A simple theory of the "cubic" laws of stock market activity*

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Abstract

We show empirically a series of sharp patterns in stock market fluctuations, trading activity and their contemporaneous relationships. We link together and explain the following facts: (i) the cubic law of returns: returns follow a power law distribution with exponent 3. This "cubic" law seems to hold both across time and internationally. Stock market "crashes" (e.g. the 1929 and 1987 crashes) are not outliers to this law; (ii) the half cubic law of volumes: volumes follow a power law distribution with exponent 3/2; (iii) the square root law of price impact (the price impact of a volume V is proportional to $V^{1/2}$); (iv) Zipf's law for mutual funds: mutual funds size follow a power law distribution with exponent 1. The model also makes predictions about the cross-conditional relationships between various trading variables. They all appear to be verified empirically. The model makes a series of other, out of sample, testable predictions. Finally, it shows that a Tobin tax, or circuit breakers, do not affect that size of extreme fluctuations. However, a tax that increases with the size of the transactions does reduce the magnitude of those very large fluctuations.

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1 Introduction: New facts and the need for an explanation

1.1 New facts on the distribution of returns, volume, number of trades

This paper offers a theory of the contemporaneous relationships and distributions of volume and volatility. It draws on some known facts, and a series of series of sharp, non-trivial and "universal" facts (see later) on volatility, trading (volumes and number of trades), uncovered by the authors and collaborators[1] (in a time-consuming project using data sets of up to 1 billion points) and proposes a simple explanation for all of those facts. Hence, it provides a precise, quantitative theory of the same-time features of stock market volume and volatility[2]. It will explain all the facts that the reader can already see in the figures of the paper.

Those facts are the following: (i) the cubic distribution of returns, (ii) the half-cubic distribution of volumes, (iii) the square root law of price impact, (iv) Zipf's law for mutual fund sizes, (v) a series of 20 cross-conditional expectations between measure of trading activity: relations of the type $E[Y \mid X]$, for X, Y=return, volume, volume imbalance, number of trades, "net" number of trades (number of "buy" minus number of "sell" orders), and at time functions of those. We will explain (i), (iii) and then show how it implies (i), (ii). The model will also all the fact that go under (v).

The first four laws are the following.

(i) *The cubic law of returns.* Returns are found to have a power law distribution with an exponent of 3. This is true at horizon of 15 min to 1 day to 1 week — beyond that, the central limit theorem imposes convergence to a Gaussian distribution[3]. More precisely, the distribution of return follows:

$$P\left(r > x\right) \sim \frac{1}{x^{\zeta_r}} \text{with } \zeta_r \simeq 3 \tag{1}$$

and we say that the "power law exponent" of returns is $\zeta_r = 3$ (here as in the rest of the paper $\sim$ means asymptotically equal, up to numerical

[1] The initial finding, the cubic distribution of return, was established by Gopikrishnan et al. (1997).

[2] We postpone the time-series extension of our framework to future research. See e.g. Gallant Rossi and Tauchen (199x) for interesting facts on the time-series structure of volatility and volume.

[3] However, long-range correlations in volatility make the convergence slower than in the central limit theorem.

constants[4]). This can be visualized in Figure 1, where $\ln \widehat{r}$ is on the $x-$axis, and $\ln P(r > \widehat{r})$ is on the $y-$axis. The fact that the slope is -3 means that we have $\ln P(r > \widehat{r}) = -3 \ln \widehat{r}$+constant, i.e. (1). There is no tautology that implies that this graph should be a straight line, or that the slope should be -3. A Gaussian would have a concave parabola, not a straight line. Distribution (1) implies the well–known result that returns have "fat tails" (their kurtosis is infinite), but expresses this in a much more precise way. The surprise is that we can do this for individual stocks, stocks of different sizes, and different time periods (see section 2 for a systematic exploration), and we always find $\zeta_r \simeq 3$. We will contend that this fact deserves an explanation.

(ii) *The half-cubic law of trading volume.* By volume we mean the number of shares traded, or the dollar value traded – those measures yield similar results. We find (Gopikrishnan et al. 2000a), that the density satisfies $f(V) \sim V^{-2.5}$, i.e. that the distribution is:

$$P(V > x) \sim \frac{1}{x^{\zeta_V}} \text{ with } \zeta_V \simeq \frac{3}{2}. \tag{2}$$

Figure 1 illustrates this (again, section 2 exposes this more systematically).

We can do the same for the number of trades, and find:

$$P\left(N > \widehat{N}\right) \sim \frac{1}{x^{\zeta_N}} \text{ with } \zeta_N \simeq 3.4 \tag{3}$$

Again, those scalings seem to be stable across different types of stocks, different time periods and time horizons etc. (see section 2).

(iii) *The square root law of price impact.* It says that the price impact Δp of a trade of size V scales as:

$$\Delta p \sim V^{\gamma} \text{ with } \gamma \simeq 1/2.$$

This stands against most microstructure models, which predict a linear price impact.

(iv) *Zipf's law for institutional investors.* Mutual funds follow also Zipf law. As section 2.5 reports, the $n-$th largest (in dollar value of the assets under management) mutual fund has a size $1/n$ (an explanation is proposed in Gabaix, Ramalho and Reuter 2002)

$$P(S > x) \sim \frac{1}{x^{\zeta_S}} \text{ with } \zeta_S \simeq 1 \tag{4}$$

[4]Formally $f(n) \sim g(n)$ means $f(n)/g(n)$ tends to a positive constant (not necessarily 1) as $n \to \infty$.

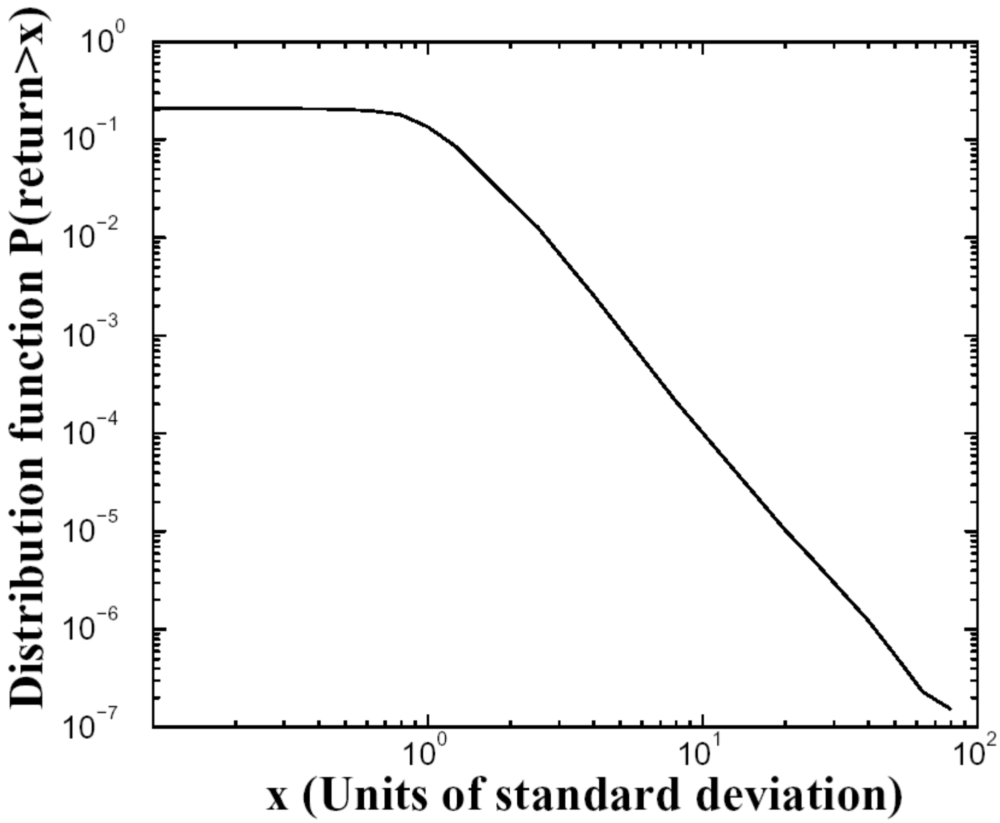


Figure 1: Empirical cumulative distribution of the absolute values of the normalized 15 minute returns of the 100 largest companies in the TAQ database for the 2-yr period 1994–1995. The solid line is a power-law fit in the region $2 < x < 80$. We find $\ln P(r > x) = -\zeta \ln x + b$, with $\zeta = 3.1 \pm 0.1$.This means that returns are distributed with a power law $P(r > x) \sim x^{-\zeta}$ for large x.

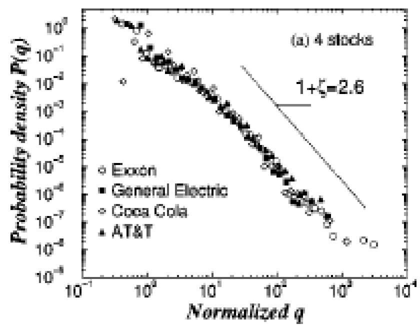


Figure 2: Probability density function of the number of shares q traded, normalized by the average value, for all transactions for the same four actively traded stocks. We find an asymptotic power-law behavior characterized by an exponent ζ_q. Fits yield values $\zeta_q = 1.87 \pm 0.13$, 1.61 ± 0.08, 1.66 ± 0.05, 1.47 ± 0.04, respectively for each of the four stocks. Source: Gopikrishnan et al. (2000a).

1.2 "Universal" laws in economics

The non-trivial challenge for a theory is not to explain some power law behavior (many mechanisms generate power laws), but the *precise (cubic or half-cubic) value* of the exponents. Are those remarkable coincidences, or are those the signs of something important and deep about the formation of prices in the market? We contend that it is the latter. Namely we think that these "cubic" (and half cubic) regularities are part of the few "universal" regularities in economics. Those are now well-documented. One of the oldest is Zipf's law for cities (Zipf 1949, Gell-Mann 1994, Gabaix 1999), which states that, with the above ranking procedure, city number n has a size (number of inhabitants) $S(n) \sim n^{-1}$. This applies in virtually all countries in the world. As such, it qualifies as a "universal" regularity, and begs for an explanation; moreover, because it holds for very different systems (e.g. both in the US in 1990 and India in 1900), its explanation cannot depend on the fine details of the production functions and the transportation costs: it thus allows room for clean theorizing: many details simply cannot matter for the explanations. Further evidence on the existence of universal relationships in economics is provided by Robert Axtell[5] (2001) has shown that Zipf's law

[5] The difference of the Axtell study and previous ones (e.g. Ijiri and Simon 1976) is that is has an essentially complete set of firms: the 5 million firms in the US census, as

holds also for the size of firms ($S(n) \sim n^{-1}$).

We thus propose the following picture: the stock market has "universal" properties with exponents of 3 and 1.5 for returns and volume (and approximately 3 for the number of trades), and these deserve an explanation. We will propose an explanation, which will be based on one power law: the power law 1 of the size of mutual funds. This is the one that generates the 3 and 1.5.

1.3 Fact on conditional expectations of measures of trading activity

Another set of facts will involve the joint distributions of returns, volumes, and their variants. More precisely, we fix a time interval Δt (for instance, Δt =15 min, 1 day, 1 week). We will consider r_t =the (log) return of the assets:

$$r_t = \ln p\,(t) - \ln p\,(t - \Delta t)$$

with the natural adjustment when there are dividends. We also define N_t to be the number of trades in the interval $(t - \Delta t, t], V_t$ (the trading volume) to be the number of shares[6] traded (the volume) in $(t - \Delta t, t]$. We also consider partitions of those quantities in buys and sells. Suppose each trade i, happening at time i, see a quantity of shares exchanged q_i, and set $\varepsilon_i = +1$ is the trade can be identified as buy initiated, $\varepsilon_i = -1$ if it was sell-initiated, $\varepsilon_i = 0$ if no identification is possible[7]. We define N' the net number of trades, i.e. number of "buy" trades minus the number of "sell" trades, and finally

opposed to just the traded firms in Compustat.

[6] The dollar value of the shares traded, or the number of the shares traded over the number of shares outstanding would give the same results.

[7] We identify buyer and seller initiated trades using the bid and ask quotes $S_{\mathrm{B}}(t)$ and $S_{\mathrm{A}}(t)$ at which a market maker is willing to buy or sell respectively . Using the mid-value $S_{\mathrm{M}}(t) = (S_{\mathrm{A}}(t) + S_{\mathrm{B}}(t))/2$ of the prevailing quote, we label a transaction buyer initiated if $S(t) > S_{\mathrm{M}}(t)$, and seller initiated if $S(t) < S_{\mathrm{M}}(t)$. For transactions occurring exactly at $S_{\mathrm{M}}(t)$, we use the sign of the change in price from the previous trade to determine whether the trade is buyer or seller initiated, while if the previous transaction is at the current trade price, the trade is labelled indeterminate.

Following the procedure of Lee and Ready (1991), we use the prevailing quote at least 5 s prior to the trade (see also Ellis et al. 2000). Lee and Ready report that 59.3% of the quotes are recorded prior to trade. They find that using the prevaling quote at least 5 s prior to the trade mitigates this problem. Quotes being recorded ahead of the trade occurs when the specialist calls out the details of the trades and new quotes, and while the trade is entered into the system by a stock exchange employee, the quotes are entered by the specialist's clerk. If the specialist's clerk is faster, the quotes can be recorded before the trade.

V' =net volume=number of shares exchanged that come from a buy order - number of shares exchanged that come from a sell order. Formally:

$$
\begin{aligned}
N_t &= \sum_{\text{trade } \#i \text{ happening in } (t-\Delta t, t]} 1 \\
V_t &= \sum_{\text{trade } \#i \text{ happening in } (t-\Delta t, t]} q_i \\
N'_t &= \sum_{\text{trade } \#i \text{ happening in } (t-\Delta t, t]} \varepsilon_i \\
V'_t &= \sum_{\text{trade } \#i \text{ happening in } (t-\Delta t, t]} \varepsilon_i q_i
\end{aligned}
$$

We plotted the graphs $E[Y \mid X]$ for many combinations of $X, Y = r$, V, V', N, N', N'/N, V/N. (The choice was guided by an assessment of how "meaningful such a graph would be", and how non-trivial the shape predicted by the theory would be). The model replicates all the shapes of those graphs, as can be seen in Figures 5 through 9 below.

Section 2 presents the "power law" empirical facts in more detail. Sections 3 and 4 expose the model we propose to explain them. Section 5, the second main empirical section, shows how the model also explains the many $E[Y \mid X]$ graphs we have found. Section 6 relates our approach to the literature. Section 7 concludes.

2 The empirical facts

2.1 Data used

We are primarily using the 1 billion points in the TAQ database: all the trades (about 1000/day for each stock), for 1000 stocks, in 4 years (about 1000 trading days) (1994-1997). This represents 70 gigabytes total. We aggregate them other different horizons: 15 minutes, 1 hr, 1 day, 1 week. (Over longer horizon, most distributions converge to a Gaussian, as they should as they have finite variance). Because we deal with power law exponents,

Trades occurring within the S_{B} and S_{A} or at the mid-quote S_{M} arise when a market buy and sell order occur simultaneously, or when the specialist or floor brokers with standing orders respond to a market order by bettering the quote. According toLee and Ready (1991) the latter is more often the case. In our case, an average of $\approx 17\%$ of the trades is left indeterminate. See also Harris (1989).

we need lots of points to get enough resolution. There is a quite sizable dispersion of measured exponents from stock to stock, and this is why we need so much data. The theory would predict this too. For instance, suppose returns are distributed according to the density proportional $1/\left(1+r^2\right)^2$, looking at the top 1% of 40,000 points, the standard deviation of measured exponents is about .5, for a mean of about 3.

To compare quantities across different stocks, we normalize them by the second moments in they exist, otherwise by the first moments. For instance, for a stock i, we consider the returns $r'_{it} := (r_{it} - r_i)/\sigma_{r,i}$, where r_i is the mean of the r_{it} and $\sigma_{r,i}$ is their standard deviation. For the volume, which has an infinite standard deviation, we consider the normalization: $V'_{it} := V_{it}/V_i$, where V_i is the mean of the V_{it}. This is explained more systematically in section 12.

2.2 The cubic law of price fluctuations: $\zeta_r \simeq 3$

In the introduction, we argued that price fluctuations had an exponent 3: $\zeta_r = 3$. We show here the robustness of this finding.

Note that this exponent of 3 rejects Mandelbrot's "Paretian" hypothesis, as well as his explanation. Mandelbrot (1963) argued that we had $\zeta_r \in (1,2)$, and this seemed to be the case for cotton prices, and gave the nice explanation that this came from a large number of shocks with infinite variance: by Lévy's theorem, this should give rise to a Lévy distribution of price fluctuation, with (necessarily) an exponent $\zeta_r < 2$. But here we firmly reject $\zeta_r < 2$.

We can look at other stock markets. Gopikrishnan et al (1999a) report also the values of ζ for the daily returns of the NIKKEI index (1984-97) and of the Hang-Seng index (1980-97), and find:

$$\zeta_r = \begin{cases} 3.05 \pm .16 \text{ (NIKKEI)} \\ 3.03 \pm .16 \text{ (Hang-Seng)} \end{cases}$$

The cumulative distributions are plotted in (Fig 9 PRE Indices)

With 10 countries (Australia, Canada, France, Germany, Japan, Honk-Kong, Netherlands, South Korea, Spain, United Kingdom), we find for the mean $\zeta_r = 2.9$ with a standard deviation of .10.[show a plot].

Having checked the robustness of the $\zeta = 3$ finding across different stock markets, we look at firms of different sizes. Small firms have a higher volatility than big firms, as is verified in Figure **??**(a). But the same Figure also

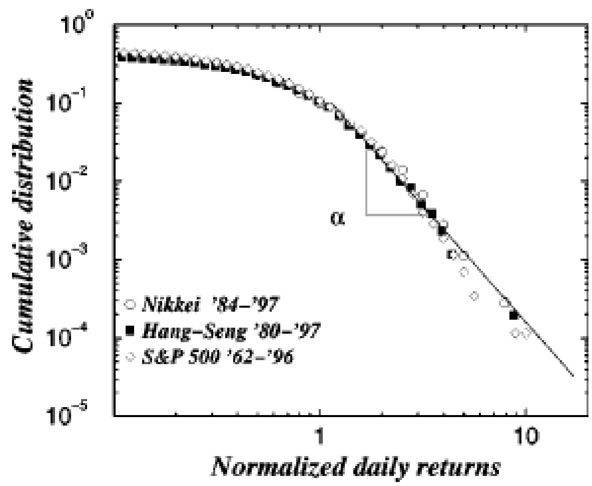
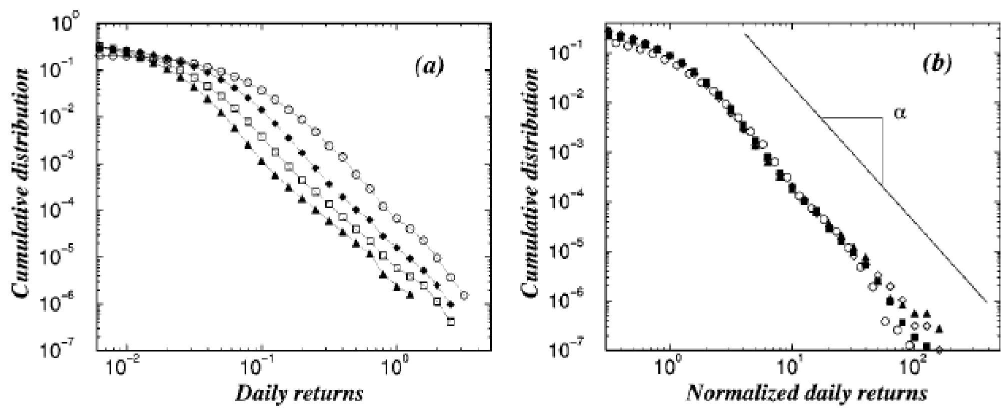


Figure 3: Zipf plot for the daily fluctuations in the Nikkei (1984-97), the Hang-Seng (1980-97), and the S&P 500 (1962-96). The apparent power-law behavior in the tails is characterized by the exponents $\alpha \simeq 3.05 \pm 0.16$ (NIKKEI), $\alpha \simeq 3.03 \pm 0.16$ (Hang-Seng), and $\alpha \simeq 3.34 \pm 0.12$ (S&P 500). The fits are performed in the region g>1. Source: Gopikrishnan et al. (1999).

shows a similar slope. Indeed, when we normalize the distribution by a common standard deviation, we see that the plots collapse, and the exponents are very similar, around $\zeta = 3$ again.



(a) Cumulative distribution of the conditional probability $P(r > x)$ of the returns for companies with starting values of market capitalization S for $\Delta t = 1$ day from the CRSP database. We define uniformly spaced bins on a logarithmic scale and show the distribution of returns for the bins, $S \in (10^5, 10^6]$, $S \in (10^6, 10^7]$, $S \in (10^7, 10^8]$, $S \in (10^8, 10^9]$.(b) Cumulative conditional distributions of returns normalized by the average volatility σ_S of each bin. The plots collapsed to an identical distribution, with $\alpha = 2.70 \pm .10$ for the negative tail, and $\alpha = 2.96 \pm .09$ for the positive tail. Source: Gopikrishnan et al. 1998.

2.2.1 Evidence on the cubic law from the foreign exchange market

There exist many studies of the power law exponent of foreign exchange fluctuations. The most comprehensive is probably Guillaume et al. (1997), whose Table 3 we reproduce here in Table 2.2.1. The standard errors on ζ are big, which makes any conclusion difficult. Someone skeptical of the cubic law wouldn't be more persuaded, but is still the ζ are tantalizingly compatible with a true $\zeta = 3$.

Rate	10m	30m	1h	6h
USD DEM	3.11 ±0.33	3.35 ±0.29	3.50 ±0.57	4.48 ±1.64
USD JPY	3.53 ±0.21	3.55 ±0.47	3.62 ±0.46	3.86 ±1.81
GBP USD	3.44 ±0.22	3.52 ±0.46	4.01 ±1.09	6.93 ±10.79
USD CHF	3.64 ±0.41	3.74 ±0.82	3.84 ±0.77	4.39 ±4.64
USD FRF	3.34 ±0.22	3.29 ±0.47	3.40 ±0.69	4.61 ±1.21
FRF DEM	3.11 ±0.41	2.55 ±0.23	2.43 ±0.23	3.54 ±1.42
DEM NLG	3.05 ±0.27	2.44 ±0.08	2.19 ±0.12	3.37 ±1.43
DEM ITL	3.31 ±0.51	2.93 ±1.17	2.54 ±0.49	2.86 ±0.98
GBP DEM	3.68 ±0.35	3.63 ±0.42	4.18 ±1.17	3.22 ±0.79
DEM JPY	3.96 ±0.41	4.18 ±0.90	4.13 ±1.05	4.71 ±1.61

Table : Estimated tail exponent ζ for exchange rate fluctuations

Estimated tail exponent and its standard error for the main FX rates against the USD and some of the main (computed) cross-rates against the DEM. The results are taken from Dacorogna et al. (1994). The bias was estimated using a bootstrap method. In contrast to quoted cross-rates, computed cross-rates are obtained via the two bilateral rates against the USD. Their spread is thus approximately twice the normal spread.

2.3 The half cubic law of volume: $\zeta_V \simeq 3/2$

Trying to understand the origins of the cubic law for returns, in Gopikrishnan et al. (2000a) we looked at the distribution of volume[8], and found[9] that we got an exponent around 1.5: $\zeta_V \simeq 1.5$. (More precisely, we found $\zeta_V = 1.53 \pm .07$). Figure 2 shows the density of of the volumes for four different stocks. Note that a cumulative distribution of $P(q > \widehat{q}) \sim \widehat{q}^{-\zeta_q}$ implies for a density function (the derivative of the cumulative) $p(q) \sim \widehat{q}^{-\zeta_q+1}$, so that a slope of 2.6 in the density implies a power law exponent of 1.6 in the cumulative. Maslov and Mills (2001) find likewise $\zeta_V = 1.4 \pm .1$ for the volume of market orders.

[8] For stock i, we look at the fluctuations of V_{it}/V_i, where V_i is the mean volume V_{it}. This makes stocks comparable, and the "number of shares" and "dollar volume" would give equivalent measures of volume.

[9] Gopikrishnan et al. (2000a) establish the exponent of 1.5 via an other method, the scaling of the moments of the volume, which is of generatl methodological interest.

2.4 The (roughly) cubic law of number of trades: $\zeta_N \simeq 3.4$

In Plerou et al., we looked (among other things) at the distribution of the number of trades N, and found an exponent $\zeta_N = 3.4 \pm .05$.Our theory will predict the exponent of 3. However, interestingly, in the simulations even under the null of our theory, an exponent large than 3 appears (typically, we found an exponent around 3.2): this suggests that empirical exponents will be biased upward. All in all, we consider that the empirical value of ζ_N is around 3,

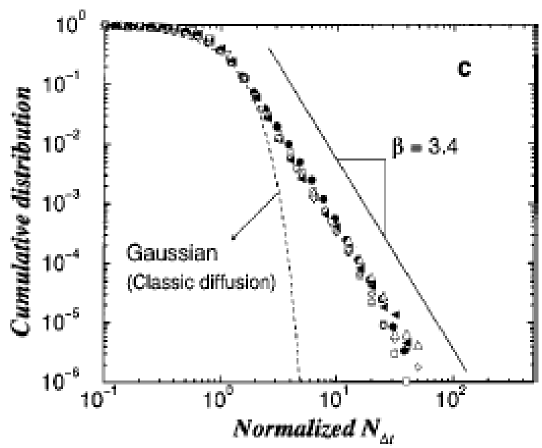


Figure 4: Cumulative distribution of the normalized number of transactions $n_{\Delta t} = N_{\Delta t}/\langle N_{\Delta t}\rangle$. Each symbol shows the cumulative distribution $P(n_{\Delta t} > x)$ of the normalized number of transactions $n_{\Delta t}$ for all stocks in each bin of stocks sorted according to size. An analysis of the exponents obtained by fits to the cumulative distributions of each of the 1000 stocks yields an average value $\zeta_n = \beta = 3.40 \pm 0.05$.

Cumulative distribution of the normalized number of transactions $n_{\Delta t} = N_{\Delta t}/\langle N_{\Delta t}\rangle$. Each symbol shows the cumulative distribution $P(n_{\Delta t} > x)$ of the normalized number of transactions $n_{\Delta t}$ for all stocks in each bin of stocks sorted according to size. An analysis of the exponents obtained by fits to the cumulative distributions of each of the 1000 stocks yields an average value $\zeta_N = \beta = 3.40 \pm 0.05$.

2.5 The unitary (Zipf) law of the size distribution of mutual funds: $\zeta_S \simeq 1$

Looking for an explanation to the above "cubic" (and half-cubic) power laws, we looked at the distribution of the size of the mutual funds. From Morningstar one gets the size (dollar value of the assets under management) of all the mutual funds[10]. Using the usual technique, we rank them by size (the largest fund being fund #1), and do a "log rank vs log size" regression. The distribution follows a power law, whose slope can be found by a linear regression:

$$\ln \text{Rank} = \underset{(.003)}{-1.04} \ln \text{Size} + \text{constant}$$
$$R^2 = .98$$

if we regression over funds the top 2000 funds. The slope of the regression gives an estimate of the power law exponent of the distribution, and we conclude

$$\zeta_S = 1 \tag{5}$$

This is the same exponent as the cities (Zipf 1949, Gabaix 1999) and business firms (Axtell 2001).

For the purpose of this paper, one can take this distribution of the sizes of mutual funds as a given. It is in fact not difficult to explain: one can transpose the explanations given for cities (Gabaix 1999, updated in Gabaix 2001b) to mutual funds: a log normal process with small perturbation to ensure convergence to a non-degenerate steady-state distribution explain the power law distribution with an exponent of 1. Gabaix, Ramalho and Reuter (2002) show that those assumptions are verified empirically.

It is only recently (say in the past 30 years) that mutual funds have come to represent a large part of the marketplace. For the earlier periods, the theory will work if the distribution of large agents still follow Zipf's law. We do not have direct evidence for this, but a very natural candidate would be the pension funds of corporations. Axtell (2001) showed the distribution of U.S. firms sizes follows Zipf's law with $\zeta_S = 1$. Takayasu (200x) has shown this for Japan. It is very likely that the size of the pension fund of a firm of S employees is proportional to S, so that their pension funds also follow Zipf's law with a slope of 1.

[10] The say 200 funds of Fidelity, for instance, count as 200 different funds, not as one big "Fidelity" fund.

2.6 The facts: conditionals

We plotted $E[Y \mid X]$ for lots of $X, Y = r$, V, V', N, N', N'/N, and powers of those. The graphs are reported in section 5 later in the paper, when we compare them to the theory's predictions.

3 The theory, assuming the square root law of price impact

3.1 Sketch of the theory

We understand the exponent of the size of mutual funds, $\zeta_S = 1$ (a fact we establish here empirically, and explained in Gabaix, Ramalho and Reuter 2002). We will show how it generates, through intelligent[11] tactical behavior of the traders, the exponents $\zeta_V = \zeta_{V'} = 3/2$ for the volume, and $\zeta_r = \zeta_N = \zeta_{N'} = 3$ for the return and number of trades.

Broadly speaking, the theory works like this: large volumes and large returns are created by the decisions to trade of large agents. The power law exponent is 1, and this is what gives rise to the cubic and half cubic laws. When a large agents wants to trade (this is due to some news that happened, or maybe results from a new "strategic orientation" he decided to take), his desired quantity is sizable compared to say the daily turnover (this is the case empirically, see section 4.4.2: say the 50th biggest mutual fund, if it wants to increase its holdings of say Apple by 50%, will have to absorb the whole daily turnover of Apple). Hence he knows that he will move the market. He knows that, if he wanted to realize his trade (say, a sell order) in the next 30 seconds, he would need to accept a big price discount. However (for many possible reasons) he doesn't want to wait too long before realizing his trade. When, while paying attention not to pay too many transactions costs over the year, he minimizes the waiting sum of execution time and transactions costs, he will trade in such a way that generates the exponent of 1.5 for volume, and an exponent of 3 for returns and the number of trades.

[11] Our traders, though intelligent (they try to avoid too high trading costs), are not necessarily hyperrational. They may trade too much, like in Daniel et al. (1999) and Odean (2000).

3.2 Linking the cubic exponent of returns and the half-cubic law of volume: he square-root law of price impact

3.2.1 Theory

We will examine the view that prices move because of trades. If we hypothesize a price impact function:

$$r = cV^{\gamma} \tag{6}$$

for some number c, and if we assume that large movements in prices are caused by large trades, then we will have, by the formula (35) in Appendix A:

$$\zeta_r = \zeta_V / \gamma \tag{7}$$

Given empirically $\zeta_r = 3$ and $\zeta_V = 3/2$, we have led to hypothesize $\gamma = 1/2$, i.e. a square-root price impact:

$$r = hV^{1/2} \tag{8}$$

For ease of reference, we gather this in the following

Proposition 1 *Suppose that we have (i) the half-cubic law of volumes $\zeta_V = 3/2$ and (ii) the square root law of price impacts. Then we have the cubic law of returns:*

$$\zeta_r = 3.$$

Proof. As $r = hV^{1/2}$,

$$\begin{aligned} P(r > x) &= P\left(hV^{1/2} > x\right) = P\left(V > (x/h)^2\right) \\ &\sim \left((x/h)^2\right)^{-3/2} \sim x^{-3}. \end{aligned}$$

■

3.2.2 Evidence on the square root law of price impact

Starting with Hasbrouck (1991), the empirical literature finds an increasing and concave price impact, i.e. supports $\gamma \in (0,1)$. This is qualitatively encouraging, but our purposes require a more direct test of $\gamma = 1/2$. To do this, we start from the benchmark were, in a given time interval, n i.i.d.

blocks are sold, with volumes $V_1, ..., V_n$, with i.i.d. signs $\varepsilon_i = \pm 1$, so that the return in the period is:

$$r = h \sum_{i=1}^{n} \varepsilon_i V_i^{1/2}$$

Then

$$\begin{aligned} E\left[r^2 \mid V\right] &= h^2 E\left[\sum_{i=1}^{n} V_i + \sum_{i \neq j} \varepsilon_i \varepsilon_j V_i V_j \mid V\right] \\ &= h^2 V + 0 \end{aligned}$$

i.e.

$$E\left[r^2 \mid V\right] = h^2 V \tag{9}$$

We get a simple conclusion that $E\left[r^2 \mid V\right]$ should be linear in V, at least for large $V's$. A more precise prediction is possible. Equation (9) predicts $\left\langle r^2 \right\rangle = h^2 \left\langle V \right\rangle$, i.e.

$$E\left[r_i^2/\sigma_r^2 \mid V_i\right] = 1 \cdot V_i / \left\langle V \right\rangle$$

i.e., in the graph $E\left[r^2 \mid V\right]$ in normalized units (normalizing V by its mean, r by its standard deviation, as we do throughout the empirical analysis of this paper), the relationship should be roughly linear with a slope of 1. Fig. ?? [insert it] shows the empirical result. Indeed we find a relationship $E\left[r^2 \mid V\right] = aV + b$ with a close to 1, and no clear sign of concavity or convexity (which would respectively imply $\gamma < 1/2$ and $\gamma > 1/2$). We thus name (8) as a good candidate for an empirical law, and dub it the "square root law of price impact".

We now a way to understand $\zeta_r = 3$. It comes, we hypothesize, from $\zeta_r = 3/2$, and the square root price impact of trades (8). We now have to try to explain each of those two regularities.

3.3 The main result

If each fund i of size S_i traded, at random, a volume V_i proportional to S_i (we write this $V_i = a_i S_i$), then the distribution of individual volumes would follow $\zeta_v = \zeta_S = 1$. The trading volume in an interval Δt would be proportional to

$$V = \sum_{i \text{ who has traded}} a_i S_i$$

hence, by summation properties mentioned in Appendix A, we would also have $\zeta_V = \zeta_S = 1$. However, $\zeta_V = 3/2$.

The distribution of volumes is less fat-tailed that the distribution of size. This means that large traders trade less often than small traders, or that, when they trade, they trade in volume less than proportional to their sizes. Why would that be the case? Intuitively, a likely reason is that large traders have large price impacts, and have to moderate their trading to avoid paying too large costs of price impacts.

This suggest that an important quantity is the amount of transaction costs paid by the funds (as a fraction of the portfolio), i.e.

$$c = \frac{\text{Amount lost by the fund in price impact}}{\text{Value of the assets under management } S \text{ of the fund}}$$

We call $c(S)$ the average proportional amount of funds of size S. For instance, if funds of size S pay on average 1% in price impact a year, $c(S) = 1\%$.

This motivates our next:

Theorem 2 *If the following conditions hold:*

- *(i) Zipf's law for institutional investors:* $\zeta_S = 1$
- *(ii) Square root price impact*

$$\Delta p \sim V^{1/2}$$

- *(iii) Funds trade in volumes*

$$V \sim S^{\delta} \tag{10}$$

  for some $\delta > 0$

- *(iv) Fund adjust trading frequency and/or volume so as to pay an transactions costs*

$$c(S) = C \tag{11}$$

  (homogenous willingness to pay transactions costs).

Then returns and volumes are power law distributed with the cubic and half-cubic exponents:

$$\begin{aligned} \zeta_r &= 3 \\ \zeta_V &= 3/2 \end{aligned}$$

We comment on the hypotheses before proceeding to the proof.

We have discussed (i) above. Gabaix, Ramalho and Reuter (2002) show this empirically. Random growth models (Gabaix 1999) offer an explanation for Zipf's law.

We have shown above the validity of (ii). In the next section we will propose an explanation for it.

(iii) is given by virtually any model. The most natural guess would be $\delta = 1$. Later we present a model with $\delta = 2/3$.

(iv) is newer. It means that large and small funds pay roughly similar price impact costs a year, e.g. 1%. An alternative would be that $c(S)$ would increase with S, e.g. $c(S) \sim S$. This would mean that if fund A is ten times as big as fund B, and $c(S_B) = 1\%$, then A pays $c(S_A) = 10\%$/year in transaction costs. We expect such a fund A to be eliminated by the market after a couple of years. We think $c(S) = C$ independent of S will be assured by evolutionary forces[12]: a fund trading so much would likely be eliminated from the market place, as most funds, in reality (see e.g. Carhart 1997) and in this model, traders do not possess superior information or insight about the markets. (Another, very related, reason might be due to "prudential guidelines" in the fund – e.g. Vanguard's funds).

We do not have direct evidence on (iv). Some indirect evidence can be gathered by talking to traders: They know that large funds impact the market more, and will have to be more "prudent" (reduce the size of their trades, trade more slowly or more infrequently) to avoid moving the market too much against them. Gabaix, Ramalho and Reuter (2002) provide some indirect evidence, as they document that, except for very small fund, small to large funds have the similar average returns.

We now proceed to the proof of the proposition.

Proof. Each time incurs a price impact proportional to $V \cdot \Delta p$. Given (ii) and (10), this cost is $V^{3/2}$. If $F(S)$ is the an annual frequency of trading, the total annual dollar lost in transactions costs is $F(S) \cdot V^{3/2}$, i.e. a fraction

$$c(S) = F(S) \cdot V(S)^{3/2} / S \tag{12}$$

[12] For lack of space, we do not offer a formal justification for this, though this could easliy be done in a richer model.

of the value S of this portfolio. Hypothesis (iii) gives that either $V(S)$ or $F(S)$ will adjust so as to satisfy:

$$F(S) \sim S \cdot V^{-3/2}. \tag{13}$$

By (i), the number of traders with size bigger than S is $G(S) \sim 1/S$. So the density of traders of size S is $\rho(S) = -G'(S) \sim 1/S^2$. We then observe that volumes $V > x$ correspond to traders of size S such that $S^\delta > x$ by (iii); that there is a number $\rho(S) \sim S^{-2}$ of traders of size S ; and that they trade with frequency $F(S)$ given in (13); so:

$$\begin{aligned} P(V > x) &\sim \int_{S^\delta > x} F(S)\,\rho(S)\,dS \sim \int_{S > x^{1/\delta}} S^{1-3\delta/2} S^{-2} dS. \\ &\sim \left[-S^{-3\delta/2}\right]_{x^{1/\delta}}^{\infty} = \left(x^{1/\delta}\right)^{-3\delta/2} = x^{-3/2}. \end{aligned}$$

so that we find a power law distribution of volumes with exponent:

$$\zeta_V = \frac{3}{2}. \tag{14}$$

Given Proposition 1, this completes the proof. ■

3.4 A model that illustrates Theorem 2

We gather the theory as it looks so far. Managers have i.i.d. "illuminations". There are uncorrelated across managers. They think that there is a mispricing M, i.e. that the asset will have excess returns of size M. We here take $M > 0$, so that the manager want to buy the asset. The same reasoning would go through with $M < 0$, and then the managers would like to sell this asset. Those illuminations arrive $F^{\max}$ times a year and managers trade on them $F \leq F^{\max}$ times a year.

The dollar profit from a trade of size V is: $V(M - \Delta p)$. As the manager trades F times a year, the annual profit is $F \cdot V(M - \Delta p)$. Note that here we abstract from the time to execute the trade, a consideration we'll focus on later. We also use the "Homogenous transaction costs" constraint (11) and (12), which becomes our constraint (16), so that the trader's program is:

$$\max_{F,V,\Delta p} F \cdot V(M - \Delta p) \tag{15}$$

$$\begin{aligned} \text{s.t. } F \cdot \Delta p \cdot V/S &\leq C && (16) \\ F &\leq F^{\max} && (17) \\ \Delta p &= hV^{1/2} \end{aligned}$$

The solution is given by

Proposition 3 *The solution of the problem gives* $V \sim S^{\delta}$ *with* $\delta = 2/3$ *in Theorem 2, hypothesis (iii).*

Proof. Immediate. Constraint (16) binds, and the square root law give $F \cdot V^{3/2} = CS$, i.e. $V \sim S^{2/3}$. ■

Other models could predict different values of δ in (10). For instance, if $M(F) = F^{-\alpha}$ (if one trades less frequently, one trades only on the best opportunities), then one can get any $V = S^{\delta}$ with $\delta \in [2/3, 1]$.

4 A stylized microstructure model yielding a square root law of price impact

We have argued that the hypothesis of Theorem 2 are valid empirically. We have economic rationales for them, except hypothesis (iii), the square root price impact (8). The basic microstructure models (e.g. Kyle 1985) give a linear[13] price impact $\Delta p \sim V$. Virtually all the empirical evidence (e.g. Hasbrouck 1991), however, shows a concave price impact. In this section, we propose an explanation for (8). It is fairy independent of the rest of the theory.

Put succintly, the argument is the following. In our model, a trader who is willing to wait for an amount of time T to realize his trade will incur a price impact:

$$\Delta p = \frac{V}{T} \tag{18}$$

so that Kyle's linearity of the price impact in V holds, *but for a given* T. But more patient traders, who are willing to wait for a larger time T, will be have a lower price impact. Suppose that asset's price is increasing at a rate μ. The total benefit from the trade of size V will be:

$$B = V\left(M - \mu T - \Delta p\right) \tag{19}$$

as the trader will enjoy the excess return μ for a time $L - T$, and will pay a price concession Δp. The optimal T satisfies

$$\max_{\Delta p} B \text{ s.t. } (18)$$

so that

$$\Delta p = \arg\max_{\Delta p} V\left(M - \mu\frac{V}{\Delta p} - \Delta p\right)$$

which gives

$$\Delta p \sim V^{1/2}$$

and $T \sim V^{1/2}$. In words, Hence the price impact of a trade of size V scales like less than V, as large traders are willing to pay more in time to execution to moderate the time to execution. When a linear cost of time to execution, we get a price impact (8). To execute a trade of size V, people are on average more patient, and wait for a time $T \sim V^{1/2}$, so that (18) gives (8).

We now proceed to fleshing out the model that gives (18).

[13] A concave price-volume is given by Seppi (1990), Barclay and Warner (1993), and Keim and Madhavan (1996). Those models do not predict a square root, however.

4.1 A stylized microstructure model, for a given large target volume

At a given point in time, a trader can be in one of 3 different states: an "active", "causal" [find better name] trader, or a liquidity provider to an active trade, or he may just be dormant – i.e. not participating in the market.

To keep it tractable, our model is very "stylized", and highlights the role of heterogeneous sizes of the agents. Other, much more refined models, include Bertsimas and Lo (1998), He and Wang (1995), Saar (2001), Vayanos (2001).

The behavior of liquidity providers A liquidity provider is willing to supply a number q of shares, for a price increase Δp, with the supply function:

$$q = \alpha \cdot s \cdot \Delta p \tag{20}$$

with α some proportionality factor. We can give two justifications for this kind of rule:

(i) This is a plausible rule, which might be derived as the first order term of the Taylor expansion of the solution of a complex optimization problem

(ii) It turns out that there is a clean justification from portfolio theory: Suppose that the provider of size s is at his optimum holding of stocks. If he sells a quantity q of shares, he will be off by an amount q, which causes, in standard portfolio theory, a second order loss of value (a dimensionless constant times) $(q/s)^2$, which gives a dollar equivalent of $(q/s)^2 s = q^2/s$. In the transaction however, he makes a profit Δpq. He is indifferent if $q^2/s = q\Delta p$, i.e. $q = s\Delta p$, which is the supply curve above[14].

To complete the model, we need to take a stand on the "permanent" part of the price impact. If the price was \$1, and the new trading price was $\$1 + \Delta p$, what is the next "reference price" (for instance the mid quote price?). In general this would depend on assumption about the information and the rationality of the traders. Here, to close the model in a simple way, we will just assume the new reference price will be $p' = p + b \cdot \Delta p$ for some

[14] This assumes that traders are, largely, close to their optimum holding. But after a big target trade has been absorbed by the market, this cannot be the case any more. In practice, liquidity providers, over the next few weeks or months, readjust their portfolio. We will not explicitly formalize this in the model – it would add a lot of complexity to the model and would be largely a side show.

(possibly stochastic) b with $E[b] > 0$ and b has not too fat[15] tails ($\zeta_b \geq 3$). For instance, the new reference price could be the last trading price ($b = 1$) or say be $p' = p + .9\Delta p$ ($b = .9$). We will not seek here to endogenize this as an optimal response to information revealed by the trade[16], though that would be an interesting question for future research.

The behavior of active traders Consider a big, "active" trader A who wants to make a trade of size V, and for that uses a limit order (maybe in the upstairs market). We will determine the optimal limit order, but for this we first have to describe the trading environment.

Say that this is a buy order for concreteness. It happens at time 0. Liquidity providers indexed by i emerge stochastically from their dormant state [another interpretation is: the active trader A has, perhaps via his broker, to call successively potential buyers, which takes some time: after a time t, one could call a number proportional to t of potential traders], and so are happy to sell the active trader a quantity $q_i = \alpha s_i \Delta p$. So after a time t, the active traders A can buy

$$Q(t) = \sum_{\text{traders } i \text{ that have appeared between } 0 \text{ and } t} \alpha s_i \Delta p.$$

The search process stops when the desired quantity is reached, i.e. at the smallest t such that $Q(t) \geq V$.

The execution will require a time T such that

$$\sum_{i \text{ arrives between } 0 \text{ and } T} \alpha s_i \Delta p = V \tag{21}$$

(rigorously, it will be the smallest T such that in the equation above the left hand side is no smaller than the right hand side). Taking expectations given $\Delta p, V$ on both sides of (21), with f being the (Poisson) frequency of arrival of liquidity providers: $fE[T \mid \Delta p, V] E[s] = V$, so the average time

[15] Indeed, if b had extremely fait tails ($b < 3$), then the biggest fluctutations of returns would be due to fluctuations in b: we would have $\zeta_r = \zeta_b < 3$.

[16] Indeed, we leave open the possibilty that those price adjustment are not a rational response to some "information" that the trader would have. A simpler interpretation might be that, given the difficulty of assessing what the "fair" price of say IBM is, to within say a 20% accuracy, using the last trade price as a benchmark is simply a good rule of thumb, or maybe focal point for the coordination of where future prices might go. These issues would obvisouly benefit from a separate treatment.

required will be

$$T = \frac{V}{\alpha f \langle s \rangle \Delta p} \tag{22}$$

$$T \sim \frac{V}{\Delta p}. \tag{23}$$

Remark: to increase descriptive realism, we could assume that the Poisson arrival rate f itself increases with the price offered $|\Delta p|$, as in $f(\Delta p)$. It is easy to verify that our results for the power laws will not change if the function f increases, for large $|\Delta p|$, like less than a power law (e.g. the arrival rate tends to saturation maximal value as $|\Delta p| \to \infty$, or say $f(\Delta p) \leq a \ln(b + |\Delta p|)^k$ for some finite a, b, k).

Traders will, in practice, want to minimize two quantities: the execution time $E[T \mid \Delta p, V]$, and the execution cost $|\Delta p|$. From above, their objective function for the given trade is:

$$B = V(M - \mu T - \Delta p) \tag{24}$$

Indeed, as above, the trader expects the asset to have excess returns μ for up to time L. But he will be able to enjoy this excess return only up from time T to L. The total dollar profit B is the realized excess return $\mu(L-T)$ minus the price concession Δp, times the dollar volume transacted V. Given (24) and (23)

$$\Delta p = \arg\min_{\Delta p} \Delta p + \mu \frac{V}{\Delta p}. \tag{25}$$

We see that $\Delta p \sim V^{1/2}$. We gather this in the following:

Proposition 4 *In the above setting, we get the square root law of price impact:*

$$\Delta p \sim V^{1/2}, \tag{26}$$

will wait for an amount of time

$$T \sim V^{1/2},$$

and will trade with a number of liquidity providers

$$N \sim V^{1/2}$$

The full expressions are:.

$$\Delta p = \left(\frac{\mu V}{\alpha f \langle s \rangle}\right)^{1/2} \tag{27}$$

$$E\left[T \mid V\right] = \left(\frac{V}{\alpha f \mu \langle s \rangle}\right)^{1/2} \tag{28}$$

$$E\left[N \mid V\right] = \left(\frac{fV}{\alpha \mu \langle s \rangle}\right)^{1/2}. \tag{29}$$

Proof. The loss function (21) gives:

$$\Delta p = \left(\frac{\mu}{f \langle s \rangle} V\right)^{1/2}$$

The expression for $E\left[T \mid V\right]$ comes from (22), and given that liquidity providers arrive with process with Poisson frequency f we have $E\left[N \mid V\right] = fE\left[T \mid V\right]$.
■

So big traders create big price impacts, and have to accept a delay in the execution of their trades: this, qualitatively, is congruent with the experience of traders, as well as the stylized facts found in the empirical literature : the price impact $|\Delta p|$ is an increase and concave function of the trade size V (Hasbrouck 1991, Holthausen et al. 2000, who find that a large part of the impact is permanent), impatient traders (high cost of time μ) will have a bigger price impact Δp (Chan and Lakonishok 1993, 1995, Keim and Madhavan 1996, 1997, Lo, MacKinlay and Zhou 1997, Breen Hodrick Korajczyk 2001). The theory here gives a precise quantitative hypothesis for this price impact and delay: is it proportional to $V^{1/2}$, the square root of the volume. Future research might examine directly whether those square root predictions (for large trades) hold in the data.

4.2 The distribution of individual trades q

Theorem 2 gave reasons to generate a distribution of target volumes $\zeta_V = 3/2$. The "upstairs" mechanism above give a way to divide them into smaller trades of size q_j, with a mean size $\langle q_j \rangle \sim V^{1/2}$. One might expect $\zeta_q = 2\zeta_V = 3$. But this is not the case, as large volumes creates also a large number of trades. This is summarized in:

Proposition 5 *With the process above, with target volumes with a distribution $\zeta_{V_T} > 1$, the distribution of individual trades follows*

$$\zeta_q = \zeta_V. \tag{30}$$

Proof. Here V_T is the target volume. To get the latter, observe that the individual trades during a trading round are due to target volume V are $q_i = s_i \Delta p = s_i V^{1/2}$, where s_i is the size of the individual liquidity provider. Individual trades of size q come from large target trades of size $V > q$. There is a density $V^{-(\zeta_V+1)}$ of them. Each large target volume V generate a number $V^{1/2}$ of trades. So we get:

$$P(q_i > q) \sim \int_{V>q} V^{-(\zeta_V+1)} \cdot V^{1/2} P\left(s_i V^{1/2} > q\right) dV.$$

As the liquidity providers have size $\zeta_s = 1$, $P\left(s_i V^{1/2} > q\right) \sim V^{1/2}/q$ and

$$\begin{aligned} P(q_i > q) &\sim \int_{V>q} V^{-(\zeta_V+1)} V^{1/2} \left(\frac{q}{V^{1/2}}\right)^{-1} dV \\ &= q^{-1} \int_{V>q} V^{-\zeta_V} dV \sim q^{-1} \cdot q^{-(\zeta_V-1)} = q^{-\zeta_V} \end{aligned}$$

so we get $\zeta_q = \zeta_V$. ■

4.3 A model that incorporates the mechanisms above

During the search of the block trade, the price increases at a rate $\mu + \nu$, where ν represents the rate of "leakage", hence should be included in the cost. There is "leakage"when an investor B who have heard that the large trader A is looking to buy shares, and investor A buys shares himself.. For instance B can think that A has perceived a mispricing. Suppose that with a probability π "liquidity provider" i above price engage in this activity. i will trade, and will have a price impact $\langle I(s_i)\rangle$. The the price pressure associated with shopping of the block is an increase of the price equal to ν per unit of time where:

$$\nu = \pi \langle I(s_i)\rangle f$$

with, as above, f is the number of liquidity providers contacted per unit of time. The fact that $\nu > 0$ is important, but the exact expression of ν does not matter.

In the trader's opinion, the dollar profit from a trade of size V is: $V(M - (\mu+\nu)\Delta p)$. So the annual profit is:$F \cdot V(M - (\mu+\nu)\Delta p)$. We also introduce the "Homogenous transaction costs" constraint (11), so that the trader's program reads:

$$\max_{F,V,\Delta p} F \cdot V \left(M - (\mu + \nu)T - \Delta p\right) \tag{31}$$

$$\begin{aligned} \text{s.t. } F \cdot (\nu T + \Delta p) \cdot V/S &\leq C \\ F &\leq F^{\max} \end{aligned} \tag{32}$$

Hence, the price impact includes not only Δt, but also the price increment due to leakage νT.

The solution is given by:

Proposition 6 *The solution of problem above gives $V \sim S^{\delta}$ with $\delta = 2/3$. Thus, using Theorem 2 and Proposition 5, we have the cubic laws stock market activity:*

$$\zeta_r = 3,\ \zeta_V = 3/2,\ \zeta_q = 3/2,\ \zeta_N = 3$$

and the square root law of price impact:

$$\Delta p \sim V^{1/2}$$

in this model.

Proof. Immediate, using the results above (to type in). First, we get the square root law, and then the cubic laws. ■

4.4 Some comments on the model

4.4.1 Arbitrage

If volume moves prices without necessarily informational reasons, this model requires limited arbitrage. We do not view this as a lethal problem. There is plenty of evidence that arbitrage is limited (see e.g. Barberis and Thaler 2001 and Shleifer 2000). The cleanest examples are probably the rejection of the basic arbitrage relationship in equity carve-outs (Lamont and Thaler 2001) and in twin stocks[17] (Dabora and Froot 1999). There are also good

[17] Incidently, our model would predict that if we regressed:

$$r_t^{\text{Shell}} - r_t^{\text{Royal Dutch}} = \alpha + \beta r_t^{\text{UK Index}} - \gamma r_t^{\text{Dutch index}} + \varepsilon_t$$

to control for stock-market wide movements, then the residual ε_t would behave very much like the returns r_t of our model, in particular would have a shape $E\left[|\varepsilon_t| \,|V_t^{\text{Shell}} + V_t^{\text{Royal Dutch}}\right]$ close to the $E\left[|r_t| \,|V_t\right]$ graphs of our model. The same would be true with signed volumes if they were available.

theoretical reasons for the existence of the limits to arbitrage (Shleifer and Vishny 1997).

4.4.2 Is this reasonable? Being more concrete about daily turnovers

The literature finds a big impact of big traders: Chan and Lakonishok (1993, 1995) find a range of 30-100 bps, Keim and Madhavan (1996), looking at small stocks, find a rather astonishing 400 bps. To see that if makes sense that a large fund will absorb a substantial liquidity of the market, and is likely to move prices, take a typical stock. Its yearly turnover is 50% (Lo and Wang 2001), so that with 250 trading days per year, its daily turnover is .5/250=0.2% of the shares outstanding. Take a moderate size fund, e.g. fund #25 (fund #1 = biggest). In June 2000, it had $8 billion, i.e. roughly 0.1% of market. So, on average, it will hold .1% of the capitalization of a given stock. Now suppose if it wants to sell its holdings, or double its holdings of this stock: it will create an additional .1% in turnover of stock, while the regular turnover is .2%/day. So, the size of the desired trade of this fund is quite sizable compared to the normal turnover. This supports the idea that big funds are indeed big for liquidity of the market, and supports the assumption that big traders will pay attention to their trading strategy to moderate their price impact.

5 Assessing some further empirical predictions of the model

The above model proposes an explanation for the power laws of financial market activity. We now show how it correctly explains another series of patterns in trading activity.

Because of analytical complexity, we recourse here to simulations to derive most of the other properties of the model. We compare the predictions of the model from a distribution of the data for various quantities: the return r, the volume V and the net volume V', the number of trades N and the net number of trades N',and possibly combinations of them, like $N'/N, V/N, r^2$. We compare the empirical and theoretical conditional expectations $E[Y \mid X]$, for $X, Y = V, N, V', N', r, N'/N, r^2, V/N$. In principle we could have done it for all $8 \cdot 7 = 56$ pairs; but given each graph takes takes a very long time to prepare, we opted to do that only for a dozen of those graphs. First, we need a word about the simulations.

5.1 Observables over a given amount of time: Aggregation over different target volumes

In practice, in a given time interval, there will be several "rounds" where a big trader creates one or more trades. Now, imagine that in the period Δt considered, there are J such trading rounds indexed by $j = 1, ..., J$. (J is, in general, a random variable). Each of them gives rise to r_j, V_j, N_j, V_j', $N_j^{'}$, and individual trades $(q_{ji})_{1 \le i \le N_i}$. The quantities we have calculated in the statistical analysis are (calling $\varepsilon_j = \text{sign}(r_j)$)

$$
\begin{aligned}
r &= \sum_{j=1}^{J} r_j = \sum_{j=1}^{J} \varepsilon_j V_j^{1/2} \qquad (33) \\
V &= \sum_{j=1}^{J} V_j \\
N &= \sum_{j=1}^{J} N_j \\
V' &= \sum_{j=1}^{J} \varepsilon_j V_j \\
N' &= \sum_{j=1}^{J} \varepsilon_j N_j
\end{aligned}
$$

and again we have $E[N_j] = V_j^{1/2}$ up to numerical prefactors.

Of course, in reality, observed r will be the r above, plus some noise due e.g. to news and other market events outside the model. However, those extraneous events do not affect the big events.

Proposition 7 *As soon as the number of events J does not have too fat tails ($\zeta_J \geq 3$), the results of Theorem 2 still hold the time intervals with different target volumes.*

Proof. As the formulaire in the Appendix A shows, power law exponents are conserved under finite addition. This extends to thin-tailed addition. ■

To sum up: the previous model explains the three power laws we set out to explain. Still it had somewhat ad hoc (in the view of the authors, not outrageously so) assumptions, so that some other way to validate the model would be welcome. This is what we present in the following section.

5.2 Conditional expectations: $E[Y \mid X]$ graphs

In Figures 5 through 9 we present the empirical values on the left panel, and on the right the theoretical prediction of the model. We obtained the theoretical prediction in the Monte Carlo simulations detailed in Appendix E.

It was very heartening for the authors to see that the model matches all the $E[Y \mid X]$ we empirically constructed. Take for instance "Hasbrouck graph" $E[r \mid V']$, which is plotted in Figure 5. We replicate in the left panel the increasing, concave shape (in the positive region) found by Hasbrouck (1991). Plotting the symmetrical version, $E[V' \mid r]$, yields a surprising finding: we see that the empirical graph, in Figure 6 is now linear. The model matches this in the right panel of Figure 6.

The model predicts that periods of high volume are periods when big traders execute trades in smaller pieces, so that, for a big buy trade, one observes lots of "buy" smaller trades. A compact way to say that is that N is high, and $|N'/N|$ is high too when the volume is high[18]. This is expressed in Figure 7.

In an earlier paper (Plerou et al. 2001) we report $E[N \mid N']$ and $E[N \mid V']$: we see that the model matches those curves quite well (Figures 8 . We tested a few other graphs, and in all cases the model matched the empirical curves quite well[19] (here, we report the curves with the most "interesting" shapes). Given that the model was not tailored to match those facts, we view this as a very reassuring feature of the model. [insert comments on the ecnomic meaning of the graphs. Insert analogues with Kyle model]

[18] A very different relationship would happen under another a priori (but not a posteriori) plausible model: suppose that there are N trades with i.i.d. signs $(\varepsilon_i)_{i=1\ldots N}$ and sizes $(q_i)_{i=1\ldots N}$, so that $V' = \sum_{i=1}^{N} \varepsilon_i q_i$. Then a large N corresponds to an $N' = \sum_{i=1}^{N} \varepsilon_i = O\left(\sqrt{N}\right)$, so that $N'/N = O\left(1/\sqrt{N}\right)$ is small. Likewise [check] a large V' corresponds (in part) to a large N, and a small $|N'/N|$, counterfactually.

[19] The graphs we tested where $E[V \mid r]$, $E[N \mid r]$, $E[r \mid V]$, $E[V' \mid N']$, $E[N' \mid V']$, $E[V/N \mid r]$, and empirical finding and theory matched very well. We do not report the results here, but they are available from the authors upon request. The printout of the 56 theoretical relationships $E[Y \mid X]$ involving $V, N, V', N', r, N'/N, r^2, V/N$ is also available from the authors.

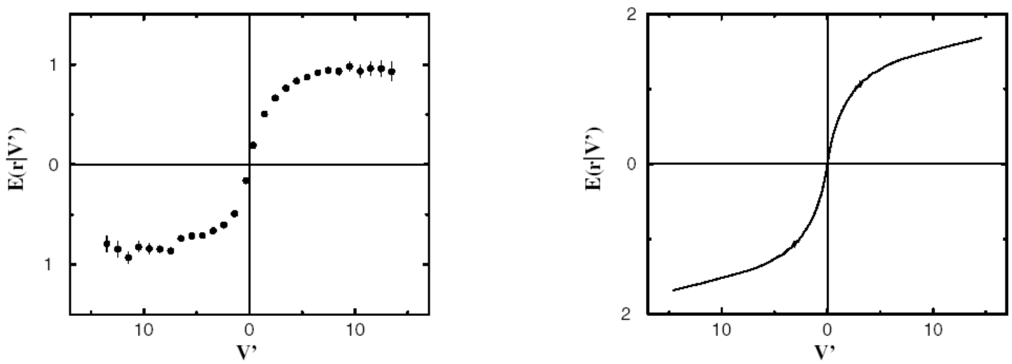


Figure 5:

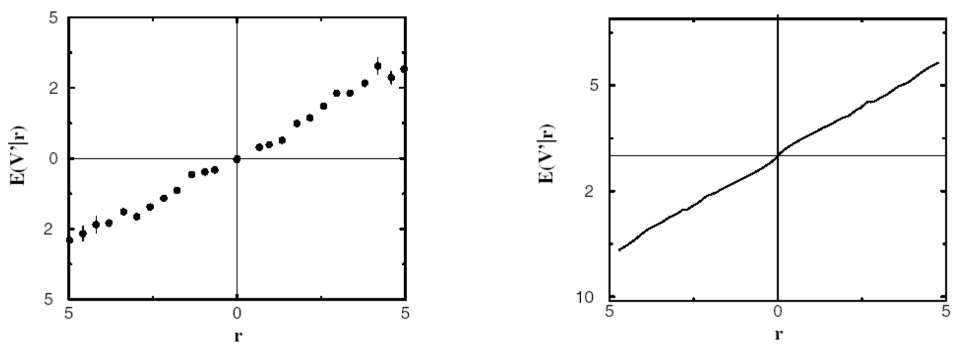


Figure 6:

In Figures 5 through 9, the left panel (solid dots) displays the empirical values, and the right panel (line) the theoretical prediction.

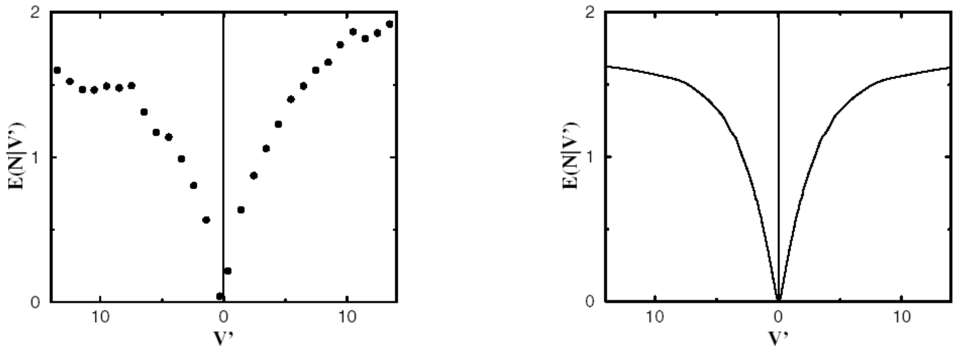


Figure 7:

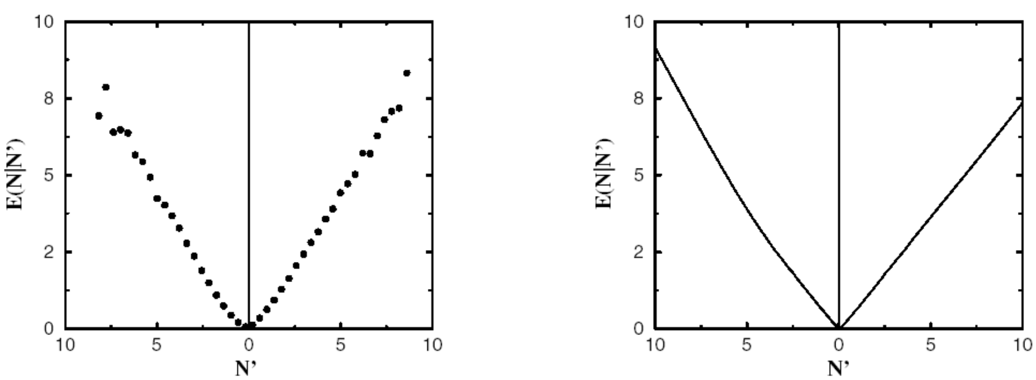


Figure 8:

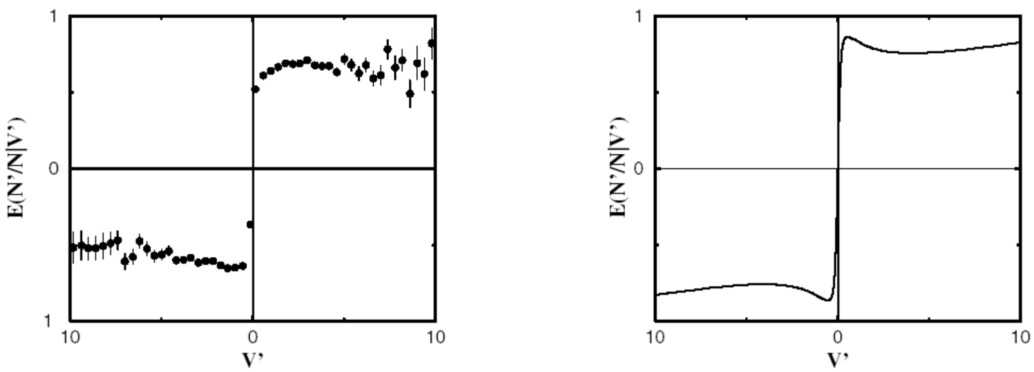


Figure 9:

5.3 Some untested predictions

The model makes a number of untested predictions, which we gather here:

- The expressions (27)-(29) for the price impact, the time to execution, and the number of trades (Proposition 4). In particular, it would be interesting to test their "square root" dependence in S, and the differential impact of μ (the impatience of the trader), α, f, $\langle s \rangle$.
- The fact that a big fund of size S will do "aggressive", "price moving" trades[20] with a frequency declining in the size (if $\delta > 2/3$) as $S^{1-3\delta/2}$.
- The model predicts many $E[Y \mid X]$ graphs that we did not build[21] (as carefully determining them empirically is a delicate and time-consuming activity). It would be nice to know whether the model holds in the data.

The model also has some untested assumptions, in particular:

- The linear supply function (20), saying that a trader willing to wait an amount of time T and to pay a price increment Δp, will get an

[20] So looking at the fund turnover will not be the right thing to do (and indeed, fund turnover is largely independent of size): to avoid looking like a "closet indexer" a large fund will maintain a fairly large turnover, though the "real" "strategical" trades will indeed happen less often (as they move prices a lot).

[21] The printout of the 56 theoretical relationships $E[Y \mid X]$ involving $V, N, V', N', r, N'/N, r^2, V/N$ is available from the authors upon request.

amount of shares $S(\Delta p, T)$ that will be proportional to $\Delta p \cdot T$ in the limit of large Δp and T.

- More generally, we anticipate the values of the 5 trading exponents α, β, γ, δ and κ in Appendix B to be 0, 1, 1, 1, and 0.

We leave the investigation of those values to future research.

6 Related literature

6.1 Some alternative theories

The general theme is that most theories (and certainly all that have been written until now) don't explain (and cannot explain with a simple modification) the cubic and half-cubic laws. They have free parameters, so one tune them to replicated the $\zeta = 3$, but this is not an explanation.

6.1.1 The public news based (efficient markets) model

One would need to assume lots of things to make this model fit the power laws. First, because returns reflects news, we have to *assume* that the distribution of news has $\zeta_r = 3$.

This is not the only difficulty. In the benchmark where volumes don't move prices, there is no reason for big traders to trade less than small ones. With no price impact, essentially all models that traders should trade in amount proportional to their sizes, so that the volume V_i traded by trader i will be equal to $V_i = a_i S_i$ for a_i a random variable that do not scale with size. Then, we would get

$$\zeta_V = \zeta_S = 1$$

This prediction of $\zeta_V = 1$ is one of the major difficulties with the news approach. We have to somehow assume $\zeta_V = 1.5$ (for instance, for some reason left unexplained, a trader of size s trades with probability[22] $s^{-1/2}$, or trades magnitudes proportional to $s^{2/3}$)..

[22] The fact that traders of size s do "aggressive trades" only with frequency $s^{-1/2}$, in our model, the natural outcome of their (fractional) trading costs being proportional to $s^{1/2}$. But in the "public news model", where trading costs are 0 or proportional to the amount traded, there is no reason any more why the frequency should be $s^{-1/2}$.

6.1.2 A mechanical "price reaction to trades" model

This model is: desired trades are i.i.d., in a given period N trades are desired, with a quantity $\varepsilon_i q_i$ for the ith trade (ε_i is the sign of the trade). The price has a reaction $f(\varepsilon_i q_i)$ to each trade. So:

$$\begin{aligned} r &= \sum_{i=1}^{N} r(\varepsilon_i q_i) \\ V &= \sum_{i=1}^{N} q_i \\ V' &= \sum_{i=1}^{N} \varepsilon_i q_i \\ N' &= \sum_{i=1}^{N} \varepsilon_i \end{aligned} \tag{34}$$

An advantage of this model (a simple variant of our preferred model) is that it will tend to generate qualitatively right $E[Y \mid X]$ curves. However there are problems with this model: (i) We need to assume $\zeta_V = 1.5$. The natural prediction (given $\zeta_S = 1$) would be $\zeta_V = 1$. (ii) we need to assume $\zeta_N \simeq 3$: it could be any other value. (iii) We also need to assume $r(V) = V^{1/2}$ to get the shape of $E[r \mid V]$.

6.1.3 Random bilateral matching

This has been proposed by Solomon and Richmond (2001). The model says that two traders of size S, S' meet, and the resulting volume is $\max(S, S')$ and the price impact is $\min(S, S')$.

Problem: as the initial size distribution is $\zeta_S = 1$, $\zeta_{\min(S,S')} = 2$, so that one would predict $\zeta_V = \zeta_r = 2$. The authors rely on a scaling exponent of the size of agent $\zeta_S = 1.5$, as is approximately the case in the US and the UK in the late 90s, but not in other periods (Feenberg and Poterba 1993 show that the exponent for the wealth of U.S. individuals had large variations from 1.5 to 2.5 in the last 30 years). They didn't look at the distribution of mutual funds with $\zeta_S = 1$, which trumps, because of its fatter power law (smaller power law coefficient), the distribution of wealth.

6.2 Related empirical findings

6.2.1 Other distributions for returns

The empirical literature has proposed other distributions. In short, the reason why we are more confident about our findings is that we have more data points, hence quantify better the tails. And we can also explain previous findings in light of ours.

Andersen et al. (2001a, 2001b) propose a lognormal distribution of return. In independent work Liu et al. (1999) show that this is true in a stock market context too, but only in the central part of the distribution: while the center is log normal, the tails are power law. We reproduce in Figure 10 one of their findings.

Gopikrishnan et al. (1999) also report how, with a small sample that looks only at the center rather than the tails of the distributions, one would find Lévy (as Mandelbrot 1963) or truncated Lévy (Mantegna and Stanley 1995) distributions.

Finally Ané and Geman (2000) propose a model which is in essence:

$$r_t = \sigma \sum_{j=1}^{N_t} u_i$$

with a constant σ and u_i are standard normal shocks (this is Clark's model, with the clock being driven by number of trades, not volume), and argue for a good fit of the model. But this has a very important empirical problem: from $\zeta_N \geq 3$, as $|r| \sim N^{1/2}$ in this model, we conclude $\zeta_r = 2\zeta_N \geq 6$, so the model misses the cubic law by a large amount. In fact, in our model, results similar to Ané-Geman would be found: for instance, $E[r^2 \mid N]$ looks almost linear (though it is not really linear). So we can see how, even under the null of our model, they would find results similar to what they report, though our mechanism is more accurate: simply put, their model is a pretty good, though ultimately inexact (in the null of our model, and according to the value of ζ_r and ζ_N) approximation of reality.

6.2.2 Buy / Sell asymmetry

Our basic model has built-in full symmetricity between positive and negative tails. This is not a bad approximation of reality: the exponents are roughly identical, around 3. However, it is true that the exponents are slightly lower (fatter tails) in the negative tail that in the positive one. More generally, one finds a bit more skewness in the negatives.

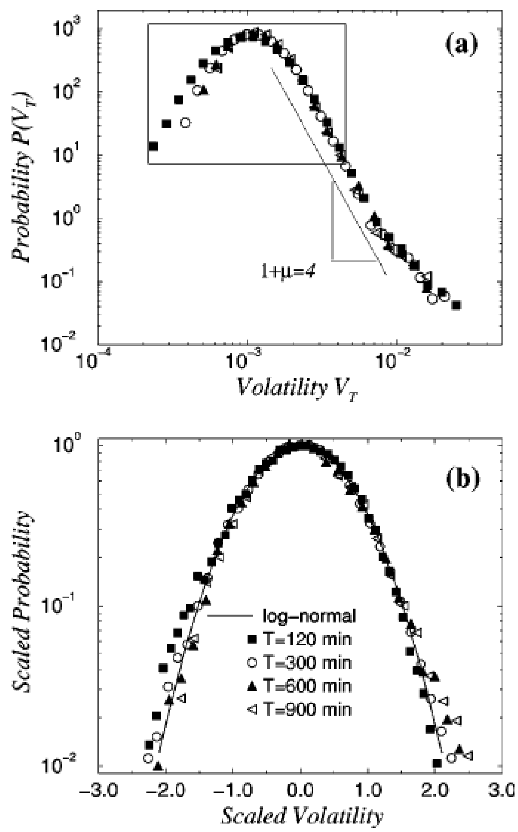


Figure 10: (a) Probability distribution of the volatility on a log-log scale with different time windows T with Δt =30 min. The center part of the distribution shows a quadratic behavior on the log-log scale. The asymptotic behavior seems consistent with a power law. (b) Center of the distribution: The volatility distribution for different window sizes T using the log-normal scaling form $\sqrt{\nu}\exp\left(a+\nu/4\right)P\left(v_T\right)$ as a function of $\left(\ln V_T - a\right)/\sqrt{\pi\nu}$, where a and ν are the mean and the width on a logarithmic scale. The scaled distributions are shown for the region shown by the box in (a). By the scaling, all curves collapse to the log-normal form with $a=0$ and $\nu=1$, $\exp\left[-\left(\ln x\right)^2\right]$ (solid line). Source: Liu et al. 1999.

We sketch here a simple hypothesis suggested by our framework. Suppose the frequency of arrival f of liquidity providers is higher for desired sells (so the liquidity providers have to buy the asset) than buys. The "Street" explanation for this (reported in Chan and Lakonishok 1993) is that for buys, the liquidity providers have to already own the asset (if short selling is time / cost consuming). So

$$f_+ < f_-$$

where f_+ =frequency of arrival of liquidity providers that are willing to be counterpart of a buy order, f_- =same for a sell order. By plugging f_+ or f_- in (33) we see that $|\Delta p_+| > |\Delta p_-|$, i.e. the theory predicts higher price movements for buys than for sells, but that the exponents are the same (which they are, to a good approximation, as shown above). Saar (2001) provides an alternative explanation.

6.3 Link with the microstructure literature

The literature to which this paper is closest is perhaps the microstructure literature. This literature is extremely vast (for reviews see e.g. O'Hara 1997 and Biais et al. 2002), so we will only mention the most directly relevant pointers. Our theory makes predictions above the relationships between volume and price impact, and we reviewed in section **??** the papers directly pertaining to our theory. Other relevant papers, that show a long run impact of trading on prices, include Evans and Lyons (2001), Lyons (2002), Easley, Hvidjkaer and O'Hara (2001), and the analysis of volume in Wang 1994, Campbell et al. 1993, Lo and Wang 2000. We adopted a fairly stylized view of trading institutions (in part, this was motivated by the fact that the cubic exponents arise for market structures, so that our explanation should not depend too finely on a specific market structure), as opposed to the fine analysis proposed in the literature, e.g. Madhavan 1992, Madhavan and Cheng 1997).

Here we want to explain the cubic power laws, and the patterns in trading activity at a fairly high degree of aggregation (1/2 hour as opposed to trade by trade). Hence we highlight some traits of reality – mainly the heterogeneity between the size of agents, and the trade-off between execution cost and execution time – that are typically not central to the microstructure literature. (Another feature is that we are agnostic about the importance of information asymmetry in the determination of asset prices). We highlight more "macro" phenomena that arise from aggregation, rather the detailed understanding of micro situations that tend to be the focus of the microstructure literature. Hence, our model and the models of this literature look quite different. Still, it would be desirable, in further research, to put together the rich informational and institutional understanding from the microstructure literature and the more macro approach of the present paper.

6.4 Link with the "economics and statistical physics" literature

This paper is part of a broader movement utilizing tools from physics in for the study of economic issues. This literature was pioneered by Mandelbrot (1963, 1997). Mantegna and Stanley (1999) provide an overview. Antecedents include Bak et al. (1993), Bouchaud and Potters (2000), Calvet, Fisher and Mandelbrot (1997), Canning et al. (1998), Gabaix (2001a) Plerou et al. (1999), Lux and Sornette (2001), Levy, Levy and Solomon (2000), Stanley et al (1999).

7 Conclusion

This paper does two things. First it presents a series of evidence on the distribution of trading activity (the variables r, V, N, V', N' and various combinations of them), both the the power law exponents and their "cubic" nature, and the joint distributions in the form of $E[Y \mid X]$ graphs. Then it provides a simple model (with some assumptions, but, we contend, a very high "number of predictions over number of assumptions" ratio). The model seems to match the evidence we gathered.

We view the model presented here as a prototype, reduced-form setup, rather than a definitive model[23] with impeccable microfoundations. On the substantive front, our model provides one possible quantitative theory of excess volatility in asset markets: it is simply due to the desire to trade of large traders (perhaps stimulated by news). More precisely is the power law 1 of the distribution of sizes that generates, through an intelligent though not hyperrational (people trade too much) trading process, the power laws of 3 in

[23] In particular, our model has very little time-series dimension. We presented it as a series of independent trading decisions. This model is silent about the time-correlations in market activity. It is straightforward to propose simple extensions of it, such as an GARCH-type model for J, so as to qualitatively account for the well-known long term memory in volatility (where J_{t+1} depends positively on past $J_t' s$ and $|r_t|$). The proper analysis of this, however, is outside the objectives of this paper. We investigate this in ongoing research.

returns and 3/2 in volume. We propose the following criterion: matching, as we do, the quantitative regularities established here (in particular *explaining* the exponents of 3 and 3/2, not merely assuming them), in particular the cubic laws, would be a *sine qua non* criterion for the admissibility of a of volume and volatility. We hope that the regularities we established will sharply constrain, and guide, future theorizing. Given its empirical success and its simple structure, the present model might a useful point of departure to think about those issues.

8 Appendix A: Some power law mathematics

We present here some basic facts about power law mathematics, and show how their great aggregation properties makes them especially interesting for both theoretical and empirical work.

We say that a random variable X has power law behavior if there is a $\zeta_X > 0$ such that:

$$P(X > x) \sim \frac{1}{x^{\zeta_X}}$$

so that the probability density $f(x)$, as (minus) the derivative of the cumulative distribution, follows:

$$f(x) \sim \frac{1}{x^{\zeta_X + 1}}$$

A more general definition is that there is a "slowly varying"[24] function $L(x)$ and a ζ_X s.t.

$$f(x) \sim \frac{L(x)}{x^{\zeta_X + 1}}$$

so that the tail follows a power law "up to logarithmic" (by some abuse of language) corrections.

$\zeta_X < \zeta_Y$ means that X has fatter tails than Y, hence the large $X's$ are (infinitely, at the limit) more frequent than large $Y's$.

The definition implies that (with $n \in \mathbf{R}^+$) $E[|X|^n] = \infty$ for $n > \zeta_X$, and $E[|X|^n] < \infty$ and for $n < \zeta_X$. As an example, if for returns $\zeta_r = 3$, then $E[|r|^n]$ for $n > 3$. In particular, the kurtosis of returns in infinite[25], and their skewness borderline infinite.

By extension, given they die out faster than power laws, the power law exponent would be $\zeta_Y = \infty$ for Y =normal, lognormal, exponential.

[24] $L(x)$ is said to be slowly varying if

$$\lim_{x \to \infty} L(tx)/L(x) = 1 \text{ for all } t > 0.$$

The prototypical example would be $\ln x$.

[25] *This makes the use of the kurtosis invalid*, not to speak of the 5th and 6th moments, that some papers use. As the theoretical kurtosis is infinite, empirical measures of it are essentially meaningless. As a symptom, according to Paul Lévy's theorem (see e.g. Durrett 1996, p.153), the median sample kurtosis of T i.i.d. demeaned variables $r_1, ..., r_T$, with $\kappa_T = \left(\sum_{i=1}^T r_i^4/T\right) / \left(\sum_{i=1}^T r_i^2/T\right)^2$, increases to $+\infty$ like $T^{1/3}$, as the sample size T increases. (The general formula would be $T^{\min(4/\zeta - 1, 1)}$ for $\zeta < 4$). The use of kurtosis should banished. As a simple diagnostic for having "fatter tail than from normality", we would recommend, rather than the kurtosis, quantile measures such as $\Pr(|(r - \langle r \rangle)/\sigma_r| > 1.96)/.05 - 1$, which is positive if tails are fatter than a normal.

A major property is that they have great aggregation properties. *The property of being power law is conserved under addition, multiplication, polynomial transformation, min, max.* The motto is that, when we combine two power law variables, "The biggest (fattest=smallest exponent) power law dominates"

Indeed, for X, Y independent variables, we have the formulaire:

$$\begin{aligned}
\zeta_{X+Y} &= \text{power law of fattest variable, i.e.} \\
\zeta_{X+Y} &= \min\left(\zeta_X, \zeta_Y\right) \\
\zeta_{X\cdot Y} &= \min\left(\zeta_X, \zeta_Y\right) \\
\zeta_{\max(X,Y)} &= \min\left(\zeta_X, \zeta_Y\right) \\
\zeta_{\min(X,Y)} &= \zeta_X + \zeta_Y
\end{aligned}$$

For instance, if X is a power law for $\zeta_X < \infty$, and Y is power law variable with an exponent $\zeta_Y \geq \zeta_X$, or even normal, lognormal or exponential variable (so that $\zeta_Y = \infty$), then $X + Y, X \cdot Y$, $\max(X, Y)$ are still power laws with the same exponent ζ_X. So *multiplying by normal variables, adding non fat tail noise, summing over i.i.d. variables preserves the exponent.* So (i) this makes theorizing with power law very streamlined; (ii) this lets the empiricist hope that those power laws can be measured, even if there is a fair amount of noise in the data. One doesn't need to carry around the additional noise, because though it will affect variances etc, it will not affect the power law exponent. PL exponent carry over the "essence" of the phenomenon: smaller order effects do not affect the PL exponent.

For instance, say a theory, for instance ours, gives a mechanism for R, with $\zeta_R = 3$. Other things are going on, so that in reality, we observe:

$$\widetilde{r_{it}} = \widetilde{a_{it}}\widetilde{R}_{it} + \widetilde{b_{it}}$$

For instance, the liquidity of the market varies, so that $\widetilde{a_{it}}$ is random, and new can affect prices $\widetilde{b_{it}}$ without affecting volume. But even then, we will have $\zeta_r = \zeta_R = 3$ if $\widetilde{a}, \widetilde{b}$ are smaller order effects, i.e. have thinner power laws $(\zeta_a, \zeta_b \geq 3)$. If the theory of R_{it} capture the 1st order effects (i.e. those with dominating power law), its predictions for the power law tails or the "noise up" empirical counterpart $\widetilde{r_{it}}$ will still be true.

Proof. See Sornette (2000) for a more systematic treatment.

For $\zeta_{\min(X,Y)}$, we have:

$$\begin{aligned}
P\left(\min\left(X, Y\right) > x\right) &= P\left(X > x \text{ and } Y > x\right) = P\left(X > x\ \right) P\left(Y > x\right) \\
&= \frac{k}{x^{\zeta_X}} \frac{k'}{x^{\zeta_y}} = \frac{kk'}{x^{\zeta_X + \zeta_Y}}
\end{aligned}$$

For $\zeta_{\max}$,

$$\begin{aligned} P(\max(X,Y)>x) &= 1-P(\max(X,Y)<x) = 1-P(X<x \text{ and } Y<x) \\ &= 1-P(X<x\ )\,P(Y<x) \\ &= 1-\left(1-\frac{k}{x^{\zeta_X}}\right)\left(1-\frac{k'}{x^{\zeta_y}}\right) \sim \frac{k''}{x^{\min(\zeta_X,\zeta_Y)}} \end{aligned}$$

where $k''=k$ if $\zeta_X<\zeta_Y, k''=k'$ if $\zeta_X>\zeta_Y$, and $k=k+k'$ if $\zeta_X=\zeta_Y$. ■

This generalizes to a finite number of independent PL variables:

$$\begin{aligned} \zeta_{X_1+\ldots+X_k} &= \min\left(\zeta_{X_1},\ldots,\zeta_{X_k}\right) \\ \zeta_{X_1\cdot\ldots\cdot X_k} &= \min\left(\zeta_{X_1},\ldots,\zeta_{X_k}\right) \\ \zeta_{\max(X_1,\ldots,X_k)} &= \min\left(\zeta_{X_1},\ldots,\zeta_{X_k}\right) \\ \zeta_{\min(X_1,\ldots,X_k)} &= \zeta_{X_1}+\ldots\zeta_{X_k} \end{aligned}$$

[extend to random k, with $\zeta_k \geq \zeta_X$].

Finally, a useful formula is:

$$\zeta_{X^\alpha} = \frac{\zeta_X}{\alpha} \text{ for } \alpha>0 \tag{35}$$

Proof. $P(X^\alpha>x)=P\left(X>x^{1/\alpha}\right)\sim\left(x^{1/\alpha}\right)^{-\zeta_X}=x^{-\zeta_X/\alpha}$. ■

9 Appendix B: The model with general trading exponents

We present here the general structure of the model. It allows to see which assumptions are crucial to get the "cubic" exponents. Also, natural quantities, the "exponents of trading", emerge. Their direct measurement is an interesting task for future research.

We call ζ_S and ζ_L the Pareto exponents of the large traders and the liquidity providers respectively. We make the following assumptions, about the behavior of the variables in the large S, $|\Delta p|$, and T limit:

- A large trader of size S will make large position of size $V=S^\delta$ at an annual frequency $S^{-\phi}$.
- If a large traders makes an order at a price concession Δp, then the frequency of arrival of liquidity traders is $f=\Delta p^\alpha$, and each liquidity provider of size s supplies a number of shares $q=s\cdot\Delta p^\beta$.

- A trader of size S will adjust the trading frequency so as to pay a proportional amount of transactions costs S^{κ}.
- We assume that the large trader, in a given trade of size V, wants to minimize a loss function:

$$\Delta p + T^{\gamma}$$

where T is the mean time needed to find enough liquidity providers to complete the trade.

So the original model corresponds to:

$$\begin{aligned} \zeta_S &= \zeta_L = 1 \\ \alpha &= 0 \\ \beta &= 1 \\ \gamma &= 1 \\ \delta &= 1 \\ \kappa &= 0 \end{aligned}$$

After a time t, the large trader has received on average $\Delta p^{\alpha+\beta} t$ shares. So the mean time needed to realize a trade of size V is:

$$T = \frac{V}{\Delta p^{\alpha+\beta}}$$

so that the objective function is:

$$\min_{\Delta p} \Delta p + \frac{V^{\gamma}}{\Delta p^{\gamma(\alpha+\beta)}}$$

which yields

$$\Delta p = V^{\eta_p} \text{ with} \tag{36}$$

$$\eta_p = \frac{\gamma}{(\alpha+\beta)\gamma + 1} \tag{37}$$

and the number of trades is $N = V/\Delta p^{\alpha}$, i.e.

$$N = V^{\frac{\alpha\gamma+1}{(\alpha+\beta)\gamma+1}}. \tag{38}$$

The probability that a target volume is $> V$ is:

$$P(\text{Target volume} > V) = \int_{S^{\delta} > V} S^{-1-\zeta_S} S^{-\phi} dS = V^{-\frac{\zeta_S+\phi}{\delta}}$$

so that the distribution of target volumes is:

$$\zeta_{V^T} = \frac{\zeta_S + \phi}{\delta}. \tag{39}$$

Combining with (36)–(38) we get:

$$\begin{aligned} \zeta_r &= \zeta_p = \frac{\zeta_S + \phi}{\delta} \frac{\gamma}{(\alpha + \beta)\gamma + 1} && (40) \\ \zeta_N &= \frac{\zeta_S + \phi}{\delta} \frac{\alpha\gamma + 1}{(\alpha + \beta)\gamma + 1} && (41) \end{aligned}$$

The distribution of individual trades is more involved. An individual trade $> q$ involves a target volume $V > q$, and a fraction equal to $P\left(s_i \Delta p^\beta > q,\right)$ of the $N(V)$ liquidity traders who meet his demand: (here we call $\Delta p(V) = V^{\eta_p}$, and recall that $N(V) = V/\Delta p^\beta)$:

$$\begin{aligned} P(q_i > q) &= \int_{V>q} V^{-1-\zeta_V} \cdot N(V)\, dV \cdot \int_{s_i \Delta p(V)^\beta > q} s_i^{-1-\zeta_L} ds_i \\ &= \int_{V>q} V^{-1-\zeta_V} \frac{V}{V^{\beta\eta_P}} \left(\frac{V^{\beta \cdot \eta_P}}{q}\right)^{\zeta_L} dV \\ &= q^{-\zeta_L} \int_{V>q} V^{-\zeta_V - \beta\eta_p + \zeta_L \cdot \beta \cdot \eta_p} \\ &= q^{-\zeta_V - (\zeta_L - 1)\left(1 - \beta\eta_p\right)} \end{aligned}$$

so that:

$$\begin{aligned} \zeta_q &= \zeta_V + (\zeta_L - 1)\left(1 - \beta\eta_p\right) && (42) \\ &= \zeta_V + (\zeta_L - 1) \frac{\alpha\gamma + 1}{(\alpha + \beta)\gamma + 1} \end{aligned}$$

Given total annual transactions costs will be (as each transaction costs $V \Delta p(V) = V^{1+\eta_p} = S^{\delta\left(\eta_p + 1\right)}$)

$$S^{-\phi} \cdot S^{\delta\left(1+\eta_p\right)} = S^{1+\kappa}$$

we will have:

$$\phi = \delta\left(1 + \eta_p\right) - \kappa - 1 \tag{43}$$

The reader can verify that with our special assumptions, we get the "cubic" exponents. It would be interesting to measure the exponents α, β, γ, δ and κ directly.

The above exercise can give interesting comparative statics. Going back to the case $\zeta_S = 1$ (which seems true empirically), $\kappa = 0$ (which seems plausible a priori), we get:

$$\begin{aligned}\zeta_r &= \frac{\zeta_S + \phi}{\delta} \frac{1}{\eta_p} \\ &= \frac{1 + \delta\left(1 + \eta_p\right) - 0 - 1}{\delta \eta_p} = \frac{1 + \eta_p}{\eta_p} \\ &= 1 + \frac{1}{\gamma} + \alpha + \beta\end{aligned}$$

So if a regulator wanted to lower the fat-tailness of returns, i.e. increase ζ_r, a cap (or a tax on large transactions) on individual transactions would be ineffective: this would correspond to a value of δ lower than 1, say, and one sees that the expression of ζ_r does not depend on δ. Economically, this means that with a curb on large trades, big institutions would do few large trades, but they would do them more often, and in the end ζ_r is not affected. If somehow (this seems very difficult, short of an outright cap on the size of a fund, given the strength of the forces that give rise to Zipf's law) the regulator could make the distribution of funds less skewed, and increase ζ_S, however, ζ_r would increase and there would be fewer extreme returns.

10 Appendix C: "Crashes" are not outliers to the cubic law

Does the cubic law apply also to the most extreme events, i.e. to the crashes? The approach we take is the following. Consider the negative returns, take their absolute value, and order them by size: We take the daily Dow Jones returns in the sample 1925-1999. Consider the returns r_t, and take absolute value of the negative returns. Order them by size: they are $r_{(1)} = 25\%$, $r_{(2)} = 13.7\%$, $r_{(3)} = 12.3\%$,..., $r_{(10)} = 7.4\%$, $r_{(100)} = 3.9\%$. There would be "crashes" if, for instance, $r_{(1)}$ or $r_{(2)}$ where "too big". Here "too big" means "too big compared to what would be predicted by the cubic law".

The following proposition establishes a convenient representation for the extreme movements in the stock market under the cubic law

Proposition 8 *If the returns are i.i.d. and follow the law* $P\left(r_t > r\right) = k/r^{\zeta}$*, for some* k*, we have:*

$$\left(\zeta\left(\ln r_{(i)} - \ln r_{(i+1)}\right)\right)_{1\leq i\leq n} =^d \left(\frac{u_i}{i}\right)_{1\leq i\leq n} \tag{44}$$

where the u_i's are i.i.d. exponential variables: $P(u_i > u) = e^{-u}$ for $u \geq 0$.

Proof. As $P(r_t > r) = kr^{-\zeta}$, we have $P(\zeta \ln r_t > x) = P\left(r_t > e^{x/\zeta}\right) = ke^{-x}$, so that $\zeta \ln r_t$ are i.i.d. exponential distributions. We then apply the Rényi representation theorem on ordered statistics (see Reiss 1989, p.36-37). ■

For instance, we can write:

$$\begin{aligned}
\ln r_{(1)} - \ln r_{(2)} &= \frac{1}{\zeta} u_1 \\
\ln r_{(2)} - \ln r_{(5)} &= \frac{1}{\zeta} \sum_{i=2}^{4} u_i / i
\end{aligned}$$

where the u_i are i.i.d. exponential variables with density $e^{-u} 1_{\{u \geq 0\}}$.

Proposition 8Taking a 5% probability cutoff, this would show up as:

$$\begin{aligned}
p_{1,2} &= P\left(\ln \widetilde{r}_{(1)} - \ln \widetilde{r}_{(2)} > \text{empirical value of } \ln r_{(1)} - \ln r_{(2)}\right) \\
&= P\left(\ln \widetilde{r}_{(1)} - \ln \widetilde{r}_{(2)} > \ln r_{(1)} - \ln r_{(2)}\right) \\
&= e^{-\zeta\left(\ln r_{(1)} - \ln r_{(2)}\right)} = e^{-3(\ln .25 - \ln .137)} = .15 > .05
\end{aligned}$$

so we cannot reject the hypothesis of "no crash" within a 5% confidence.

We can do the same for

$$p_{m,n} = P\left(\ln \widetilde{r}_{(m)} - \ln \widetilde{r}_{(n)} > \text{empirical value of } \ln r_{(m)} - \ln r_{(n)}\right) \tag{45}$$

We find:

$$\begin{aligned}
p_{1,10} &= .21 > .05 \\
p_{1,100} &= .34 > .05 \\
p_{2,3} &= .83 > .05 \\
p_{2,10} &= .48 > .05 \\
p_{2,100} &= .76 > .05 \\
p_{3,10} &= .36 > .05 \\
p_{3,100} &= .72 > .05
\end{aligned}$$

We conclude that there is no evidence for abnormally large negative returns, in excess to what the cubic law would predict.

11 Appendix D: Cubic laws and the Tobin tax

Suppose that a regulator would like to dampen extreme price fluctuations; what kind of tax might he use? We will see that a Tobin tax (a simple proportional tax on transactions) is ineffective. However, a tax that affects more than proportionally large trades that go "in the direction of the market", and with large price impact, would work.

Dampening extreme price movement will mean, here, increasing ζ_r : there are fewer large price movements when ζ_r is higher. We shall not here examine the legitimacy of objective, but will just examine some means of achieving it.

First, observe that a "Tobin tax", a simple proportional tax on transactions, would be ineffective as it would not change ζ_r. Indeed, with a tax Tobin tax τ independent of Δp and V, the objective function of the trader is $\min_{\Delta p} \Delta p + \mu V/\Delta p + \tau$, so that his policy $\Delta p \sim V^{1/2}$ does not change. With $V \sim S^{\delta}$, for some unimportant δ, the frequency of trading large positions, F now satisfies $F\left(S^{3\delta/2} + \tau S^{\delta}\right) = \kappa S$, so we still have $F \sim S^{1-3\delta/2}$ for large traders. By the reasoning in the paper we get $\zeta_V = 3/2$ and $\zeta_r = 3$. Economically, the Tobin tax does not work because for large traders, the proportional tax τ is a negligible fraction of their true cost, which is the price impact $V^{1/2}$. For large traders, hence large volumes, $V^{1/2} \gg \tau$.

A more appropriate tax would discourage large trades that go in the direction of the markets (e.g. large sell trades when the market is going down). Specifically, consider a tax per share $\tau = \tau_1 V^{\xi} (\Delta p)^{\rho}$, perhaps applying only above a total amount traded V or a total Δp, so that only the largest transactions in the most volatile environment are taxed. The large trader's objective is now $\min_{\Delta p} \Delta p + \mu V/\Delta p + \tau_1 V^{\xi} (\Delta p)^{\rho}$ with $\xi, \rho \geq 0$ and $\rho + 2\xi > 1$ for the tax to be effective. This yields

$$\Delta p \sim V^{(1-\xi)/(1+\rho)} \tag{46}$$

and condition $F(\Delta p + \tau) V = \kappa S$ now gives

$$F \sim \frac{S}{V(\Delta p + \tau)} \sim \frac{S}{V^{1+\xi+\rho(1-\xi)/(1+\rho)}}$$

i.e. with $V \sim S^{\delta}$

$$F(S) \sim S^{1-\delta\frac{1+2\rho+\xi}{1+\rho}}.$$

The probability that a volume is $> x$ is now:

$$P(V > x) = \int_{S^{\delta} > x} \rho(S) F(S) \, dS$$

with a density $\rho(S) \sim S^{-2}$ of mutual funds (Zipf's law), i.e.

$$\begin{aligned} P(V > x) &\sim \int_{S > x^{1/\delta}} S^{-2} S^{1-\delta \frac{1+2\rho+\xi}{1+\rho}} dS \\ &\sim x^{-\frac{1+2\rho+\xi}{1+\rho}} \end{aligned}$$

i.e.

$$\zeta_V = \frac{1+2\rho+\xi}{1+\rho}. \tag{47}$$

And (46) gives (using the fact, derived in Appendix A, that $\zeta_{X^\alpha} = \zeta_x/\alpha$):

$$\zeta_r = \frac{1+2\rho+\xi}{1-\xi}. \tag{48}$$

While we keep $\zeta_r = 3$ without taxes ($\xi = 0$, $\rho = 1$), now we can get any value $\zeta_r > 3$ by adjusting the tax exponents ξ, ρ. Like in all tax schemes, details of implementation and enforcement would be crucial in the use of the tax we have outlined. Still the lesson that one should tax large trades that go in the direction of the market, with a large price impact, is a robust one.

12 Appendix E: A simplified algorithm for the simulations

We will simulate T time intervals: for a given $t = 1, ..., T$ (representing, in our real data, some interval Δt of trading activity)

1. One fixes an integer value J_t drawn at random. We took J_t =Integer part of $10e^{u_t}$, with u_t Normal(0,1).

2. One draws

$$\begin{aligned} v_t &= e^{.5u_t} \\ \eta_t &= e^{.5n_t} \end{aligned}$$

where u, n are i.i.d. standard normal variables. And for $j = 1...J_t$, one draws V_t with $\zeta_V = 1.5$. One also draws $\varepsilon_j = \pm 1$ with probability 1/2, and sets (this is an approximation of the model, but this simplifies a lot the simulations)

$$r_j := \varepsilon_j \nu_t \eta_t^{-1} V_j^{1/2} \tag{49}$$

$$N_j := \nu_t \eta_t V_j^{1/2} \tag{50}$$

$$N_j' := \varepsilon_j N_j \tag{51}$$

Note that ν_t, η_t are constant across the J_t rounds j of the time interval.

3. Applying (33), one gets a five values (r, V, N, V', N'), which get stored as $(r_t, V_t, N_t, V_t', N_t')$.

Then, we do the statistical analysis (e.g. calculations of $E[N \mid V']$ etc.) on the data set made up of those values $(V_t, N_t, V_t', r_t = N_t')_{t=1...T}$.−after rescaling of all those variables.

A few remarks are in order.

- Equations (49) and (50) are intended to be the analogues to the theoretical (27) and (29) resp. r and N vary like $V^{1/2}$, but there are varying "market conditions" $\nu_t = (\alpha\langle s\rangle)^{-1/2}$ (a measure of the "depth" of each liquidity provider of the market at the given point in time), and $\eta_t = \mu^{1/2}$ (the arrival rate of liquidity providers). While those quantities must have some randomness, we do not have strong prior on the extent of their randomness, and any way they play only a minor role in the model. So, largely arbitrarily (but the results are largely insensitive to that choice), we chose make them equal to $e^{.5u}$, with u a normal(0,1). The ".5" factor captures a relatively "small" randomness.

- The value of J around 10 was so that the average number of trades in a given half hour match roughly the empirical one, which depends on the stock, and is around 30 for large stocks). No further effort to "calibrate" the model was done here.

- Because of simplicity considerations, we do not simulate the "full", "nanoscopic" part of the model and the search process: in particular we did not simulate the gradual arrival of the liquidity suppliers, do did not simulate the full stochasticity of N. Rather, we took it to be its mean value $V_t^{1/2}$.

- We implicitly assume that the number of big "causal" orders in a given Δt (think of an hour, or a day), is independent of the size of those. This choice was made again to simplify the Monte Carlos.

To facilitate comparisons, we rescale the variable – demean them, and divide them by their standard deviation if this standard deviation exists[26], otherwise by their absolute mean[27].

This means, for instance, for the normalized volume, that we take:

$$\widehat{V_t} := \frac{V_t - E\left[V_{t'}\right]}{E\left[\left|V_{t'} - E\left[V_{t''}\right]\right|\right]}$$

and the normalized number of trades:

$$\widehat{N_t} := \frac{N_t - E\left[N_{t'}\right]}{E\left[\left(N_{t'} - E\left[N_{t''}\right]\right)^2\right]^{1/2}}$$

[26] So we took the 1st moment for $N, N', r, N'/N$, and the second moment for $V, V/, r^2, V/N$.

[27] [technical note to suppress in final draft] Rescaling: the variables of interest are the component of $D = \left(V, N, V', N', r, N'/N, r^2, V/N\right)$. Call n^i =normalization for component i = 2nd moment of D^i if the 2nd moment exists, otherwise 1st moment. More explicitly:

$$\begin{aligned} n_i &= stddev\left(D_i\right) \text{ for } i = 2, 4, 5, 6 \\ &= E\left[\left|D^i - E\left[D^i\right]\right|\right] \text{ for the other } i's \end{aligned} \tag{52}$$

and one defines:

$$\widehat{D}_{it} = \frac{D_{it} - E\left[D_i\right]}{n_i} \tag{53}$$

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