AGING 7/29/02
DAVIDOFF/BROWN/DIAMOND
1:00PM

# Annuities and Individual Welfare

Thomas Davidoff, Jeffrey Brown, Peter Diamond *

June 29, 2002

**Abstract**

This paper extends the theory of annuitization with no bequest motive in two directions. First, we derive sufficient conditions, in a more general setting than Yaari (1965), under which complete annuitization is optimal, and weaker conditions under which partial annuitization is better than zero annuitization. Second, we explore how incremental and complete annuitization affect consumer welfare in these more general conditions. When markets are complete, all savings are optimally annuitized as long as there is no bequest motive and annuitized assets have greater returns than conventional assets. Consumers' utility need not satisfy intertemporal additive separability nor the expected utility axioms, and annuities need not be actuarially fair. The result is weakened if annuities markets are incomplete, so that there are some assets which do not exist in annuitized form: as long as trade occurs all at once and consumption is positive in every state of nature, a small degree of annuitization is better than no annuitization. When conventional asset markets are incomplete, if annuities are illiquid, then it is possible that no savings are annuitized. We present numerical calculations of the financial benefit and optimal degree of annuitization for consumers with standard CRRA preferences, and compare these results to results where otherwise identical consumers have utility that depends both on present consumption and a standard of living to which they have grown accustomed. In our specification, the effect of adding intertemporal dependence hinges on the size of initial standard of living relative to resources.

*Davidoff, Haas School of Business, UC Berkeley; Diamond, MIT; Brown, University of Illinois at Urbana-Champaign and NBER. Davidoff is grateful for financial support from the Center for Retirement Research at Boston College pursuant to a grant from the U.S. Social Security Administration funded as part of the Retirement Research Consortium. The opinions and conclusions are those of the authors and should not be construed as representing the opinions or policy of the Social Security Administration or any agency of the Federal Government, or the Center for Retirement Research.

# 1 Introduction

Annuities play a central role in the theory of a life-cycle consumer with an unknown date of death. Yaari (1965) shows that certain consumers should annuitize all savings, and evaluates the welfare gain for such consumers if they move all of their savings from unannuitized to annuitized assets. These consumers satisfy several restrictive assumptions: they are von Neumann-Morgenstern expected utility maximizers with intertemporally separable utility, they face no uncertainty other than time of death, the markets in which they trade are complete, actuarially fair annuities are available, and they have no bequest motive.[1]. The subsequent literature has typically relaxed one or two of these assumptions, but has universally retained expected utility and additive separability, the latter dubbed "not a very happy assumption" by Yaari.[2]

This paper extends the theory of annuitization with no bequest motive in two directions: first, we derive sufficient conditions in a more general setting than Yaari's under which complete annuitization is optimal, and yet weaker conditions under which partial annuitization is better than zero annuitization. Second, we explore how incremental and complete annuitization affect consumer welfare in these more general conditions.

Section 2 considers annuitization when trade takes place all at once. In section 2.1, we consider a two period setting with no uncertainty other than individuals' date of death. Here, all savings are placed in annuities as long as there is no bequest motive and annuities pay a greater return, net of transaction costs, than conventional assets in the event that the consumer survives. This is a weaker assumption than actuarial fairness. The intuition behind this result is that consumers who do not care about wealth after death will prefer any asset which pays out more in every period of life than some other asset, regardless of how small the difference in payouts. Neither expected utility nor additive separability are required for these results. We are thus able, in section 2.2 to extend the result to the Arrow-Debreu case with arbitrarily many future periods with aggregate uncertainty, as long as conventional asset and annuities markets are complete.

In section 2.3, we provide a weaker result for the case where conventional asset markets are complete, but annuities markets are incomplete. In this case, as long as trade occurs all at once and preferences are such that consumers avoid zero consumption in every state

[1]It must be noted that Yaari does not suggest that any of these assumptions (other than absence of bequest motives) are required.

[2]Kotlikoff and Spivak (1981), Bernheim (1991), Hurd (1989) Jousten (2001) and Walliser (2001) consider incomplete markets and bequest motives. Brown (2001b) and Palmon and Spivak (2001) consider actuarially unfair pricing. Milevsky (2001) explores the consequences of uncertainty concerning asset returns.

of nature, some positive degree of annuitization is better than zero annuitization. All that is required for this result is that zero consumption in any state of nature is always avoided and that the annuities that exist pay greater returns than any conventional asset which pays off in all the same states of nature. Also, if trade in unannuitized assets occurs all at once, we have the result that an annuitized version of any conventional asset always dominates the underlying asset for consumers with no bequest motive. This result holds even if the asset in question does not pay off in every state of nature. An important consequence is that the result that annuities dominate conventional assets extends past riskless bonds to risky securities including mutual funds and certificates of deposit. For example, suppose the provider of some family of mutual funds doubles the set of available funds by offering a matching, "annuitized" fund which periodically takes the accounts of investors who die and distributes the proceeds proportionally across the accounts of surviving investors. With a large number of investors (and small additional administrative costs), the returns to this annuitized fund would strictly exceed the returns of the underlying funds for investors as long as they live. Since we assume investors do not care about wealth after death, the annuitized funds must dominate the underlying funds as an investment.

When conventional markets are incomplete, so that trade occurs more than once, it is possible that zero wealth is optimally annuitized, but only if annuities are illiquid relative to conventional assets. This possibility is discussed in section 3.

Given these results, the empirically observed dearth of private supply of annuitized assets is even more puzzling than previous analysis suggests.[3] Taking this shortage of annuitized assets as given, it is interesting to consider the welfare consequences of moving savings from conventional assets, which allow complete choice over payout trajectories, to a single annuitized asset, which imposes a particular, and possibly unattractive payout trajectory. One common approach to measuring the benefits of annuitization is to compare expenditures with zero annuitization to expenditures with complete annuitization holding utility constant (as in Yaari (1965), Kotlikoff and Spivak (1981), Mitchell, Poterba, Warshawsky and Brown (1999), Brown (2001b)). This is an equivalent variation, the standard measure of the welfare consequences of policies with economic impact.[4] We may also be interested in the effect

[3]While Poterba (1997) notes that variable annuities are becoming increasingly popular, these insurance products are tax-favored savings devices that include an option to annuitize, an option which rarely appears to be activated.

[4]Mitchell et al. (1999) calculate a measure of "wealth equivalence," which is the ratio of annuitized to non-annuitized wealth that achieves the no annuitization utility level. Other studies often calculate the ratio of non-annuitized to annuitized wealth required to achieve the full annuitization utility level, and this is sometimes referred to as the "annuity equivalent wealth."

on expenditures of a differential increase in annuitization: Mitchell et al. (1999) estimate a sort of marginal valuation of annuities by determining the difference between the change in expenditures from the move from zero to complete annuitization and from the move from half annuitization to complete annuitization. Brown (2001a) calculates the effect of a similar move from partial to complete annuitization for individuals in the Health and Retirement Survey.

Results on the sign of welfare effects follow from those on optimal choice of annuitization: completeness of markets is sufficient to guarantee that incremental (and hence complete) annuitization is welfare improving. With incomplete markets, the welfare effects of incremental annuity purchases are ambiguous (although we know that a move from zero to a small degree of annuitization is welfare increasing). We also analyze the demand characteristics that determine the size of welfare benefits of complete and incremental annuitization.

In section 4, following the literature, we apply the welfare analysis to the provision of fixed real annuities using two particular utility functions. We start with the case of a 65-year old single male with CRRA utility, and provide theoretical results on the optimal degree of annuitization as well as numerical estimates of the welfare consequences. We also relate the gain from increased annuitization to parameters of the utility function. Consistent with previous research, we find that annuitization is equivalent to a large increase in retirement wealth. We also find that the optimal fraction of savings placed in a constant real annuities is very large: 100 percent when the rate of time preference equals the interest rate (this result generalizes to all additively separable preferences), and 72 percent when the rate of time preference is large relative to the interest rate (10 percent vs. 3 percent). We then examine how the results change when this same 65-year old single male has preferences that are intertemporally dependent. Specifically, we consider a utility function (similar to one presented in Deaton (1991)) in which instantaneous utility is a function of the ratio of present consumption to a standard of living, which is itself derived from past consumption. We find that the value of annuitization increases with the introduction of a standard of living effect when the initial (age 65) standard of living is small relative to wealth. Conversely, for cases where the initial standard of living effect is large relative to initial retirement wealth, the welfare effects of annuitization are much smaller, and sometimes even negative. In these cases, consumers prefer to slowly adjust consumption downwards, but a constant annuity requires a large initial jump downwards. In what we regard as an extremely bad case for annuitization however, approximately 60 percent of wealth is optimally annuitized. We also calculate the strictly positive welfare changes associated with a move from zero annuitization to complete annuitization when consumers are free to choose the trajectory of annuity payments. Freedom of choice over payments naturally has the greatest benefit relative to the constant annuity for the households who optimally consume earlier in retirement.

To the extent that the population exhibits heterogeneity in the importance of standard of living effects, the results suggest that there may be important heterogeneity in the value placed on annuitization. However, the large fraction of savings placed in annuities for all preferences reinforces the empirical puzzle of low annuitization rates and suggests that absent bequest motives and markets for private annuitization, some mandatory annuitization may be social welfare improving.

# 2 Annuitization When Trade Occurs All At Once

Analysis of intertemporal consumer choice is greatly simplified if resource allocation decisions are made all at once. Consumers will be willing to commit to a fixed plan of expenditures at the start of time under either of two conditions. The first condition, standard in the complete market Arrow-Debreu model is that, at the start of time, consumers are able to trade goods across time and all states of nature. Alternatively, first period asset trade obviates future trade across states of nature if consumers live for only two periods. In this case, absent a bequest motive, there is no reason to do anything with available wealth in the second period other than consume it.[5]

## 2.1 Two Periods, No Aggregate Uncertainty

Yaari considers annuitization in a continuous time setting where consumers are uncertain only about the time at which they will die. Some important results can be seen more simply by dividing time into two discrete periods: the present, period 1, when the consumer is definitely alive, and period 2, when the consumer is alive with probability $1 - q$. We maintain the assumption that there is no bequest motive, and for the moment assume that only survival to period 2 is uncertain. In this case, lifetime utility is defined over first period consumption $c_1$ and planned consumption in the event that the consumer is alive in period 2, $c_2$. By writing

$$U = U(c_1, c_2)$$

we allow for the possibility that the effect of second-period consumption on utility depends on the level of first period consumption. This formulation does not require that preferences satisfy the axioms for $U$ to be an expected value.

[5] With multiple consumption goods, trade would generally be optimal, but here we consider a single good.

We approach both optimal decisions and the welfare evaluation of the availability of annuities by taking a dual approach. That is, we analyze consumer choice in terms of minimizing expenditures subject to attaining a minimal level of utility. We measure expenditures in units of first period consumption. Assume that there is a bond available which returns $R_B$ units of consumption in period 2, whether the consumer is alive or not, in exchange for each unit of the consumption good in period 1. Assume in addition the availability of an annuity which returns $R_A$ in period 2 if the consumer is alive and nothing if the consumer is not alive. Whereas the bond requires the supplier to pay $R_B$ whether or not the saver is alive, the annuity pays out only if the saver is alive. If the annuity were actuarially fair, then we would have $R_A = \frac{R_B}{1-q}$. Adverse selection and transaction costs may drive returns below this level (the relevant definition of $R_A$ for our purposes is net of transaction costs and inclusive of the effects of adverse selection). However, because any consumer will have a positive probability of dying between now and any future period, thereby relieving borrowers' obligation, we regard the following as a weak assumption [6]:

**Assumption 1** $R_A > R_B$

Denoting by $A$ savings in the form of annuities, and by $B$ savings in the form of bonds, if there is no income in period 2 (e.g. retirees), then

$$c_2 = R_A A + R_B B, \tag{1}$$

and expenditures for lifetime consumption are

$$E = c_1 + A + B. \tag{2}$$

The expenditure minimization problem can thus be defined as a choice over first period consumption and bond and annuity holdings:

$$\min_{c_1,A,B} c_1 + A + B \tag{3}$$
$$s.t.\ U(c_1, R_A A + R_B B) \geq \bar{U}$$

By Assumption 1, purchasing annuities and selling bonds in equal numbers would cost nothing and yield positive consumption when alive in period 2 but leave a debt if dead. However, such an arbitrage would imply that lenders would be faced with losses in the event that such a trader failed to live to period 2. The standard Arrow-Debreu assumption is that

[6] That $R_B < R_A < \frac{R_B}{1-q}$ is supported empirically by Mitchell et al. (1999). If the first inequality were violated, annuities would be dominated by bonds.

planned consumption is in the consumption possibility space. For someone who is dead, this would require that the consumer not be in debt. In this simple setting the restriction is therefore that

$$B \geq 0.$$

This setup yields two important results. The first considers improving an arbitrary allocation while the second refers to the optimal plan.

**Result 1** *(i) If* $B > 0$, *then (i) annuitization can be increased while reducing expenditures and holding the consumption vector constant. (ii) The solution to problem (3) sets* $B = 0$.

***Proof.*** *For (i) a sale of* $\frac{R_A}{R_B}$ *of the bond and purchase of 1 annuity works by Assumption 1 and definition of* $c_2$. *For (ii), by(i), a solution with* $B > 0$ *fails to minimize expenditures. Solutions with the inequality reversed are not permitted.* ■

In this two period setting, Part (ii) of Result 1 is an extension of Yaari's result of complete annuitization to conditions of intertemporal dependence in utility, preferences that may not satisfy expected utility axioms and actuarially unfair annuities. All that is required is that there is no bequest motive, and that the payout of annuities dominates that of conventional assets.

Part (i) of Result 1 implies that the introduction of annuities reduces expenditures for constant utility, thereby generating increased welfare (a positive equivalent variation or a negative compensating variation). We might be interested in two related calculations: the reduction in expenditures associated with allowing consumers to annuitize a larger fraction of their savings (particularly from a level of zero), and the benefit associated with allowing consumers to annuitize all of their savings. That is, we want to know the effect on the expenditure minimization problem of loosening or removing an additional constraint on problem (3). To examine this issue, we restate the expenditure minimization problem with a constraint on the availability of annuities as:

$$\min_{c_1, A, B} : c_1 + A + B \tag{4}$$

$$s.t. : U(c_1, R_A A + R_B B) \geq \bar{U}$$

$$A \leq \bar{A}. \tag{5}$$

$$B \geq 0 \tag{6}$$

We know that utility maximizing consumers will take advantage of an opportunity to annuitize as long as second-period consumption is positive. This is ensured by the plausible condition that zero consumption is extremely bad:

**Assumption 2**

$$\lim_{c_t \to 0} \frac{\partial U}{\partial c_t} = \infty \quad for \;\; t = 1, 2$$

We can see from the optimization that allowing consumers previously unable to annuitize any wealth to place a small amount of their savings into annuities (incrementing $\bar{A}$ from zero) leaves second period consumption unchanged (since the cost of the marginal second-period consumption is unchanged, and so too, therefore, is the level of consumption in both periods). By Result 1, in this case, a small increase in $\bar{A}$ generates a very small substitution of the annuity for the bond proportional to the prices

$$\frac{dA}{d\bar{A}} = 1$$

$$\frac{dB}{d\bar{A}} = -\frac{R_A}{R_B},$$

leaving consumption unchanged: $dc_2 = R_A - R_A \frac{R_B}{R_B} = 0$.

The effect on expenditures is equal to $1 - \frac{R_A}{R_B} < 0$. This is the welfare gain from increasing the limit on available annuities for an optimizing consumer with positive bond holdings.

If constraint (5) is removed altogether, the price of second period consumption in units of first period consumption falls from $\frac{1}{R_B}$ to $\frac{1}{R_A}$. With a change in the cost of marginal second-period consumption, its level will adjust. Thus the cost savings is made up of two parts. One part is the savings while financing the same consumption bundle as when there is no annuitization and the second is the savings from adapting the consumption bundle to the change in prices. We can measure the welfare gain in going from no annuities to unlimited annuities by integrating the derivative of the expenditure function between the two prices:

$$E|_{\bar{A}=0} - E|_{\bar{A}=\infty} = -\int_{R_A^{-1}}^{R_B^{-1}} c_2(p_2) dp_2, \tag{7}$$

where $c_2$ is compensated demand arising from minimization of expenditures equal to $c_1 + c_2 p_2$ subject to the utility constraint without a distinction between asset types.

Equation (7) implies that consumers who save more (have larger second-period consumption) benefit more from the ability to annuitize completely:

**Result 2** *The benefit of allowing complete annuitization (rather than no annuitization) is greater for consumer i than for consumer j if consumer i's compensated demand for second period consumption (equivalently, compensated savings) exceeds consumer j's for any price of second period consumption.*

## 2.2 Many Future Periods and States, Complete Markets

The result of the optimality of complete annuitization survives subdivision of the aggregated future defined by $c_2$ into many future periods and states. A particularly simple subdivision would be to add a third period, so that survival to period 2 occurs with probability $1 - q_2$ and to period 3 with probability $(1-q_2)(1-q_3)$. In this case, bonds and annuities which pay out separately in period 2 with rates $R_{B2}$ and $R_{A2}$, and period 3 with rates $R_{B3}$ and $R_{A3}$ are sufficient to obviate trade in periods 2 or 3. That is, defining bonds and annuities purchased in period 1 with the appropriate subscript[7],

$$E = c_1 + A_2 + A_3 + B_2 + B_3$$

$$c_2 = R_{B2}B_2 + R_{A2}A_2,$$

$$c_3 = R_{B3}B_3 + R_{A3}A_3.$$

If Assumption 1 is modified to hold period by period, Result 1 extends trivially. Note that we have set up what we will call "Arrow bonds" (here $B_2$ and $B_3$ by combining two states of nature that differ in no other way except whether this consumer is alive. "Arrow annuities" which also recognize whether this consumer is alive complete the set of true Arrow securities of standard theory.

In order to take the next logical step, we can continue to treat $c_1$ as a scalar, and interpret $c_2$, $B_2$, and $A_2$ as vectors with entries corresponding to arbitrarily many (possibly infinity) future periods ($t \leq T$), within arbitrarily many states of nature ($\omega \leq \Omega$). $R_{A2}$ and $R_{B2}$ are then $T\Omega \times T\Omega$ matrices with columns corresponding to annuities (bonds) and rows corresponding to payouts by period and state of nature. Thus, the assumption of no aggregate uncertainty can be dropped. Multiple states of nature might refer to uncertainty about aggregate issues such as output, or individual specific issues beyond mortality such as health.[8] In order to extend the analysis, we need to assume that the consumer is sufficiently

[7]Implicitly, we are assuming that if markets reopened, the relative prices would be the same as are available in the initial trading period

[8]For a discussion of annuity payments that are partially dependent on health status, see Warshawsky, Spillman and Murtaugh (Forthcoming).

"small" that for each state of nature where the consumer is alive, there exists a state where the consumer is dead and the allocation is otherwise identical. Completeness of markets still allows construction of Arrow bonds which represent the combination of two Arrow securities.

Annuities with payoffs in only one event state are contrary to our conventional perception of (and name for) annuities as paying out in every year until death. However, with complete markets, separate annuities with payouts in each year can be combined to create such securities. It is clear that the analysis of the two-period model extends to this setting, provided we maintain the standard Arrow-Debreu assumptions that do not allow an individual to die in debt. In addition to the description of the optimum, the formula for the gain from allowing more annuitization holds for state-by-state increases in the level of allowed level of annuitization. Moreover, by choosing any particular price path from the prices inherent in bonds to the prices inherent in annuities, we can measure the gain in going from no annuitization to full annuitization. This parallels the evaluation of the price changes brought about by a lumpy investment (see Diamond and McFadden (1974)).

Thus we have extended the Yaari result of complete annuitization to conditions of aggregate uncertainty, actuarially unfair (but positive) annuity premiums and intertemporally dependent utility that need not satisfy the expected utility axioms. Moreover, we have the result that increasing the extent of available annuitization increases welfare if there is positive holding of Arrow bonds.[9]

## 2.3 Many Future Periods and States, Complete Bond Markets, Incomplete Annuities Markets, Trading Only Once

Annuities are frequently assumed to require a particular time path of payouts, thereby combining in a single security a combination of Arrow securities. Such annuities, for example with a constant real payout, have been evaluated in the literature. The United States Social Security system works this way (ignoring the role of the earnings test). Private annuities have generally been fixed in nominal terms rather than real. Variable annuities make the payoff depend on returns on some given portfolio, and combine Arrow securities in that way. CREF annuities also vary the payoff with mortality experience for the class of investors, which is also a combination of Arrow securities(see Poterba (1997)). To consider such lifetime annuities in

[9]The generalization of Result 2 to this case requires the very strong condition that after the present, consumption for agent $i$ exceeds that of agent $j$ state of nature by state of nature. That $i$'s consumption grows at a greater rate than $j$'s is not sufficient: allowing complete annuitization may yield reduction in many different prices by increasing any of many ratios $\frac{A_{t\omega}}{A_{t\omega}+B_{t\omega}}$. In general, these price changes are non-monotonic in time past period 1.

this setup, we continue to assume a double set of states of nature, differing only in whether the particular consumer we are analyzing is alive. We continue to assume a complete set of Arrow bonds and consider the effect of the availability of particular types of annuities. We also need to consider whether the return from annuities and bonds can be reinvested (markets are open) or must be consumed (markets are closed) In general, we will lose the result that complete annuitization is optimal. Nevertheless, we will get a similar result for real annuities provided that optimal consumption is rising over time and markets are open. In a more general setting we examine sufficient conditions for the result that the optimal holding of annuities is not zero.

To illustrate these points, we consider a three-period model with no aggregate uncertainty and a complete set of bonds. Then we will show how the results generalize. If there are no annuities, then the expenditure minimization problem is:

$$\min_{c_1,A,B} : c_1 + B_2 + B_3 \qquad (8)$$
$$s.t. : U(c_1, R_{B2}B_2, R_{B3}B_3) \geq \bar{U}$$

That is, we have:

$$c_2 = R_{B2}B_2,$$
$$c_3 = R_{B3}B_3.$$

With the assumption of infinite marginal utility at zero consumption, all three of $c_1$, $B_2$, and $B_3$ are positive. Now assume that there is a single available annuity, $A$, that pays given amounts in the two periods. Assume further that there is no opportunity for trade after the initial contracting. The minimization problem is now

$$\min_{c_1,A,B} : c_1 + B_2 + B_3 + A \qquad (9)$$
$$s.t. : U(c_1, R_{B2}B_2 + R_{A2}A, R_{B3}B_3 + R_{A3}A) \geq \bar{U}$$
$$c_2 = R_{B2}B_2 + R_{A2}A,$$
$$c_3 = R_{B3}B_3 + R_{A3}A.$$

Before proceeding, we must revise the assumption that $R_{At\omega} > R_{Bt\omega} : \forall t\omega$. A more appropriate formulation for the return on a complex security that combines Arrow securities to exceed bond returns is that for any quantity of the payout stream provided by the annuity, the cost is less if bought with the annuity than if bought through bonds. Define by $\ell$ a row vector of ones with length $T\Omega$, let the set of bonds continue to be represented by a $T\Omega \times 1$ vector with elements corresponding to the columns of the $T\Omega \times T\Omega$ matrix of returns $R_B$, and let $R_A$ be a $T\Omega \times 1$ vector of annuity payouts multiplying the scalar $A$ to define state-by-state payouts.

**Assumption 3** *For any annuitized asset $A$ and any collection of conventional assets $B$, $R_A A = R_B B \Rightarrow A < \ell B$.*

For example, if there is an annuity that pays $R_{A2}$ per unit of annuity in the second period and $R_{A3}$ per unit of annuity in the third period, then we would have $1 < \left(\frac{R_{A2}}{R_{B2}} + \frac{R_{A3}}{R_{B3}}\right)$. By linearity of expenditures, this implies that any consumption vector that may be purchased strictly through annuities is less expensive when financed strictly through annuities than when purchased by a set of bonds with matching payoffs.[10]

Given the return assumption and the presence of positive consumption in all periods, it is clear that the cost goes down from the introduction of the first small amount of annuity, which can always be done without changing consumption. Thus we can also conclude that the optimum (including the constraint of not dying in debt) always includes some annuity purchase. It is also clear that full annuitization may not be optimal if the implied consumption pattern with complete annuitization is worth changing by purchasing a bond. That is, optimizing first period consumption given full annuitization, we would have the first order condition

$$U_1(c_1, R_{A2}A, R_{A3}A) = R_{A2}U_2(c_1, R_{A2}A, R_{A3}A) + R_{A3}U_3(c_1, R_{A2}A, R_{A3}A)$$

Denoting partial derivatives of the utility function with subscripts, purchasing a bond would be worthwhile if we satisfy either of the conditions:

$$U_1(c_1, R_{A2}A, R_{A3}A) < R_{B2}U_2(c_1, R_{A2}A, R_{A3}A) \tag{10}$$

or

$$U_1(c_1, R_{A2}A, R_{A3}A) < R_{B3}U_3(c_1, R_{A2}A, R_{A3}A) \tag{11}$$

By our return assumption, we can not satisfy both of these conditions, but we might satisfy one of them. That is, the optimum will involve holding some of the annuitized asset and may involve some bonds, but not all of them.

It is clear that these results generalize to a setting with complete Arrow bonds and some compound Arrow annuities with many periods and many states of nature. We show below that expenditure minimization requires that there must be positive purchases of at least one annuity.

[10]This assumption leaves open the possibility considered below that both bond and annutiy markets are incomplete and some consumption plans can be financed only through annuities.

**Lemma 1** *Consider an asset $A*$ with finite, non-negative payouts $R_{A*}$. Any consumption plan $[c_1 \, c_2]'$ with positive consumption in every state of nature can be financed by a combination of first period consumption, a positive holding of $A*$, and another strictly non-negative consumption plan.*

***Proof.*** *Define $\bar{R}_{A*} = [\frac{1}{R_{A*21}}, ..., \frac{1}{R_{A*t\omega}}, \cdots \frac{1}{R_{A*T\Omega}}]'$, and define the scalar $\alpha = \min(c_2 \cdot \bar{R}_{A*})$. Now $c_2 = R_{A*}\alpha + Z$, where $Z$ is weakly positive.* ■

We now have a weaker version of Result 1:

**Result 3** *If Assumption 2 holds, and there exist annuities with non-negative payouts which satisfy Assumption 3, then (i) when no annuities are held, a small increase in annuitization reduces expenditures, holding utility constant. Also, then (ii) expenditure minimization implies $A > 0$.*

***Proof.*** *Suppose that the optimal plan $(c_1, A, B)$ features $A = 0$. Then there are two possibilities: first, consumption might be zero in some future state of nature. By Assumption 2 this implies infinitely negative utility and fails to satisfy the utility constraint. If consumption is positive in every state of nature, then consumption is a linear combination of all strictly positive linear combinations of the Arrow bonds. But then since some strictly positive consumption plan can be financed by annuities, by Assumption 3 and Lemma 1, expenditures can be reduced holding consumption constant by a trade of some linear combination of the bonds for some combination of annuities with strictly positive payouts. This contradicts optimality of the proposed solution.* ■

Part (i) of Result 3 states that if consumers are willing to commit to lifetime expenditures all at once, then starting from a position of zero annuitization, a small purchase of *any* annuity increases welfare. This applies to any annuity with returns in excess of the underlying asset, no matter how distasteful the payout stream. Part (ii) is the corollary that optimal annuity holdings are always positive. Lemma 1 shows that up to some point, annuity purchases do not distort consumption, so that their only effect is to reduce expenditures, as in the case where annuities markets are complete. When a large fraction of savings is annuitized, if the supply of annuitized assets fails to match demand, annuitization distorts consumption and conventional assets may be preferred. From the proof of Result 3, it follows that the annuitized version of any conventional asset that might be part of an optimal portfolio dominates the underlying asset.

# 3 Annuitization with Trade in Many Periods

## 3.1 Many Future Periods and States, Complete Liquid Bond Markets, Incomplete Illiquid Annuities Markets, Trading More than Once

The setup so far has not allowed a second period of trade. Now assume that trade in bonds is allowed after the first period, with bond prices consistent with the returns that were present for trade before the first period. To begin we assume that there is not an annuity available at the second trading time and that the consumer can save the second period annuity payout, but can not sell the remaining portion of the annuity. Since there would be no further trade without an annuity purchase at the start, the optimum without any annuity is unchanged. Utility at the optimum, assuming some annuity purchase, is at least as large as it was without the further trading opportunity. Thus we conclude that the result that some annuity purchase is optimal (Result 3) carries over to the setting with complete bond markets at the start and further trading opportunities in bonds that involve no change in the terms of bond transactions.

Returning to the three period example with no aggregate uncertainty, a sufficient condition for complete annuitization at the start, even if one of the inequalities (10) or (11) is violated, is that the consumption stream associated with full annuity purchase at the first trading point was such that saving (rather than dissaving) was attractive. To examine this issue, we now set up the expenditure minimization problem with retrading, denoting saving at the end of the first period by $Z$.

$$\min_{c_1,A,B} : c_1 + B_2 + B_3 + A \tag{12}$$

$$s.t. : U(c_1, R_{B2}B_2 + R_{A2}A - Z, R_{B3}B_3 + R_{A3}A + (R_{B3}/R_{B2})\, Z) \geq \bar{U}.$$

The restriction of not dying in debt is the nonnegativity of consumption if $A$ is set equal to zero [11]

$$B_2,\ B_3,\ Z \geq 0$$

[11] $B_3$ can be negative if $Z$ is positive. However, a budget-neutral reduction in $Z$ and increase in $B_3$, holding $A$ constant, then yields equivalent consumption, so there is no restriction in disallowing negative $B_3$. If $B_3$ is non-negative, then $Z$ must be zero as long as $B_2$ is positive, or else constant consumption with reduced expenditures could be obtained at a lower price by reducing $B_2$ and increasing $A$. That is, there are no savings out of bonds.

$$R_{B2}B_2 \geq 0$$

$$R_{B3}B_3 + (R_{B3}/R_{B2})\, Z \geq 0$$

The assumption that dissaving would not be attractive given full annuitization is

$$R_{B2}U_2(c_1, R_{A2}A, R_{A3}A) \leq R_{B3}U_3(c_1, R_{A2}A, R_{A3}A) \tag{13}$$

This condition can be readily satisfied for preferences satisfying a suitable relationship between (implicit) utility discount rates and interest rates. The result extends with many future periods, as long as trade is allowed in each. However, once we introduce uncertainty, the sufficient condition along these lines would need to hold in every state of nature.

Absent uncertainty, the presence of future opportunities to purchase annuities would increase the set of circumstances where full initial annuitization is optimal. With stochastic availability of more favorable annuity returns, annuitization may become relatively unattractive, as in Milevsky (2001).

## 3.2 Incomplete Markets With Illiquid Annuities and Future Trade

The saving condition (13) required for complete annuitization becomes implausible once it is recognized that people receive information over time about life expectancy and that such states (like individual life and death) are not distinguished by existing Arrow bonds. A similar issue would arise with other noninsurable events.

Indeed, with incomplete markets, even absent annuities, future trade typically occurs. In this case, the value of an asset is not equivalent to the sum of scheduled payouts multiplied by expected marginal utility in the period of payouts, as with complete markets. Now, assets' values also depend on their trading value in all future periods. If annuitized assets were as liquid as their underlying conventional asset, then it would remain the case that the annuitized version always dominates the underlying version. As a practical matter, the payment of survivorship premia for annuitized assets will require a minimal holding period and a possible penalty for resale, so that the annuitized asset is not as liquid as the underlying asset. As a consequence, it is possible that zero annuitization may be optimal, even under the assumption that annuities provide the cheapest means of purchasing any cashflow they can reproduce (this is Assumption 3, above).[12]

To see this possibility, returning to the three period case with future trade and no aggregate uncertainty, suppose that in period 1, a consumer expects to survive to period 2 with

[12]The result of partial annuitization survives if there are incomplete markets with no future trade, which technically occurs if people live for only two periods.

probability $1 - q_2$ and to period 3 with probability $(1 - q_2)(1 - q_3)$. However, the consumer knows that in period 2, the conditional probability of survival to period 3 will be updated to zero ("bad health news") with probability $\alpha$ or to $\frac{1-q_3}{1-\alpha}$ ("good health news") with probability $1 - \alpha$. In the likely case that neither the annuity nor the bonds distinguish between the two health conditions, the consumer will sell whatever bonds pay off in period 3 on obtaining bad health news, but will be unable to cash out the illiquid third period annuity claim.

Without annuitization, consumption in period two is thus given by $R_{B2}B_2$ if there is good health news, and $R_{B2}B_2 + \frac{R_{B2}}{R_{B3}}R_{B3}B_3$ if the health news is bad. Assuming the consumer is an expected utility maximizer, zero annuity purchase is thus optimal as long as:

$$\alpha R_{A2}U_2(c_1, R_{B2}(B_2+B_3))+(1-\alpha)(R_{A2}U_2(c_1, R_{B2}B_2, R_{B3}B_3)+R_{A3}U_3(c_1, R_{B2}B_2, R_{B3}B_3)) \leq$$

$$R_{B2}(\alpha U_2(c_1, R_{B2}(B_2 + B_3)) + (1 - \alpha)U_2(c_1, R_{B2}B_2, R_{B3}B_3)).$$

In this case, the superior return condition for annuities is $R_{A2} + R_{A3} > R_{B2} + R_{B3}$. The zero annuitization condition above can be consistent with this relationship if the annuities' payouts are sufficiently graded towards future payouts relative to the bonds. Hence, with incomplete markets, zero annuitization, partial annuitization, and complete annuitization are all consistent with utility maximization without further assumptions.

# 4 Valuation of a constant real annuity in special cases

Here, we consider a world with $T - 1$ future periods and no uncertainty except individual mortality, so that future consumption conditional on survival can be described by a vector $c_2$ with one element for each period up to $T$, beyond which no individual survives: $c_2 = [c_2, c_3 ... c_T]'$. We consider the welfare consequences of a policy whereby consumers either choose or are forced to purchase some quantity of an actuarially fair annuity which pays out a constant sum $R_A$ in every future period, when annuities are otherwise not available. That is, consumers minimize expenditures in a world with "Arrow" bonds and no annuity products, having already made an irreversible expenditure of $A$ units and a commitment to take in $R_A A$ per period, where $A$ is the amount of required annuity spending. We assume that no annuities are available after the first period, but that future bond trades are allowed. By completeness of bond markets, we can consider the set of bonds to be described by $T - 1$ securities, each of which pays out at a rate of $(1 + r)^{t-1}$ at date $t$ only.

With a constant real interest rate of $r$, without the annuity, expenditures are given by

$$E(c, 0) = c_1 + \sum_{t=2}^{T} c_t R_{Bt}^{-1} = c_1 + \sum_{t=2}^{T} c_t (1 + r)^{1-t}. \tag{14}$$

With annuities, the cost of a consumption plan is equal to the cost of annuitized consumption plus the difference between annuitized consumption and actual consumption in every period:

$$E(c,A) = c_1 + A + \sum_{t=1}^{T}(c_t - R_A A)(1+r)^{1-t},$$

where $R_A$ is the per-period annuity payout. For $t > 1$, if consumption is less than the annuity payout, the difference can be used to purchase consumption at later dates, with the relative prices given by bond returns. If consumption is greater than the annuity payout, then a bond with maturity at date $t$ must be purchased.

If $1 - m_t \equiv \Pi_{s=2}^{t}(1 - q_s)$ is the probability of survival to period $t$, then actuarial fairness implies that the cost per unit of the annuity is equal to the survival-adjusted present discounted value of bond purchases yielding the same unit per period:

$$1 = \sum_{t=2}^{T}(1 - m_t)\frac{R_A}{(1+r)^{t-1}}$$

$$\Rightarrow R_A = \frac{1}{\sum_{t=2}^{T}(1 - m_t)(1+r)^{1-t}}. \tag{15}$$

Assumption 3 applies as long as there is a positive probability of death by the end of $T$ periods because the cost of consuming any plan $R_A A$ per period past period 1 with annuities is $\frac{A}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}$. This is less than $\frac{A}{\sum_{t=2}^{T}(1+r)^{1-t}}$, the cost of purchasing $A$ per period with conventional securities.

As discussed above, a small increase in $A$ from zero has no effect on consumption, so that the CV from incremental annuitization from 0 to a small number $\epsilon$ is equal to the difference between $E(c,0)$ and $E(c,\epsilon)$:

$$\frac{dE}{dA}|_{A=0} = 1 - \sum_{t=2}^{T}(R_A(1+r)^{1-t}) < 0. \tag{16}$$

The inequality follows from equations (14) and (15) as long as $m_T > 0$.

Larger increases in annuitization are more difficult to sign because they may constrain consumption. Below, we consider the effects for particular utility functions.

## 4.1 Additively Separable Preferences, Constant Real Annuity Constraint, Actuarially Fair Annuities, No Uncertainty Other Than Length of Life

### 4.1.1 Optimal Choice of Annuitization

Here, we assume that utility is given by:

$$U(c_1, c_2) = \sum_{t=1}^{T} \delta^{t-1}(1 - m_t)u(c_t), \tag{17}$$

Where $u' > 0$, $u'' < 0$; $\lim_{c_t \to 0} u' = \infty$, and $\delta$ is the rate of time preference.

Because Assumptions 2 (infinite disutility from zero consumption in any future period) and 3 (any consumption plan that can be financed by annuities alone is financed most cheaply by annuities alone) are met:

**Claim 1** *The solution to the expenditure minimization problem features $A > 0$.*

***Proof.*** *Follows immediately from Result 3.* ■

By the no bankruptcy constraint, consumers may undo annuitization by saving if annuitization renders consumption too weighted towards early periods, but not by borrowing if annuitization renders consumption too weighted to later periods. The liquidity constraint given a constant real annuity requires that expenditures on consumption up to any date $\tau$ must be less than total planned expenditures less expenditures committed to future annuity payments. This constraint can be written as:

$$\sum_{t=1}^{\tau} c_t(1+r)^{1-t} \leq c_1 + A + \sum_{t=2}^{T} B_t - R_A A \sum_{t=\tau+1}^{T} (1+r)^{\tau - t}\, \forall \tau. \tag{18}$$

This induces one constraint for every period in which consumption is bound from above by the required annuity. Annuities are costly in optimization terms because they contribute to these constraints.

The expenditure minimization problem becomes:

$$\min_{c_1, A, B} c_1 + A + B \tag{19}$$

$$s.t.\ U(c_1, c_2(A, B)) \geq \bar{U}$$

$$s.t. \text{ equation (18) is satisfied.}$$

Under these circumstances, consumers whose discount rates are no greater than the interest rate annuitize fully [13]:

[13] annuitization is full in the sense that the present value of bond purchases are equal to zero.

**Claim 2** *If optimal consumption is weakly increasing over time, then complete initial annuitization is optimal.*

***Proof.*** *With non-decreasing consumption, constraint 18 reduces to a definition of expenditures. Hence, if net bond holdings are greater than zero, expenditures can be reduced and utility increased by an additional purchase of* $\epsilon$ *units of* $A$ *and sale of* $\epsilon \frac{R_A}{R_{B2}} > \epsilon$ *units of* $B_2$. ■

**Claim 3** *For the dual utility maximization problem with fixed expenditures, if the optimal level of annuitization* $A$ *is less than savings, so that there are positive expenditures on bonds, an increase in* $\delta$ *yields an increase in optimal* $A$ *relative to savings.*

***Proof.*** *With an increase in* $\delta$*, for any periods* $s' > s$*, the ratio of consumption induced by initial period consumption and investment* $\frac{c_{s'}}{c_s}$ *must increase. This follows since the ratio of marginal utilities increases with* $\delta$*, and the ratio can be increased with a small budget-neutral exchange of* $B_s$ *for* $B_{s'}$*. Hence, planned consumption with the increase in* $\delta$ *must be equal to the original consumption plan plus a weakly increasing sequence with negative elements for all dates up to some date* $t$*. By the result above, the old consumption plan is revised with minimal expenditures by selling bonds with maturity less than* $t$ *and increasing* $A$*.* ■

**Claim 4** *If* $\delta(1+r) \geq 1$*, complete initial annuitization is optimal.*

***Proof.*** *By Claim 3, it is sufficient to show that this is true for* $\delta(1+r) = 1$*. For complete annuitization to be suboptimal, it must be the case that there exists some* $t$ *for which purchasing a bond with maturity at date* $t$ *provides greater marginal utility than purchase of the real annuity, or:*

$$\exists t > 1 \,:\, \delta^{t-1}(1+r)^{t-1}u'(R_A A)(1-m_t) > \frac{\sum_{t=2}^{T} \delta^{t-1}(1-m_t)u'(R_A A)}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}.$$

$$\Rightarrow \delta^{t-1}(1+r)^{t-1}(1-m_t) > \frac{\sum_{t=2}^{T} \delta^{t-1}(1-m_t)}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}.$$

*If* $\delta(1+r) = 1$*, then this is impossible, because the left hand side is less than or equal to one (by non-negative mortality) and the right hand side equals one in the simplified equation.* ■

The "dual" to Claim 4 follows from the proof:

**Claim 5** *If* $\delta(1+r) \geq 1$*, then any increase in annuitization in the range* $A \in [0, E - c_1]$ *is welfare enhancing.*

For more impatient consumers, we solve for the optimal fraction of savings put into annuities numerically. Results are detailed below.

Beyond the results we have above, making statements about the size of EV for a move from complete annuitization to zero annuitization is difficult, because in general, this calculation must take into the period-by-period positive wealth constraints summarized in equation (18). That said, a plausible conjecture, based on Claim 3 is that valuation will increase in the patience parameter $\delta$, which should push consumption later in life. Further, in cases where optimal consumption is decreasing over time, increased smoothing should increase valuation. Hence, for $\delta(1+r) \leq 1$, we should expect valuation to increase with any parameter of risk aversion, because the desire for decreasing consumption, which makes the $\mu$ constraints bind, would then be tempered by a desire for consumption smoothing. We confirm these intuitions below with numerical examples.

### 4.2 Utility Dependent on Standard of Living

Additive separability of utility does not sit well with intuition. For example, life in a studio apartment with no car is surely more tolerable for someone used to living in a studio apartment without a car than for someone who was forced by a negative income shock to abandon a four bedroom house and a Lexus for a studio apartment and no car. In this section, we revisit the analysis above with an extreme, and hence illustrative, example of intertemporal dependence in the utility function, taken from Diamond and Mirrlees (2000). The intuition behind this formulation is that it is not the level of present consumption, but the level relative to past consumption that matters. We consider the ratio of present to past consumption, but the difference has also been considered in the literature. In choosing how to allocate resources across periods, consumers with such utility trade off immediate gratification from consumption not only against a lifetime budget constraint, but also against the effects of consumption early in life on the standard of living later in life.

$$U(c_1, c_2) = \sum_{t=1}^{T} \delta^{t-1}(1-m_t)u(\frac{c_t}{s_t}), \tag{20}$$

where

$$s_t = \frac{s_{t-1} + \alpha c_{t-1}}{1+\alpha}.$$

Note that if $\alpha = 0$, so that individuals have no control over their standard of living, we are in the additively separable case. A positive value of $\alpha$ indicates that past consumption makes individuals less satisfied with a given level of present consumption.

In the absence of the positive wealth constraints (18), the marginal utility of consumption in any period incorporates two effects not present in the additively separable case: (1) the effect of the present standard of living on present marginal utility and (2) the effect of present consumption on future periods' utility through subsequent standards of living. Under this specification, the marginal benefit of present consumption is given by:

$$\frac{\partial U}{\partial c_t} = \frac{1}{s}u'(\frac{c_t}{s_t}) - \sum_{k>t} \frac{\alpha}{(1+\alpha)^{k-t}} \frac{c_k}{s_k^2} u'(\frac{c_k}{s_k}).$$

We note that if $\lim_{\frac{c_t}{s_t} \to 0} u'(c_t) = \infty$, then Assumption 2 holds, and Claim 1 applies for finite $s_1$.

To obtain results, we assume that $u(\frac{c}{s}) = \frac{(\frac{c}{s})^{1-\gamma}}{1-\gamma}$ and that $\gamma \geq 1$. Hence:

$$\frac{\partial U}{\partial c_t} = c_t^{-\gamma} s_t^{\gamma-1} - \sum_{k>t} \frac{\alpha}{(1+\alpha)^{k-t}} c_k^{1-\gamma} s_k^{\gamma-2}.$$

For $\gamma > 1$, effect (1) will tend to push consumption towards later periods relative to the no standard ($\alpha = 0$) case if the standard of living is increasing over time. If the standard of living is decreasing over time, and $\gamma \geq 2$, then this will push consumption to earlier periods. For $\gamma < 2$, the effect is ambiguous.

Effect (2) will unambiguously push consumption towards later periods in life. Hence, the result of complete annuitization when the discount rate is less than the interest rate, Claim 4, continues to hold if $s$ is constant or decreasing over the period of annuitization. This occurs if the initial value of $s$ is small and the required level of utility, $\bar{U}$, is large. If the initial value $s_1$ is sufficiently large relative to the expenditures required to attain $\bar{U}$, then the smoothing implied by risk aversion may undo the result by rendering optimal consumption relatively decreasing over time.

With the constraint that the only annuity available pays out a constant real sum, relative valuations are particularly difficult to calculate with standard of living effects, because the intertemporal effects compound the difficulty of the multiple positive wealth constraints. However, we can conjecture that parameter changes that tend to defer optimal consumption will tend to increase valuation. Hence, simulated valuations should tend to be increasing in $\delta$. Further, large $s_1$ should yield decreasing valuation, and small $s_1$ increasing valuation, with both effects magnified by $\gamma$.

### 4.3 Numerically Estimated Magnitudes of Welfare Effects

To estimate numerical valuations of annuitization, we specify $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for both the additively separable and standard of living exponential discounting, flat yield curve cases considered above. In the separable case, this gives constant relative risk aversion and an intertemporal rate of substitution $\frac{\partial U/\partial c_t}{\partial U/\partial c_k} = (\frac{c_t}{c_k})^{-\gamma}$. In the standard of living case, both risk aversion and intertemporal substitution are complicated by the intertemporal utility linkage.

We calculate five values: first, the CV for a mandatory purchase of a very small quantity of a constant real annuity when $A = 0$. This value is identical and negative for all consumers, as shown above. Second, and of more interest for determining parameter effects, we calculate EV for a move from putting all expenditures in such an annuity to putting nothing in the annuity ($EV_R$). Third, we determine the optimal fraction of savings $A*$ placed in the real annuity as opposed to in bonds. Fourth, we calculate the EV of a move from complete annuitization with complete annuity markets to zero annuitization ($EV_C$). This number will be greater than the EV associated with complete real annuitization, or equal in the knife-edge case where optimal consumption is constant with actuarially fair prices (in the additively separable case, this occurs when $\delta(1+r) = 1$). Finally, we compute an intermediate equivalent variation, $EV_O$: that associated with a move from annuitizing the optimal fraction $A*$ of wealth to zero annuitization.

We perform these calculations for a single 65 year old male in 1999 with survival probabilities taken from the US Social Security mortality tables, modified (to ease computation) so that death occurs for sure at age 100. We use a real interest rate $r$ of 0.03, and vary $\delta$. We consider coefficients of relative risk aversion $\gamma$ of 1 (log utility) and 2, which are on the low end of plausible values. We use the same values for the standard of living case (where these cannot be interpreted as coefficients of relative risk aversion). For the case where the discount rate is $1.03^{-1}$ and there is no standard of living, our results are very close to those found in the existing literature despite the truncation of life by 10 or 15 years.

We "reverse engineer" the expenditure minimization problem to have minimal expenditures of 100 with complete annuitization, and exclude consumption in the year of retirement (so utility is defined only over the 35 element vector $c_2$). Changes in parameters will change not only the optimal allocation of savings, but also the level of savings. Comparative static analysis of the effect of parameter changes on the welfare consequences for an individual with constant lifetime wealth would thus require solution of a lifetime problem, not just a retirement consumption problem. Adding consumption at age 65 does not fix this problem.

The graphs depict and summarize results for each of 9 cases. Each graph plots optimal consumption with and without annuitization. A positive EV of annuitization indicates that the level of utility achieved with optimal consumption subject to complete actuarially fair annuitization and real annuity constraint (18) when 100 units of the annuity are purchased requires a greater level of expenditures to attain when consumption is financed through conventional bonds with no trajectory constraint. A rough estimate of the magnitude of EV can be obtained by observing the difference in trajectories between the two consumption plans: when optimal consumption is sharply decreasing, the constraints implied by (18) bind consumption away from the optimal path; in these cases the price benefit of annuitization is largely offset by the constraints. When optimal consumption is hump shaped, and less steeply decreasing, the constraints impose less costs, so the net benefit to annuitization is greater.

### 4.3.1 Results

#### Magnitudes

Pursuant to equation (16), CV for all consumers is equal to -125.39 when there is no annuitization. This means that for a very small increase in annuitization $dA$, total expenditures are equal to $100 - 125.39dA$.

Consistent with past results, the EV for a change from complete to zero annuitization is 44 in the case of log utility with no standard of living effect, and the discount and interest rates equalized (case 1). The positive EV in this case is guaranteed by Claim 4. Consistent with our expectations, and past results, EV increases in risk aversion (to 56 with $\gamma = 2$ (case 7), and decreases when discounting is heavier (to 15 with a discount rate of 0.1 and log utility (case 3)). Note that the value of annuitization is increasing as the trajectory of optimal consumption with no constraints approaches the flat (or upward sloping) optimal annuitized consumption path. This is because the positive wealth constraints have less bite when optimal consumption is nearly flat.

As expected, EV increases with the introduction of the standard of living effect when the initial standard of living is small relative to wealth. The externality of present consumption on future consumption leads consumption to be upward sloping over some range, with large mortality eventually bringing consumption to lower than initial levels. This occurs around age 80 for $\delta = 1.03^{-1}$ and around 70 for $\delta = 1.1^{-1}$. Comparable to Case 1, Case 2 has a small standard of living and large valuation of 67, approximately 50 percent greater. The initial value of 5 is "small" because with expenditures of 100, a constant real annuity pays out 8.5 per period. Case 4 is comparable to case 3, and here valuation almost doubles. Case 8 is comparable to Case 7, and valuation increases, although not as sharply from 56 to 70.

Assuming rationality and retirement with assets sufficient to sustain the standard of living enjoyed going into retirement, we see that this particular relaxation of additivity has an economically very significant effect on valuation.

For the cases where the initial standard of living is large, there is an economically significant effect on valuation in the other direction. Cases 5, 6 and 9 are comparable to cases 1 and 2, 3 and 4, and 7 and 8, respectively. In this case, the consumption smoothing effect of the externality of early consumption on subsequent utility is overcome by the desire to smooth the ratio of consumption to the standard of living. The standard of living must fall over time because the initial standard of living (50 in cases 5, 6 and 9) are not sustainable throughout retirement.

Perhaps the most striking result is the consistently very large fraction of savings placed in the annuity. The minimal annuitized fraction is 60, and this is for a consumption plan (case 9) with very sharply decreasing consumption. For the familiar additively separable case, the minimal annuitized fraction is 72 percent. Following Claim 3, optimal annuitization increases in patience. Following Claim 4, complete annuitization is optimal in the separable case when $\delta(1+r)=1$. The last column, showing equivalent variations associated with the optimal level of constant real annuitization can be interpreted as stating that for all of our simultions, annuitizing 60 percent of wealth is equivalent to an increase in wealth of at least 20 percent.

For some patient consumers, the presence of a constant real annuity is as good as the presence of complete annuities. This outcome, however, is a knife-edge. Adding a savings motive through the standard of living effect, we find a benefit to the move to complete annuities markets, because upward sloping consumption can be financed through initial annuity purchases, rather than future savings and bond purchases. For more impatient consumers, the move to complete annuities markets introduces considerable further gains relative to the introduction of the real annuity.

Table 1: Summary of simulations

| Case | $\delta$ | $\gamma$ | $s_1$ | $\alpha$ | $EV_R$ | $A*_R$ | $EV_C$ | $EV_O$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $1.03^{-1}$ | 1 | 1 | 0 | 44 | 100% | 44 | 44 |
| 2 | $1.03^{-1}$ | 1 | 5 | 1 | 67 | 100% | 82 | 67 |
| 3 | $1.1^{-1}$ | 1 | 1 | 0 | 15 | 72% | 24 | 20 |
| 4 | $1.1^{-1}$ | 1 | 5 | 1 | 36 | 99% | 37 | 36 |
| 5 | $1.03^{-1}$ | 1 | 50 | 1 | 36 | 84% | 49 | 46 |
| 6 | $1.1^{-1}$ | 1 | 50 | 1 | 3 | 63% | 24 | 21 |
| 7 | $1.03^{-1}$ | 2 | 1 | 0 | 56 | 100% | 56 | 56 |
| 8 | $1.03^{-1}$ | 2 | 5 | 1 | 70 | 100 % | 87 | 70 |
| 9 | $1.03^{-1}$ | 2 | 50 | 1 | negative | 60% | 30 | 27 |

# 5 Conclusion

With complete markets, the result of complete annuitization survives the relaxation of several standard, but restrictive assumptions. Utility need not satisfy the von Neumann-Morgenstern axioms and need not be additively separable. Further, annuities must only offer positive net premia over conventional assets; they need not be actuarially fair. We have retained the abstractions of no bequest motive, and no learning about health status or other liquidity concerns. Exploring the consequences of dropping these assumptions in the context of non-separable preferences and unfair annuity pricing will be an important generalization, but obtaining results will require strong assumptions both on annuity returns and on the nature of bequests and liquidity needs.

When annuities are restricted to be constant in real terms, we find that adding a particular form of intertemporal dependence in utility reinforces the benefits of annuitization as long as the standard of living entering retirement is not too large relative to resources. Even in what we regard to be extremely unfavorable conditions for annuitization, 60 percent of savings are optimally placed in the annuity. For consumers who benefit less from constant real annuitization, there are substantial gains to completion of annuity markets.

Figure 1: Case 1

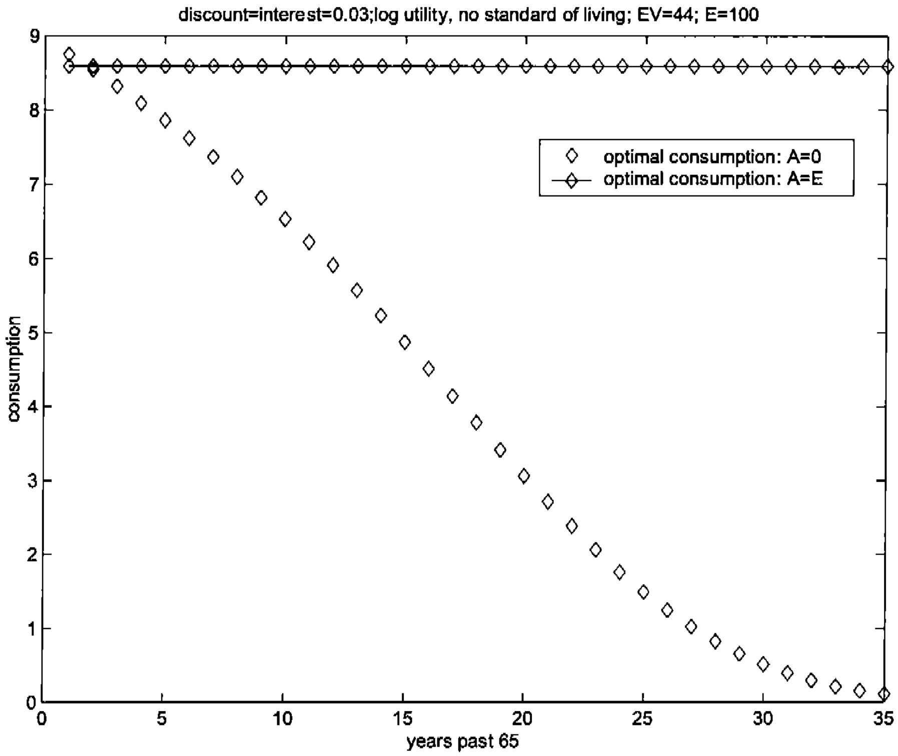


Figure 2: Case 2

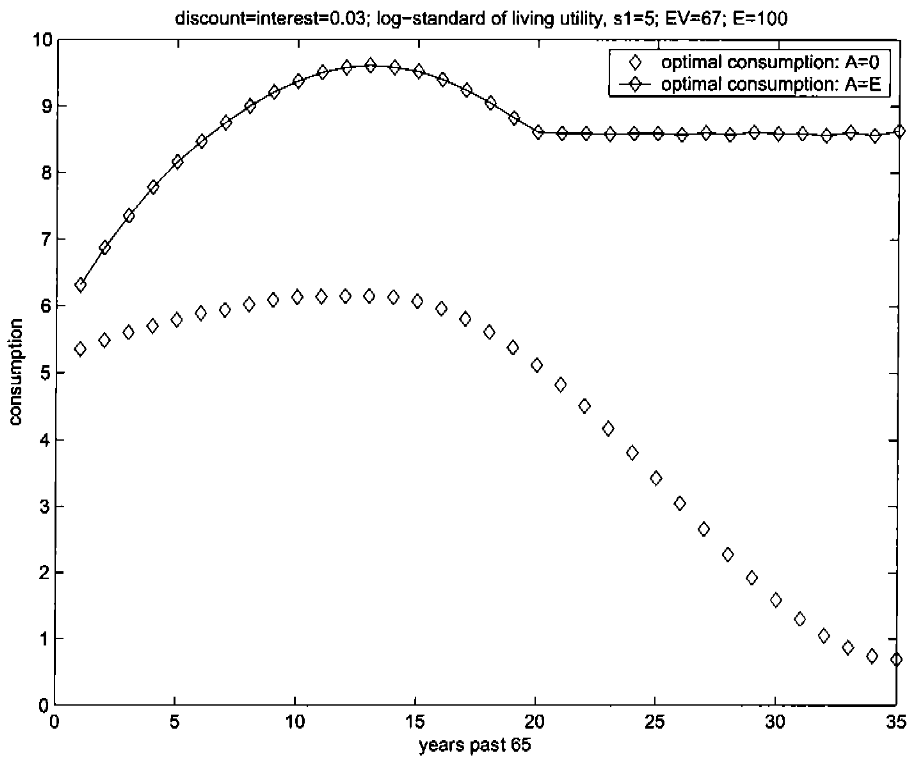


Figure 3: Case 3

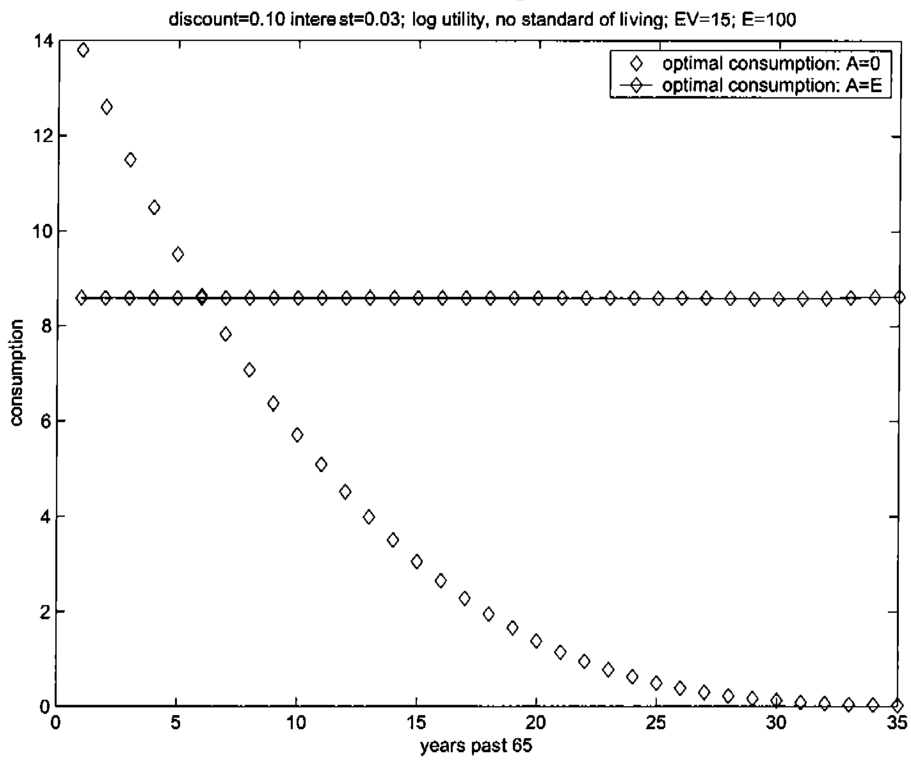


Figure 4: Case 4

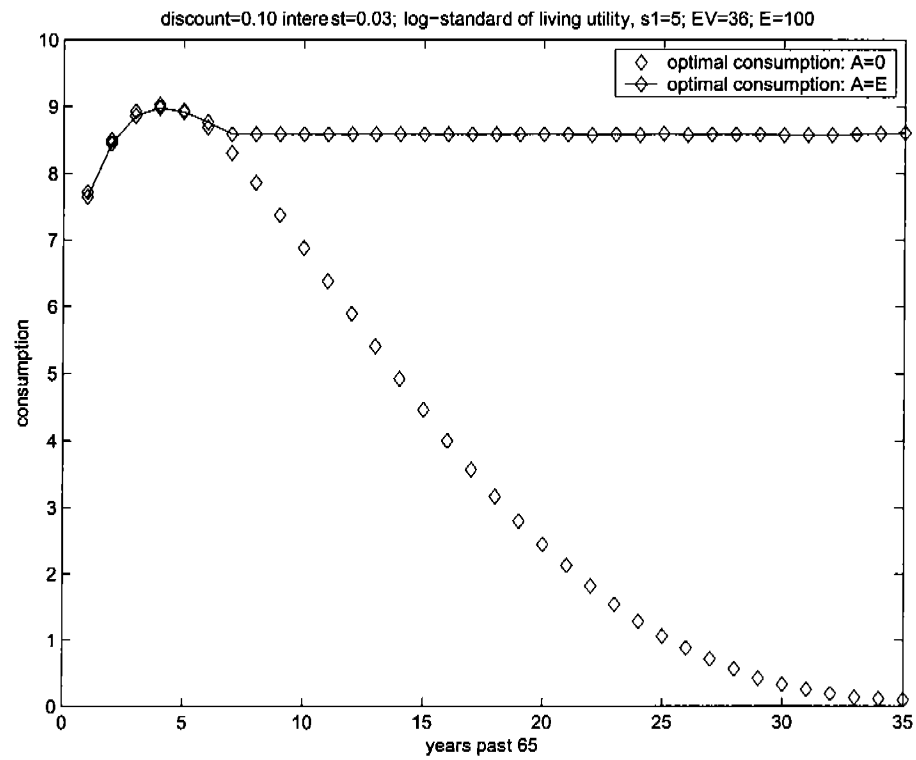


Figure 5: Case 5

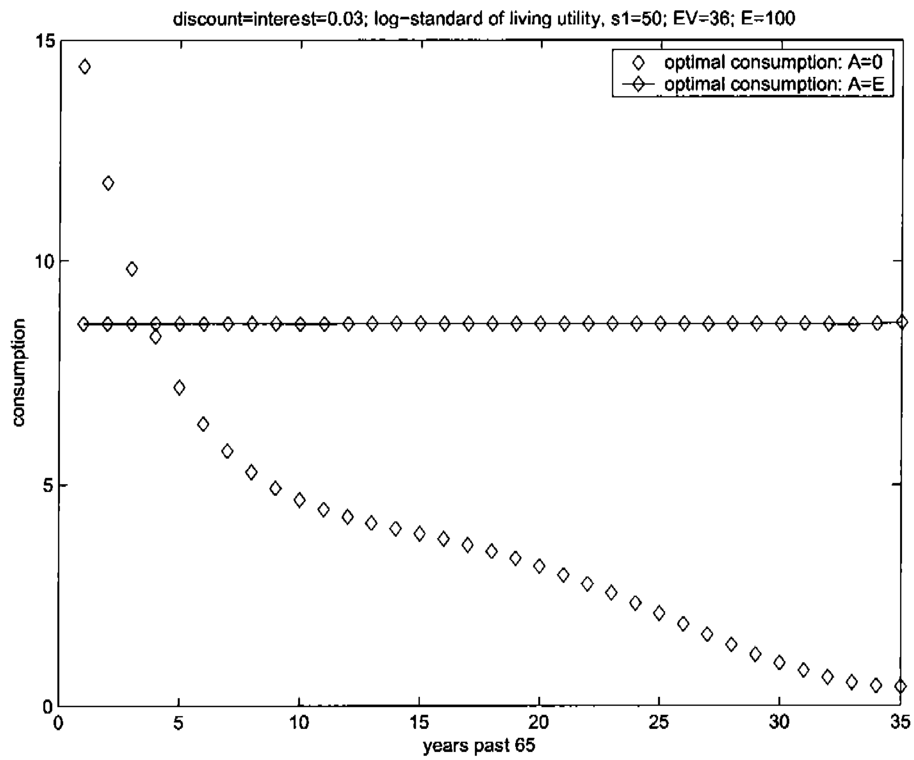


Figure 6: Case 6

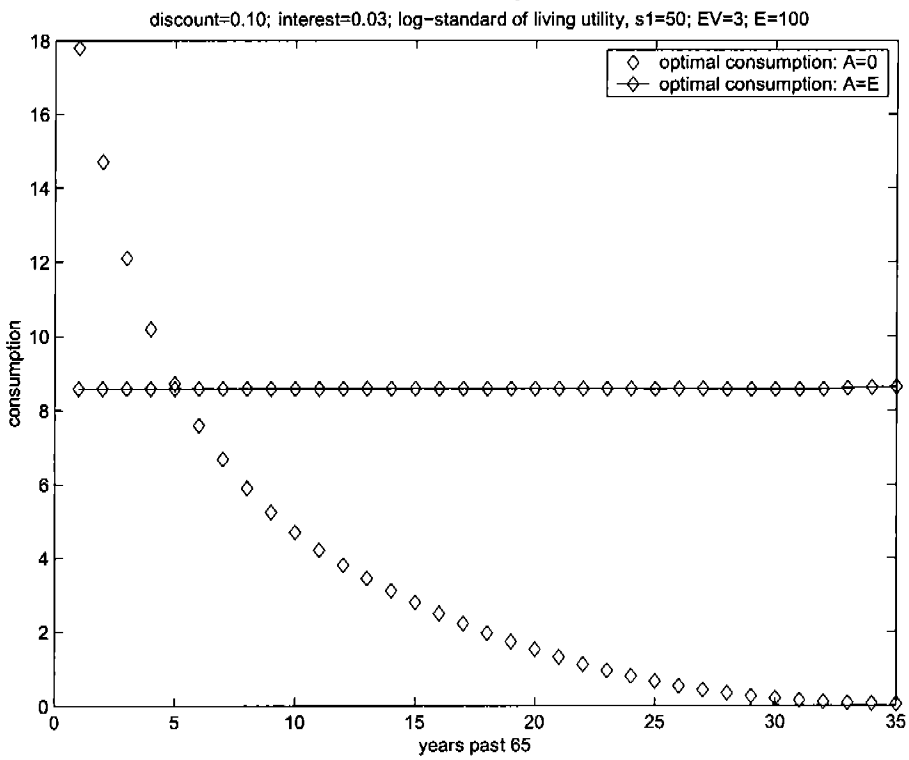


Figure 7: Case 7

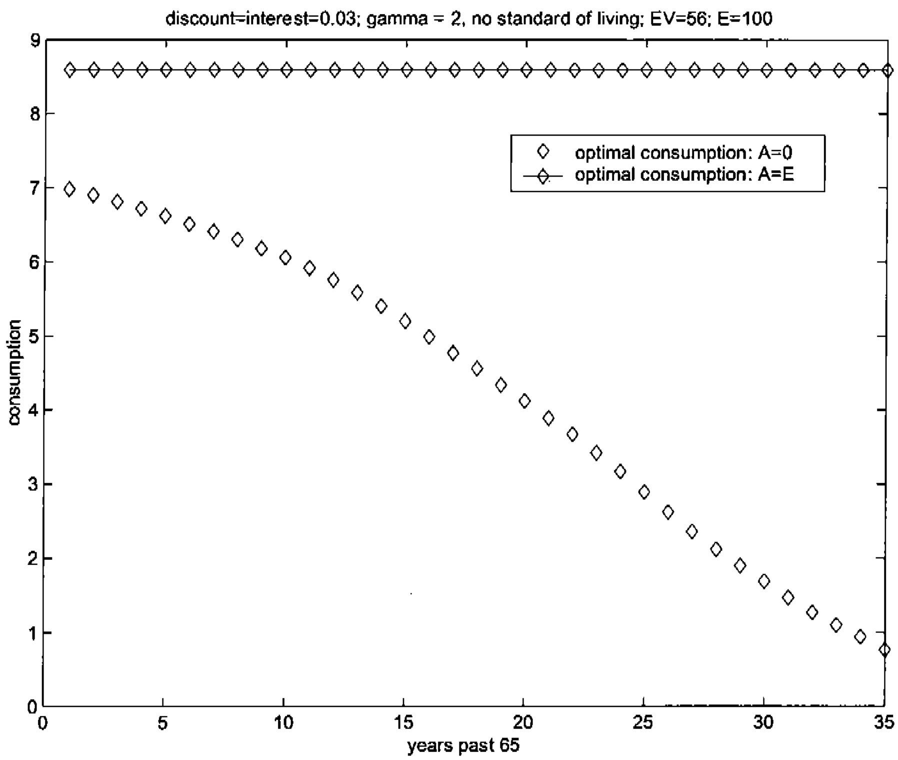


Figure 8: Case 8

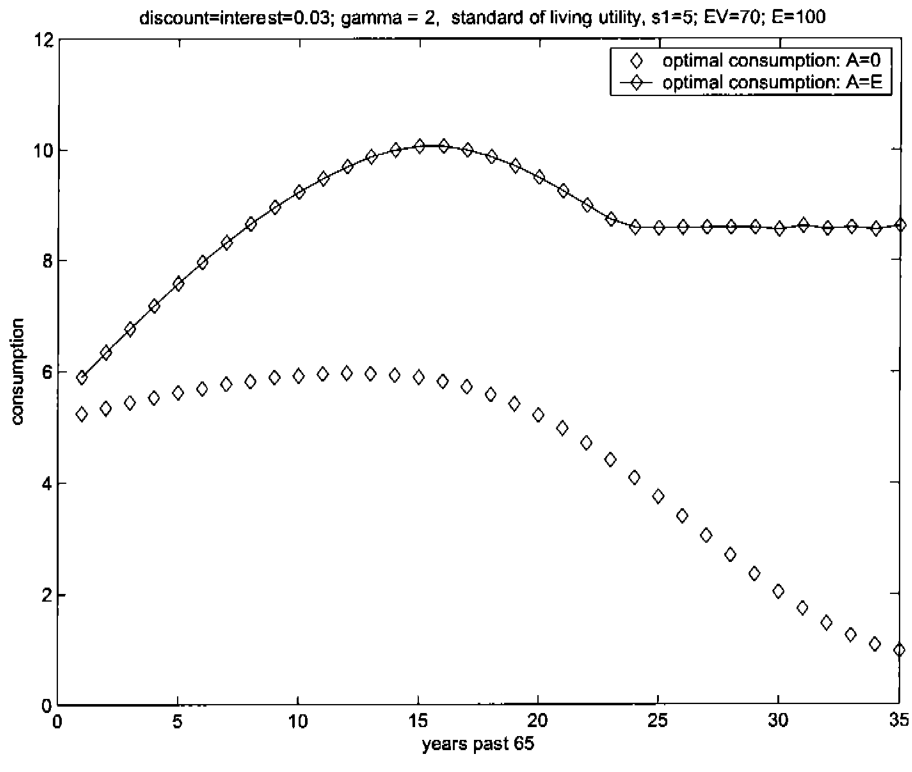
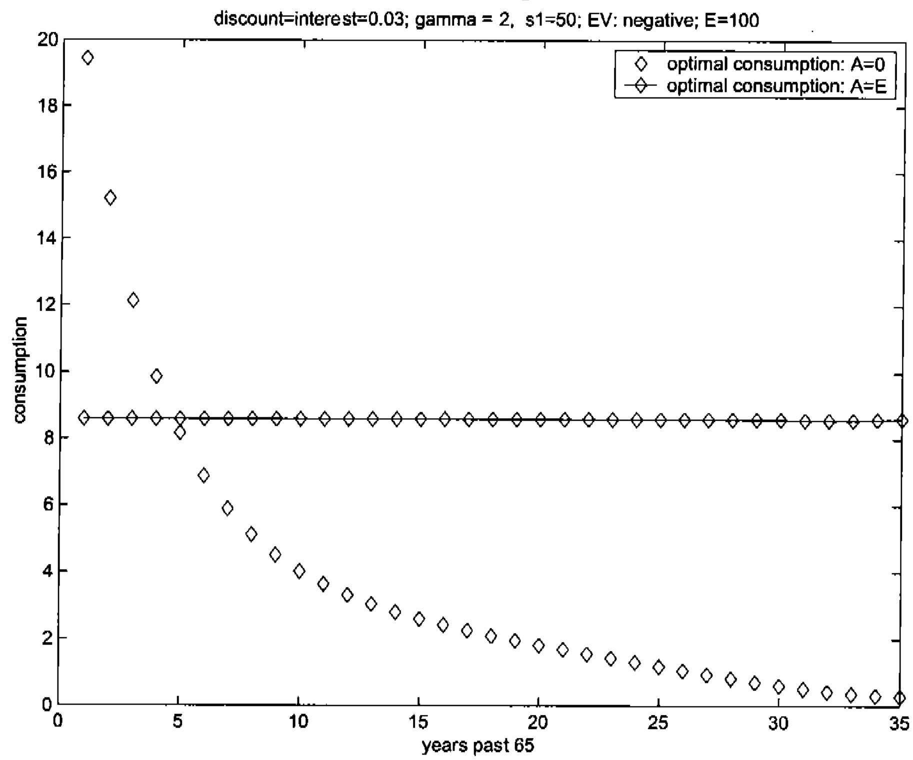


Figure 9: Case 9



## References

Bernheim, B. Douglas (1991) 'How strong are bequest motives? evidence based on estimates of the demand for life insurance and annuities.' *Journal of Political Economy* 99(5), 899–927

Brown, Jeffrey R. (2001a) 'Private pensions, mortality risk, and the decision to annuitize.' *Journal of Public Economics* 82(1), 29–62

Brown, Jeffrey R. (2001b) 'Redistribution and insurance: Mandatory annuitization with mortality heterogeneity.' Technical Report, Boston College Center for Retirement Research. Working Paper 2001-02

Deaton, Angus (1991) *Understanding Consumption* (New York: Oxford University Press)

Diamond, Peter, and Daniel McFadden (1974) 'Some uses of the expenditure function in public economics.' *Journal of Public Economics* 3, 3–21

Hurd, Michael (1989) 'Mortality risk and bequests.' *Econometrica* 57(4), 779–813

Jousten, Alain (2001) 'Life-cyclemodelling of bequests and its implications for annuity valuation.' *Journal of Public Economics* 79, 149–177

Kotlikoff, Laurence J., and Avia Spivak (1981) 'The family as an incomplete annuities market.' *Journal of Political Economy* 89(21), 372–391

Milevsky, Moshe A. (2001) 'The real option to delay annuitization: Its not now or never.' Working paper presented at 2001 Meetings of the American Economic Association

Mitchell, Olivia, James M. Poterba, Mark Warshawsky, and Jeffrey R. Brown (1999) 'New evidence of the money's worth of individual annuities.' *The American Economic Review* 89(5), 1299–1318

Palmon, Oded, and Avia Spivak (2001) 'Information and the market for annuities.' Mimeo

Poterba, James M. (1997) 'The history of annuities in the united states.' Working Paper, NBER, April. W6001

Walliser, Jan (2001) 'Regulation of withdrawals in individual account systems.' In *New Ideas About Old Age Security: Toward Sustainable Pension Systems in the 21st Century,* ed. Robert Holzmann and Joseph Stiglitz (The World Bank) pp. 367–390

Warshawsky, Mark J., Brenda Spillman, and Chris Murtaugh (Forthcoming) 'Integrating the life annuity and long-term care insurance: Theory, evidence, practice and policy.' In *Financial Innovations in Retirement Income,* ed. Olivia Mitchell, Zvi Bodie, Brett Hammond, and Steve Zeldes

Yaari, Menahem E. (1965) 'Uncertain lifetime, life insurance, and the theory of the consumer.' *Review of Economic Studies* 32, 137–150