Optimal Valuation of Claims on Noisy Real Assets: Theory and an Application

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A theory for valuing claims on noisy real assets is developed and applied. Central to the theory is determination of the dynamics for the best estimate of real asset value. The dynamics of the value estimate are shown to differ from the dynamics of the true asset value only in the arrival rate of information. The rate of information arrival in the value estimate can be faster or slower than information arrival in the true asset value, which can lead to unexpected outcomes in the valuation and exercise of options on noisy real assets. The theory we develop is illustrated through an application. An imperfectly competitive market for real estate development is examined, in which agents compete over the timing of lead investment. Information spillover and free-rider incentives are shown to cause significant delay in lead investment. Delay together with a competitive response once lead investment has occurred explain observed patterns of development in gentrified urban land markets and multi-stage development projects.

We develop a theory for valuing claims that are contingent on noisy real asset values. This is an important topic—options on noisy real assets are ubiquitous and have substantial economic value. Examples of noisy contingent-claims in a real estate setting include the valuation and exercise of proprietary development options in untested local real estate markets, strategic interaction among developers when investment outcomes are a public good, the exercise of leasing options, and the exercise of default options on debt that is securitized by real property. Other applications include the general process of research and development, and the valuation of corporate assets such as mines, factories and human capital.

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We directly extend Childs et al. (2002) (hereafter COR) by developing a theory for valuing claims on noisy assets when noise is mean-reverting. A major difference between the two papers is the role of time and information. COR (2002) show how to optimally value a noisy real asset using current and historical information. In contrast, contingent-claims pricing is explicitly forward-looking. Agents form expectations as to future asset values and calculate a variance around those expected values. Contingent-claim valuation of noisy real assets thus requires a shift from an historical (*ex post*) perspective to a forward-looking (*ex ante*) perspective that explicitly depends on rational expectations.

The central result of this paper is the determination of forward-looking dynamics for the *time-filtered value*, which is the best estimate of the true asset value when true values are observed with noise.[1] These dynamics are shown to be identical to those associated with the true asset value, except that the variance rate of the time-filtered value depends crucially on the rate of time change in the variance of the estimation error (we call the estimation error the *residual variance*). The variance rate of the time-filtered value—which is a measure of the instantaneous rate of information arrival—may be higher or lower than the variance rate of the true asset value.

We show that *revealed variance*, which is the total variance of the time-filtered value as measured over a specified time interval, is bounded from above by the total information in the system. Revealed variance differs from the upper bound by the residual variance, and may equal or even exceed total variance associated with the true asset value.

Futures and options values that are contingent on noisy real asset values follow directly from the time-filtered value dynamics. A noisy asset's forward value is simply its expected value conditioned on the current value estimate. The value and exercise policy for an option on a noisy real asset are shown to depend critically on the revealed variance. Option value may exceed, equal, or be less than the full information option value depending on the arrival rate of information. Discretionary option exercise also depends on the rate of information arrival, where a low (high) rate of information arrival tends to diminish (increase) the value of waiting to exercise a proprietary real option.

We apply the theory to explore the implications of noise in an imperfectly competitive market for real estate development. In this market agents compete over the timing of lead investment, where the precise location of the pre-investment rental rate demand curve is observable only with noise. However, once lead development has occurred, its relative success or failure is revealed to the market *vis-à-vis* realized demand. This information externality creates a free-rider problem, the consequence of which is to cause delay in lead investment. A tendency toward delay together with fully revealed demand once lead investment has occurred explains boom and bust in localized real estate markets.

Real world examples that reflect the described market structure include well located but blighted urban land, urban-fringe residential and retail development, and certain types of large-scale multi-stage development projects such as DisneyWorld in Florida. Redevelopment of blighted urban land is particularly plagued by information-related free-rider problems. Pioneering developers worry that profits associated with successful development will be eroded by rapid follow-on entry when land use regulators are eager to facilitate additional investment. They also worry that losses are likely if there is unexpectedly low realized demand. Concerns as to information spillover and a desire to internalize benefits from lead investment explain why developers often wish to assemble large tracts of contiguous land before they undertake their multi-stage development projects.

We extend the literature by modeling the strategic development problem as a continuous-time Cournot competition game of incomplete information. In this game, agents compete over the timing rather than the absolute quantity of new supply. A *conditional expected perfect Markov equilibrium* is shown to obtain when agents are indifferent between leading and following at the time initial investment occurs.[2]

An important outcome of the analysis is an extreme "no investment" proposition that follows when product demand is infinitely elastic. The intuition for this result is that free-rider incentives completely dominate business-stealing effects when there are no monopoly rents associated with moving first. Delay is less extreme, but occurs nonetheless, when temporary monopoly rents can be realized from assuming the role of the lead investor.

We analyze characteristics of the lead investment threshold as a function of several important explanatory variables, including the elasticity of product demand, asset value noise and time. The probability of immediate follow-on development is also quantified, and is shown to be significant for realistic parameter values. High lead investment thresholds coupled with high probabilities of immediate follow-on investment explain why localized development is often delayed for long periods of time, but that, once lead investment occurs and information is revealed as to the market's economic viability, a development boom quickly follows.

The main body of the paper is organized as follows. The next section develops the theory of claim valuation when the underlying real asset value is observed with noise. The penultimate section addresses the issue of strategic real estate development with information spillover. A concluding section summarizes our main results.

Theory

Basic Model Structure

To begin the formal analysis, we describe the continuous-time stochastic processes governing the true asset value, noise and the observed asset value. Assume that the true asset value follows a lognormal stochastic process, specified as follows:

$$dX(t) = \mu_X X(t)dt + \sigma_X X(t)dW_X \tag{1}$$

For ease of comparison, the notation follows that in COR (2002). One important difference is that we consider lognormal distributions here instead of the normal distributions in COR (2002). A standard transformation is all that is required to move from a lognormal process to a normal process, and vice versa.

The true value is obscured by noise, meaning that its value is estimated with error. Noise exists and persists because real assets are unique in their physical, locational and contractual-relational characteristics, implying market incompleteness in financial payoffs (COR (2001,2002)). We model noise as a lognormal process that reverts toward its long-run, neutral value of one, with dynamics as follows:

$$dY(t) = \left(-\kappa \ln Y(t)\right)Y(t)dt + \sigma_Y Y(t)dW_Y \tag{2}$$

Noise accumulates at a rate determined by the noise volatility term, σ_Y. Mean reversion in the drift term partially offsets the cumulative effect of noise volatility. The rate of mean reversion, or noise dissipation, is summarized by κ.

In a lognormal setting, the observed value is defined as the product of the true asset value and noise; i.e., $Z(t) \equiv X(t)Y(t)$. Using this relation, the stochastic process that describes the observed value is

$$dZ(t) = \left(\mu_X - \kappa \ln\left(\frac{Z(t)}{X(t)}\right)\right)Z(t)dt + \sigma_X Z(t)dW_X + \sigma_Y Z(t)dW_Y \tag{3}$$

The observed value is cointegrated with the unobserved true asset value, in the sense that *Z(t)* reverts toward *X(t)* at a rate determined by κ. The correlation between instantaneous changes in the observed value and the true value is

$$\rho \equiv Corr\left(dX(t), dZ(t)\right) = \frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_Y^2}} = \frac{\sigma_X}{\sigma_Z} \tag{4}$$

which increases as noise volatility, σ_Y, diminishes. We refer to this measure as the *instantaneous information ratio.*

The time-filtered asset value, *M(t),* provides an efficient unbiased estimate of the true asset value that is conditional on all available information through time *t*. Its dynamics are determined by the optimal filtering techniques of Lipster and Shiryayev (1978), which produce a stochastic differential equation similar to equation (16) in COR (2002). The solution to the stochastic differential equation with lognormal true and observed asset values is

$$M(t) = Z(0)\left(e^{\frac{1}{2}\left(\gamma(t)-\sigma_0^2\right)}\right)\left(e^{\mu_X \int_0^t (1-\xi(s))ds}\right)\left(e^{\int_0^t \xi(s)\left[dz(s)+\kappa \ln\left(\frac{Z(s)}{M(s)}+\frac{1}{2}\gamma(s)\right)ds\right]}\right)\left(e^{\frac{1}{2}\kappa\int_0^t \gamma(s)ds}\right) \tag{5}$$

This quantity provides a best estimate of the unobserved true asset value, *X(t),* by optimally weighting current and historical market data. The intuition underlying the determination of this value is similar to that described in COR (2002), except that relations are here multiplicative rather than additive.

For easy reference, in Table 1 we restate several important formulas found in COR (2002). COR (2002) contains definitions of all parameters and values used in equations (1) through (5) here, and for the formulas shown in the table.

Table 1 About Here

Forward-Looking Dynamics of the Time-Filtered Asset Value and the Revealed Variance

We are interested in contingent-claims whose values are determined by noisy real assets. For example, a firm may have a development option on raw land or a consumer product in an untested market. In this case, the asset does not even exist until option exercise occurs, so estimating product demand will naturally occur with error.

To focus ideas, suppose the claimholder will choose to either exercise or not exercise the product development option at a particular time *T.* Further suppose that the current time is *t,* $t<T$. The best available estimate of the true asset value—namely, the time-filtered value, *M(T)*—will be used to determine the *expected* payout of the claim at time *T.* At the current time *t,* the claimholder does not have the benefit of knowing the time *T* time-filtered value. Rather, knowing only the current time-filtered value, *M(t),* the claimholder must look forward to time *T* and calculate the mean and variance of the distribution for *M(T).* In order to do this, it is necessary to know the stochastic process for *M(t)* that can be used to calculate an expected value and variance at all possible future dates.

The forward-looking dynamics of the time-filtered value are summarized in proposition 1:

<u>Proposition 1 (Forward-Looking Dynamics of M(t))</u>: *Conditional on the information set, I(t), as it determines M(t), the forward-looking dynamics for the time-filtered expected value can be written as*

$$dM(t) = \mu_X M(t)dt + \xi(t)\sigma_Z M(t)dW_Z \tag{6}$$

where $\sigma_Z dW_Z = \sigma_X dW_X + \sigma_Y dW_Y$

Proof: *See Appendix*

The forward-looking dynamics are such that returns to *M(t)* follow a random walk with time-varying volatility. Realized paths of the time-filtered value will generally revert toward the unobserved true asset value*, X(t)*. Because *X(t)* is unobservable, however, it is *expected* that *M(t)* equals *X(t)* at any given time, which follows because errors in the estimation of the true asset value are by definition idiosyncratic. This implies that the instantaneous expected return to *M(t)* is simply μ_X.

Thus, we can determine the expected value of *M(T), T≥t, as if* the drift term were unaffected by noise and mean reversion between values. The volatility term is not unaffected by mean reverting noise, however. Specifically, the instantaneous variance rate in (6), $\xi^2(t)\sigma_Z^2$, differs from the instantaneous variance rate of the true return, σ_X^2, because noise systematically affects the rate of information arrival in the time-filtered value estimate.

The instantaneous variance of the time-filtered value can be reexpressed as follows:

$$\xi^2(t)\sigma_Z^2 = \sigma_X^2 - \frac{d\gamma(t)}{dt} \tag{7}$$

The variance rate of the time-filtered value differs from the variance rate of the true asset value by the time *t* rate of change in the residual variance of the time-filtered value (see Table 1 here and COR (2002) for further detail on the residual variance). When residual variance is increasing (decreasing) over time, the instantaneous arrival rate of information in the time-filtered value is less than (greater than) the instantaneous information arrival rate in the true asset value. This is because increases (decreases) in residual variance indicate deterioration (improvement) in the precision of the true asset value estimate, which inhibits (intensifies) information arrival relative to the arrival rate of information in the true asset value.

The cumulative variance of *M(T)* from time *t* to *T*, *Var[M(T)|I(t)]*, can be determined. This variance, which we refer to as the *revealed variance,* and denote as $\nu(T|I(t))$, is

$$\nu\left(T \mid I(t)\right) = \int_t^T \xi^2(h)\sigma_Z^2 dh = \sigma_x^2(T-t) - \left(\gamma(T) - \gamma(t)\right) \qquad (8)$$

for $0 \le t < T$.[3]

The revealed variance measures the total amount of information that is expected to arrive from time *t* to time *T* with respect to the unobserved true asset value. An upper bound on information arrival over the time interval *[t,T]* is $\sigma_X^2(T-t)+\gamma(t)$. This upper bound establishes the *law of conservation of information*; namely, that accumulated information flow in the best estimate of the true asset value cannot exceed the accumulated information flow in the system as a whole. Note, however, that it is possible for the revealed variance to exceed the total quantity of information that would arrive in a full information market setting, $\sigma_X^2(T-t)$. This happens when beginning residual variance, $\gamma(t)$, is positive and decreases over time to result in $\gamma(T)<\gamma(t)$ for all $T>t$.

Given these observations, we can restate equation (8) as follows:

$$\underbrace{Total\ Information}_{\gamma(t)+\sigma_X^2(T-t)} \equiv \underbrace{\text{Re}vealed\ Variance}_{\nu(T|I(t))} + \underbrace{\text{Re}sidual\ Variance}_{\gamma(T)} \tag{8a}$$

We can use this relation to consider several special cases. When $\sigma_0^2 > 0$ and $\kappa\to\infty$ (initial noise that immediately dissipates), $\nu(t \mid I(0)) = \sigma_0^2 + \sigma_x^2 t$ for $t>0$. At $t=0$, revealed variance is zero. In the instant that t exceeds zero, the observed value jumps to the true value to result in an instantaneous information arrival rate of σ_0^2. Then, for all other $t>0$, information arrives at a rate of σ_X^2, which is the information arrival rate in the true asset value. Hence, initial estimation error combined with strong mean reversion produces a revealed variance that exceeds the total variance of the true value process—even though the processes for $Z(t)$ and $X(t)$ are identical for $t>0$.

The case of $\kappa=0$ (strictly accumulating noise) results in revealed variance of $\nu(t \mid I(0)) = \rho^2\sigma_x^2 t$ for $t\geq 0$. The flow of asset value information is slowed by a factor of ρ^2 compared to when asset value is perfectly observed. Note that revealed variance does *not* depend on the initial variance, σ_0^2. Without information acquisition through mean reversion in the observed value, initial noise provides no additional information to the claimholder (see COR (2001) for additional discussion).

Revealed variance in the constant residual variance case, $\sigma_0^2 = c$, is $\nu(t \mid I(0)) = \sigma_x^2 t$ for $t \geq 0$. Interestingly, the rate of information arrival in this case is identical to the information arrival rate that obtains in a full-information market setting. This is true even though there is error in the estimation of the true value, *X(t),* and even though adjustments are made to correct for *ex post* estimation error in the calculation of *M(t)*. A revealed variance of $\sigma_x^2 t$ follows directly from the law of conservation of information: there is an exact balance between the rate of information arrival *vis-à-vis* mean reversion in the observed value and the information dulling effect of accumulating noise.

Forward Values

We assume henceforth that all agents in the economy are risk-neutral. This assumption allows us to avoid complicating issues associated with the effects of risk aversion on derivative asset prices in a noisy rational expectations economy. Risk neutrality implies that $\mu_X = r - \delta_X$, where *r* denotes the constant risk-free rate of interest and δ_X is a dividend rate or convenience yield.[4]

From Proposition 1, the value of a forward contract that is based on *M(t)* can be determined. The following corollary states the result.

Corollary 1 (Forward Valuation based on M(t)): *Consider a forward contract with a time T delivery date, in which the (cash settled) delivery value equals the time T estimate of true asset value, M(T). Denote the time t, t<T, value of the forward contract as* $\hat{M}(t \mid I(t), T)$. *The value of the forward contract is*

$$\hat{M}\left(t \mid I(t), T\right) = M(t)e^{\mu_X (T-t)} \tag{9}$$

Proof: *Follows immediately from the dynamics expressed in equation (6) and established results in forward and futures valuation.*

A best estimate of the future true asset value is simply the time T expected value extrapolated forward from the current value estimate, *M(t)*.

Options on Real Assets

Now consider an option whose value is contingent on the value of a noisy real asset. Options on noisy real assets are commonplace and have significant value in terms of their impact on the real economy. Examples include development options on land, R&D options in new product markets, and default and related liability-based options.

Because the true asset value is unobservable, the time-filtered value and its dynamics as expressed in Proposition 1 are relevant in the estimation of the real option value. As we have noted, the stochastic process describing time-filtered value dynamics differs from the process describing the full-information asset value *only* in that the variance rate is $\xi^2(t)\sigma_Z^2$ instead of σ_X^2. This observation produces Corollary 2:

Corollary 2 (Proprietary Option Valuation): *Consider the time t valuation of a proprietary (non-strategic) option on a noisy real asset whose time-filtered value evolves according to equation (6). All parameter values used in the determination of option value are identical to those associated with an otherwise equivalent noiseless asset, except that the variance rate in the time-filtered value replaces the variance rate in the true asset value. In the case of a European* *option with expiration date T, the revealed variance of asset prices,* $\nu\big(T \mid I(t)\big)$*, is used in place of the total noiseless asset variance,* $\sigma_X^2(T-t)$*, wherever it appears in the closed-form expression of value.*

Proof: *Follows immediately from the dynamics expressed in equation (6).*

The intuition for this result lies in understanding that only information that can be *acted on*, as summarized marginally by $\xi(t)\sigma_Z$ and in total by $\nu\big(T \mid I(t)\big)$, are useful in making exercise decisions.

To better understand the implications of this proposition, consider the time t=0 valuation of a standard (convex payoff) European call or put option on a noisy real asset with expiration date T. When initial noise is low compared to the steady-state residual noise value $(\sigma_0^2 < c)$, then noisy option value is less than its corresponding full information value. This follows because the residual variance, $\gamma(t)$, increases with t; i.e., the probability of making an option exercise error (exercising the option when it is out-of-the-money or not exercising the option when it is in-the-money) increases over time.

Alternatively, when initial noise variance is high ($\sigma_0^2 > c$), then noisy option value actually *exceeds* the full-information option value. This surprising result follows from the information production characteristic of mean-reverting noise. A high rate of mean reversion implies that t>0 observed values are quite informative compared to the initial value estimate, which increases option value. This happens even though there may be residual errors in the exercise of the call or put option.

The inability to precisely observe the underlying asset value will typically impact American option exercise policy. Corollary 3 summarizes the major insight:

<u>Corollary 3 (American Option Exercise Policy)</u>: *Consider the exercise timing associated with a proprietary (non-strategic) convex payoff option when exercise timing is discretionary. When* $\sigma_0^2 < c$ *(*$\sigma_0^2 > c$*), optionholders will exercise earlier (later) than would be the case when asset value information is fully available.*

<u>Proof</u>: *Follows immediately from the dynamics expressed in equation (6).*

When initial noise levels are relatively low, the value of waiting for additional information to arrive is lessened due to the information-dampening effect of accumulating noise. Alternatively, when initial noise levels are high, there may be the opposite incentive to delay exercise for longer than would be the case with full information. This happens because strong reversion in the observed value toward the true value creates an incentive to wait for more information to arrive in order to improve the precision of the value estimate.

These results extend findings of Williams (1995), who considers exercise policy associated with a perpetual development option. In Williams' model, costly search and negotiation between developers and landowners introduces noise into the land valuation problem. He shows that developers exercise their development option earlier than they would when the matching process is instantaneous and costless. The value of waiting to develop and the land value are reduced with increasing noise (average search time) in the Williams' model.

This result is directly analogous to our model in the special case of $\kappa=0$ (noise is strictly cumulative). When we allow noise to dissipate over time, exercise timing effects are shown to be more complex, depending on, among other factors, initial conditions and the speed of mean reversion. For example, constant residual variance results in development timing that is identical to that in a full-information setting, and diminishing residual variance results in delays in development beyond a full-information setting.

The potential for error in the exercise decision may result in incentives for the optionholder to seek more precise information as to the true asset value. The due diligence process associated with the exercise of the development option is an important example of information acquisition in a noisy asset value setting. The basic tradeoff is to compare the marginal costs of information acquisition with the marginal benefits of greater precision in the exercise of an option. The net benefits of information acquisition will be greatest when option values are high compared to the costs of acquiring information and when the optionholder is effectively indifferent between exercise options. See COR (2001) for further analysis of this issue and similar issues associated with costly information acquisition and real option exercise decisions.

Application: Strategic Development and Information Externalities in Real Estate Markets

Economic Setting

To demonstrate the implications of this theory, we provide a detailed application of development strategy in an imperfectly competitive real estate market. Prior to any land development, agents are unable to deduce the exact location of the built property rental rate demand curve. Once initial development occurs, and rental demand is revealed, it is possible to assess the relative success or failure of the lead developer. The option to undertake lead development in this market is therefore written on a demand function that is observable only with noise.

In a dynamic setting, a potential lead developer will consider the tradeoff between capturing temporary monopoly rents through early investment (a "business-stealing" effect) with the possibility of miscalculating product demand. An additional cost to moving first is that investment provides information to competitors regarding the position of the demand curve. This latter effect suggests that incentives for delay will exist in order to free-ride on information generated by a first-mover. When free-rider incentives dominate business-stealing effects, there may be long lags in lead development. Once initial investment does occur and product demand is revealed, rapid follow-on investment may be realized due to information spillover and high realized demand.

We model strategic development as a continuous-time Cournot duopoly game in which suppliers have the option to invest capital to create one unit of production capacity (space). A Cournot modeling approach (as opposed to a Bertrand approach) is appropriate in the case of real estate development, as the demand for space is derived from local economic activity. This implies that developers will not compete on price when they formulate their business plans. Instead, competition over quantity is the crucial strategic variable in development markets.

Quantity competition in our model is different from that characterized in the standard Cournot model, however. In real estate development, land use regulations typically constrain choices along the capacity dimension through maximum density restrictions. As a result, developers typically take capacity as given, and instead compete over the *timing* of product delivery in markets where space is a strategic substitute.

Once developed, a built asset will produce a stream of real-time observable, but dynamically uncertain, cash flows for an indefinite time period. Product price is observable only with noise prior to product development. Consequently, from an *ex ante*, pre-development perspective, determining a best estimate of the true inverse demand is crucial in the formulation of competitive investment policy.

An efficient unbiased estimate of inverse product demand is described by the relation:

$$p(q,t) = M(t)D(q) \tag{10}$$

where *p(q,t)* is the time *t* estimated product price per unit (net of operating costs) as a function of realized production capacity, *q*. Inverse demand is separable into *M(t)*, the value of the multiplicative demand shock term, and *D*(*q*), which maps realized production capacity *q* = 1, 2 (units of space) into a positive value.

The pre-development product price depends on the best estimate of the unobservable true demand shock term, *X(t)*. We assume that dynamics associated with *X(t)* are as described in equation (1). We also assume that noise follows mean-reverting dynamics as described in equation (2), which in turn produces an observed value, *Z(t)*, as stated in equation (3). In this case, the best estimate of the current *X(t)* is the time-filtered value, *M(t),* as defined in equation (5).

Furthermore, conditional on all the information available as of time *t,* proposition 1 shows that the forward-looking dynamics for *M(t)* evolve according to equation (6). The dynamics of *M(t)* are identical to the dynamics associated with the full-information shock term, *X(t)*, except that the variance rate in the time-filtered value dynamics is modified to reflect a different arrival rate of information.

It is also important to recognize that *M(t)* now measures per unit revenue flow, while we originally specified *M(t)* as an asset price estimate. The original results are easily modified to address the current situation, since the functional relation between revenue flow and price is homogeneous of degree one. Finally, we require that $D(2) \leq D(1)$, which means that space is a weak strategic substitute.

Our strategy is to first value the option to develop for the follower, and then determine the optimal investment policy for the leader. Although our primary focus is on the effects of noise on competitive lead investment decisions, we also provide the leader's full information optimal exercise policy in order to gauge the relative effects of noise on investment decisions.

Investment by the Follower

Equilibrium is determined by backward induction. The follower's optimal exercise policy and investment value are determined as a function of the full-information demand term, *X(t),* rather than *M(t),* since previous investment by the leader fully reveals the demand curve.

Assume that the unit cost of investment is K for both competitors. Further assume that once the investment decision is made, development is instantaneous and irreversible. Conditional on lead investment having occurred, the follower holds a proprietary option on the development of the second project. Given that demand is fully observable, the critical value, $p^{f*} = X^{f*}D(2)$, at which follow-on investment is triggered is

$$X^{f*}D(2) = \frac{\beta}{\beta - 1}\delta_X K \tag{11}$$

where

$$\beta = \frac{\frac{1}{2}\sigma_X^2 - \mu_X + \sqrt{\left(\frac{1}{2}\sigma_X^2 - \mu_X\right)^2 + 2r\sigma_X^2}}{\sigma_X^2} \tag{12}$$

and other parameter values are as defined previously.[5]

Note that $\frac{X^{f*}D(2)}{\delta_X} = \frac{\beta}{\beta - 1}K$ is the capitalized value of expected cash flows (the built asset value) at the time of follow-on investment. For properly defined parameter values, it is easily shown that $1<\beta<\infty$. If we further define the measure θ as the ratio of investment value to construction (replacement) cost at the time follow-on investment is optimal, then $\theta = \frac{\beta}{\beta - 1} > 1$. Investment is delayed beyond the time at which built asset value equals the cost of construction. This happens because of the combined effects of irreversible investment, uncertainty and opportunity capital costs.

The follower's value function conditional on lead investment having occurred is

$$V^f(X(t)) = \begin{cases} \frac{X(t)D(2)}{\delta_X} - K, & X(t) \geq X^{f*} \text{ for some } t > 0 \\ \left(\frac{X(t)}{X^{f*}}\right)^{\beta}\left[\frac{X^{f*}D(2)}{\delta_X} - K\right], & X(t) < X^{f*} \text{ for all } t \geq 0. \end{cases} \tag{13}$$

When $X(t) \geq X^{f*}$ for some $t>0$, follow-on investment will have occurred, and investment value is simply the capitalized expected cash flow less the sunk investment cost. Alternatively, when $X(t) < X^{f*}$ for all $t\geq 0$, (and given that lead investment has previously occurred), the follower holds a proprietary investment option on the second development project. In this case, β is a normalized measure of the price elasticity of demand for the proprietary development option, and is inversely related to the propensity to develop as it determines the critical investment value, X^{f*}.

Lead Investment Equilibrium When Demand is Fully Observable

Before analyzing lead investment equilibrium under noisy rational expectations, consider the special case when the leader can fully observe the inverse demand curve (i.e., $M(t) \equiv X(t)$ for all $t \geq 0$). If $X(0) < X^{f*}$ and if $X(t) \geq X^{f*}$ at some time $t>0$, the leader will already have invested with current value $V^l(X(t)) = V^f(X(t))$. Alternatively, given that lead investment has occurred and if $X(t) < X^{f*}$ for all time $t \geq 0$, follow-on investment has not yet occurred and the leader currently realizes monopoly flows from investment.

Specifically, conditional on lead investment having occurred and knowing the bound at which follow-on investment will occur, the leader's asset value is:

$$V^l(X(t)) = \begin{cases} \dfrac{X(t)D(2)}{\delta_X} - K & X(t) \geq X^{f*} \; \textit{for some } t > 0 \\ \dfrac{X(t)D(1)}{\delta_X}\left[1 - \left(\dfrac{X(t)}{X^{f*}}\right)^{\beta-1}\right] + \dfrac{X(t)D(2)}{\delta_X}\left(\dfrac{X(t)}{X^{f*}}\right)^{\beta-1} - K, & X(t) < X^{f*} \; \textit{for all } t \geq 0 \end{cases} \tag{14}$$

Given that $X(t) < X^{f*}$ for all $t>0$, the leader's investment value is a weighted average of the value of the monopoly cash flows, which are received until the random time when $X(t)$ hits the boundary, X^{f*}, and the value of cash flows received after the follower invests.

Let X^{l*} denote the critical threshold value of $X(t)$ at which the leader invests. A symmetric perfect Markov equilibrium obtains when $V^l(X^{l*}) = V^f(X^{l*})$ (see Maskin and Tirole (1997) and

Vives (1999) for further background on this equilibrium concept). That is, in a dynamic setting where competitors are otherwise identical in terms of production capabilities and access to (full) information, equilibrium is such that agents are indifferent between leading and following at the time lead investment occurs.

Equating the appropriate product values from equations (13) and (14) establishes the leader's critical investment value. After simplifying, the critical relation is

$$\frac{X^{l*}D(1)}{\delta_X}\left[1-\left(\frac{X^{l*}}{X^{f*}}\right)^{\beta-1}\right]=K\left[1-\left(\frac{X^{l*}}{X^{f*}}\right)^{\beta-1}\right] \tag{15}$$

where X^{l*} is determined implicitly.

If $D(1)=D(2)$, development occurs simultaneously at $X(t)=X^{f*}=X^{l*}$. This happens because monopoly rents are unobtainable when product demand is infinitely elastic, thereby creating no first-mover incentives. However, when $D(1)>D(2)$, Pareto efficiency requires that $X^{l*}<X^{f*}$; i.e., the leader invests strictly before the follower.[6] This is a dynamic version of the "business-stealing" effect identified by Mankiw and Whinston (1986), in that preemptive incentives cause lead investment to occur earlier than it would in a non-competitive setting. Space is thus dynamically oversupplied in this market (at least temporarily), where the leader trades off risks associated with early investment with the realization of temporary monopoly rents that follow when space is a strategic substitute.

When $X^{l*} < X^{f*}$ the model predicts discrete time intervals between investments. The expected time to investment by the follower, given that the leader has just invested, is $E_{L \to F}[t \mid X^{l*}, X^{f*}] = \dfrac{ln(X^{l*}) - ln(X^{f*})}{\mu_X - \frac{1}{2}\sigma_X^2}$ if $\mu_X - \frac{1}{2}\sigma_X^2 > 0$. Otherwise, if $\mu_X - \frac{1}{2}\sigma_X^2 \le 0$, the expected time is infinite.

For Figure 1 we have chosen parameter values so that the expectation is finite. As the figure shows, the expected times between development may be rather lengthy. The empirical implication is that development in (finitely demand-elastic) local real estate markets will in general follow an orderly sequenced process. This is not what the data show, however, as supply tends to clump in real estate markets; see, e.g., Wheaton (1987, 1999), Wheaton and Torto (1990).

Figure 1 About Here

Lead Investment Equilibrium When Demand is Observable With Noise

We extend the basic model in an attempt to explain the data. Assume that the inverse demand curve is as specified in equation (10). That is, assume that true demand is unobservable to agents prior to lead investment. Once lead investment occurs, however, current demand is fully revealed. This complete price revelation assumption, while extreme, reflects the spirit of the leader-follower conundrum when there are information externalities in competitive markets for new development.

To determine the leader's investment threshold under incomplete price information, competitors will equalize the *conditional expected values* of their payoffs that result from pioneering investment. In other words, when there is pre-investment uncertainty regarding the true product value, equilibrium requires competitors to be indifferent between leading and following in an expected value rather than a deterministic sense.

Conditional on lead investment having just occurred, the realized investment payoffs to competitors depend on whether the revealed *X(t)* is above or below the follower's investment threshold, X^{f*}. If the revealed *X(t)* is at or above the threshold, the follower invests immediately and both competitors realize an investment value of $X(t)D(2)/\delta_X - K$. If *X(t)* is below the threshold, however, the follower refrains from investment and holds a call option with exercise price *K,* as described in equation (13). Similarly, conditional on the realized *X(t),* value for the leader is as described in equation (14).

The follow-on investment threshold, X^{f*}, is therefore crucial in terms of identifying a symmetric *conditional expected* perfect Markov equilibrium in lead investment. After some algebraic simplification, the following equilibrium relation must hold at the point at which lead investment occurs:

$$\int_0^{X^{f*}} \frac{X(t)D(1)}{\delta_X}\left[1-\left(\frac{X(t)}{X^{f*}}\right)^{\beta-1}\right] g\big(X(t)\big|I(t)\big)dX(t) = \int_0^{X^{f*}} K\left[1-\left(\frac{X(t)}{X^{f*}}\right)^{\beta}\right] g\big(X(t)\big|I(t)\big)dX(t) \quad (16)$$

where $g(X(t)|I(t))$ denotes the probability distribution of the unobservable true value, $X(t)$, conditional on the information available at that time, $I(t)$.

The integrand on the left-hand side of equation (16) is the present value of the leader's monopoly rents, conditional on delay in follow-on development. Integrating this over the conditional density function yields the expected value of the monopoly rents. The integrand on the right-hand side is the present value of the follower's cost savings from deferring investment, conditional on delay in follow-on development. Integrating this quantity over the conditional density function provides the expected cost savings. Thus, at the lead investment threshold, competitors trade off the expected value of temporary monopoly rents from assuming a leader position in the market with the expected cost savings from following.

Note the similarity between equations (15) and (16). When the demand curve is fully observable, as in Equation (15), there is no *ex ante* uncertainty as to the payoff from investing at X^{l*} and this threshold value is explicit in the investment equation. Conversely, when the demand curve is observable only with noise, as in Equation (16), exercise policy is determined by payoffs that are uncertain prior to lead investment.

All possible realized payoffs from lead investment must therefore be considered when product demand information is incomplete, where probabilities depend on the distribution of $X(t)$ conditioned on the information set, $I(t)$. Proposition 2 states a closed-form expression for equation (16) that makes the lead investment threshold explicit in $M(t)$:

<u>Proposition 2 (Investment Boundary for the Leader)</u>: *Consider a symmetric perfect Markov equilibrium in conditional expected values between agents. The lead investor threshold value,* $M^{l*}(t)$*, is determined by the relation:*

$$\frac{M(t)D(1)}{\delta_x}\left[\Phi\left[-d\left(X^{f*},1,\gamma(t)\right)\right]-\left(\frac{M(t)e^{\frac{1}{2}\beta\gamma(t)}}{X^{f*}}\right)^{\beta-1}\Phi\left[-d\left(X^{f*},\beta,\gamma(t)\right)\right]\right]$$

$$=K\left[\Phi\left[-d\left(X^{f*},0,\gamma(t)\right)\right]-\left(\frac{M(t)e^{\frac{1}{2}(\beta-1)\gamma(t)}}{X^{f*}}\right)^{\beta}\Phi\left[-d\left(X^{f*},\beta,\gamma(t)\right)\right]\right] \tag{17}$$

where M(t) is solved implicitly in equation (17) to determine $M^{l*}(t)$*, and where Φ[•] denotes the cumulative standard normal distribution with d defined as*

$$d\left(X^{f*},b,\gamma(t)\right)=\frac{\ln\left(\frac{M(t)}{X^{f*}}\right)+\left(b-\frac{1}{2}\right)\gamma(t)}{\sqrt{\gamma(t)}} \tag{18}$$

<u>Proof</u>: *See Appendix.*

Determining $M(t)=M^{l*}(t)$ so that equation (17) holds is a straightforward application of iterative numerical methods. Note that $M^{l*}(t)$ is a function of time, while the exercise policy in the noiseless case is time-homogeneous. Time dependence results because the residual variance, $\gamma(t)$, is, in general, time-dependent. Also note that there is a convexity correction for $\beta>1$, as seen in the $e^{1/2\beta(\beta-1)\gamma(t)}$ terms in equation (17). This correction is required because of the presence of noise and the fact that payoff functions to the leader and follower are non-linear in the range [*0,* X^{f*}]. Finally, observe that the equilibrium relation stated in equation (17) is Markov in the sense that solution depends only on current time *t*, although *M(t)* is itself a specific composite of historical values as defined in equation (5).

Properties of the lead investment boundary can be illustrated by the special cases of $\kappa \to \infty$ (no noise), $\kappa = 0$ (accumulating noise), and $\sigma_0^2 = c$ (a constant rate of noise over time). As $\kappa \to \infty$, the observed value, *Z(t),* reverts to the true value, *X(t)*, immediately. This implies that $\gamma(t) \to 0$ for all $t>0$ and the lead investment threshold relation expressed in equation (17) reduces to the certainty relation expressed in equation (15).

For $\kappa = 0$, noise steadily accumulates at a rate of $\rho^2 \sigma_Y^2$. In this case, as well as when $\gamma(t)$ is increasing in *t,* it can be shown that the lead investment threshold becomes higher over time. This follows because, as residual variance grows over time, the probability of making an exercise error also increases. Increasing error probabilities and information spillover are strictly to the advantage of follow-on investors, which creates incentives to further delay lead investment.

Finally, when $\sigma_0^2 = c$, residual variance is constant over time to result in a time-homogeneous lead investment threshold.

For realistic parameter values, the lead investment boundary, M^{l*}, often exceeds the follow-on investment threshold of X^{f*}. This is because information generated by lead investment is a public good, thereby creating powerful incentives for delay. In direct relation to our claim that

$M^{l*}(t)$ can exceed X^{f*}, we state a result for the special case of constant returns to industry scale:

<u>Corollary 4 (No Investment Outcome)</u>: *If D(1) = D(2) and demand is observable only with noise, investment <u>never</u> occurs.*

<u>Proof</u>: *See Appendix*

The intuition for the no-investment result is that, if marginal returns to capital investment are constant in scale, there is no first-mover advantage with respect to capturing temporary monopoly rents. Because there is a positive probability that the leader will make a mistake—i.e., investment might provide subnormal returns *ex post*—and because the information externality generated by lead investment is strictly a benefit to the follower, free-rider incentives will dominate, and neither competitor will be willing to move first. Long delays of this type would appear to be costly from a social perspective, and suggest a potential role for coordination in the development of new product markets.

The result stated in corollary 4 is similar to the no-investment result obtained by Geltner et al. (1996). They consider the perpetual investment option on the best of two assets in a full information setting, and show that investment never occurs when the two asset values equal one another. The result holds no matter how deep the investment option is in-the-money. The intuition for this finding is that, when the claimholder is currently indifferent between outcomes, the opportunity cost of waiting to see which investment dominates the other is always less than the risk of immediate investment.

In our model, delay is for strategic reasons. Investment *never* occurs in our model because of the one-sided spillover costs of moving first, while investment always occurs within a finite period of time in the Geltner et al. model.

To analyze the characteristics of the lead investment threshold, $M^{l*}(t)$, in greater detail, in Figures 2 through 4 we vary key parameter values as they affect the relation between the lead threshold value (on the y-axis) and the noise volatility parameter, σ_Y (on the x-axis). These figures directly illustrate that the lead investment threshold rises as the residual variance increases. A positive relation occurs because $\frac{\partial \gamma(t)}{\partial \sigma_Y} > 0$ for all time *t*.

Figures 2 and 3 About Here

Incentives for the leader to delay are derived from two complementary sources of uncertainty: waiting for the arrival of information in order to resolve *future* price uncertainty, as well as waiting to resolve *current* price uncertainty. A higher level of current price uncertainty, as summarized by σ_Y, interacts with future price uncertainty to induce competitors to set higher threshold values in order to compensate for information spillover costs that follow from lead investment.

Figure 2 shows that the lead investment rises with the price elasticity of demand. As *D(2)* approaches *D(1)* from below, a flatter-sloped demand "curve" obtains to provide relatively lower expected monopoly rents, which causes the leader to invest at a higher threshold value. Figure 3 illustrates that, as the observed value is more tightly cointegrated with the true value *vis-à-vis* a higher κ parameter, the lead investment threshold drops. This follows because an increase in κ results in a higher relative rate of information arrival, which reduces errors in exercise policy to mitigate incentives for delay.

Figure 4 illustrates the effects of current time on the lead investment threshold value. Interestingly, for low values of σ_Y, the lead investment threshold is higher for *t=2* than for *t=10.* The relation changes when σ_Y is sufficiently high. This follows from the complicating effects of initial estimation error, as summarized by σ_0. For examples used to generate Figure 4, σ_0=.10. The value of σ_Y at which residual variance, $\gamma(t)$, equals initial variance, σ_0, and therefore produces a constant residual variance through time, is σ_Y=.056. For σ_Y< (>).056, residual variance reduces (increases) over time. Reduced residual variance implies a relatively high rate of information arrival for small *t,* and therefore a stronger tendency for delay in order to realize the benefits of the new information. Increased residual variance has the opposite effect.

Figure 4 About Here

Figure 5 graphs the lead investment threshold against the level of initial noise, σ_0. In this case, the value toward which the residual variance asymptotes (which is the parameter *c* as defined in Table 1) is constant, regardless of σ_0. As can be seen, increasing the level of initial noise raises the lead investment threshold. This follows because $\frac{\partial \gamma(t)}{\partial \sigma_0} > 0$; i.e., all else equal, higher initial noise increases the residual variance, which increases the likelihood of an error in exercise policy. This in turn leads to further delay in lead investment.

Figure 5 About Here

These figures clearly illustrate that the leader's investment boundary can exceed the follower's boundary. Rational development cascades can be characterized by calculating the probability that the revealed $X(t)$ exceeds the follower's investment threshold, X^{f*}, conditional on the leader investing at $M^{l*}(t)$. The probability of immediate follow-on investment in this case is given by $\Phi\left[-d\left(X^{f*},0,\gamma(t^*)\right)\right]$, where t^* corresponds to the time at which lead investment occurs. If $X(t^*)\geq X^{f*}$, there will be a rush to develop in order to harvest high realized market rents. Conversely, if revealed demand is such that $X(t^*)< X^{f*}$, the market leader may have to go it alone for a period of time until the market can support further entry.

Probabilities of immediate follow-on investment are shown in Figures 6 and 7. Observe that the probability of a development cascade increases with σ_Y and $D(2)$, and decreases with κ. The intuition for these comparative statics follows directly from comparative statics associated with the lead investment threshold. These relations and others that can be obtained from a further examination of equation (17) provide empirically testable hypotheses regarding local development outcomes when irreversibility, value uncertainty, an imperfectly competitive market structure, and information spillover interact to affect investment incentives.

Figures 6 and 7 About Here

It is interesting to compare our model with Grenadier's (1999) model of option-based investment cascades. In Grenadier's model, developers receive private signals of the true built asset value, which vary in their precision. In the absence of agent-generated information, noise remains constant over time, and value signals are independent of one another. Heterogeneous value signals result in a precision-based ordering of development.

In our model, value signals are commonly observed by developers, and hence are perfectly correlated. Noise variance increases or decreases over time, depending on initial levels of noise relative to the steady-state. Because of the symmetry of information, developers are equally likely to assume the role of lead developer in our model.

Information is revealed by the actions of developers in both models. Information revelation may be partial or complete in Grenadier's model, where developers with less precise value signals learn by observing the exercise or delay decisions of agents who have more precise value signals. Developers do not learn by observing revenue flows from built projects, however. Learning by observing may lead developers to throw away their own information and jump on the investment bandwagon, as relatively better informed agents commit to development. Significant underinvestment is not a crucial issue in the Grenadier model, since the information structure is such that inaction gradually reveals the true asset value.

In our model, information revelation is complete because of the observation of realized revenue flows from lead development. Before that, however, no information is revealed by a lack of action among agents. Symmetric *ex ante* value signals and complete *ex post* value revelation are responsible for the long delays in lead development in our model. Thus, useful information that is tossed away *after the fact* may be responsible for over-investment and herding in Grenadier's model, but no information is lost as a result of investment in our model.

It is the incentive to delay to avoid mistakes, and the inability to deduce information *prior* to lead investment, that results in initial under-investment and then potential herding in our model. That is, information is potentially underused after development occurs in the Grenadier model, while under-utilization problems occur in the pre-investment stage in our model.

Summary and Conclusion

We develop a theory of contingent-claims valuation when the underlying real asset value is observed with noise. Our central result is the determination of the forward-looking dynamics of the time-filtered value. These dynamics are shown to be identical to the dynamics associated with the true asset value, except that the measures of the rate of information arrival—the variance rate of the processes—are different.

Most important, the rate of information arrival in the conditional expected asset value at a particular time can be faster or slower than the rate of information arrival in the true asset value. We use this rate to derive the *revealed variance*, which is the total amount of information that arrives over a particular time interval. Revealed variance is used in the determination of value and exercise policy for options on noisy real assets. Time-filtered value dynamics are also used in the determination of forward contract values.

This theory allows us to examine the effects of competition in the timing of real estate development when, before lead investment, the precise location of the demand curve is uncertain. This economic setting describes the persistence of blighted urban land, gentrification and large-scale multi-stage developments such as DisneyWorld.

Information spillover combined with irreversibility and uncertainty as to future rental rates create strong incentives for delay in order to free ride on information generated by the first mover. If there is infinitely elastic product demand, we show that investment never occurs. We also show that, conditional on the lead investment threshold value, the likelihood of a development cascade is significant. We thus provide an information-based explanation for both supply booms and busts in localized markets for real estate development.

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Appendix

Proof of Proposition 1: By applying a general formulation from COR (2001), dynamics for the lognormally distributed time-filtered value can be written as

$$dM(t) = \left(\mu_X + \xi(t)\left(X(t) - M(t)\right)\right)M(t)dt + \xi(t)M(t)\sigma_Z dW_Z \tag{A1}$$

The drift term suggests that, *ex post*, the time-filtered value reverts toward the unobserved true value. We are, however, concerned with *ex ante* dynamics. That is, we are conditioning on information available at time *t* to determine the instantaneous change in the time-filtered value, *M(t)*. Because *M(t)* an unbiased estimate of *X(t),* and because *X(t)* is unobservable, the drift term in (A1) simplifies to the drift term of the true value, as in equation (6). It is important to recognize that *ex post* reversion of *M(t)* toward *X(t)* has no effect on variance of the stochastic process (see COR (2001) for a demonstration of this result). Independence allows one to simply integrate the variance term in (6) to obtain the total variance over a particular time interval. Independence obtains because noise is by definition idiosyncratic, so that there is no systematic relation between differences in *M(t)* and *X(t)* over time. This is different from standard mean-reverting specifications such as the Vasicek (1977) interest rate process, since noise does not impose any systematic effects on the drift of *M(t)* to limit the cumulative impact of random movements in the time-filtered value.

Proof of Proposition 2: Proving this proposition amounts to showing that

$$\int_0^{X^{f*}} X^b(t)\, g\left(X(t) \middle| I(t)\right) dX(t) = \left(M(t) e^{\frac{1}{2}(b-1)\gamma(t)} \right)^b \Phi\left[-d\left(X^{f*}, b, \gamma(t)\right)\right] \tag{A2}$$

Recognize that *g[ln(X(t)|I(t)]* is normally distributed with mean $\ln(M(t)) - \frac{1}{2}\gamma(t)$ and variance $\gamma(t)$. Now define the transformations $U(t) = \frac{\ln\left(\frac{X(t)}{M(t)}\right) + \frac{1}{2}\gamma(t)}{\sqrt{\gamma(t)}}$ and $W(t) = U(t) - b\sqrt{\gamma(t)}$. Use the defined relation between *U(t)* and *X(t),* complete the square, and then rewrite the left-hand side of equation (A2) as follows:

$$\left(M(t)e^{\frac{1}{2}(b-1)\gamma(t)}\right)^{b} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d\left(X^{f*},0,\gamma(t)\right)} e^{-\frac{1}{2}\left(U(t)-b\sqrt{\gamma(t)}\right)} dU(t)$$

Substitute *W(t)* for *U(t),* and the right-hand side of equation (A2) is verified. Finally, equation (A2) can be used to obtain equation (17) from equation (16).

<u>Proof of Corollary 4</u>: If *D(1)=D(2),* product demand is insensitive to supply. In this case, it is a simple exercise to show that the leader's expected value conditional on investment is $M(t)D(2)/\delta_X - K$. Once the leader enters the market, the true demand becomes known. If demand is revealed to be above X^{f*}, the follower enters and product value for both suppliers is $X_t D(2)/\delta_X - K$. Alternatively, if the true demand is below X^{f*}, the follower defers investment. In this case the follower's product value is $(X(t)/X^{f*})^{\beta}(X^{f*}D(2)/\delta_X - K)$, which is greater than the product value of the leader since $X(t) < X^{f*}$ and $\beta > 1$. Now, for any value *M(t)* for which $\gamma(t)>0$, there is some positive probability that the true value, *X(t),* is below X^{f*}. Therefore, given any *M(t),* the expected product value of the follower exceeds the expected product value of the leader. Since investment occurs only if the expected product values of the leader and the follower are equal, neither competitor wishes to lead and investment never occurs.

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Table 1: Key Definitions and Formulas

Formula	Equation Number COR (2001a)
Initial Noise: $\sigma_0^2 = Var[x(0) \mid \vartheta(0)]$	(5)
Residual Variance Dynamics: $\frac{d\gamma(t)}{dt} = \sigma_X^2 - \frac{\rho^2\left(\sigma_X^2 + \gamma(t)\kappa\right)^2}{\sigma_X^2}$	(17)
Weighting Term: $\xi(t) = \rho^2\left(1 + \frac{\kappa\gamma(t)}{\sigma_X^2}\right)$	(18)
Residual Variance: $\gamma(t) = Var[x(t) \mid I(t)]$ $= \frac{2\sigma_X^2\left(\sigma_0^2 - c\right)}{e^{2\rho\kappa t}\left(2\sigma_X^2 + \rho\kappa\left(\sigma_0^2 - c\right)\right) - \rho\kappa\left(\sigma_0^2 - c\right)} + c$	(19)
Steady-State Residual Variance: $c = \frac{\sigma_X^2(1-\rho)}{\rho\kappa} = \frac{\sigma_X\left(\sigma_Z - \sigma_X\right)}{\kappa}$	(20)

Figure 1: Expected Time to Follow-on Development. Parameter Values: μ_X*=.12;* δ_X*=.09; r=.06; D(1)=1.0; D(2)=.90; K=10.*

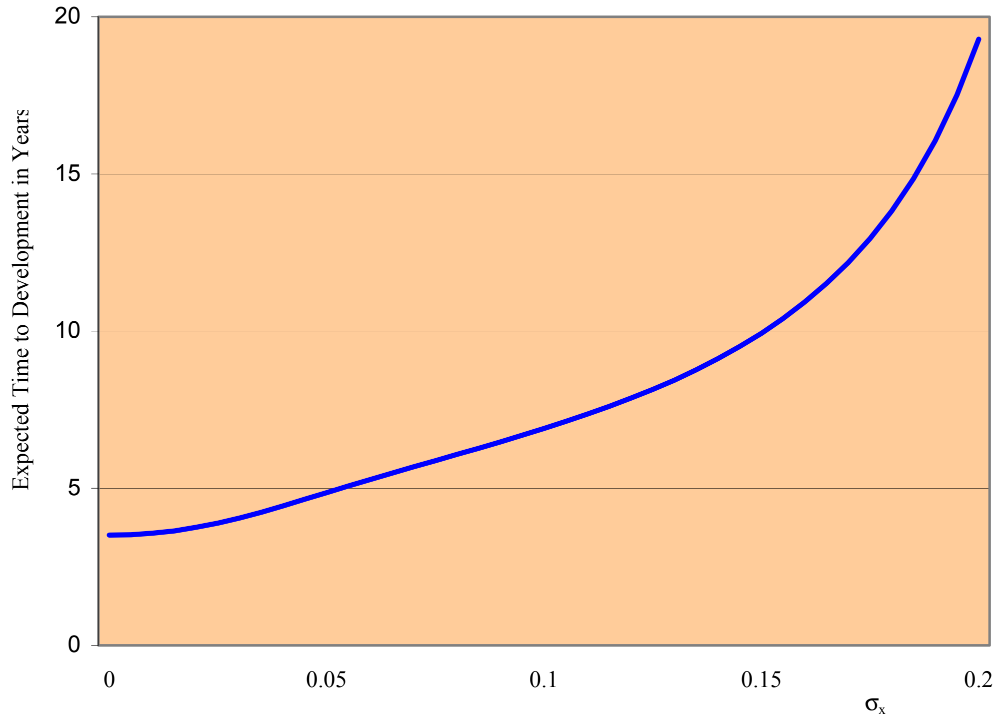


Figure 2: Lead Investment Threshold as a Function of Noise Volatility—Alternative Demand Elasticities, *D(2)*. Parameter Values: r=.06; δ_X=.08; σ_X=.15; σ_0=.10; κ=.15; *D(1)=1.0; K=10; t=5.*

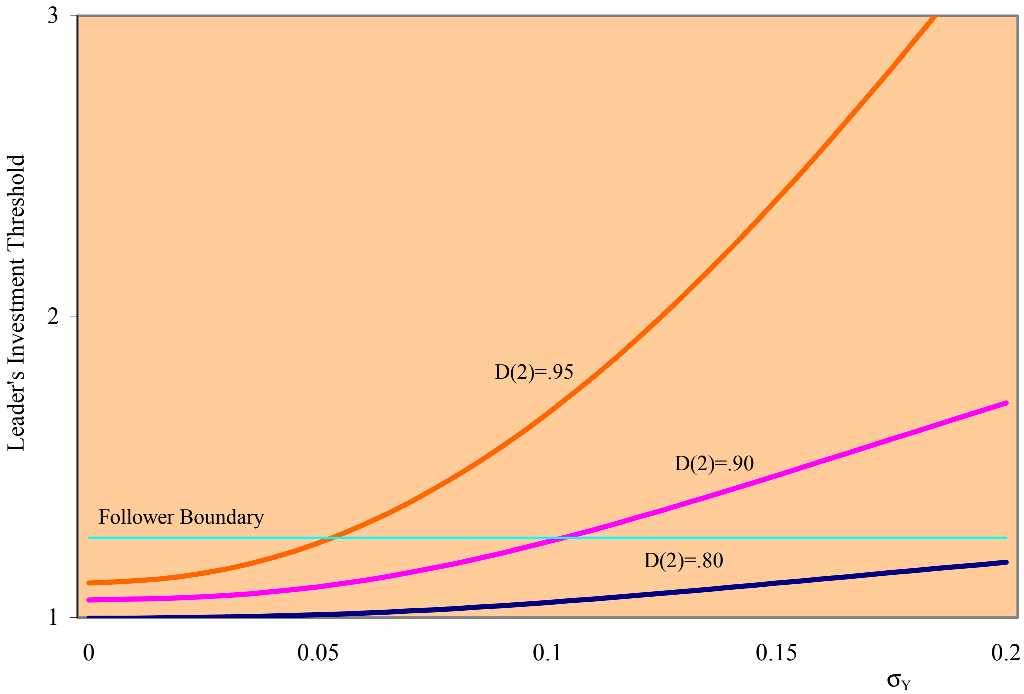


Figure 3: Lead Investment Threshold as a Function of Noise Volatility—Alternative Rates of Mean Reversion in Noise, κ: Parameter Values: $r=.06$; $\delta_X=.08$; $\sigma_X=.15$; $\sigma_0=.10$; $D(1)=1.0$; $D(2)=.90$; $K=10$; $t=5$.

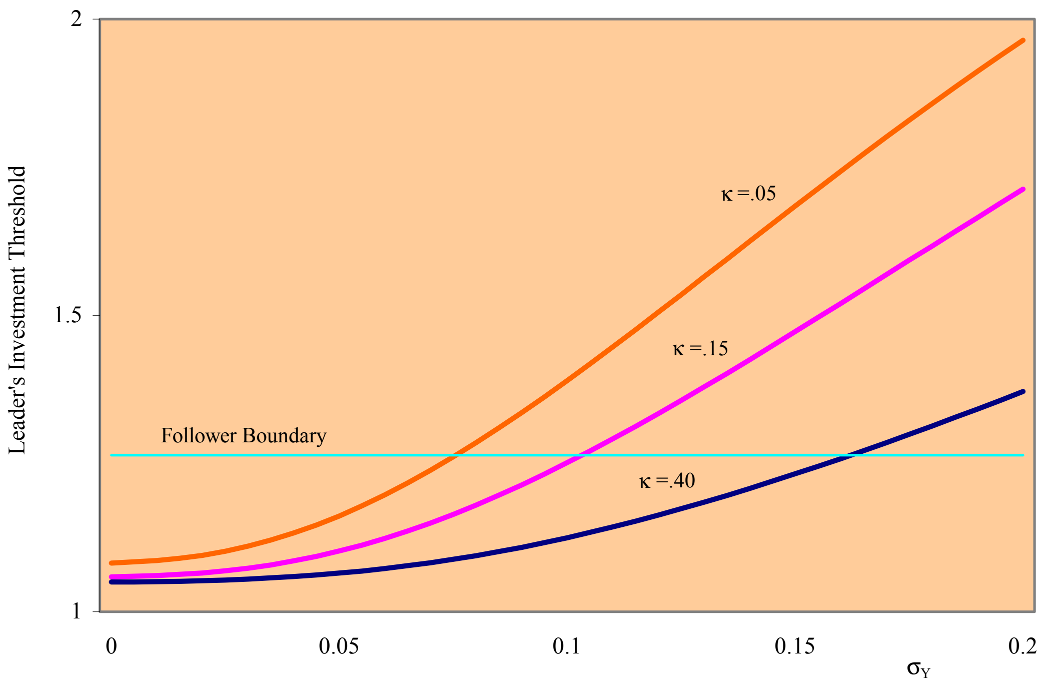


Figure 4: Lead Investment Threshold as a Function of Noise Volatility—Alternative Times from Initial Observation, *t*. Parameter Values: $r=.06$; $\delta_X=.08$; $\sigma_X=.15$; $\sigma_0=.10$; $\kappa=.15$; $D(1)=1.0$; $D(2)=.90$; $K=10$.

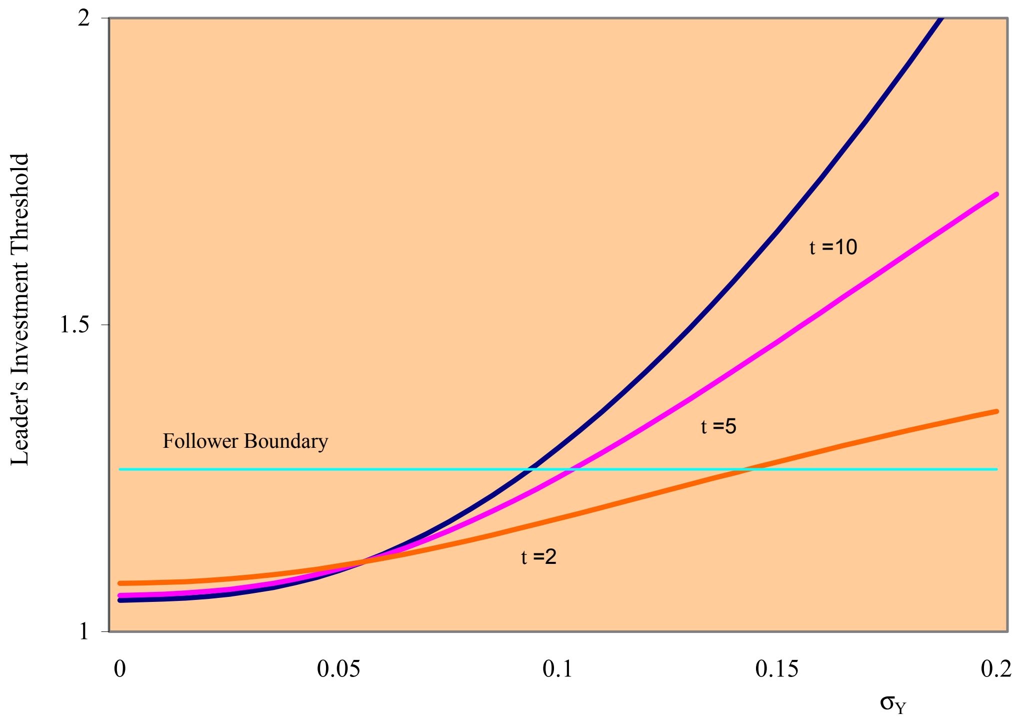


Figure 5: Lead Investment Threshold as a Function of Initial Noise Volatility, σ_0. Parameter Values: r=.06; δ_X=.08; σ_X=.15; κ=.15; σ_Y=.10; $D(1)$=1.0; $D(2)$=.90; K=10; t=5.

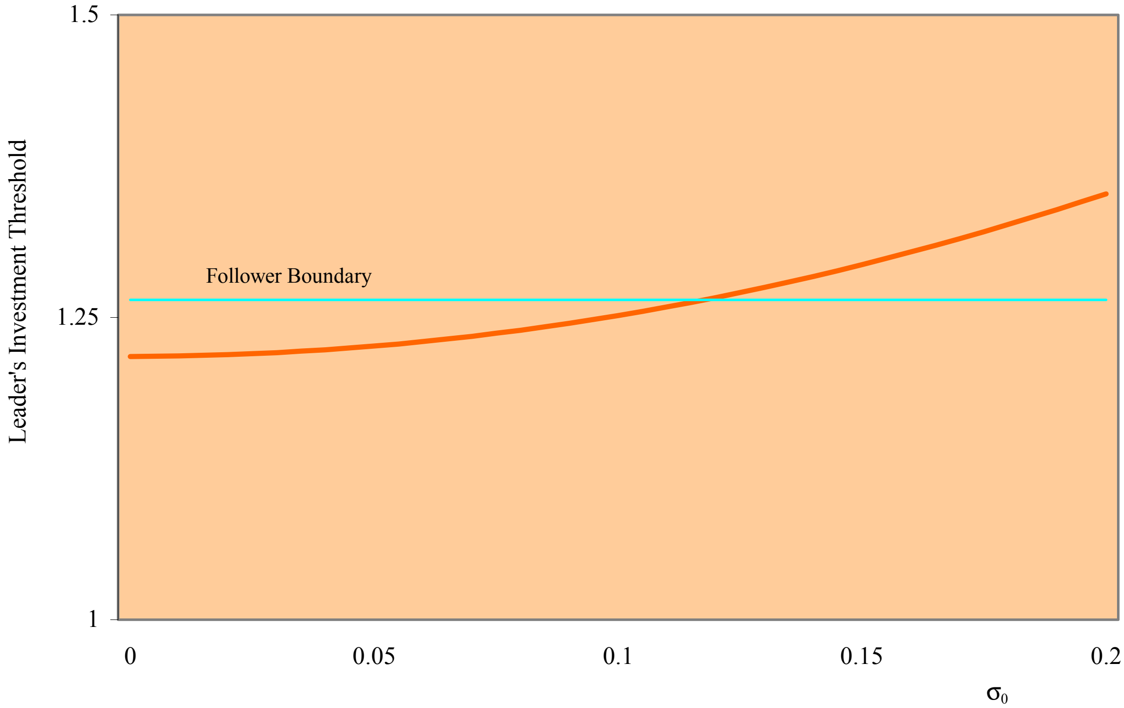


Figure 6: Probability of Immediate Follow-On Development as a Function of Noise Volatility—Alternative Demand Elasticities, *D(2)*. Parameter Values: r=.06; δ_X=.08; σ_X=.15; σ_0=.10; κ=.15; *D(1)=1.0; K=10; t=5.*

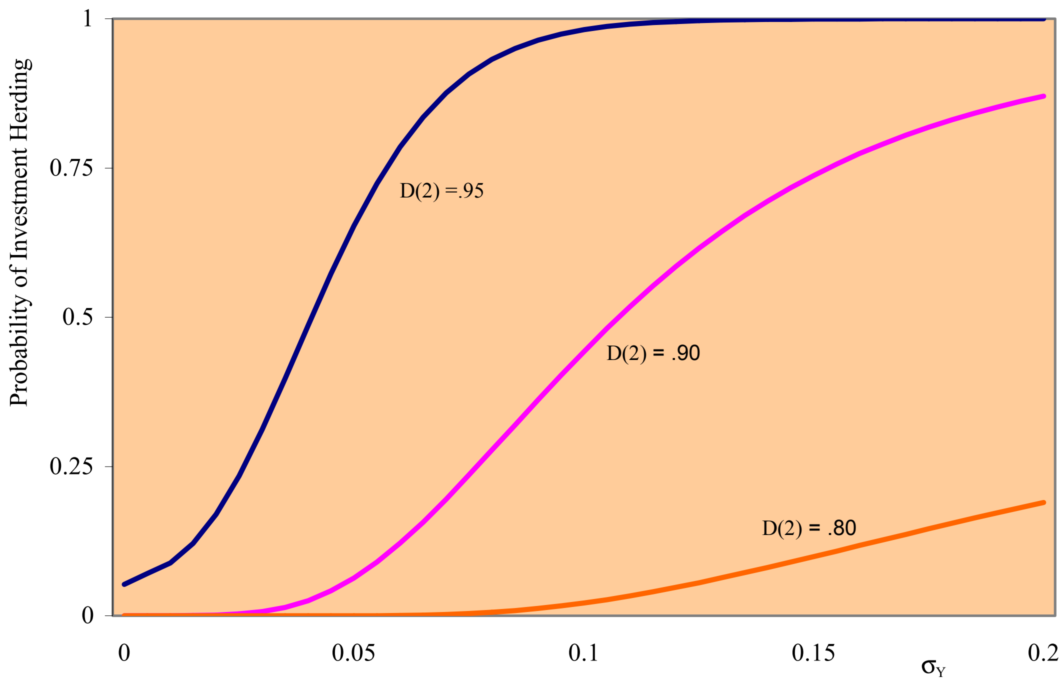
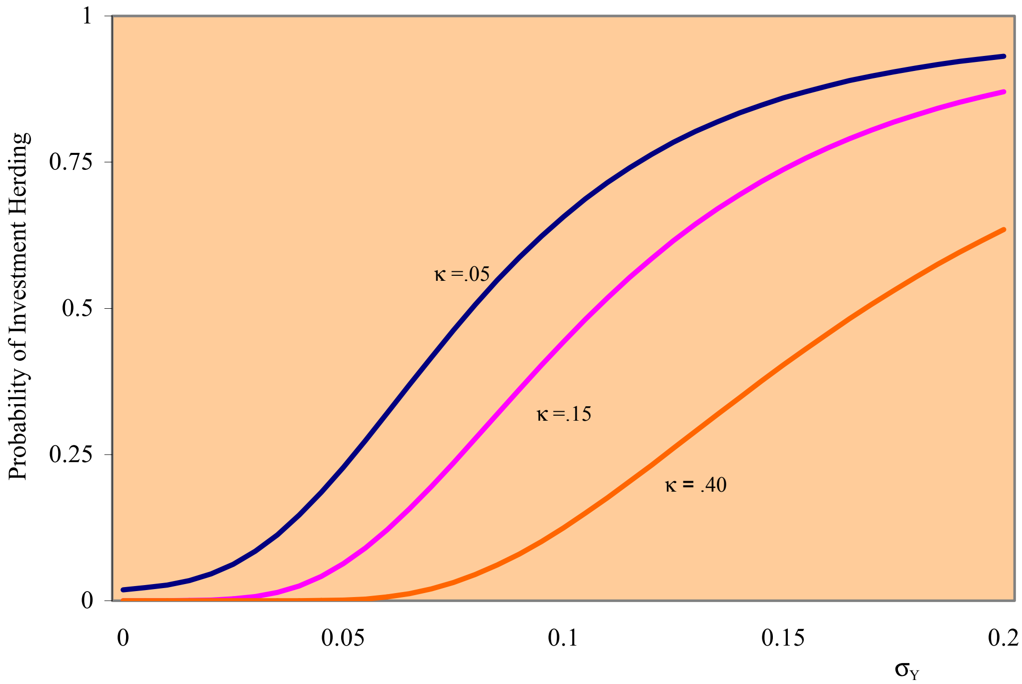


Figure 7: Probability of Immediate Follow-On Development as a Function of Noise Volatility—Alternative Rates of Mean Reversion in Noise, κ: Parameter Values: $r=.06;\ \delta_X=.08;\ \sigma_X=.15;\ \sigma_0=.10;\ D(1)=1.0;\ D(2)=.90;\ K=10;\ t=5.$



Endnotes

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[1] COR (2001) outlines a general theory for contengent-claim valuation with noisy real assets. The theory is applied to the special case of mean-reverting noise in this paper. We extend COR (2001) by deriving additional theoretical results and applying the theory to a real world development setting.

[2] Our model directly relates to at least three strands of literature. One has to do with the strategic exercise of options that are not in zero net supply (e.g., Emanuel (1983), Spatt and Sterbenz (1988), Williams (1993), and Grenadier (1996)). A second strand of literature in industrial organization considers preemption under uncertainty (e.g., Spatt and Sterbenz (1985) and Fudenberg and Tirole (1985)). A third strand of literature considers herding behavior and information cascades as explanations for the rational 'clumping' of investments. See in particular Rob (1991), Banerjee (1992), Bikhchandani, Hirshleifer and Welch (1992), Caplin and Leahy (1995), and Grenadier (1999). Our application is perhaps most similar to Caplin and Leahy (1995) in its emphasis on information externalities as a rationale for long waiting times in new product markets, and for potentially rapid market development once initial investment is made.

[3] See COR (2001) for a formal proof of equation (8).

[4] A more general economic setting could, with appropriate qualification, allow for the existence of an equivalent martingale, so that, after a risk adjustment, the cash flows could be discounted at the risk-free rate of interest. See, for example, Rubinstein (1976) for a utility-based application of equivalent martingale pricing in incomplete markets.

[5] A more detailed presentation of the noiseless version of the model is in Dixit and Pindyck (1994, pp. 309-314).

[6] See proposition 2 in Grenadier (1996) for a proof that $X^{l*} < X^{f*}$. It is relevant to note that this equilibrium requires communication and coordination among competitors. Absent such conditions, both competitors will invest at X^{l*}.