case in Prop. 3.8. Similarly, it is also the social value of a marginal change in initial stock $s(0)$.

At the same time, however, the formula for $q^*(t)$ in Prop. 4.4 reflects the intuition developed at the beginning of this section. Again we can set $\mu = 1$ by choice of numeraire. From Prop. 4.4,

$$q^*(0) = (\gamma - \widehat{\gamma})^{-1} \times p^*(0) = (\gamma - \widehat{\gamma})^{-1} \times U'(c^*(0)),$$

so that the multiplier on $p^* = U'$ is now $(\gamma - \widehat{\gamma})^{-1}$, not $(\gamma - \widehat{\gamma})^{-1} \times \gamma$. The numerator γ from the discrete-time Prop. 3.7 no longer appears. This highlights the first-period effect in the discrete-time

model, where no further, within-period actions could be taken after setting $U'(c_0^*) > 0$. In the continuous-time version, as $\gamma \uparrow \infty$, the multiplier $(\gamma - \widehat{\gamma})^{-1}$ converges to zero rather than unity. So too then does asset price q converge to zero, even with $p = U'$ remaining strictly positive.[10] As γ rises, the time interval when prices p remain appreciably positive becomes shorter and shorter. Asset price q, which involves a product of both the price level and the appropriate covering timespan, then converges to zero, rather than remain strictly positive.[11]

Elsewhere in this paper I describe the results of this continuous-time formulation using evocative language like "economic agents taking actions progressively frequently as technology advances". Some have objected that this is not strictly consistent with a model where time is continuous, rather than one where, say, the time interval shrinks with γ. At the expense of some cumbersome notation this dissonance is easy to repair. Re-use the discrete-time model in section 3 but now let the time interval have length γ^{-1}. Rewrite equations (1), (3), (5), and (6) appropriately so that $t + j$ becomes $(t + j)/\gamma$ in indexing and taking powers of β. The only substantive change then is to the firm's value equation (6) describing revenue flow where now individual summands on the right side of this equation are multiplied by time-period length γ^{-1}. (A similar change appears in (1) but that is irrelevant to the argument.) The resulting SE and CE descriptions remain unaltered except the formula for q^* in Prop. 3.7

[10] Example 4.6 in the Technical Appendix contains an explicit parametric treatment for $U'(c^*(0))$, showing how it can remain bounded even when $\gamma \uparrow \infty$. In general equilibrium, as γ changes, both substitution rates and production possibilities vary, and so the effect of γ on c^* need not be transparent.

[11] An argument might be made that the zero price q is, indeed, the appropriate, socially efficient outcome, as zero is, literally, what the time integral of marginal utilities turns out to be. In this view, at the margin society as well as putative enterpreneurs do not value such innovations. But this is just the market failure identified by Arrow (1962) and Nordhaus (1969) reworded.

is now divided by γ. This adjustment cancels the same term in the numerator, and gives the multiplier in Prop. 4.4. More generally, the same reasoning works for a time interval having length *any* function that has order of magnitude $O(\gamma^{-1})$. The numerator in the expression for q^* in Prop. 3.7 is always the same magnitude, and so always gets appropriately cancelled.

5 Completely unrestricted dissemination

In the analysis thus far, as in Boldrin and Levine (2002b), property rights cover the asset s. Although the model economy envisions no restrictions on how purchasers of s can subsequently use s—to compete, say, with the original owner—it does restrict the behavior of purchasers of the consumption flow c. In particular, those users cannot themselves become competing producers. Whether this restriction is legal or technical is not made explicit in Boldrin and Levine (2002b). Arrangements such as SDMI, for instance, combine both legal and technical dimensions. This section relaxes that restriction and asks what happens with pure untramelled competition.

The analysis in two parts. First, section 5.1 alters the technology to consider goods that are nonrival, in the sense described in the introduction, but at the same time remain finitely expansible. Second, section 5.2 reverts to the rival and finitely expansible technology from section 3. In both cases competitive equilibrium achieves social efficiency, but the equilibrium allocations differ in interesting ways. In section 5.1 social efficiency calls for the widest possible dissemination of the intellectual asset. The price system supports that in equilibrium. Technological feasibility, not the price system, becomes the binding constraint.

Section 5.2 has, under current legal systems, activity that might be identified as bootlegging. Nevertheless, perfectly competitive markets achieve the socially efficient outcome, requiring only an adjustment in prices to those obtained in section 3 above and in Boldrin and Levine (2002b). Here, however, prices do curtail dissemination to levels below that in section 5.1.

5.1 Unrestricted dissemination with nonrivalry but finite expansibility

Nonrivalry means that one use does not detract from another. This says nothing about the magnitudes of rates of copying. Therefore, reconsider the assumption that generating consumption flow c off the stock of intellectual asset s degrades the copying potential of that stock. In Sections 3–4 that degradation is $\gamma - \widehat{\gamma} > 0$. When an s-holder puts aside $c \leq s$ to provide that amount of consumption services, the remainder $(s - c)$ reproduces at rate γ but c, the amount set aside, reproduces only at rate $\widehat{\gamma} < \gamma$.

Suppose now that when s is used to provide consumption services c, the consumer who purchased c can, while enjoying $U(c)$, also turn c into asset stock $s' = \sigma \times c$ that she can use for further copying, with $\sigma > 0$. (The mnemonic is σ for *subsidiary producer* or *user* or *consumer*.) If, in Boldrin and Levine (2002b), this action is legally prohibited, then what I have just described might be regarded as making bootleg copies—under a particular legal regime. If, on the other hand, Boldrin and Levine (2002b) intend this to be a technical restriction, then what I have just described is simply an acknowledgement of advancing information and media technologies.[12]

Unlike the degraded rate $\widehat{\gamma}$, the copying rate σ can bear any relation to γ. If consumers own copying machines that are not as good as professionals', then it might be reasonable to take $\sigma < \gamma$. If, however, consumers are themselves professionals in converging industries, then there is no reason why we could not have $\sigma > \gamma$. That products grow faster by disseminating as widely as possible to all potential users has long been an operating tenet of, for instance, the Open Source Software community—the formulation with σ positive and potentially exceeding γ allows exactly this.

I proceed with the analysis as before, first characterizing social

[12] This formulation suggests also that the typical identification of nonrivalry with infinite expansibility can be misleading. What Boldrin and Levine (2002b) study is finite expansibility, not near-nonrivalry.

efficiency and then decentralizing the model with a price system under perfect competition. Society's resource constraints are now altered from (3) to:

$$\begin{aligned} c_t &\in [0, s_t] \\ s_{t+1} &= \gamma s_t + \sigma c_t. \end{aligned} \tag{12}$$

The coefficient on c_t has changed from the negative $-(\gamma - \widehat{\gamma})$ to the positive σ. Notice, however, that technology (12) remains constant returns to scale.

Because of the nonrivalry in consumption and production, the feasible consumption set $\mathsf{C}(s_0)$ is no longer obtained by simply iterating equation (12). Instead, $\mathsf{C}(s_0)$'s upper boundary—the only part that will matter when consumers show nonsatiation—has the following characterization. Set c_t to its upper bound s_t so that

$$c_{t+1} = s_{t+1} = [\gamma + \sigma]s_t = [\gamma + \sigma]^{t+1} \times s_0.$$

Then, on $\mathsf{C}(s_0)$'s upper boundary,

$$\frac{dc_t}{ds_0} = [\gamma + \sigma]^t.$$

Definition 5.1 (SE – CUD) *At initial stock $s_0 > 0$, an allocation is socially efficient (SE) when it is feasible relative to technology* (12) *and maximizes the representative consumer's preferences W in* (1). *Define the value function $V : \mathbb{R}_+ \to \mathbb{R}$*

$$V(s_0) \overset{def}{=} \sup_{\mathsf{C}(s_0)} \sum_{t=0}^{\infty} \beta^t U(c_t). \quad ∎$$

Here, social efficiency differs in essential features from those previously obtained in Sections 3 and 4. Conditions such as $\mathcal{G}$ no longer restrict c away from the boundary at s. Indeed, since $\sigma > 0$, the greater is consumption, the greater is next period's stock. It is therefore optimal to set consumption to the boundary at s, achieving widest possible dissemination of the intellectual asset at the same time maximizing its growth rate.

Proposition 5.2 *The SE allocation $\{ (c_t^*, s_{t+1}^*),\ t = 0, 1, 2, \dots \}$ satisfies*

$$\begin{aligned} c_0^* &= s_0 \\ c_t^* &= s_t^* = (\gamma + \sigma)s_{t-1}^* \\ &= (\gamma + \sigma)^t s_0, \quad t = 1, 2, \dots. \end{aligned}$$

The value function can be explicitly written

$$V(s_0) = \sum_{t=0}^{\infty} \beta^t U(c_t^*) = \sum_{t=0}^{\infty} \beta^t U([\gamma + \sigma]^t s_0). \quad ∎$$

This allocation satisfies, simultaneously, two potentially competing considerations. First, the allocation achieves widest feasible dissemination of the intellectual asset, consistent with suggestions in, e.g., Richard Stallman's GNU Manifesto and many writings in the Open Source movement. Second, the allocation respects the finite capacity constraint emphasized in Boldrin and Levine (2002b). By contrast, the allocations studied in Sections 3–4, socially efficient under assumption (3), optimally never had use c achieve the feasible boundary s.[13]

That the SE allocation can be attained under perfect competition comes from two features of the model. One, the technology (12) displays constant returns to scale. Two, the payoff structure of asset s resembles those of the dividend-paying trees in Lucas (1978), where, we will see below, here the dividend equals the price of the asset next period multiplied by the growth rate σ. That there is constant returns to scale means that it is irrelevant whether the economy has one firm, acting competitively, or multiple firms, all competing with each other. That the model is close to that in Lucas (1978) means that asset prices here behave the same way as there.

Turn next to markets and competitive equilibrium. While the SE allocation is obviously that given in Prop. 5.2—as long as utility

[13] Boldrin and Levine (2002b) rule out this boundary solution. See also footnote 6 above.

is monotone increasing—that allocation could well imply unbounded welfare, with consumption growing at rate $\gamma + \sigma > \gamma > 1$. This is problematic for characterizing the consumer's choice problem. To get around this, a sufficient condition is

Condition $\mathcal{C}$: *Technology (γ, σ) and preferences (β, U) jointly satisfy:*

$$\lim_{t\to\infty}\left[([\gamma+\sigma]\beta)^t \times \sum_{j=0}^{\infty} ([\gamma+\sigma]\beta)^j \times U'\left([\gamma+\sigma]^{t+j}\right)\right] = 0. \quad \blacksquare$$

Condition $\mathcal{C}$ states that marginal utility U' must decline sufficiently rapidly as growth occurs. The higher is $\gamma + \sigma$ relative to β, the faster must U' fall. While the condition might seem intricate, it is implied by consumer preferences W displaying low constant intertemporal elasticity of substitution:

Lemma 5.3 *Suppose that for preferences W given in* (1) *the function U satisfies* (7). *Then condition $\mathcal{C}$ holds whenever the intertemporal elasticity of substitution $R^{-1} \leq 1$.* ∎

The inequality $R^{-1} \leq 1$ is sufficient but not necessary for condition $\mathcal{C}$—see the proof of Lemma 5.3 and equation (36) in particular. Thus, log utility or $R^{-1} = 1$—which will play a special role in Props. 5.5 and 6.2—implies condition $\mathcal{C}$ but so will utility functions with even higher intertemporal elasticities of substitution.

Despite the specifics in Lemma 5.3, that marginal utility have the form $U'(c) = c^{-R}$ is inessential to condition $\mathcal{C}$. What matters is that, asymptotically, marginal utility decline faster than c^{-1}.

When the representative consumer purchases c_t she now pays p_t upfront but then can, if she wishes, allow c_t to grow at rate σ and sell the result on the market q_{t+1}. The consumer's budget constraint at time t is therefore changed from (5) to:

$$q_{t+1}s_{t+1} \leq (q_t s_t - p_t c_t) + q_{t+1}\sigma c_t. \tag{13}$$

In competitive equilibrium the representative consumer takes as given prices p and q, and at time t solves

$$\begin{aligned}\sup_{\{c_{t+j},\ s_{t+1+j}\}_{j=0}^{\infty}} \ & \sum_{j=0}^{\infty} \beta^j U(c_{t+j}) \\ \text{s.t. } & p_{t+j}c_{t+j} + q_{t+j+1}s_{t+j+1} \\ & \quad \leq q_{t+j}s_{t+j} + q_{t+j+1}\sigma c_{t+j}, \quad j = 0, 1, 2, \ldots, \\ & \text{given } s_t.\end{aligned}$$

The representative firm at the beginning of period t sells consumption flow c_t at price p_t, and then if it so wishes, at the end of the period, buys the consumer-generated stock $\sigma \times c_t$ at price q_{t+1} to increment its holdings of s. The firm's revenue flow for period t is therefore $(p_t - q_{t+1}\sigma) \times c_t$, while its s holdings follow:

$$s_{t+1} = \gamma s_t + \sigma c_t.$$

Its value then is given by:

$$q_t s_t = \sum_{j=0}^{\infty} (p_{t+j} - q_{t+1+j}\sigma) \times c_{t+j}. \tag{14}$$

In competitive equilibrium the representative firm maximizes value (14) taking p and q as given. At each time period t, therefore, the firm solves

$$\begin{aligned}\sup_{\{c_{t+j},\ s_{t+1+j}\}_{j=0}^{\infty}} \ & q_t s_t = \sum_{j=0}^{\infty} (p_{t+j} - q_{t+1+j}\sigma)\, c_{t+j} \\ \text{s.t. } & c_{t+j} \leq s_{t+j} \\ & s_{t+j+1} = \gamma s_{t+j} + \sigma c_{t+j}, \quad j = 0, 1, 2, \ldots, \\ & \text{given } s_t.\end{aligned}$$

The firm's production set coincides with that for society given in (12).

Competitive equilibrium achieves social efficiency through the following market-clearing prices:

Proposition 5.4 *Assume condition $\mathcal{C}$ and set $s_0^* = s_0$. For any positive constant μ, the SE allocation*

$$\left\{ (c_t^*, s_{t+1}^*),\ t = 0, 1, 2, \ldots \right\}$$

of Prop. 5.2 gives market-clearing quantities at prices

$$q_t^* = (s_t^*)^{-1}\beta^t \times \left[\sum_{j=0}^{\infty} \beta^j U'(s_{t+j}^*) s_{t+j}^*\right] \times \mu$$
$$p_t^* = \beta^t U'(s_t^*) \times \mu + q_{t+1}^* \sigma.$$

The resulting $\left\{ p_t^*, q_t^*, c_t^*, s_{t+1}^* \right\}_{t=0}^{\infty}$ is a CE. Taking $\mu = 1$, the initial asset value q_0^* further satisfies $q_0^* = V'(s_0)$. ∎

Equilibrium asset price q^* again has a present value form. However, it now has time profile that is no longer a smooth geometric decline $q_{t+1}^*/q_t^* = \gamma^{-1}$, but instead one that varies with consumer preferences and all the other parameters of the economy. Similarly, the equilibrium time profile for the consumption flow or rental price p^* is no longer the same geometric decline but now depends also on the dynamics of a forward-looking present value term. In Prop. 5.4, p^* is no longer only contemporaneous marginal utility but has added to that the resale value from the consumer creating an appropriate extra number of copies.

The competitive outcome continues to be socially efficient even as goods approach infinite expansibility. Writing

$$q_0^* = s_0^{-1} \times \left[U'(s_0)s_0 + \sum_{t=1}^{\infty} \beta^t U'(s_t^*) s_t^*\right] \times \mu$$
$$p_0^* = U'(s_0) \times \mu + q_1^* \sigma,$$

both q_0^* and p_0^* are bounded from below by the positive quantity $U'(s_0) \times \mu$, independent of copying rates. All socially efficient innovations, therefore, continue to occur, even when $\gamma + \sigma \uparrow \infty$, re-confirming the Boldrin-Levine insight that perfectly competitive markets achieve social efficiency even as the degree of expansibility grows without bound.

We can get further intuition in more precise formulas if we specialize preferences to that earlier used to motivate condition $\mathcal{C}$.

Proposition 5.5 *Assume the hypotheses in Lemma 5.3. Then CE $\left\{ p_t^*, q_t^*, c_t^*, s_{t+1}^* \right\}_{t=0}^{\infty}$ exists where, taking an appropriate numeraire, we have:*

$$c_t^* = s_t^* = (\gamma + \sigma)^t s_0$$
$$q_0^* = \left[1 - \frac{\beta}{(\gamma+\sigma)^{R-1}}\right]^{-1} \times s_0^{-R}$$
$$q_t^* = \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^t \times q_0^*$$
$$p_t^* = \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^t \times \left[s_0^{-R} + q_0^* \frac{\sigma\beta}{(\gamma+\sigma)^R}\right].$$ ∎

Asset and consumption prices decline at rate $\beta/(\gamma+\sigma)^R$, and thus have time profiles that depend on not just technology (γ, σ) but also preference parameters (β, R).

Consider the effects of copying rates γ and σ on equilibrium in Prop. 5.5. When $R^{-1} < 1$ so that demand is inelastic, then both period-0 prices converge to the same limiting value, $q_0^* \to s_0^{-R}$ and $p_0^* \to s_0^{-R}$ if either γ or σ increase without bound. However, when $R^{-1} = 1$, i.e., U is log utility, then $q_0^* = (1-\beta)^{-1} \times s_0^{-1}$ is invariant with respect to the copying rates. Now, if $\gamma \to \infty$ at least as quickly as σ, then the period-0 consumption price $p_0^* \to s_0^{-1}$, whereas if $\sigma \to \infty$ while γ remains bounded (or increases slower than σ), then consumption price

$$p_0^* \to s_0^{-1} + \beta q_0^* = \left(1 + \frac{\beta}{1-\beta}\right) s_0^{-1} = q_0^*.$$

Thus, except with log utility and $\gamma \uparrow \infty$ no slower than σ, as *some* copying rate gets arbitrarily large the distinction between the asset s and its consumption flow c disappears, in both prices converging towards each other.

This peculiar feature of unit intertemporal elasticity has already appeared elsewhere in dynamic macroeconomics, and should be relatively familiar. As one might already suspect from the discussion contrasting the discrete and continous time findings from section 4, unit elasticity manifests as an interesting special case in continuous time. We turn to that in section 6 further below.

5.2 Unrestricted dissemination with rivalry and finite expansibility

Now, instead, I discuss what happens if the same kind of unrestricted copying is allowed, maintaining the rival and finitely expansible technology in section 3 and Boldrin and Levine (2002b). With discrete time, competitive equilibrium remains optimal and the asset price q is unchanged. Only the consumption price p is affected, and in the obvious and intuitive way. Thus, the Boldrin and Levine (2002b) insight that perfectly competitive markets achieve social efficiency obtains even under weaker restrictions on trading than they described.

The analysis combines the discussions from sections 3.6 and 5.1. Since neither preferences nor society-wide technology has changed, the feasible consumption set $\mathsf{C}(s_0)$ and the value function V are unchanged from those in Props. 3.6–3.8.

Consumers maximize preferences (1) subject to (13) with $\widehat{\gamma}$ replacing σ, i.e.,

$$q_{t+1}s_{t+1} \leq (q_t s_t - p_t c_t) + q_{t+1}\widehat{\gamma}c_t.$$

Because goods are rival here, when a firm rents out consumption services c at price p, it loses access to those assets and so is no longer able to make copies at rate $\widehat{\gamma}$. What the firm continues to hold, $s - c$, can be copied at the usual rate γ. To increase s next period beyond this, the firm can purchase $\widehat{\gamma}c$ from the consumer at price q. Or, the consumer can now compete with the firm, using her freshly-constituted holdings $\widehat{\gamma}c$. Since, however, the technology is constant returns to scale, the outcome is invariant to which of these occurs.

The firm's revenue flow is, paralleling section 5.1,

$$(p_t - q_{t+1}\widehat{\gamma}) \times c_t,$$

so that its value, modifying equation (14), is:

$$q_t s_t = \sum_{j=0}^{\infty}(p_{t+j} - q_{t+1+j}\widehat{\gamma}) \times c_{t+j}, \tag{15}$$

while now its s holdings follow

$$s_{t+1} = (s_t - c_t)\gamma + \widehat{\gamma}c_t. \tag{16}$$

But, rewritten, (16) is exactly the same as the dynamic equation in (3). Firms therefore maximize value (15) subject to (3).

Proposition 5.6 *Assume $\mathcal{G}'$ and fix $s_0 > 0$ from Prop. 3.6. Then CE $\{\, p_t^\dagger, q_t^\dagger, c_t^\dagger, s_{t+1}^\dagger \,\}_{t=0}^{\infty}$ exists and achieves the SE allocation of Prop. 3.6,*

$$c_t^\dagger = c_t^* \quad \textit{and} \quad s_{t+1}^\dagger = s_{t+1}^*, \quad t = 0, 1, 2, \ldots$$

at prices

$$p_t^\dagger = q_t^\dagger = q_t^*, \quad t = 0, 1, 2, 3, \ldots .$$

Taking $\mu = 1$, the initial asset value $q_0^\dagger$ satisfies $q_0^\dagger = V'(s_0)$. ∎

Competitive markets continue to achieve social efficiency, even with no restrictions on what consumers can do with copies of the asset that they have merely rented. Exactly the same amount of innovation continues to take place—asset price q is unchanged. Only the rental price p changes. It rises to reflect the increased demand when consumers can now make copies and therefore enter the market themselves to compete with the incumbent. And, intuitively, it equals the asset price q itself since now the two no longer differ.

It might be useful to discuss a real-world example relating the equilibrium in Prop. 5.6 to the Boldrin-Levine one in Props. 3.6–3.8. Example 5.7 provides intuition for this comparison by looking at the evolution of technology, law, and Internet music dissemination. Since this takes the flow of argument from the more rigorous and analytical, some readers might wish to skip the rest of this section and proceed directly to section 6.

Example 5.7 *Suppose that corporation $\mathfrak{S}$ uses Internet streaming technology that purports to distribute music for one-time listening only. The music digital bits pass through a Media Player on my Internet node, I enjoy the music, and then—so $\mathfrak{S}$ engineers think—the bits simply vanish. Corporation $\mathfrak{S}$ charges me price p for this. However, they charge the value-added retailer $\mathfrak{R}$ a higher price $q > p$ because $\mathfrak{R}$ wants not just to listen but to make legal copies and sell them on. In equilibrium I am indifferent between paying p for just listening and paying q for becoming a value-added retailer (VAR).*

If $\mathfrak{S}$ can enforce this technical restriction then provided that only finite reproduction occurs, competitive equilibrium is possible with $q > p > 0$. This is what happens in Boldrin and Levine (2002b) and in Props. 3.6–3.8 above.

Now, however, suppose I write a software program that breaks $\mathfrak{S}$'s protection scheme. So, when $\mathfrak{S}$ thinks I only listen, I could actually be writing the music (bits) to hard disk at the same time. However, even if the technology has changed, provided that all agents obey the law against illegal copying, the competitive equilibrium of Boldrin and Levine (2002b) and Props. 3.6–3.8 continues to apply.

Next, suppose that corporation $\mathfrak{S}$ suddenly realizes advancing technology has broken their protection scheme, and suppose further that the law either has been repealed or can no longer be enforced. What happens in equilibrium?

The price p charged consumers rises until they are again indifferent between being consumers and VARs. What price is that? It will be q, the same price that VARs pay, because now both VARs and I can do the same things with the music we buy. The line between consumers and producers vanishes. Nothing has changed for the VARs. Provided that I can't reproduce infinite copies instantaneously the economy will still have $q > 0$ at the same value as before. But this is just Prop. 5.6. ∎

Music holds no special features that make Example 5.7 work. The principle applies to a broad range of cases where technology, law, and expansible assets are evolving in the way described.

6 Unrestricted dissemination in continuous time

In continous time consumer preferences remain as earlier in (1′), while the economy's technology constraint becomes

$$\begin{aligned} c(t) &\in [0, s(t)] \\ \dot{s}(t) &= \gamma s(t) + \sigma c(t). \end{aligned} \qquad (12')$$

Then the socially efficient (SE) allocation maximizes (1′) subject to (12′), with solution:

$$\forall\ t \geq 0: \quad c^*(t) = s^*(t) = e^{(\gamma+\sigma)t} s(0). \qquad (17)$$

The set $\mathsf{C}(s(0))$ of feasible consumption profiles has upper boundary given by equation (17) and the value function $V : \mathbb{R}_+ \to \mathbb{R}$ is

$$V(s(0)) = \sup_{\mathsf{C}(s(0))} \int_{t=0}^{\infty} e^{-\rho t} U(c(t))\, dt = \int_{t=0}^{\infty} e^{-\rho t} U(e^{[\gamma+\sigma]t} s(0))\, dt.$$

For competitive equilibrium (CE), the representative consumer's budget constraint is now

$$\frac{d}{dt}[qs] = -pc + q\sigma c = -(p - q\sigma)c, \qquad (13')$$

following the reasoning surrounding its discrete-time version (13). Similarly, the representative firm's value is

$$q(t)s(t) = \int_{r=0}^{\infty} [p(t+r) - q(t+r)\sigma] \times c(t+r)\, dr. \qquad (14')$$

The firm maximizes value (14′) subject to constraint (12′). At initial stock $s(0) > 0$ competitive equilibrium is, as before, a time profile of prices and quantities

$$\{\, p^*(t), q^*(t), c^*(t), s^*(t) :\ t \geq 0 \,\}$$

with $s^*(0) = s(0)$, such that when consumers and firms take CE prices as given, then markets clear at CE quantities, with consumers maximizing preferences $(1')$ subject to $(13')$ and firms maximizing value $(14')$ subject to $(12')$.

For exactly the same reason as before, we need the continuous-time counterpart to condition $\mathcal{C}$:

Condition $\mathcal{C}'$: *Technology (γ, σ) and preferences (ρ, U) jointly satisfy:*

$$\lim_{t\to\infty}\left[e^{-(\rho-[\gamma+\sigma])t} \times \int_{r=0}^{\infty} e^{-(\rho-[\gamma+\sigma])r} U'\left(e^{[\gamma+\sigma](t+r)}\right) dr\right] = 0. \quad ▮$$

As with the previous condition $\mathcal{C}$ the preferences given in Lemma 5.3 imply $\mathcal{C}'$.

Lemma 6.1 *Suppose that for preferences W given in $(1')$ the function U satisfies (7). Then condition $\mathcal{C}'$ holds whenever the intertemporal elasticity of substitution $R^{-1} \leq 1$.* ▮

The principal result of this section then is:

Proposition 6.2 *Assume condition $\mathcal{C}'$. Competitive equilibrium exists and achieves social efficiency. The CE and SE allocations coincide at (17), and, taking any positive constant μ, the CE prices are:*

$$q^*(t) = s^*(t)^{-1} e^{-\rho t} \times \left[\int_{r=0}^{\infty} e^{-\rho r} U'(s^*(t+r)) s^*(t+r)\, dr\right] \times \mu$$
$$p^*(t) = e^{-\rho t} U'(s^*(t)) \times \mu + q^*(t)\sigma.$$

Taking $\mu = 1$, the period-0 asset value $q^*(0)$ satisfies $q^*(0) = V'(s(0))$. ▮

Finally, give the special case with constant intertemporal elasticity of substitution.

Proposition 6.3 *Assume the hypotheses in Lemma 6.1. Then CE prices, taking an appropriate numeraire, are:*

$$q^*(0) = [\rho + (\gamma+\sigma)(R-1)]^{-1} \times s(0)^{-R}$$
$$q^*(t) = e^{-[\rho+(\gamma+\sigma)R]t} \times q^*(0)$$
$$p^*(t) = e^{-[\rho+(\gamma+\sigma)R]t} \times \left[s(0)^{-R} + q^*(0)\sigma\right],$$

so that

$$\dot{q^*}/q^* = \dot{p^*}/p^* = -[\rho + (\gamma+\sigma)R] < 0. \quad ▮$$

As in section 4 the generic case has $q^*(0) \to 0$ as $\gamma \uparrow \infty$, so that competitive markets then fail to deliver socially efficient innovation. However, as in Prop. 5.5, the case of log utility, $R^{-1} = 1$, is special. Here, even in continuous time and even as $(\gamma + \sigma) \uparrow \infty$ we have $q^*(0)$ remaining bounded away from 0. Competitive innovation continues, 24/7 and with completely unrestricted dissemination of infinitely expansible intellectual assets—an extremely special case but nonetheless hypothetically possible.

7 Conclusions

This paper has considered competitive markets as a mechanism for creating and desseminating nonrival or near-nonrival goods. The traditional view, since at least Arrow (1962), Machlup (1962), and Nordhaus (1969) has been that markets fail and therefore mechanisms other than perfect competition are necessary. While not socially optimal, such alternative institutions improve economic performance over market failure.

Boldrin and Levine (2002b) have recently proposed that under reasonable approximations to nonrivalry, perfectly competitive markets perform their usual efficient allocation functions and *continue to do so even as the degree of expansibility grows without bound*. That competitive markets achieve the optimal outcome is never in dispute with only rival goods. Boldrin and Levine's important and profound contribution is to show that markets continue to work well, even in the nonrival limit.

This paper has, first of all, confirmed the Boldrin-Levine analysis, providing a first-principles, easily accessible exposition. Following that, this paper did three things. First, it considered when demanders and suppliers speed up their actions as goods become progressively expansible. The limit under this "24/7 Internet time" approximation restores the conventional wisdom that perfectly competitive markets do not function with nonrival or near nonrival goods. Second, it removed the distinction between consumption and production of an intellectual asset, so that ordinary users can legally make copies and compete with asset holders. Third, this paper considered when nonrivalry means consumption not detracting from production, but potentially *increasing* productivity.

The last formalization suggests that the typical identification of nonrivalry with infinite expansibility can be misleading. It is possible to have an intellectual asset used simultaneously in consumption and production, in a nonrival way, while its rate of copying nevertheless remains bounded. The socially efficient outcome then has the intellectual asset disseminated as widely as technologically possible.

As long as time remains discrete competitive markets continue to achieve social efficiency—even with consumers engaged in what, under current legal systems, might be interpreted as bootleg copying. Although in 24/7 continuous time, markets fail in general, a special case continues to allow socially efficient competitive innovation even as the degree of expansibility increases without bound and market participants act in continuous time.

In summary, parts of the analysis in this paper call for institutions like IPRs, to overcome failure in markets for intellectual assets. In other parts, the opposite conclusion obtains. The critical distinction, in the analysis above, is the speed with which economic agents can act.

When perfectly competitive markets suffice to achieve social efficiency, IPR institutions are damaging in two distinct ways. First, by creating unnecessary monopolies, they restrict consumption and dissemination—lowering these below socially optimal levels. Second, because they increase the flow of rents to an intellectual asset, they

cause excessive innovation.[14]

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[14] To be clear, when competitive markets fail, IPR institutions don't guarantee social optimality either. IPRs might produce too much or too little innovation. This last conclusion comes not from the analysis above but instead from a range of other studies, for instance, looking across alternative IP systems (Wright, 1983), or even with optimal IPRs creating *necessary* monopolies in dynamic general equilibrium (Quah, 2002b).

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8 Technical Appendix

This section holds all proofs to results in the paper. It also treats, in Example 4.6, an explicit functional form in U, illustrating how $U'(c^*(0))$ varies as γ changes.

Proof of Prop. 3.3 *The socially efficient allocation solves*

$$\max_{\{c_0,\ s_1,\ c_1\}} U(c_0) + \beta U(c_1)$$
$$\text{s.t. } c_0 \leq s_0$$
$$s_1 = \gamma s_0 - (\gamma - \widehat{\gamma})\, c_0$$
$$c_1 \leq s_1.$$

Since the economy ends after $t = 1$, set

$$c_1^* = s_1^* = \gamma s_0 - (\gamma - \widehat{\gamma})\, c_0^*.$$

From Inada condition (2) we must have $c_0^* > 0$. The necessary FOC for the optimization is then

$$U'(c_0^*) \geq \beta U'(c_1^*) \cdot (\gamma - \widehat{\gamma}) \text{ with equality if } c_0 < s_0.$$

Define the function

$$M(c) \stackrel{\text{def}}{=} (\gamma - \widehat{\gamma})\beta U'(\gamma s_0 - [\gamma - \widehat{\gamma}]\, c), \quad c \in [0, s_0].$$

From U concave, the function M is increasing in c. At $c = s_0$, condition $\mathcal{G}$ gives

$$M(c) = (\gamma - \widehat{\gamma})\beta U'(\widehat{\gamma} s_0) > U'(s_0).$$

But then we must have $c_0^* \lneqq s_0$ so that

$$U'(c_0^*) = \beta U'(c_1^*) \cdot (\gamma - \widehat{\gamma}). \qquad \textit{Q.E.D.}$$

Proof of Prop. 3.4 *First show that at CE (p_0^*, p_1^*) firms maximize value by supplying (c_0^*, c_1^*), and that that implies values (q_0^*, q_1^*). Since the economy ends after period 1 firms optimally supply $c_1 = s_1$. We can therefore restrict attention to (c_0, c_1) such that*

$$0 \leq c_0 \leq s_0$$
$$(\gamma - \widehat{\gamma})c_0 + c_1 \leq \gamma s_0. \tag{18}$$

From (18), if $p_0 p_1^{-1} > \gamma - \widehat{\gamma}$, firms optimally supply $c_0 = s_0$. If, conversely, $p_0 p_1^{-1} < \gamma - \widehat{\gamma}$, then optimal supply is $c_0 = 0$. However, when $p_0 p_1^{-1} = \gamma - \widehat{\gamma}$, as in (p_0^*, p_1^*), any (c_0, c_1) satisfying (18) with equality maximizes value. Then

$$p_0^* c_0 + p_1^* c_1 = [(\gamma - \widehat{\gamma})c_0 + c_1]\,(\gamma - \widehat{\gamma})^{-1} p_0^*$$
$$= s_0 \gamma \cdot (\gamma - \widehat{\gamma})^{-1} p_0^*,$$

so that $q_0^* = \gamma \cdot (\gamma - \widehat{\gamma})^{-1} p_0^*$. Moreover, since $c_1^* = s_1^*$, we also have $q_1^* = p_1^* = \gamma^{-1} q_0^*$. Next, verify that consumers optimally demand (c_0^*, c_1^*) at the hypothesized prices $\{\,(p_t^*, q_t^*) :\ t = 0, 1\,\}$. At an interior optimum, guaranteed by condition $\mathcal{G}$, consumers' first-order conditions are:

$$U'(c_0) = \beta U'(c_1) \times p_0 p_1^{-1} \tag{19}$$
$$p_0 c_0 + p_1 c_1 = q_0 s_0$$
$$p_1 c_1 = q_1 s_1.$$

At the hypothesized CE prices, Prop. 3.3 asserts (19) is satisfied at the CE quantities (c_0^*, c_1^*). Finally, the consumption sequence (c_0^*, c_1^*) is affordable since

$$p_0^* c_0^* + p_1^* c_1^* = [(\gamma - \widehat{\gamma})c_0^* + c_1^*]\,(\gamma - \widehat{\gamma})^{-1} p_0^*$$
$$= s_0 \cdot \gamma \times (\gamma - \widehat{\gamma})^{-1} p_0^* = s_0 \times q_0^*. \qquad \textit{Q.E.D.}$$

Proof of Prop. 3.5 *Divide the last line in the Proof of Prop. 3.4, by s_0 to get:*

$$q_0^* = s_0^{-1} \times (p_0^* c_0^* + p_1^* c_1^*)\,.$$

From Prop. 3.3,

$$\begin{aligned} s_0 &= (\gamma - \widehat{\gamma})\gamma^{-1}c_0^* + \gamma^{-1}c_1^* \\ \Longrightarrow \frac{dc_0^*}{ds_0} &+ (\gamma - \widehat{\gamma})^{-1}\frac{dc_1^*}{ds_0} = (\gamma - \widehat{\gamma})^{-1}\gamma. \end{aligned}$$

Then using the optimality characterization in Prop. 3.3 in the direct calculation,

$$\begin{aligned} V'(s_0) &= U'(c_0^*)\frac{dc_0^*}{ds_0} + \beta U'(c_1^*)\frac{dc_1^*}{ds_0} \\ &= \left[\frac{dc_0^*}{ds_0} + (\gamma - \widehat{\gamma})^{-1}\frac{dc_1^*}{ds_0}\right] U'(c_0^*) \\ &= (\gamma - \widehat{\gamma})^{-1}\gamma U'(c_0^*) \\ &= q_0^*\mu^{-1} = q_0^*. \end{aligned}$$

Q.E.D.

Proof of Prop. 3.6 *The Lagrangean for maximizing W in (1) subject to technology constraint (3) is*

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[U(c_t) - (s_{t+1} - \gamma s_t + (\gamma - \widehat{\gamma})c_t) \times \lambda_t\right],$$

where $\{\lambda_t\}_{t=0}^{\infty}$ is a sequence of non-negative multipliers. The first-order conditions are

$$\begin{aligned} U'(c_t^*) &\geq (\gamma - \widehat{\gamma})\lambda_t, \quad \text{with equality if } c_t < s_t \\ \lambda_t &= \beta\lambda_{t+1} \times \gamma. \end{aligned} \tag{20}$$

To verify equation (8), suppose conversely that $c_t = s_t$ so that the first component in (20) has the inequality strict at time t but otherwise holds with equality at $c_{t'} = c_{t'}^* < s_{t'}$. Then $c_t = s_t$ and $s_{t+1} = \widehat{\gamma}s_t$, giving

$$U'(c_t) > \beta U'(c_{t+1}^*) \times \gamma \implies \gamma < \beta^{-1}\frac{U'(s_t)}{U'(c_{t+1}^*)} < \beta^{-1}\frac{U'(s_t)}{U'(\widehat{\gamma}s_t)}.$$

But by $\mathcal{G}'$ this is impossible. Thus, every c_t^* chosen is strictly less than s_t^* giving

$$U'(c_t^*) = \beta U'(c_{t+1}^*) \times \gamma,$$

i.e., equation (8) of the Prop. Next, iterating (3) forwards from $t = 0$,

$$\begin{aligned} s_0 &= (\gamma - \widehat{\gamma})\gamma^{-1}c_0 + \gamma^{-1}s_1 \\ &= (\gamma - \widehat{\gamma})\gamma^{-1}c_0 + (\gamma - \widehat{\gamma})\gamma^{-2}c_1 + \gamma^{-2}s_2 \\ &= (\gamma - \widehat{\gamma})\gamma^{-1}\sum_{t=0}^{\infty}\gamma^{-t}c_t + \lim_{t\to\infty}\gamma^{-t}s_t. \end{aligned}$$

Whenever $\lim_{t\to\infty}\gamma^{-t}s_t > 0$ an allocation can be strictly improved, allowing greater c_t while maintaining (8). Thus, at the socially efficient allocation, $\lim_{t\to\infty}\gamma^{-t}s_t = 0$, verifying (9). Using this in (3) then gives (10). *Q.E.D.*

In (20) the term $(\gamma - \widehat{\gamma})$ simply scales the sequence of marginal utilities, it does not affect their slopes in time. The two-period case in Prop. 3.3, by contrast, conflates the two.

Proof of Prop. 3.7 *First show that at CE $\{p_t^*\}_{t=0}^{\infty}$ firms maximize value by selecting $\{c_t^*, s_{t+1}^*\}_{t=0}^{\infty}$, with result $\{q_t^*\}_{t=0}^{\infty}$. From arbitrary $s_t > 0$ feasible supplies of c are described by*

$$\begin{aligned} s_t &= (\gamma - \widehat{\gamma})\gamma^{-1}c_t + \gamma^{-1}s_{t+1} \\ &= (\gamma - \widehat{\gamma})\gamma^{-1}\sum_{j=0}^{\infty}\gamma^{-j}c_{t+j} + \lim_{j\to\infty}\gamma^{-j}s_{t+j}. \end{aligned} \tag{21}$$

At nonnegative prices p the firm will always choose c to zero out the limiting term on the right side of (21). At the hypothesized CE prices the firm, when it supplies any feasible $\{c_{t+j}\}_{j=0}^{\infty}$, has value

$$\sum_{j=0}^{\infty} p_{t+j}^* c_{t+j} = p_t^* \times \sum_{j=0}^{\infty}\gamma^{-j}c_{t+j}. \tag{22}$$

Evaluate equations (21) and (22) at CE $\{c^*_{t+j}\}_{j=0}^{\infty}$. *The resulting limit term* $\lim_{j\to\infty}\gamma^{-j}s^*_{t+j}$ *equals zero, by the proof of Prop. 3.6, and therefore vanishes from the right side of (21). Comparing (21) and (22) at CE we see that any feasible perturbation in c either lowers the firm's value or keeps that value invariant. Thus, the firm maximizes value by supplying CE* $\{c^*_{t+j}\}_{j=0}^{\infty}$, *with resulting asset prices at CE values* $\{q^*_t\}_{t=0}^{\infty}$ *and asset stocks* $\{s^*_t\}_{t=0}^{\infty}$ *at the SE allocation. Next, show that at CE prices,* $\{p^*_t, q^*_t\}_{t=0}^{\infty}$, *consumers optimize by choosing* $\{c^*_t, s^*_{t+1}\}_{t=0}^{\infty}$. *At time t the consumer's problem has Lagrangean:*

$$\mathcal{L}_t = \sum_{j=0}^{\infty}\beta^j\left[U(c_{t+j}) - \left(q^*_{t+j+1}s_{t+j+1}\right.\right.$$
$$\left.\left. + p^*_{t+j}c_{t+j} - q^*_{t+j}s_{t+j}\right)\times\lambda_{t+j}\right],$$

for $\{\lambda_{t+j}\}_{j=0}^{\infty}$ *a sequence of non-negative multipliers. The first-order conditions are:*

$$U'(c_{t+j}) = \lambda_{t+j}p^*_{t+j}$$
$$\lambda_{t+j}q^*_{t+j+1} = \beta\lambda_{t+j+1}q^*_{t+j+1},$$

satisfied with equality from condition $\mathcal{G}'$. *They imply:*

$$U'(c_{t+j}) = \beta U'(c_{t+j+1})\frac{p^*_{t+j}}{p^*_{t+j+1}} = \beta U'(c_{t+j+1})\times\gamma. \tag{23}$$

Iterating on budget constraint (5) gives

$$q^*_t s_t = \sum_{j=0}^{\infty}p^*_{t+j}c_{t+j} + \lim_{j\to\infty}q^*_{t+j}s_{t+j}$$
$$\Longrightarrow\ (\gamma-\widehat{\gamma})^{-1}\gamma p_t s_t = p_t\sum_{j=0}^{\infty}\gamma^{-j}c_{t+j} + (\gamma-\widehat{\gamma})^{-1}\gamma p_t\times\lim_{j\to\infty}\gamma^{-j}s_{t+j},$$

or

$$s_t = (\gamma-\widehat{\gamma})\gamma^{-1}\sum_{j=0}^{\infty}\gamma^{-j}c_{t+j} + \lim_{j\to\infty}\gamma^{-j}s_{t+j}. \tag{24}$$

By Prop. 3.6 equation (23) is satisfied with sequence c at its SE allocation value. Moreover, Prop. 3.6 guarantees that at the SE allocation, equation (24) holds, with the resulting limiting term $\lim_{j\to\infty}\gamma^{-j}s^*_{t+j}$ *equal to zero, so that the SE allocation* $\{c^*_t, s^*_t\}_{t=0}^{\infty}$ *is affordable.*

Q.E.D.

Proof of Prop. 3.8 *By direct calculation,*

$$q_t = (\gamma-\widehat{\gamma})^{-1}\gamma p_t = (\gamma-\widehat{\gamma})^{-1}(\gamma s_t)p_t s_t^{-1}$$
$$= (\gamma-\widehat{\gamma})^{-1}\left[(\gamma-\widehat{\gamma})\sum_{j=0}^{\infty}\gamma^{-j}c^*_{t+j}\right]p_t s_t^{-1}$$
$$= \left[\sum_{j=0}^{\infty}p^*_{t+j}c^*_{t+j}\right]s_t^{-1},$$

using $\lim_{j\to\infty}\gamma^{-j}s^*_{t+j} = 0$. *Next, from (23) in the Proof of Prop. 3.7,*

$$\frac{\beta^t U'(c^*_t)}{U'(c^*_0)} = \gamma^{-t}.$$

Therefore,

$$V'(s_0) = \sum_{t=0}^{\infty}\beta^t U'(c^*_t)\frac{dc^*_t}{ds_0} = U'(c^*_0)\sum_{t=0}^{\infty}\gamma^{-t}\frac{dc^*_t}{ds_0}$$
$$= U'(c^*_0)\times(\gamma-\widehat{\gamma})^{-1}\gamma$$
$$= q^*_0\mu^{-1} = q^*_0,$$

where the second line above uses equation (4) characterizing the feasible consumption set $\mathsf{C}(s_0)$.

Q.E.D.

Proof of Prop. 4.3 *The SE allocation solves*

$$\sup_{\{c(t):t\geq 0\}}\int_0^{\infty}e^{-\rho t}U(c(t))\,dt$$
$$\text{s.t. } \dot{s}(t) = \gamma s(t) - (\gamma-\widehat{\gamma})c(t)$$
$$\text{given } s(0).$$

The Hamiltonian at time t is then

$$\mathcal{H} = e^{-\rho t}\left[U(c(t)) + \{\, \gamma s(t) - (\gamma - \widehat{\gamma})c(t)\,\} \cdot \lambda(t),\right]$$

for some nonnegative time profile $\{\, \lambda(t) : t \geq 0 \,\}$. *The dynamic first-order condition is:*

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial s(t)} &= -\frac{d}{dt}\left[e^{-\rho t}\lambda(t)\right] \\ \Longrightarrow\ e^{-\rho t}\gamma\lambda(t) &= -\left[e^{-\rho t}\dot{\lambda}(t) - \rho e^{-\rho t}\lambda(t)\right] \\ &= e^{-\rho t}\left[\rho - \dot{\lambda}/\lambda\right]\lambda(t), \end{aligned}$$

so that

$$\dot{\lambda}/\lambda = \rho - \gamma, \tag{25}$$

while the static first-order condition $\partial\mathcal{H}/\partial c(t) = 0$ *implies*

$$U'(c(t)) \geq (\gamma - \widehat{\gamma})\lambda(t), \quad \text{with equality if } c(t) < s(t). \tag{26}$$

To verify the first equation in the Prop., consider a perturbation with $c(t) = s(t)$ *so that* $U'(c(t)) - (\gamma - \widehat{\gamma})\lambda(t) > 0$ *but then converges back to zero. For this, we must have*

$$\begin{aligned} &-R(s(t))\dot{c}(t)/c(t) < \rho - \gamma \\ \Longleftrightarrow\ &\gamma < \rho + R(s(t))\dot{c}(t)/c(t) \leq \rho + \widehat{\gamma}\,R(s(t)). \end{aligned}$$

But by Assumption $\mathcal{G}''$ *this is impossible. Thus, such a perturbation with* $c(t) = s(t)$ *cannot be optimal, and relation (26) must therefore hold with equality. Finally, rewrite the dynamic constraint on s to get*

$$\begin{aligned} &\gamma s(t) - \dot{s}(t) = (\gamma - \widehat{\gamma})c(t) \\ \Longrightarrow\ &-\frac{d}{dt}\left[e^{-\gamma t}s(t)\right] = (\gamma - \widehat{\gamma})c(t)e^{-\gamma t} \\ \Longrightarrow\ &s(0) - \lim_{t\to\infty} e^{-\gamma t}s(t) = (\gamma - \widehat{\gamma})\int_0^\infty e^{-\gamma t}c(t)\,dt \\ \Longrightarrow\ &s(0) = (\gamma - \widehat{\gamma})\int_0^\infty e^{-\gamma t}c(t)\,dt + \lim_{t\to\infty} e^{-\gamma t}s(t). \end{aligned}$$

But if $\lim_{t\to\infty} e^{-\gamma t}s(t) > 0$, *the allocation cannot be maximizing as all* $c(t)$ *can then be increased while maintaining (26) and (25). Therefore,*

$$s(0) = (\gamma - \widehat{\gamma})\int_0^\infty e^{-\gamma t}c(t)\,dt. \qquad \textit{Q.E.D.}$$

The proof for the continuous-time equivalence result that follows uses all the same ideas as its discrete-time counterpart Prop. 3.7.

Proof of Prop. 4.4 *Verify first that firms are maximizing value at the hypothesized* c^* *when consumption and asset prices are set at* (p^*, q^*). *For the firm the technology constraint*

$$\dot{s}(t) = \gamma s(t) - (\gamma - \widehat{\gamma})c(t)$$

means it can supply consumption stream constrained by

$$s(t) = (\gamma - \widehat{\gamma})\int_0^\infty e^{-\gamma t}c(t)\,dt + \lim_{t\to\infty} e^{-\gamma t}s(t). \tag{27}$$

At the hypothesized CE prices p^*,

$$q(t)s(t) = \int_0^\infty p^*(t+r)c(t+r)\,dr = p^*(t)\int_0^\infty e^{-\gamma r}c(t+r)\,dr. \tag{28}$$

At nonnegative prices p the firm will always select c to zero out the limiting term on the right side of (27). Evaluate (27) and (28) at the SE allocation quantities c^*. *Every feasible perturbation in c satisfying (27) either lowers the value (28) or leaves it invariant. Thus, the firm maximizes value at* c^*, *implying* $s(t) = s^*(t)$. *Turn next to show that the consumer's optimal choice is* (c^*, s^*) *given the hypothesized prices* (p^*, q^*). *Rewrite the consumer's budget constraint* (5′) *as*

$$\dot{s} = -\left(\frac{p}{q}c + \frac{\dot{q}}{q}s\right). \tag{29}$$

The Hamiltonian for the consumer's problem is then

$$\mathcal{H} = e^{-\rho t}\left[U(c(t)) - \left(\frac{p(t)}{q(t)}c(t) + \frac{\dot{q}(t)}{q(t)}s(t)\right) \times \nu(t)\right]$$

for some non-negative time profile $\{\nu(t) : t \geq 0\}$*. The first-order conditions* $\partial\mathcal{H}/\partial c = 0$ *and* $\partial\mathcal{H}/\partial s = -\frac{d}{dt}\left[e^{-\rho t}\nu(t)\right]$ *imply:*

$$U'(c) = \frac{p}{q} \times \nu \tag{30}$$

$$\dot{\nu}/\nu = \rho + \dot{q}/q, \tag{31}$$

satisfied with equality from condition $\mathcal{G}''$*. Moreover, (29) implies that the desired consumption rental stream satisfies*

$$s(t) = p(t)q(t)^{-1}\int_0^\infty e^{-\int_0^r \frac{\dot{q}(u)}{q(u)}\,du} \times c(t+r)\,dr. \tag{32}$$

But at the hypothesized CE we have

$$p/q = \gamma - \widehat{\gamma} \quad \text{and} \quad \dot{q}/q = -\gamma$$

so that (30)–(32) become

$$\begin{aligned} U'(c(t)) &= (\gamma - \widehat{\gamma})\nu(t) \\ \dot{\nu}/\nu &= \rho - \gamma \\ s(t) &= (\gamma - \widehat{\gamma})\int_{r=0}^\infty e^{-\gamma r}c(t+r)\,dr, \end{aligned}$$

Setting $\nu = \lambda$ *and using the characterization from Prop. 4.3 these conditions are satisfied at* $c = c^*$ *and* $s = s^*$*.* *Q.E.D.*

Proof of Prop. 4.5 *By direct calculation*

$$\begin{aligned} q(0) &= (\gamma - \widehat{\gamma})^{-1}p(0) = (\gamma - \widehat{\gamma})^{-1}s(0)p(0)s(0)^{-1} \\ &= (\gamma - \widehat{\gamma})^{-1}\left[(\gamma - \widehat{\gamma})\int_{t=0}^\infty e^{-\gamma t}c^*(t)\,dt\right]p(0)s(0)^{-1} \\ &= \left[\int_{t=0}^\infty e^{-\gamma t}p^*(0)c^*(t)\,dt\right]s(0)^{-1} \\ &= \left[\int_{t=0}^\infty p^*(t)c^*(t)\,dt\right]s(0)^{-1}. \end{aligned}$$

Next, from equations (26) and (25) in the Proof of Prop. 4.3, we have

$$\frac{U'(c^*(t))}{U'(c^*(0))} = e^{(\rho-\gamma)t}.$$

Therefore, directly calculating

$$\begin{aligned} V'(s(0)) &= \int_{t=0}^\infty e^{-\rho t}U'(c^*(t))\frac{dc^*(0)}{ds(0)}\,dt \\ &= U'(c^*(0))\int_{t=0}^\infty e^{-\gamma t}\frac{dc^*(0)}{ds(0)}\,dt \\ &= U'(c^*(0)) \times (\gamma - \widehat{\gamma})^{-1} \\ &= q^*(0)\mu^{-1} = q^*(0). \end{aligned}$$

Q.E.D.

The discussion in the text following Prop. 4.5 takes $U'(c^*(0))$ finite. This is, arguably, the interesting case. Having $q^*(0)$ be positive only because its constituent U' diverges to infinity would be too much an artifact arising from a specific functional form. Example 4.6 shows how $U'(c^*(0))$ can remaind bounded even as the copying rate γ grows arbitrarily large.

Example 4.6 (U' bounded as $\gamma \uparrow \infty$) *Suppose that* $R(s)$ *from condition* $\mathcal{G}''$ *is constant, i.e., repeating from Section 3.5,*

$$U(c) = \frac{c^{1-R} - 1}{1 - R}. \tag{7}$$

Repeat the conditions from Prop. 4.3:

$$U'(c^*(t)) = (\gamma - \widehat{\gamma})\lambda(t) \tag{33}$$

$$\dot{\lambda}/\lambda = \rho - \gamma < 0 \tag{34}$$

$$s(0) = (\gamma - \widehat{\gamma})\int_{t=0}^\infty e^{-\gamma t}c^*(t)\,dt. \tag{35}$$

Equation (34) gives

$$\lambda(t) = e^{-(\gamma-\rho)t}\lambda(0),$$

so that from (7) *and* (33) *we have*

$$\frac{U'(c^*(t))}{U'(c^*(0))} = \left[\frac{c^*(0)}{c^*(t)}\right]^R = \frac{\lambda(t)}{\lambda(0)} = e^{-(\gamma-\rho)t},$$

or

$$c^*(t) = c^*(0) \times \exp\left([\gamma - \rho]R^{-1} \times t\right).$$

Using this in (35) *gives*

$$s(0) = (\gamma - \widehat{\gamma})\ c^*(0) \times \int_{t=0}^{\infty} \exp(-\gamma t) \cdot \exp\left([\gamma - \rho]R^{-1} \times t\right)\ dt.$$

The integral on the right converges if:

$$\gamma > \frac{\gamma - \rho}{R} \iff (1 - R^{-1})\gamma > -R^{-1}\rho.$$

This inequality is always satisfied when $R^{-1} \leq 1$, i.e., when demand is intertemporally inelastic. Otherwise, if $R^{-1} > 1$, convergence occurs provided that:

$$\gamma < (R^{-1} - 1)^{-1}R^{-1}\rho,$$

i.e., the copying rate γ must be bounded above. When $R^{-1} > 1$, demand is intertemporally elastic, so that overly high γ has consumption constantly being postponed, falling to zero in the immediate present. In either case, provided the integral converges, equilibrium consumption at time 0 *is:*

$$c^*(0) = (\gamma - \widehat{\gamma})^{-1}\left[(1 - R^{-1})\gamma + R^{-1}\rho\right] \times s(0).$$

When $R^{-1} < 1$, i.e., demand is inelastic, the limit of $c^*(0)$ when $\gamma \uparrow \infty$ is $(1 - R^{-1}) \times s(0) \in (0, s(0))$, so that $U'(c^*(0))$ is strictly positive and bounded. If $R^{-1} = 1$, then $c^*(0)$ converges to 0 when $\gamma \uparrow \infty$. Finally, when demand is elastic, with $R^{-1} > 1$, then as the copying rate increases, i.e., $\gamma \uparrow (R^{-1} - 1)^{-1}R^{-1}\rho$, equilibrium consumption $c^*(0) \downarrow 0$. In either of the last two cases then, $U'(c^*(0))$ grows without bound, implying that $q^*(0) = (\gamma - \widehat{\gamma})^{-1}U'(c^*(0))$ too increases to

infinity. Aside from $R = 1$, however, this occurs when γ remains finite, and so is not a description of infinite expansibility. Perhaps the most that should be said of $R^{-1} \geq 1$ is that competitive equilibrium is not well defined when asset s is close to infinitely expansible. But that is of course also the conclusion from Arrow (1962) and Nordhaus (1969). ∎

Proof of Prop. 5.2 *The characterization is immediate from technology* (12) *and Defn. 5.1.* *Q.E.D.*

Proof of Lemma 5.3 *When $U'(c) = c^{-R}$,*

$$\begin{aligned}\sum_{j=0}^{\infty} ([\gamma + \sigma]\beta)^j \times U'\left([\gamma + \sigma]^{t+j}\right) \\ &= \sum_{j=0}^{\infty} ([\gamma + \sigma]\beta)^j \times (\gamma + \sigma)^{-(t+j)R} \\ &= (\gamma + \sigma)^{-tR} \times \sum_{j=0}^{\infty} \left([\gamma + \sigma]^{1-R} \times \beta\right)^j.\end{aligned}$$

The infinite sum on the right side of the last equation converges when

$$\beta < [\gamma + \sigma]^{R-1}. \tag{36}$$

Since $\beta < 1$ and $\gamma + \sigma \geq \gamma > 1$ condition (36) *holds whenever $R^{-1} \leq 1$. Moreover, since the expression in condition $\mathcal{C}$ can now be rewritten:*

$$\begin{aligned}([\gamma + \sigma]\beta)^t \times \sum_{j=0}^{\infty} ([\gamma + \sigma]\beta)^j \times U'\left([\gamma + \sigma]^{t+j}\right) \\ &= \left([\gamma + \sigma]^{1-R} \times \beta\right)^t \times \sum_{j=0}^{\infty} \left([\gamma + \sigma]^{1-R} \times \beta\right)^j,\end{aligned}$$

condition (36) *also implies the expression's convergence to zero as $t \to \infty$.* *Q.E.D.*

Proof of Prop. 5.4 *Take first the firm's decision. Increasing the supply of c only expands the firm's production set (12). At the hypothesized prices, per unit revenue is*

$$p_t^* - q_{t+1}^*\sigma = \beta^t U'(s_t^*) \times \mu \geq 0.$$

Thus, the firm maximizes value by setting $c^* = s^*$. Next, consider the consumer. At time t the consumer's problem has Lagrangean

$$\mathcal{L}_t = \sum_{j=0}^{\infty} \beta^j \left[U(c_{t+j}) - \left(q_{t+j+1}^* s_{t+j+1} + p_{t+j}^* c_{t+j} \right.\right. \\ \left.\left. - q_{t+j}^* s_{t+j} - q_{t+j+1}^* \sigma c_{t+j} \right) \times \lambda_{t+j} \right],$$

for $\{ \lambda_{t+j} \}_{j=0}^{\infty}$ a sequence of non-negative multipliers. The first-order conditions are:

$$U'(c_{t+j}) = \left(p_{t+j}^* - q_{t+j+1}^* \sigma \right) \lambda_{t+j}$$
$$\lambda_{t+j} q_{t+j+1}^* = \beta \lambda_{t+j+1} q_{t+j+1}^*.$$

Define user cost $\widetilde{p}_t^* = p_t^* - q_{t+1}^*\sigma$ so that the first-order conditions can then be written:

$$\forall\ j = 1, 2, 3, \ldots : \quad \frac{\beta^j U'(c_{t+j})}{U'(c_t)} = \frac{\widetilde{p}_{t+j}^*}{\widetilde{p}_t^*}.$$

From the consumer's budget constraint (13), write:

$$q_t^* s_t = (p_t^* - q_{t+1}^*\sigma) c_t + q_{t+1}^* s_{t+1} = \widetilde{p}_t^* c_t + q_{t+1}^* s_{t+1}$$
$$= \sum_{j=0}^{\infty} \widetilde{p}_{t+j}^* c_{t+j} + \lim_{j \to \infty} q_{t+j}^* s_{t+j}.$$

At the hypothesized CE prices and quantities, however,

$$q_t^* s_t^* = \beta^t \times \left[\sum_{j=0}^{\infty} \beta^j U'(s_{t+j}^*) s_{t+j}^* \right] \times \mu$$
$$= s_0 \left[([\gamma + \sigma]\beta)^t \times \sum_{j=0}^{\infty} ([\gamma + \sigma]\beta)^j \times U'\left([\gamma + \sigma]^{t+j} \times s_0 \right) \right] \times \mu$$
$$\to 0 \text{ as } t \to \infty$$

by condition $\mathcal{C}$, so that $q_t^* s_t^* = \sum_{j=0}^{\infty} \widetilde{p}_{t+j}^* c_{t+j}^*$. Substituting into the above conditions the SE allocation $c_t^* = s_t^* = (\gamma + \sigma)^t s_0$ and prices (p^*, q^*), we see that the SE allocation is affordable and maximizes consumer preferences at the hypothesized CE prices. The collection $\{ p_t^*, q_t^*, c_t^*, s_{t+1}^* \}_{t=0}^{\infty}$ is therefore a CE. Next, from condition $\mathcal{C}$, the value function is bounded, so that its derivative can be calculated explicitly,

$$V'(s_0) = \sum_{t=0}^{\infty} \beta^t U'(c_t^*)\ [\gamma + \sigma]^t = s_0^{-1} \sum_{t=0}^{\infty} \beta^t U'(s_t^*)\ s_t^*$$
$$= q_0^* \mu^{-1} = q_0^*.$$

Q.E.D.

Proof of Prop. 5.5 *Apply Prop. 5.4 and choose numeraire so that μ there equals 1. With $U'(s) = s^{-R}$ and $R^{-1} \leq 1$, we have*

$$\sum_{j=0}^{\infty} \beta^j U'(s_{t+j}^*) \times s_{t+j}^* = \sum_{j=0}^{\infty} \beta^j \left([\gamma + \sigma]^j s_t^* \right)^{-R} \times [\gamma + \sigma]^j s_t^*$$
$$= (s_t^*)^{1-R} \times \sum_{j=0}^{\infty} \left([\gamma + \sigma]^{1-R} \times \beta \right)^j$$
$$= (s_t^*)^{1-R} \times \left[1 - \frac{\beta}{(\gamma + \sigma)^{R-1}} \right]^{-1}.$$

From Prop. 5.4 we can write:

$$q_t^* = (s_t^*)^{-1} \beta^t \times \left[\sum_{j=0}^{\infty} \beta^j U'(s_{t+j}^*) s_{t+j}^* \right],$$

so that

$$q_0^* = \left[1 - \frac{\beta}{(\gamma+\sigma)^{R-1}}\right]^{-1} \times s_0^{-R}$$
$$q_t^* = \beta^t \times \left[1 - \frac{\beta}{(\gamma+\sigma)^{R-1}}\right]^{-1} \times (s_t^*)^{-R}$$
$$= \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^t \times q_0^*.$$

Finally, by direct substitution,

$$p_t^* = \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^t \times s_0^{-R} + \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^{t+1} q_0^*\sigma$$
$$= \left[\frac{\beta}{(\gamma+\sigma)^R}\right]^t \times \left[s_0^{-R} + q_0^* \frac{\sigma\beta}{(\gamma+\sigma)^R}\right]. \qquad \text{Q.E.D.}$$

Proof of Prop. 5.6 *Define user cost to be* $\widetilde{p}_t^\dagger = p_t^\dagger - q_{t+1}^\dagger\widehat{\gamma}$*, and notice that the problems consumers and firms solve here coincide with those in Prop. 3.7, provided user cost* $\widetilde{p}$ *is everywhere substituted for rental price* p*. Thus, CE from Prop. 3.7 gives CE quantities and market-clearing prices here, with* $\widetilde{p}_t^\dagger = p_t^*$*. But then*

$$p_t^* = p_t^* + q_{t+1}^*\widehat{\gamma} = p_t^* + (\gamma - \widehat{\gamma})^{-1}\gamma\gamma^{-1}p_t^*\widehat{\gamma}$$
$$= \left[1 + \frac{\widehat{\gamma}}{\gamma - \widehat{\gamma}}\right] p_t^* = (\gamma - \widehat{\gamma})^{-1}\gamma p_t^* = q_t^*.$$

Finally, since preferences and society-wide technology—and therefore feasible consumption $\mathsf{C}(s_0)$ *and the value function* V*—are unchanged from Props.3.6–3.8, that asset prices* $q^\dagger$ *remain at* q^* *immediately implies* $q_0^\dagger = V'(s_0)$*.* Q.E.D.

Proof of Lemma 6.1 *When* $U'(c) = c^{-R}$*,*

$$\int_{r=0}^{\infty} e^{-(\rho-[\gamma+\sigma])r} U'\left(e^{[\gamma+\sigma](t+r)}\right) dr$$
$$= \int_{r=0}^{\infty} \exp\left[-(\rho - [\gamma+\sigma])r - [\gamma+\sigma](t+r)R\right] dr$$
$$= \exp\left[-(\gamma+\sigma)tR\right] \times \int_{r=0}^{\infty} \exp\left[-(\rho - [\gamma+\sigma](1-R))r\right] dr.$$

The integral on the right side of the last equation converges when

$$-\rho < [\gamma+\sigma](R-1)\ . \tag{37}$$

Since ρ*,* γ*, and* σ *are all positive, condition (37) holds whenever* $R^{-1} \le 1$*. Moreover, since the expression in condition* $\mathcal{C}'$ *can now be rewritten:*

$$e^{-(\rho-[\gamma+\sigma])t} \times \int_{r=0}^{\infty} e^{-(\rho-[\gamma+\sigma])r} U'\left(e^{[\gamma+\sigma](t+r)}\right) dr$$
$$= e^{-(\rho-[\gamma+\sigma](1-R))t} \times \int_{r=0}^{\infty} \exp\left[-(\rho - [\gamma+\sigma](1-R))r\right] dr,$$

condition (37) also implies the expression's convergence to zero as $t \to \infty$*.* Q.E.D.

Proof of Prop. 6.2 *Take first the firm's decision. Increasing the supply of* c *only expands the firm's production set (12′). At the hypothesized prices, revenue flow per unit of* c *supplied is*

$$p^*(t) - q^*(t)\sigma = e^{-\rho t}U'(s^*(t)) \times \mu \ge 0.$$

Thus, the firm maximizes value by setting $c^* = s^*$*. Next, rewrite the consumer's budget constraint (13′) as*

$$\dot{s} = -\left(\frac{p - q\sigma}{q}c + \frac{\dot{q}}{q}s\right). \tag{38}$$

At time t the consumer's problem has Hamiltonian

$$\mathcal{H} = e^{-\rho t}\left[U(c(t)) - \left(\frac{p(t)-q(t)\sigma}{q(t)}c(t) + \frac{\dot{q}(t)}{q(t)}s(t)\right) \times \nu(t)\right]$$

for some non-negative time profile $\{\nu(t) : t \geq 0\}$. The first-order conditions $\partial\mathcal{H}/\partial c = 0$ and $\partial\mathcal{H}/\partial s = -\frac{d}{dt}\left[e^{-\rho t}\nu(t)\right]$ imply:

$$U'(c) = [p - q\sigma] \times \frac{\nu}{q}$$
$$\dot{\nu}/\nu = \rho + \dot{q}/q \implies \frac{\nu(t)}{q(t)} = e^{\rho t}\frac{\nu(0)}{q(0)}.$$

Together, these imply

$$\frac{p(t)-q(t)\sigma}{p(0)-q(0)\sigma} = \frac{e^{-\rho t}U'(c(t)}{U'(c(0))}. \tag{39}$$

Integrate the flow budget constraint (13′) to get

$$q(t)c(t) = \int_{r=0}^{\infty}[p(t+r) - q(t+r)\sigma] \times c(t+r)\,dr + \lim_{r\to\infty} q(t+r)c(t+r).$$

Substitute in (39), evaluate the result at $c = s^*$, and apply condition $\mathcal{C}'$ to get

$$q(t)s^*(t) = \left[\frac{p(0)-q(0)\sigma}{U'(s(0))}\right]e^{-\rho t} \times \left[\int_{r=0}^{\infty} e^{-\rho r}U'(s^*(t+r))s^*(t+r)\,dr\right].$$

For any positive constant μ, set

$$p(0) - q(0)\sigma = U'(s(0)) \times \mu.$$

Then the previous equation gives

$$q(t) = s^*(t)^{-1}e^{-\rho t} \times \left[\int_{r=0}^{\infty} e^{-\rho r}U'(s^*(t+r))s^*(t+r)\,dr\right] \times \mu.$$

The consumer's plans are, therefore, feasible at the hypothesized CE prices p^* and q^*. Evaluate (39) at those prices and verify that $c = s^*$ satisfies the consumer's first-order conditions. Thus, at the hypothesized prices the allocation $c = s^*$ maximizes the consumer's preferences across affordable allocations. Prices (p^*, q^*) together with $c^*(t) = s^*(t) = e^{[\gamma+\sigma]t}s(0)$ therefore constitute CE. Next, from condition $\mathcal{C}'$, the value function is bounded, so that its derivative can be calculated explicitly,

$$\begin{aligned} V'(s(0)) &= \int_{t=0}^{\infty} e^{-\rho t}U'(e^{[\gamma+\sigma]t}s(0))\,e^{[\gamma+\sigma]t}\,dt \\ &= s(0)^{-1}\int_{t=0}^{\infty} e^{-\rho t}U'(s^*(t))\,s^*(t)\,dt \\ &= q^*(0)\mu^{-1} = q^*(0). \end{aligned}$$

Q.E.D.

Proof of Prop. 6.3 *Apply Prop. 6.2 and choose numeraire so that μ there equals 1. With $U'(s) = s^{-R}$ and $R^{-1} \leq 1$, we have*

$$\begin{aligned} &\int_{r=0}^{\infty} e^{-\rho r}U'(s^*(t+r)) \times s^*(t+r)\,dr \\ &= \int_{r=0}^{\infty} e^{-\rho r}\left(e^{[\gamma+\sigma]r}s^*(t)\right)^{-R} \times e^{[\gamma+\sigma]r}s^*(t)\,dr \\ &= s^*(t)^{1-R} \times \int_{r=0}^{\infty} e^{-(\rho+[\gamma+\sigma](R-1))r}\,dr \\ &= s^*(t)^{1-R} \times [\rho + (\gamma+\sigma)(R-1)]^{-1}. \end{aligned}$$

From Prop. 6.2 we can write:

$$q^*(t) = s^*(t)^{-1}e^{-\rho t} \times \left[\int_{r=0}^{\infty} e^{-\rho r}U'(s^*(t+r)) \times s^*(t+r)\,dr\right],$$

so that

$$\begin{aligned} q^*(0) &= [\rho + (\gamma+\sigma)(R-1)]^{-1} \times s(0)^{-R} \\ q^*(t) &= e^{-\rho t} \times [\rho + (\gamma+\sigma)(R-1)]^{-1}\,s^*(t)^{-R} \\ &= e^{-[\rho+(\gamma+\sigma)R]t} \times q^*(0). \end{aligned}$$

Finally, by direct substitution,

$$\begin{aligned} p^*(t) &= e^{-[\rho+(\gamma+\sigma)R]t} s(0)^{-R} + e^{-[\rho+(\gamma+\sigma)R]t} q^*(0)\sigma \\ &= e^{-[\rho+(\gamma+\sigma)R]t} \times \left[s(0)^{-R} + q^*(0)\sigma \right]. \end{aligned}$$

Q.E.D.