Overconfidence and Speculative Bubbles*

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Abstract

Motivated by the behavior of internet stock prices in 1998-2000, we present a continuous time equilibrium model of bubbles where overconfidence generates disagreements among agents regarding asset fundamentals. With short-sale constraints, an asset owner has an option to sell the asset to other overconfident agents who have more optimistic beliefs. This re-sale option has a recursive structure, that is, a buyer of the asset gets the option to resell it. This causes a significant bubble component in asset prices even when small differences of beliefs are sufficient to generate a trade. Agents pay prices that exceed their own valuation of future dividends because they believe that in the future they will find a buyer willing to pay even more. The model generates prices that are above fundamentals, excessive trading, excess volatility, and predictable returns. However, our analysis shows that while Tobin's tax can substantially reduce speculative trading when transaction costs are small, it has only a limited impact on the size of the bubble or on price volatility. We give an example where the price of a subsidiary is larger than its parent firm. Finally, we show how overconfidence can justify the use of corporate strategies that would not be rewarding in a "rational" environment.

Comments are welcome.

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1 Introduction

The behavior of market prices and trading volumes of internet stocks during the period of 1998-2000 presents a challenge to asset pricing theories. Several studies have shown that it is difficult to match prices to the underlying fundamentals: The prices were too high and too volatile, the value of parent companies were less than the value of its holdings of an "internet" subsidiary, and the trading volume of internet stocks was excessive when compared to that of more traditional companies.[1]

In this paper, we propose a model of asset trading based on short-sale constraints and heterogeneous beliefs generated by agents' overconfidence. The model can generate equilibria that broadly fit these observations. We also provide explicit links between certain parameter values in the model, such as trading cost and information, and the behavior of equilibrium prices. In particular, this allows us to discuss the effects of trading taxes and information on prices and trading volume. In addition, we examine how overconfidence makes profitable corporate strategies that would not be rewarding in a "rational" environment.

More generally, our model provides a flexible framework to study speculative trading that can be used to analyze asset pricing puzzles, such as the excess volatility and large trading volume, that are observed in several asset markets.

The presence of short-sale constraints is important in our set up, since it prevents arbitrageurs from eliminating the bubbles,[2] and provides an opportunity (option) for the asset owner to profit from other investors' over-valuation. Recent empirical studies document that although limited shorting of internet stocks did occur, it was very expensive at the margin, restricting the ability of arbitrageurs or other investors to sell short.[3] In

[1] Lamont and Thaler (2001), Ofek and Richardson (2001), and Cochrane (2002) provide excellent survey and many references on the market behavior of stocks in the internet sector during this period. In particular they point out that "pure internet firms represented as much as 20% of the dollar volume in the public equity market, even though their market capitalization never exceeded 6%."

[2] Shleifer and Vishny (1997) argued that in practice, arbitrage involves capital and that the capital available to arbitrageurs is limited. This can cause arbitrage to fail. See also Xiong (2001), Kyle and Xiong (2001), and Gromb and Vayanos (2002) for studies linking the dynamics of arbitrageurs' capital with asset price dynamics.

[3] Several recent empirical studies, *e.g.* Jones and Lamont (2002), Geczy, Musto and Reed (2002), and D'Avolio (2002), have found evidence linking stock mispricing with short-sale constraints. Duffie, Garleanu, Pedersen (2001) provide a search model for the short-sale process.

the model, we take the extreme view that short sales are not permitted, although our qualitative results should survive the presence of limited short sales as long as the asset owners can expect to make a profit when others have higher valuations.

Our model follows the insight of Harrison and Kreps (1978), that, when agents agree to disagree and short selling is not possible, asset prices may exceed their fundamental value. This difference was called the speculative component by Harrison and Kreps. In their model, agents trade because they disagree about the probability distributions of dividend streams. The reason for the disagreement is not made explicit. We study overconfidence, the belief of an agent that his information is more accurate than what it is, as a source of disagreement. Although overconfidence is only one of the many ways by which disagreement among investors may arise, it is, as we summarize in the next section, strongly supported by experimental studies of human behavior, and allows us to specifically analyze the properties of the bubble and to link the dynamics of the equilibrium to observables.[4]

We study a market for a single risky-asset with limited supply and many risk-neutral agents in a continuous time model with infinite horizon. The current dividend of the asset is a noisy observation of a fundamental variable that will determine future dividends. In addition to the dividends, there are two other sets of information available at each instant. The information is available to all agents, however, agents are divided in two groups and each group has more confidence in one of the two sets of information. As a consequence, when forecasting future dividends, each group of agents place different weights in the three sets of information, resulting in different forecasts. Although agents in our model know exactly the amount by which their forecast of the fundamental variable exceeds that of agents in the other group, behavioral limitations lead them to agree to disagree. As information flows, the forecasts by agents of the two groups fluctuate, and the group of agents that is at one instant relatively more optimistic, may become in a future date less optimistic than the agents in the other group. These changes in relative opinion generate trades.

[4] Behavioral biases of investors have also been used in recent papers, *e.g.* Barberis, Shleifer and Vishny (1998), Daniel, Hirshleifer and Subrahmanyam (1998), Hong and Stein (1999), to study asset prices. What distinguishes our paper is the analysis of the role of overconfidence in generating speculative behavior.

Each agent in the model understands that the agents in the other group are placing different weights on the different sources of information. When deciding the value of the asset, agents consider their own view of the fundamental as well as the fact that the owner of the asset has an option to sell the asset in the future to the agents in the other group. This option can be exercised at any time by the current owner, and the new owner gets in turn another option to sell the asset in the future. These characteristics makes the option "American" and gives it a recursive structure. The value of the option is the value function of an optimal stopping problem. Since the buyer's willingness to pay is a function of the value of the option that he acquires, the payoff from stopping is, in turn, related to the value of the option. This gives us a fixed point problem that the option value must satisfy.

We show that in equilibrium an asset owner will sell the asset to agents in the other group, whenever his view of the fundamental is surpassed by the view of agents in the other group by a critical amount. We call this difference the critical point. When there are no trading costs, we show that the critical point is zero - it is optimal to sell the asset immediately after the valuation of the fundamentals of the asset owner is "crossed" by the valuation of agents in the other group. This results in a trading frenzy. Our agents' beliefs satisfy simple stochastic differential equations and it is a consequence of properties of Brownian motion, that once the beliefs of agents cross, they will cross infinitely many times in any finite period of time right afterwards. Although agents' profit from exercising the resale option is infinitesimal, the net value of the option is large because of the high frequency of trades. Since the option value component in the asset price fluctuates with the difference in agents' beliefs, it contributes to the excess volatility of the asset prices. In this way, our model captures excessive trading and excess volatility observed in internet stocks during the period of 1998-2000.

We show that when a trade occurs the buyer has the highest fundamental valuation among all agents, and because of the re-sale option the price he pays exceeds his fundamental valuation. Agents pay prices that exceed their own valuation of future dividends, because they believe that in the future they will find a buyer willing to pay even more. This difference between the transaction price and the highest fundamental valuation can be reasonably called a bubble.[5] The size of this bubble increases with the degree of the agents' overconfidence and the fundamental volatility of the asset, because as these parameters increase beliefs become more heterogeneous.[6] A calibrated example shows that the magnitude of the bubble component can be large relative to the fundamental value of the asset. The same exercise shows that an increase in the information content of the non-dividend signals, which are the signals that agents disagree over precision, may increase the size of the bubble.

The existence of heterogeneous beliefs and bubbles can cause the asset returns to be predictable from the perspective of a (rational) econometrician. We show that the asset returns can be predicted by the difference of beliefs between the overconfident asset owner and the econometrician. This is consistent with the recent empirical evidence that stock returns are perhaps predictable, using variables that are related to the ratio between stock prices and their fundamental values. Our analysis also indicates an interesting possibility that, if given the opportunity to trade, the econometrician is willing to pay more than the reservation price of the current asset holder, even though he has exactly the same beliefs about future dividends as the current (overconfident) owner. This happens because the overconfident traders underestimate the volatility of beliefs and thus undervalue the resale option.[7]

The bubble proposed in our model, based on the recursive expectations of traders to take advantage of the mistakes of each other, is very different from the rational bubbles studied in the previous literature including Blanchard and Watson (1982) and others.[8] Since investors in the models of rational bubbles all have the same rational expectations, in order to make the rational bubbles sustainable, it is required that assets have infinite

[5] An alternative would be to measure the bubble as the difference between the asset price and the fundamental valuation of the dividends by a rational agent. Since both types of overconfident overweight certain signal, it is possible that a rational observer would attribute a higher value to the asset than all overconfident agents. In this case, at some point the bubble could become negative. We opted for our definition because it highlights the difference between beliefs about fundamentals and trading price.

[6] A recent experimental study by Ackert et al. (2001) shows that price bubbles are larger for assets with lottery characteristics. We interpret this result as stating that an increase in the fundamental volatility increases the size of the bubble, and thus consistent with our model.

[7] This is analogous to the observation by De Long et al. (1990) that rational arbitrageurs may want to front run positive-feedback traders.

[8] In fact, as we discuss in Section 5 of our model there are also equilibria that exhibit, in addition, a rational bubble much like those in the Blanchard and Watson model.

maturity and that many variables, such as the asset prices and the changes of asset prices, must have explosive conditional expectations. These requirements are restrictive and/or sometimes inconsistent with empirical evidence. In contrast, in our model asset prices and changes in asset prices may possess a stationary distribution. Furthermore, although we chose, for mathematical simplicity, to treat the infinite horizon case, as we argue in Section 6.7, the bubble in our model does not require infinite maturity.

Our model often exhibits a stationary bubble and, at first glance, does not seem appropriate to analyze the appearance of bubbles or crashes. In Section 6.4, we discuss how to accommodate fluctuations in parameter values that can generate fluctuations in the average size of the bubble. In this way, we may accommodate crashes and the appearance of bubbles.

When there is a trading cost, our model shows that the critical point for trade increases monotonically with the cost. Consequently, the trading frequency, asset price volatility, and the option value are all reduced. This effect is very significant for trading frequency when the cost of trading is small. At zero cost, an increase in the cost of trading has an infinite impact in the critical point and in the trading frequency. However, the impact on price volatility and on the size of the bubble is much more modest. As the trading cost increases, the increase in the critical point also raises the profit of the asset owner from each trade, thus partially offsetting the decrease in the value of the re-sale option caused by the reduction in trading frequency.

Our analysis suggests that a transaction tax, such as proposed by Tobin (1978), would, in fact, substantially reduce the amount of speculative trading in markets with small transaction costs, such as foreign exchange markets. However, our analysis also predicts that a transaction tax would have a limited effect on the size of the bubble or on price volatility. Although it is difficult to estimate the exact numerical impact of a trading tax, we provide an estimate based on a calibration exercise. According to our calibration, a trading tax in excess of 1% causes a reduction of roughly 10% in the magnitude of the bubble or in excess volatility. Since a Tobin tax will no doubt also deter trading generated by fundamental reasons that are absent from our model,[9] the

[9] See Dow and Rahi (2000) and references therein for studies of effects of taxes on trading with fundamental

limited impact of the tax on the size of the bubble and on price volatility cannot serve as an endorsement of the Tobin tax. The limited effect of transaction costs on the size of the bubble is also compatible with the observation of Shiller (2000) on the existence of bubbles in the real estate market, where transaction costs are high.

A calibrated example shows that when trading costs are present, an increase in the information content of the non-dividend signals, may decrease the average performance of traders. Intuitively, the increase in the informational content of these signals leads investors to underweight the dividend signal and, in certain cases this increases the difference in agents' beliefs. In turn, this increase in the difference of opinions causes higher trading frequency and bigger trading costs.

The existence of the option component in the asset price creates potential violations to the law of one price. Through a simple example, we illustrate that the bubble may cause the price of a subsidiary to be larger than that of its parent firm. The intuition behind the example is that if a firm has two subsidiaries with fundamentals that are perfectly negatively correlated, there will be no differences in opinion, and hence no option component on the value of the parent firm, but possibly strong differences of opinion about the value of a subsidiary. This nonlinearity of the option value may help explain the mispricing of carve-outs that occurred in the late 90's such as the 3Com-Palm case.[10]

The presence of overconfidence makes it profitable for managers to exploit corporate strategies that would not be used in a more "rational" world. In Section 10 we suggest how the model in this paper can be used to explain the use of corporate strategies such as IPO underpricing or name changes. We argue that because these strategies lead to an increase in analysts coverage and media attention, they may cause an increase in the precision of the information contained in the non-dividend signals. This increase in precision, as we argued above, may increase the difference in agents' beliefs, and this leads to a higher option value and a higher price for the stock.

There is a large literature on the effects of heterogeneous beliefs. Harris and Raviv

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reasons.

[10] Lamont and Thaler (2001), Mitchell, Pulvino and Stafford (2001), Schill and Zhou (2000), and Ofek and Richardson (2001) empirically analyze mispricings in these recent carve-outs.

(1993) show that heterogeneous beliefs can generate speculative trading. They derive a model in which there are no trading costs and agents trade when their beliefs cross each other.[11] However, they do not study the bubble associated with this type of speculative behavior and many of the other properties we discuss. Kyle and Lin (2001) study the trading volume caused by overconfident traders in a model without short-sale constraints. Miller (1977) discusses that when investors have heterogeneous beliefs, asset will be overvalued and returns will be predictable because short-sale constraints cause the prices to reflect the highest beliefs among investors.[12] Morris (1996) shows that heterogeneous beliefs and short-sale constraint can lead to IPO long-run under-performance. Detemple and Murthy (1997), and Basak and Croitoru (2000) study the effect of heterogeneous beliefs on assets prices through position constraints. Hong and Stein (2001) study asymmetric price movements generated by heterogeneous beliefs. Viswanathan (2001) analyzes the strategic behavior of traders in a model with heterogeneous beliefs and short-sale constraints.

Other types of models have been proposed to explain bubbles. Allen and Gorton (1993) study the incentives of fund managers to churn bubbles. Allen, Morris, and Postlewaite (1993) provide a mechanism of bubbles through higher order beliefs among agents. Abreu and Brunnermeier (2001) show that the inability of arbitrageurs to coordinate their selling strategies can allow bubbles to persist. Horst (2001) provides a mathematical framework to study bubbles caused by the social interaction among agents. Duffie, Garleanu and Pedersen (2001) study a model in which investors have heterogeneous beliefs and short-selling of assets requires a searching and bargaining process. In their model a bubble in asset prices results from the lending fee which the asset owner can collect.

The structure of the paper follows. Section 2 briefly reviews the literature related to overconfidence and financial markets. Section 3 describes the structure of the model. Section 4 derives the evolution of agents' beliefs. Section 5 sets up a recursive Bellman

[11] In addition, Kandel and Pearson (1995) provide both theoretical analysis and some empirical evidence for heterogeneous beliefs as a driving force for trading.

[12] Along this line, Chen, Hong and Stein (2001) provide both theoretical analysis and empirical evidence linking stock returns with the breadth of stock ownership, a measure of heterogeneous beliefs.

equation for the optimal exercise of the asset owner's re-sale option. Section 6 discusses several of the characteristics of the equilibrium dynamics. In Section 7, we focus on the effect of trading costs on the equilibrium dynamics. Section 8 shows that more information can lead to worse trading performance of investors. In section 9, we construct an example where the price of a subsidiary is larger than its parent firm. In section 10, we discuss the possible strategies that firm managers can adopt to take advantage of bubbles. Section 11 concludes the paper.

2 Overconfidence and financial markets

Psychology studies suggest that people are overconfident. Alpert and Raiffa (1982), and Brenner et al. (1996) and other calibration studies find people overestimate the precision of their knowledge. Camerer (1995) argues that even experts can display overconfidence. Hirshleifer (2001) and Barber and Odean (2002) contain extensive reviews of the literature.

In finance, researchers have developed theoretical models to analyze the implications of overconfidence on financial markets. Kyle and Wang (1997) show that overconfidence can be used as a commitment device over competitors to improve one's welfare. Daniel, Hirshleifer and Subrahmanyam (1998) use overconfidence to explain the predictable returns of financial assets. Odean (1998) demonstrates that overconfidence can cause excessive trading. Bernardo and Welch (2001) discuss the benefits of overconfidence to entrepreneurs through the reduced tendency to herd. In these studies, overconfidence is modelled as overestimation of the precision of one's information. We follow a similar approach in this paper.

Overconfident investors believe more strongly in their own assessments of assets' value than that of others. In this way, overconfidence leads to heterogeneous beliefs or differences in opinions. In fact, we show that each overconfident investor believes that the beliefs of other investors fluctuate randomly around his own beliefs according to a linear mean-reverting diffusion process. Varian (1989) and Harris and Raviv (1993) study speculative trading caused by heterogeneous beliefs. Odean (1999), and Barber and

Odean (2002) provide empirical evidence that individual investors decrease their welfare by trading too much.

3 The model

There exists a single risky asset with a dividend process that is the sum of two components. The first component is the fundamental variable that will determine future dividends. The second is "noise". More precisely, the cumulative dividend process D_t satisfies:

$$dD_t = f_t dt + \sigma_D dZ_t^D, \tag{1}$$

where Z^D is a standard Brownian motion and σ_D is a constant volatility parameter. The fundamental variable f is not observable. However, it satisfies:

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f, \tag{2}$$

where $\lambda \geq 0$ is the mean reversion parameter, $\bar{f}$ is the long-run mean of f, σ_f is a constant volatility parameter and Z^f is a standard Brownian motion. The asset is in finite supply and we normalize the total supply to unity.

There are two sets of risk-neutral agents. The assumption of risk neutrality not only simplifies many calculations, but also serves to highlight the role of information in the model. Since our agents are risk-neutral, the dividend noise in equation (1) has no direct impact in the valuation of the asset. However, the presence of dividend noise makes it impossible to infer f perfectly from observations of the cumulative dividend process. Agents will use the observations of D and any other signals that are correlated with f to infer current f and to value the asset. In addition to the cumulative dividend process, all agents observe a vector of signals s^A and s^B that satisfy:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s dZ_t^A \quad (3) \\ ds_t^B &= f_t dt + \sigma_s dZ_t^B, \quad (4) \end{aligned}$$

where Z^A and Z^B are standard Brownian motions. We assume that all four processes Z^D, Z^f, Z^A and Z^B are mutually independent. Agents in group A (B) think of s^A (s^B)

as their own signal although they can also observe s^B (s^A). Heterogeneous beliefs arise because each agent believes that the precision of his own signal is larger than its true precision. Agents of group A (B) believe that the volatility of noise to the signal s^A (s^B) is $\frac{\sigma_s}{\phi}$ instead of σ_s, where $\phi \geq 1$ measures the degree of overconfidence. In addition, we assume that the beliefs of each group concerning the evolution of cumulative dividends, the fundamental variable and the signals are common knowledge. In particular, each agent of group A (B) understands that agents of the other group have a different opinion concerning the precision of the signals.

One way to summarize the model structure is to state that agents in group A believe that $(Z^D, Z^f, \frac{\phi(s^A - \int_0^t f_u du)}{\sigma_s}, \frac{s^B - \int_0^t f_u du}{\sigma_s})$ is a four dimensional Brownian motion, whereas agents of group B believe that $(Z^D, Z^f, \frac{s^A - \int_0^t f_u du}{\sigma_s}, \frac{\phi(s^B - \int_0^t f_u du)}{\sigma_s})$ is a four dimensional Brownian motion. Agents in both groups are "irrational" in the sense that they do not infer the precision of their signals through the observations of the signals, even though they could do it.[13] This is a behavioral assumption that is well supported by experimental studies. Alternatively one can imagine that the agents know the correct volatility of their signal, but simply use the wrong weights when solving their filtering problem (see section 4), overweighing their own signal.

Modelling overconfidence through the exaggeration of the precision of signals is standard in the finance literature such as Kyle and Wang (1997), and Odean (1998). It has the advantage of matching the notion of overconfidence that comes from the psychology literature. It is of course possible that overconfidence results from excessively tight initial priors and is not the consequence of overestimation of the precision of signals. However, in this case overconfidence would eventually disappear. In addition, we assumed that all signals are public to avoid the problem of inference from prices, which would greatly complicate the mathematics.[14]

[13] It is perhaps more satisfactory to assume instead that agents in group $C \in \{A, B\}$ believe that $ds_t^C = \psi f_t dt + \sigma_s dZ_t^C$, since it is much harder to infer the drift than to infer a diffusion coefficient from data. However, while in our formulation everything depends only on the difference of beliefs (see Proposition 1 below), in this alternative formulation one must keep track of the evolution of beliefs for each group. Consequently the formulas for the trading times etc... are much more complicated. In any case, the qualitative picture should not change.

[14] See Diamond and Verrecchia (1987) and Cao and Zhang (2002) for studies of trading with asymmetric information and short-sale constraints.

Each group is large and there is no short selling of the risky asset. We assume the market to be perfectly competitive in the sense that agents in each group value the asset at their reservation price. To value future cash flows, we may either assume that every agent can borrow and lend at the same rate of interest r, or equivalently that agents discount all future payoffs using rate r, and that each class has infinite total wealth. These assumptions will facilitate the calculation of equilibrium prices.

4 Evolution of beliefs

The model that we described in the previous section implies that the evolution of a trader's view of the difference in beliefs among traders in the two groups has a particularly simple structure (see Proposition 1 below). The presence of overconfidence has two effects. On the one hand, it makes each agent believe that even if today the difference in beliefs is positive, it may become negative in the future. On the other hand, it increases the mean reversion of the difference in beliefs. This is the content of Proposition 1.

Since all variables involved are Gaussian, the filtering problem of the agents is standard. With Gaussian initial conditions, the conditional beliefs of agents in group $C \in \{A, B\}$ is Normal with mean $\hat{f}^C$ and variance γ^C. We will characterize the stationary solution. According to section VI.9 in Rogers and Williams (1987),

$$\gamma^A = \gamma^B = \gamma = \frac{\sqrt{\lambda^2 + \sigma_f^2\left(\frac{1}{\sigma_D^2} + \frac{1+\phi^2}{\sigma_s^2}\right)} - \lambda}{\frac{1}{\sigma_D^2} + \frac{1+\phi^2}{\sigma_s^2}}, \tag{5}$$

and the conditional mean of the beliefs of agents in group A satisfies:

$$d\hat{f}^A = -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi^2\gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt) + \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt). \tag{6}$$

Since f mean-reverts, the conditional beliefs also mean-reverts. The other three terms represent the effects of "surprises" in the three sources of information. These surprises can be represented as standard mutually independent Brownian motions for agents in group A:

$$dW_A^A = \frac{\phi}{\sigma_s}(ds^A - \hat{f}^A dt), \tag{7}$$

$$
\begin{aligned}
dW_B^A &= \frac{1}{\sigma_s}(ds^B - \hat{f}^A dt), && (8)\\
dW_D^A &= \frac{1}{\sigma_D}(dD - \hat{f}^A dt). && (9)
\end{aligned}
$$

Note these processes are only Wiener processes in the mind of group A agents.

Similarly, the conditional mean of the beliefs of agents in group B satisfies:

$$
d\hat{f}^B = -\lambda(\hat{f}^B - \bar{f})dt + \frac{\gamma}{\sigma_s^2}(ds^A - \hat{f}^B dt) + \frac{\phi^2\gamma}{\sigma_s^2}(ds^B - \hat{f}^B dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^B dt). \quad (10)
$$

These surprise terms can be represented as standard mutually independent Wiener processes for agents in group B:

$$
\begin{aligned}
dW_A^B &= \frac{1}{\sigma_s}(ds^A - \hat{f}^B dt), && (11)\\
dW_B^B &= \frac{\phi}{\sigma_s}(ds^B - \hat{f}^B dt), && (12)\\
dW_D^B &= \frac{1}{\sigma_D}(dD - \hat{f}^B dt). && (13)
\end{aligned}
$$

Again, we emphasize that these processes form a standard 3-d Wiener process only for agents in group B.

Since the beliefs of all agents have constant variance, we refer their beliefs to the conditional mean of the beliefs, and let g_A and g_B denote the differences in beliefs:

$$
\begin{aligned}
g^A &= \hat{f}^B - \hat{f}^A && (14)\\
g^B &= \hat{f}^A - \hat{f}^B. && (15)
\end{aligned}
$$

Agents in group A believe that signal s^A is more precise than signal s^B, and their updating rule reflects this difference in precision. They also "know" that agents in group B mistakenly believe that s^A is less precise than s^B. Over time they expect that future dividends will reflect more of the behavior of s^A, and therefore they expect that the beliefs of agents in group B will mean-revert towards their own belief. The next proposition states this property formally:

Proposition 1

$$
dg^A = -\rho g^A dt + \sigma_g dW_g^A, \quad (16)
$$

where

$$\rho = \lambda + (1+\phi^2)\frac{\gamma}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2} > 0, \tag{17}$$

$$\sigma_g = (\phi^2 - 1)\sqrt{1 + 1/\phi^2}\frac{\gamma}{\sigma_s}, \tag{18}$$

where W_g^A is a standard Wiener process for agents in group A, and it is independent to innovations to $\hat{f}^A$.

Proof: see appendix.

The dynamics of g^A in the mind of group A agents exactly captures the essence of their overconfidence. On the one hand, the presence of overconfidence makes $\sigma_g > 0$. Agents of group A think that group B agents put too little weight on s^A and too much weight in s^B. This causes the difference in their beliefs to fluctuate over time as information flows in from the dividend and the signals. On the other hand, a larger ϕ leads to a forecast of faster mean reversion in the difference of beliefs. Although the reaction of agents in each group to their own signal is not optimal, their over-reaction to the signal actually makes their beliefs converge faster.

In an analogous fashion, g^B satisfies:

$$dg^B = -\rho g^B dt + \sigma_g dW_g^B, \tag{19}$$

where W_g^B is a standard Wiener process for agents in group B, and it is independent to innovations to $\hat{f}^B$.

We are also interested in the beliefs of a rational econometrician who processes all the information objectively. We use a superscript of R to denote the rational econometrician. His beliefs are also normal with mean $\hat{f}^R$ and variance γ^R. Similarly, the variance of the rational beliefs is

$$\gamma^R = \frac{\sqrt{\lambda^2 + \sigma_f^2\left(\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}\right)} - \lambda}{\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}}, \tag{20}$$

and the conditional mean of the rational beliefs satisfies:

$$d\hat{f}^R = -\lambda(\hat{f}^R - \bar{f})dt + \frac{\gamma^R}{\sigma_s^2}(ds^A - \hat{f}^R dt) + \frac{\gamma^R}{\sigma_s^2}(ds^B - \hat{f}^R dt) + \frac{\gamma^R}{\sigma_D^2}(dD - \hat{f}^R dt). \tag{21}$$

These surprise terms can be represented as standard mutually independent Wiener processes for the rational econometrician:

$$dW_A^R = \frac{1}{\sigma_s}(ds^A - \hat{f}^R dt), \tag{22}$$

$$dW_B^R = \frac{1}{\sigma_s}(ds^B - \hat{f}^R dt), \tag{23}$$

$$dW_D^R = \frac{1}{\sigma_D}(dD - \hat{f}^R dt). \tag{24}$$

From the perspective of the rational econometrician, the difference of beliefs among the overconfident agents would also mean revert to zero, except the process has a different volatility parameter:

$$dg^A = -\rho g^A dt + \sigma_g' dW_g^R, \tag{25}$$

where W_g^R is a standard Wiener process for the rational econometrician and

$$\sigma_g' = \sqrt{2}(\phi^2 - 1)\frac{\gamma}{\sigma_s}. \tag{26}$$

Note that $\sigma_g' > \sigma_g$, i.e., the econometrician anticipates more volatility in the difference of beliefs between the overconfident agents because he knows that there is more noise in the signals than each group of overconfident agents.[15]

5 Trading

In our set-up, trading is costly - a seller pays c per unit of the asset sold. This cost may represent an actual cost of transaction or a tax. Fluctuations in the difference of beliefs across agents in different groups will induce trade. It is natural to expect that investors that are more optimistic about the prospects of future dividends will bid up the price of the asset and eventually hold the total (finite) supply.

At each t, agents in group $C = A, B$ are willing to pay p_t^C for a unit of the asset. The presence of the short-sale constraint, a finite supply of the asset, and an infinite number of

[15] There is an analogue of Proposition 1 when $\phi < 1$. In this case, agents also exaggerate the noise in one signal relative to that in the other, but it is perhaps more natural to refer to this case as underconfidence. As Dimitri Vayanos pointed out to us, if $\phi < 1$, an econometrician would anticipate less volatility of beliefs than the agents.

prospective buyers, guarantee that any successful bidder will pay his reservation price.[16] The amount, that an agent is willing to pay, reflects that agent's fundamental valuation and the fact that he may be able to sell his unit at a later date at the demand price of the other group for a profit. If we let $o \in \{A, B\}$ denote the group of the current owner, $\bar{o}$ be the other group, and E_t^o be the expectation of members of group o, conditional on the information they have at t, then:

$$p_t^o = \sup_{\tau \geq 0} \mathrm{E}_t^o \left[\int_t^{t+\tau} \mathrm{e}^{-r(s-t)} dD_s + \mathrm{e}^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right], \tag{27}$$

where τ is a stopping time, and $p_{t+\tau}^{\bar{o}}$ is the reservation value of the buyer at the time of transaction $t+\tau$. Note that $p_{t+\tau}^{\bar{o}} - p_{t+\tau}^{o} - c$ represents the trading profit to the seller.

Since, $dD = \hat{f}_t^o dt + \sigma_D dW_D^o$, we have, using the equations for the evolution of the conditional mean of beliefs (equations (6) and (10) above) that:

$$\int_t^{t+\tau} \mathrm{e}^{-r(s-t)} dD_s = \int_t^{t+\tau} \mathrm{e}^{-r(s-t)} [\bar{f} + \mathrm{e}^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + M_{t+\tau}, \tag{28}$$

where $\mathrm{E}_t^o M_{t+\tau} = 0$. Hence, we may rewrite equation (27) as:

$$p_t^o = \max_{\tau \geq 0} \mathrm{E}_t^o \left\{ \int_t^{t+\tau} \mathrm{e}^{-r(s-t)} [\bar{f} + \mathrm{e}^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + \mathrm{e}^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right\}. \tag{29}$$

To characterize equilibria we will start by postulating a particular form for the equilibrium price function, equation (30) below. Proceeding in a heuristic fashion, we derive properties that our candidate equilibrium price function should satisfy. We then construct a function that satisfies these properties, and verify that in fact we have produced an equilibrium.[17]

Since all the relevant stochastic processes are Markovian and time-homogeneous, and traders are risk-neutral, it is natural to look for an equilibrium in which the demand

[16] This observation simplifies our calculations, but is not crucial for what follows. We could partially relax the short sale constraints or the division of gains from trade, provided it is still true that the asset owner expects to make speculative profits from other investors.

[17] The argument that follows will also imply that our equilibrium is the only one within a certain class. However, there are other equilibria. In fact, given any equilibrium price p_t^o and a process M_t that is a martingale for both groups of agents, if $\tilde{p}_t^o = p_t^o + e^{rt} M_t > 0$, then, it is easy to verify that $\tilde{p}_t^o$ is also an equilibrium using equation (29). This suggests that our model also admits rational bubbles. But in our analysis, we rule out the rational bubbles to focus on the speculative bubbles.

price of the current owner satisfies

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda} + q(g_t^o). \tag{30}$$

with $q > 0$ and $q' > 0$. This equation states that prices are the sum of two components. The first part, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda}$, is the expected present value of future dividends from the viewpoint of the current owner. The second is the value of the resale option, $q(g_t^o)$, which depends on the current difference between the beliefs of the other group's agents and the beliefs of the current owner. We call the first quantity the owner's fundamental valuation and the second the value of the resale option. Applying equation (30) to evaluate $p_{t+\tau}^{\bar{o}}$, and collecting terms we may rewrite the stopping time problem faced by the current owner, equation (29) as:

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda} + \sup_{\tau \geq 0} \mathrm{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r+\lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) \mathrm{e}^{-r\tau} \right]. \tag{31}$$

Equivalently, the resale option value satisfies

$$q(g_t^o) = \sup_{\tau \geq 0} \mathrm{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r+\lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right) \mathrm{e}^{-r\tau} \right]. \tag{32}$$

Hence to show that an equilibrium of the form (30) exists, it is necessary and sufficient to construct an option value function q that satisfies equation (32). This equation is similar to a Bellman equation. A candidate function q when plugged into the right hand side must yield the same function on the left hand side. The current asset owner chooses an optimal stopping time to exercise his re-sale option. Upon the exercise of the option, the owner gets the "strike price" $\frac{g_{t+\tau}^o}{r+\lambda} + q(g_{t+\tau}^{\bar{o}})$, the amount of excess optimism that the buyer has about the asset's fundamental value and the value of the resale option to the buyer, minus the cost c of exercising the option. Different from a typical optimal exercise problem of American options, the "strike price" in our problem depends on the re-sale option value function itself. This makes the problem more difficult.

Intuitively, the value of the option $q(x)$ should be at least as large as the gains realized from an immediate sale. The region where the value of the option equals that of an immediate sale is the stopping region. The complement is the continuation region. The discounted value of the option $\mathrm{e}^{-rt}q(g_t^o)$ should be a (local) martingale in the continuation region, and a (local) supermartingale in the stopping region. These conditions can be stated as:

$$q(x) \geq \frac{x}{r+\lambda} + q(-x) - c \tag{33}$$

$$\frac{1}{2}\sigma_g^2 q'' - \rho x q' - rq \leq 0, \text{ with equality if (33) holds strictly.} \tag{34}$$

In addition, the function q should be continuously differentiable (smooth pasting). We will derive a smooth function q that satisfies equations (33) and (34) and then use these properties and a growth condition on q to show that in fact the function q solves (32).

To construct the function q, we guess that the continuation region will be an interval $(-\infty, k^*)$, with $k^* > 0$. k^* is the minimum amount of difference in opinions that generates a trade. As usual, we begin by examining the second order ordinary differential equation that q must satisfy, albeit only in the continuation region:

$$\frac{1}{2}\sigma_g^2 u'' - \rho x u' - ru = 0 \tag{35}$$

The following proposition helps us construct an "explicit" solution to equation (35).

Proposition 2 *Let*

$$h(x) = \begin{cases} U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x \leq 0 \\ \frac{2\pi}{\Gamma\left(\frac{1}{2}+\frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)} M\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) - U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x > 0 \end{cases} \tag{36}$$

where $\Gamma(\cdot)$ is the Gamma function, and $M : R^3 \to R$ and $H : R^3 \to R$ are two Kummer functions described in the appendix. $h(x)$ is continuously differentiable at $x = 0$ with a value of

$$h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{37}$$

Then any solution $u(x)$ to equation (35) that is strictly positive and increasing in $(-\infty, 0)$ must be of the following form: $u(x) = \beta_1 h(x)$ with $\beta_1 > 0$.

Proof: see appendix.

We will also need properties of the function h that are summarized in the following Lemma.

Lemma 1 *h, which is strictly positive and increasing in $(-\infty, 0)$, is strictly positive in R and satisfies $h' > 0$, $h'' > 0$, $h''' > 0$, $\lim_{x \to -\infty} h(x) = 0$, and $\lim_{x \to -\infty} h'(x) = 0$.*

Proof: see appendix.

Since q must be positive and increasing in $(-\infty, k^*)$, we know from Proposition 2 and Lemma 1 that

$$q(x) = \begin{cases} \beta_1 h(x), & \text{for} \quad x < k^* \\ \frac{x}{r+\lambda} + \beta_1 h(-x) - c, & \text{for} \quad x \geq k^*. \end{cases} \tag{38}$$

Since q is continuous and continuously differentiable at k^*,

$$\beta_1 h(k^*) - \frac{k^*}{r+\lambda} - \beta_1 h(-k^*) + c \quad = \quad 0 \tag{39}$$

$$\beta_1 h'(k^*) + \beta_1 h'(-k^*) - \frac{1}{r+\lambda} \quad = \quad 0. \tag{40}$$

These equations imply that

$$\beta_1 = \frac{1}{(h'(k^*) + h'(-k^*))(r+\lambda)}, \tag{41}$$

and k^* satisfies

$$[k^* - c(r+\lambda)](h'(k^*) + h'(-k^*)) - h(k^*) + h(-k^*) = 0. \tag{42}$$

The next theorem shows that for each c there exists a unique k^*, that solves equation (42), and a unique β_1 as a consequence of equation (41). Hence, the smooth pasting conditions are sufficient to fully determine the function q and the "trading point" k^*.

Theorem 1 *For each trading cost $c \geq 0$, there exists a unique k^* that solves (42). If $c = 0$ then $k^* = 0$. If $c > 0$, $k^* > c(r+\lambda)$.*

Proof: see appendix.

When a trade occurs, the buyer has the highest fundamental valuation. The difference between what a buyer pays and his fundamental valuation can be legitimately be named a bubble. In our model this difference is given by

$$b = q(-k^*) = \frac{1}{(r+\lambda)} \frac{h(-k^*)}{(h'(k^*) + h'(-k^*))}. \tag{43}$$

Using equation (43), we can write the value of the re-sale option as

$$q(x) = \begin{cases} \frac{b}{h(-k^*)} h(x), & \text{for} \quad x < k^* \\ \frac{x}{r+\lambda} + \frac{b}{h(-k^*)} h(-x) - c, & \text{for} \quad x \geq k^*. \end{cases} \tag{44}$$

The next theorem establishes that in fact q solves (32). The proof consists of two parts. First, we show that (33) and (34) hold and that q' is bounded. We then use a standard argument (see *e.g.* Kobila (1993) or Scheinkman and Zariphopoulou (2001) for similar arguments) to show that in fact q must solve equation (32).

Theorem 2 *The function q constructed above is an equilibrium option value function. The optimal policy consists of exercising immediately if $g^o > k^*$, otherwise wait until the first time in which $g^o \geq k^*$.*

Proof: see appendix.

To facilitate our discussion on the duration between trades, we define

$$u(x,k) = E^o[\mathrm{e}^{-r\tau(k)}|x], \quad \text{with} \;\; \tau(k) = \inf\{s : g^o_{t+s} > k\}, \quad x \leq k. \tag{45}$$

$u(x,k)$ is the discount factor for cashflow received in the future when the difference in beliefs reaches the level of k for the first time given the current difference in beliefs is x. Standard arguments (*e.g.* Karlin and Taylor (1981), page 243) show that u is a non-negative and strictly monotone solution to:

$$\frac{1}{2}\sigma_g^2 u_{xx} - \rho x u_x = ru, \quad u(k,k) = 1. \tag{46}$$

Therefore, Proposition 2 implies that

$$u(x,k) = \frac{h(x)}{h(k)}. \tag{47}$$

Note that the free parameter β_1 in the h function has no effects on u.

Using the discount factor $u(x,k)$, we can interpret the optimal stopping problem in equation (32) as choosing the optimal trading point k^* that solves

$$\sup_{k \geq 0} \left[\left(\frac{k}{r+\lambda} + q(-k) - c \right) u(x,k) \right], \tag{48}$$

where x is the current difference in agents' beliefs. The optimal trading point k^* represents the compromise between larger trading profits $\frac{k}{r+\lambda} + q(-k) - c$ and the smaller discount factor $u(x, k)$ from waiting longer for the difference of opinions to hit a larger k. Solving this optimization problem gives exactly the same optimal trading point k^* as the one obtained from the smooth pasting conditions.

In the following sections, we discuss several properties of the equilibrium pricing function and the associated bubble.

6 Properties of equilibria without trading cost

In this section, we discuss several of the characteristics of the equilibrium dynamics in the absence of trading cost. This serves as a benchmark for our discussions. Except for the results on the trading frenzy which is obtained near $c = 0$, these properties continue to hold for positive trading costs.

6.1 The bubble and trading frenzy

When $c = 0$, Theorem 1 shows that $k^* = 0$, that is a trade occurs each time traders' fundamental beliefs "cross". Nonetheless, the bubble is strictly positive, since

$$b \quad = \quad \frac{1}{2(r+\lambda)} \frac{h(0)}{h'(0)}. \tag{49}$$

Owners do not expect to sell the asset at a price above their own valuation, but the option has a positive value. This result may seem counterintuitive. To clarify it, it is worthwhile to examine the value of the option when trades occur whenever the absolute value of the differences in fundamental valuations equal an $\epsilon > 0$. An asset owner in group A (B) expects to sell the asset when agents in group B (A) have a fundamental valuation that exceeds the fundamental valuation of agents in group A (B) by ϵ, that is $g^A = \epsilon$ ($g^A = -\epsilon$). If we write b_0 for the value of the option for an agent in group A that buys the asset when $g^A = -\epsilon$, and b^1 for the value of the option for an agent of group B that buys the asset when $g^A = \epsilon$, then

$$b_0 = \left[\frac{\epsilon}{r+\lambda} + b_1\right] \frac{h(-\epsilon)}{h(\epsilon)}, \tag{50}$$

where $\frac{h(-\epsilon)}{h(\epsilon)}$ is the discount factor from equation (47). Symmetry requires that $b_0 = b_1$ and hence

$$b_0 = \frac{\epsilon}{(r+\lambda)} \frac{h(-\epsilon)}{[h(\epsilon) - h(-\epsilon)]}. \tag{51}$$

Another way of deriving b_0 is to note that by symmetry:

$$b_1 = \left[\frac{\epsilon}{r+\lambda} + b_0\right] \frac{h(-\epsilon)}{h(\epsilon)}, \tag{52}$$

and hence we may derive an expression for b_0 that reflects the value of all future options to sell, properly discounted:

$$\begin{aligned} b_0 &= \frac{\epsilon}{r+\lambda} \left[\frac{h(-\epsilon)}{h(\epsilon)} + \left(\frac{h(-\epsilon)}{h(\epsilon)}\right)^2 + \left(\frac{h(-\epsilon)}{h(\epsilon)}\right)^3 + \cdots \right] \\ &= \frac{\epsilon}{(r+\lambda)} \frac{h(-\epsilon)}{[h(\epsilon) - h(-\epsilon)]}. \end{aligned} \tag{53}$$

As $\epsilon \to 0$,

$$b_0 \to \frac{1}{2(r+\lambda)} \frac{h(0)}{h'(0)} = b. \tag{54}$$

In this illustration, as $\epsilon \to 0$, trading occurs with higher frequency and the waiting time goes to zero. In the limit, traders will trade infinitely often and the small gains in each trade compound to a significant bubble. This situation is similar to the cost from hedging an option using a stop-loss strategy studied in Carr and Jarrow (1990).

It is a property of Brownian motion that if it hits the origin at t, it will hit the origin at an infinite number of times in any non-empty interval $[t, t + \Delta t)$. In our limit case of $c = 0$, this implies an infinite amount of trade in any non-empty interval that contains a single trade. However, frequent trading is not essential in causing the bubble. As we will show in Section 7, trading costs can greatly reduce the trading frequency, but have a more modest effect on the bubble.

6.2 Excess volatility

The option component also introduces another source of return volatility in addition to the fundamental volatility. According to Proposition 1, the innovations in the asset owner's beliefs $\hat{f}^o$ and the innovations in the difference of beliefs g^o are independent. Therefore, the total return volatility is the sum of the fundamental value volatility and the volatility of the option component.

Proposition 3 *The volatility from the option value component is given by* $\frac{1}{(r+\lambda)}\frac{(\phi^2-1)\gamma}{\sqrt{2}\sigma_s}\frac{h'(g_t^o)}{h'(0)}$.

Proof: see appendix.

Since $h' > 0$, and in equilibrium $g^o \leq 0$, the volatility of the option value is maximum at the trading point $g^o = 0$. The volatility of the option value at the trading point, $\frac{1}{(r+\lambda)}\frac{(\phi^2-1)\gamma}{\sqrt{2}\sigma_s}$, increases when the interest rate or the degree of mean reversion decreases. Equation 5 guarantees that γ increases when the volatility of fundamentals σ_f increases. Hence an increase in the volatility of fundamentals has an additional effect on return volatility at trading points, through an increase in the volatility of the option value.

In this way, our model captures excessive trading and excess volatility observed in internet stocks during the period of 1998-2000.

6.3 Comparative statics

In this subsection, we present results on the effect of certain parameter changes on the option value function q and the value of the bubble b. Let

$$\alpha = \frac{\sigma_g^2}{2\rho}, \quad \beta = \frac{r}{\rho}. \tag{55}$$

The parameters α and β determine the coefficients of the differential equation that h solves. We start by establishing the effect of changes in α and β on b and q.

Lemma 2

$$b = \frac{\sqrt{2\alpha}}{4(r+\lambda)}\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)}. \tag{56}$$

b increases with α and decreases with β. For all $x < 0$, $q(x) = b\frac{h(x)}{h(0)}$ increases with α and decreases with β.

Proof: See appendix.

Proposition 1 allows us to write α and β using the parameters: ϕ, λ, σ_f, $i_s = \frac{\sigma_f}{\sigma_s}$ and $i_D = \frac{\sigma_f}{\sigma_D}$. i_s and i_D measure the information in each of the two types of signals and the dividend flow respectively. To simplify mathematics, we set $\lambda = 0$, then

$$\alpha = \frac{(\phi^2-1)^2(\phi^2+1)i_s^2\sigma_f^2}{2\phi^2\left[(1+\phi^2)i_s^2+i_D^2\right]^{3/2}} \tag{57}$$

$$\beta = \frac{r}{\sqrt{(1+\phi^2)i_s^2+i_D^2}} \tag{58}$$

Differentiating these equations, one can show the following:

Fixing all the other parameters, as ϕ increases, α increases and β decreases. Therefore, from Lemma 2, b and $q(x)$, for $x < 0$, increase. The option value and the bubble increase with the degree of overconfidence.

Fixing all the other parameters, as σ_f increases, α increases and β is unchanged. Therefore, using Lemma 2, b and $q(x)$, for $x < 0$, increase. The option value and the bubble increase with the volatility of the fundamental.

Fixing all the other parameters, as r increases, α is unchanged and β is increased. Therefore, using Lemma 2, b and $q(x)$, for $x < 0$, decrease. An increase in the interest rate decreases the option value and the bubble.

6.4 Crashes

There are several ways in which we can imagine a change in equilibrium that brings the bubble b to zero. The over-confident agents may correct their over-confidence. The fundamental volatility of the asset may disappear. The public information (the type of information that all agents can agree on) may become infinitely precise. For concreteness, imagine that agents in group A (B) believe that agents in the other group will at some point change their opinion and agree with them on the precision of the signals s^A and s^B. This type of beliefs is again overconfidence! Suppose further that agents in A (B) believe that this change of mind happens according to a Poisson process Θ^A (Θ^B). Finally, suppose that these Poisson processes have a common Poisson parameter θ and that they are independent of each other and of the four Brownian motions that describe the model.

In this case, it is easy to see that the option value is

$$q(x) = \max_k \left[\left(\frac{k}{r+\lambda} + q(-k) \right) \mathrm{E}_t^o \mathrm{e}^{-(r+\theta)\tau} \right]. \tag{59}$$

Effectively, a higher discount rate $r+\theta$ is used for the profits from exercising the option, but all the reasoning of the earlier sections hold. In particular, the results from section 6.3 show that an increase in θ decreases b and $q(x)$. When agents realize that bubble might burst in the future, the bubble becomes smaller. However the bubble persists until the (random) moment in which it crashes.

More generally, we may postulate that some parameter, σ_f or ϕ, changes according to Poisson times that are independent of all the other relevant uncertainty. The model will then produce results that are qualitatively similar to the case in which these parameters are constant, except that the average size of the bubble at any time will depend on the current value of the parameter. In this way, we can model the appearance of bubbles and market crashes.

6.5 A calibration exercise

We give a numerical example to illustrate the magnitude of the bubble component for certain parameter values that are inspired by the recent internet stock bubble. By substituting α and β into the expression for b in Lemma 2, we obtain

$$b = \frac{\sigma_f}{(r+\lambda)} \frac{i_s(\phi^2-1)\sqrt{(\phi^2+1)}}{4\phi\left[(1+\phi^2)i_s^2+i_D^2\right]^{3/4}} \frac{\Gamma\left(\frac{r+\theta}{2\sqrt{(1+\phi^2)i_s^2+i_D^2}}\right)}{\Gamma\left(\frac{1}{2}+\frac{r+\theta}{2\sqrt{(1+\phi^2)i_s^2+i_D^2}}\right)}. \tag{60}$$

The first term in this expression, $\frac{\sigma_f}{r+\lambda}$ is exactly the volatility of the fundamental value of the asset. Because we assumed that the fundamentals are normally distributed, this volatility is measured in "dollars" as opposed to percentages. We can use this dollar amount of fundamental volatility as a numeraire of the bubble component. To determine the rest of the bubble component, we need to know six parameters: r, λ, θ, ϕ, i_s and i_D.

The mean-reverting parameter of the fundamental variable λ has been set to be 0 in equation (60). We set the interest rate $r = 5\%$ and the bubble burst rate $\theta = 0.1$. The overconfidence parameter ϕ can be calibrated from psychology studies. According to an

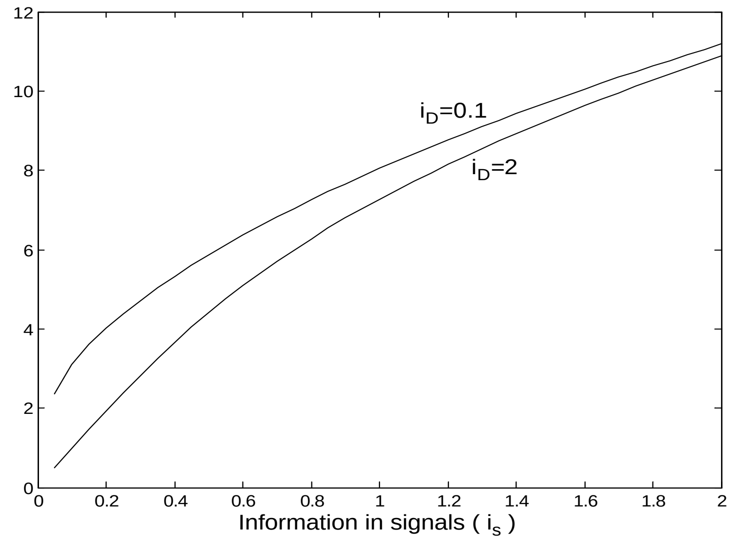

Figure 1: Bubble measured by multiples of fundamental volatility. The following parameters have been specified: $r = 5\%, \lambda = 0, \phi = 2.77, \theta = 0.1$.

experiment reported by Alpert and Raiffa (1982), the 98% confidence intervals projected by a group of individuals only cover 60% of the realizations. If a symmetric interval around the mean contains 60% of the mass of a $N(\mu, \sigma)$, it will contain 98% of a $N(\mu, \sigma')$ if $\sigma' = \frac{\sigma}{2.77}$. Hence, we set $\phi = 2.77$. In our model, the dividend volatility measures the amount of public information that all agents can agree on. From our numerical exercises, the bubble component is not very sensitive to i_D. So we have chosen two values $i_D = 0.1$ and $i_D = 2$ for illustration. The bubble component depends crucially on i_s, the amount of information that agents disagree about.

In Figure 1, we plot the bubble component b as a function of i_s. b is measured as a multiple of the fundamental volatility $\frac{\sigma_f}{r+\lambda}$. A few observations can be made about the bubble component. First, it increases with i_s. The bubble becomes larger when there is more information for agents to disagree. Second, the bubble component decreases with i_D, although this dependence is less dramatic. The bubble becomes smaller when there is more information that agents can agree on. Third, the bubble can be very significant. If we postulate $i_s = 1$, that is the noise in the agents' information is as volatile as the fundamental variable, the bubble component is about eight times fundamental volatility,

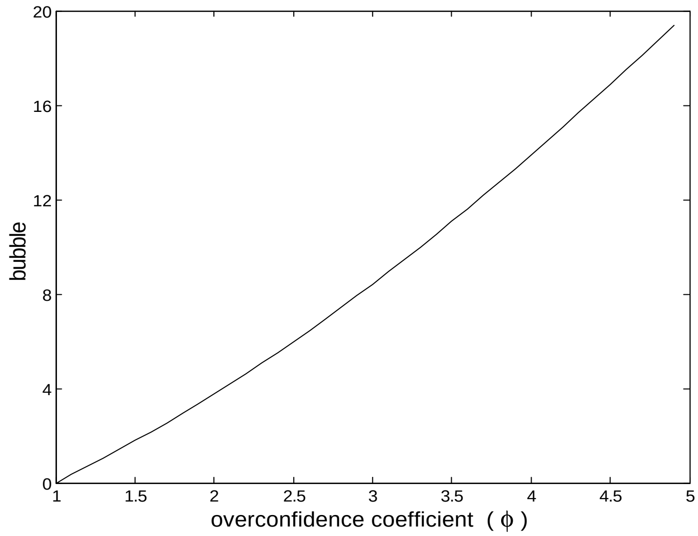

Figure 2: Bubble vs. Overconfidence Coefficient. The following parameters have been specified: $r = 5\%, \lambda = 0, \theta = 0.1, i_s = 1, i_D = 2$.

and this value can be much larger than the fundamental value of the asset.[18]

In Figure 2, we plot the bubble b versus the overconfidence coefficient ϕ. The bubble is of the same order as the asset's fundamental volatility even with a relative small overconfidence coefficient $\phi = 1.5$.

6.6 Expected returns

In this subsection, we discuss the expected returns of the asset with the presence of heterogeneous beliefs and bubbles. From the perspective of the overconfident asset holder, the expected return is always the risk free rate from the construction of the equilibrium. The expected return can be very different from the perspective of a (rational) econometrician. We denote

$$dQ = dp + dD - rpdt \tag{61}$$

as the instantaneous excess return from holding the asset. The following proposition gives the expected excess return from the perspective of a rational econometrician.

[18] Although risk neutrality of investors may have inflated the bubble size, the presence of only two groups of overconfident investors leads to an under-estimation of the bubble.

Proposition 4 *For the econometrician, the expected excess return for the asset holder is*

$$\mathrm{E}^R[dQ] = -\left[1 + \frac{\gamma}{r+\lambda}\left(\frac{1+\phi^2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)\right]\left(\hat{f}^o - \hat{f}^R\right)dt + \frac{1}{2}\left({\sigma'}_g^2 - \sigma_g^2\right)q''(x)dt, \qquad (62)$$

where $\hat{f}^o$ and $\hat{f}^R$ are the beliefs of the asset owner and the econometrician respectively.

Proof: see appendix.

From Proposition 4, there are two components in the expected excess return. The first part is generated from the difference of expected value of the fundamentals between the asset owner and the econometrician. If the asset owner has a higher mean, the econometrician expects a negative excess return. Note that although the asset owner has the highest expected value for f among all overconfident agents, this expected value might be lower than that of a rational observer.

The second part of the expected excess return is generated from the option component of the asset price. Since, as we argued in Section 4, x has a larger volatility in the mind of the econometrician (${\sigma'}_g^2 > \sigma_g^2$), the econometrician expects the re-sale option to be exercised quicker than the asset owner, and therefore expects a positive excess return from the option component.

According to Proposition 4, the expected asset return changes over time, and can be predictable from the difference between the rational beliefs and the "irrational" beliefs of the asset owner. This is consistent with some empirical work that argues that stock returns are in fact predictable using variables that are related to the ratio between stock prices and their fundamental values (See Fama and French (1992) for detail).

The positive expected excess return from the option component gives rise to the possibility that, if given the opportunity to trade, an econometrician may be willing to pay more than the reservation price of the current owner even if he has slightly less positive beliefs about future dividends. The econometrician understands that the volatility of beliefs is higher than the estimates of the overconfident owners and therefore attributes a higher value to the resale option.[19] This situation counters the common intuition that

[19]This result is only indicative since the rational econometricians are not present in the model. Their presence will affect the price dynamics and change the strategies of overconfident traders. This result also depends on

rational traders would always trade to reduce bubbles, and in this sense is analogous to the trading strategy of rational arbitrageurs discussed in De Long et al. (1990) to front run positive feedback traders.

6.7 Comparison with rational bubbles

There has been a large literature studying rational bubbles including Blanchard and Watson (1982) and others. In these studies, all agents have rational expectations, and the asset prices can be decomposed into two parts, a fundamental component and a bubble component which is expected to grow at a rate equal to the risk free rate. Although this type of bubble models is consistent with rational expectations and constant expected returns, it is based on investors' expectation that other investors would drive prices even higher in the future, independent of any fundamental reasons. Campbell, Lo, and MacKinlay (1997, pages 258-260) provide a detailed discussion on the properties of the rational bubbles. To make this type of bubbles sustainable, the asset must have an infinite maturity and there cannot exist any limit in the asset's price, so that agents can always expect the bubble to grow. An important property of the rational bubbles is that many variables, such as the asset price and the change of the asset price, have explosive conditional expectations, but this property is not consistent with empirical evidence.

The bubble in our model is generated by the expectation of investors to take advantage of future mistakes by other investors. In contrast to rational bubbles, the bubble in our model does not need to be explosive. Consequently, variables such as the asset's price and the change of this price may have stationary distributions. While our model has an infinite horizon, the bubble could still exist for an asset with only finite maturity, which is not possible for rational bubbles.[20] In addition, the bubble in our model can generate predictable returns that are consistent with some empirical evidence.

the way we model overconfidence, i.e., traders under-estimate noise of one signal while correctly estimating the noise of another signal (see footnote 15).

[20]We chose an infinitely lived asset because the complete analysis of a model with a finitely lived asset is much more difficult, due to the lack of time invariance.

7 Effects of trading costs

We now examine the effects of trading costs. According to Theorem 1, the trading barrier k^* becomes nonzero when there is a trading cost, and k^* satisfies equation (42). The bubble b and the option value $q(x)$ are determined in equations (43) and (44) respectively. Let $\eta(x)$ denote the volatility of the option value component. $\eta(x)$ represents the excess volatility caused by speculative trading, and by Ito's Lemma it must satisfy

$$\eta(x) = \frac{\sqrt{2}(\phi^2 - 1)\gamma}{\sigma_s(r+\lambda)} \frac{h'(x)}{(h'(k^*) + h'(-k^*))}, \quad \forall x \leq k^*. \tag{63}$$

The following proposition shows that increasing the trading cost c raises the trading barrier k^* and reduces b, $q(x)$ and $\eta(x)$.

Proposition 5 *If c increases, the optimal trading barrier k^* increases. Furthermore, the bubble b, the option component $q(x)$ and the excess volatility $\eta(x)$ ($\forall x \leq k^*(c)$) decrease. As $c \to 0$, $\frac{dk^*}{dc} \to \infty$, but the derivatives of b, $q(x)$, and $\eta(x)$ are always finite.*

Proof: see appendix.

In order to illustrate the effects of trading costs, we use the following parameter values from our previous calibration exercise,

$$r = 5\%,\ \phi = 2.77,\ \lambda = 0,\ \theta = 0.1,\ i_s = 1.0,\ i_D = 2.0. \tag{64}$$

Figure 2 shows the relations of the trading barrier $\frac{k^*}{r+\lambda}$, expected duration between trades, the bubble b, and $\eta(0)$ (the excess volatility when beliefs coincide), with respect to the trading cost c. The expected duration between trades is measured in years. The trading barrier $\frac{k^*}{r+\lambda}$, the excess volatility $\eta(0)$, the bubble b, and the trading cost c are all measured in multiples of the fundamental volatility $\frac{\sigma_f}{r+\lambda}$. For the illustration, we also adopt a value of $\sigma_f = 0.66$ which can be translated into a 66% fundamental volatility per annum. This value is consistent with the price volatility of internet stocks after the crash of 2000.

Panel A of Figure 3 shows the optimal trading barrier $\frac{k^*}{r+\lambda}$ together with the barrier c which represents the minimum level in difference of beliefs that allows the asset owner to cover the trading cost. The difference between these two barriers represents the profits to the asset owner for exercising his option. When the trading cost is zero, the asset owner

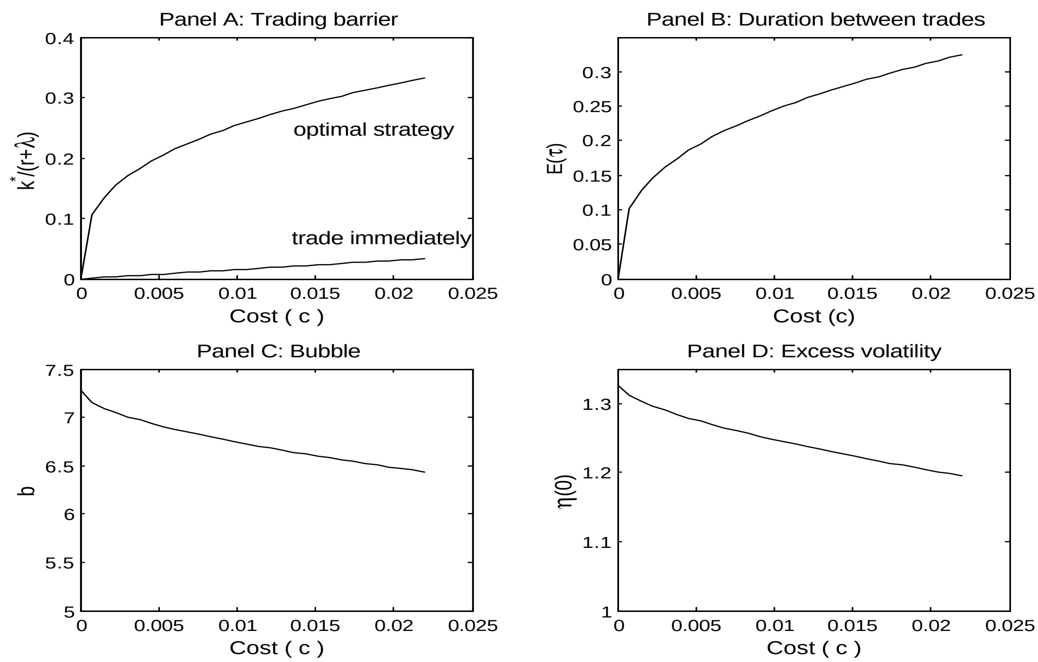

Figure 3: Effects of trading costs. The following parameters have been specified: $r = 5\%, \phi = 2.77, \lambda = 0, \theta = 0.1, i_s = 1.0, i_D = 2.0$.

sells the asset immediately when it is profitable and the profits are infinitely small. As the trading cost increases, the optimal trading barrier increases, and the rate of increase near $c = 0$ is dramatic, since the derivative $\frac{dk^*}{dc}$ is infinite at the origin. As a result, the trading frequency is greatly reduced by the increasing trading cost as shown in Panel B. Note that the profits from each trade $\frac{k^*}{r+\lambda} - c$ also increase dramatically with the trading cost near $c = 0$.

As shown in Panels C and D, the trading cost also reduces the bubble and the excess volatility, but only at a limited rate even near $c = 0$. Although we expect that the great reduction in trading frequency, caused by the increase in the trading cost, should greatly reduce the bubble, this effect is partially offset by the the increase in profits in each trade.[21] Similar intuition applies to the effect of the trading cost on excess volatility. According to our calibration exercise, increasing the trading cost by 2%, measured in

[21] Vayanos (1998) provides an interesting analysis of the effects of transaction cost on asset prices in a life-cycle model. He shows that an increase of transaction cost can reduce the trading frequency and therefore may even cause asset prices to be higher.

multiples of fundamental volatility, from the case with zero trading cost would cause a reduction of about 10% to the magnitudes of both the bubble and the excess volatility. This increase in the trading cost represents an ad valorem tax in excess of 1.3%.

The effectiveness of a trading tax to reduce speculative trading has been hotly debated since James Tobin's (1978) initial proposal for a transaction tax in the foreign currency markets. Shiller (2000, pages 225-228) provides an overview of the current status of this debate. Our model implies that trading costs are crucial for determining the trading frequency, at least when we start from an initial position of low costs, but trading costs have a more limited impact in excess volatility or the magnitude of the price bubble. Our calibration shows that an ad valorem tax of 1.3%, a proportion that exceeds the proportion contemplated in most "Tobin tax" proposals, decreases bubbles and excess volatility by only 10%. In reality, trading also occurs for other reasons, such as liquidity shocks or changes in risk bearing capacity, that are not considered in our analysis and, for this reason, the limited impact of transaction costs on volatility and price bubbles cannot serve as an endorsement of Tobin's proposal. Our calibration can also be used to answer a question raised by Shiller (2000) of why bubbles can exist in real estate markets, where the transaction costs are typically high.

8 Trading performance

Although the agents in our model are all risk neutral price takers, their trading performance, i.e., the returns of their trading account, does not equal the risk free rate. This is due to two reasons. The first one is overconfidence and is present even in the absence of transaction costs. The second reason is transaction costs. Trading costs make trading a negative sum game, and therefore worsens performance. In this subsection, we analyze the impact of transaction costs on trading performance.

Since the two groups are symmetric, we only need to analyze the aggregate trading cost. If the current difference of beliefs among agents is $g^o = x \leq k$, the first trade occurs when g^o hits k, the trading point. The discount factor for cashflows received at that time is $u(x, k)$ as shown in equation (45). The second trade occurs when g^o moves

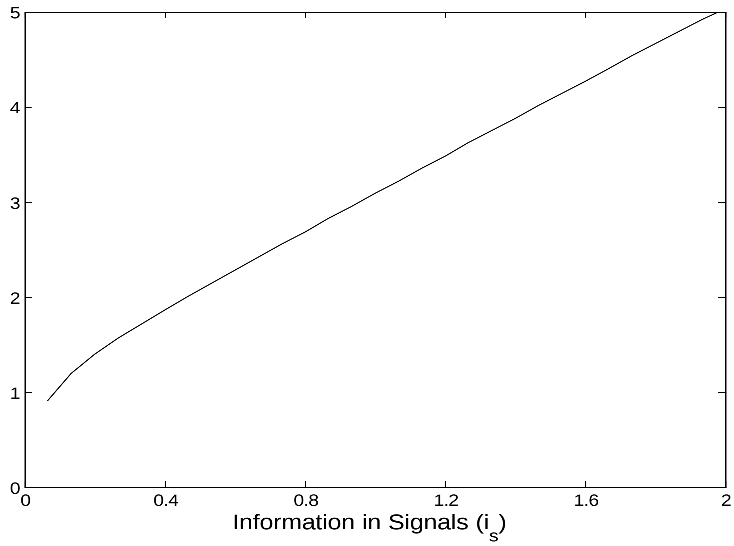

Figure 4: Aggregate present value of future trading costs. The following parameters have been specified: $r = 5\%, \phi = 2.77, \lambda = 0, \theta = 0.1, i_D = 2.0, c = 0.1$.

from $-k$ to k after the first trade, and the discount factor for cashflows received then is $u(x,k)u(-k,k)$. Similar arguments apply for subsequent trades. Therefore, the total present value of all the future trading costs is

$$K = \sum_{i=1}^{\infty} cu(x,k)u(-k,k)^{i-1} = \frac{cu(x,k)}{1-u(-k,k)} = \frac{ch(x)}{h(k)-h(-k)} \tag{65}$$

where the last equation follows from equation (47).

Our model provides an example where more information can cause investors to do worse in their trading. We model the increase in information as a decrease in the diffusion coefficient σ_s of the (non-dividend) signal. Holding the volatility of fundamentals σ_f constant, a decrease in σ_s is equivalent to an increase in the information coefficient i_s. An increase in i_s can cause investors to trade more and consequently incur higher trading cost, whenever $c > 0$. This result echoes the discussion by Barber and Odean (2001) who argue that the greater volume and variety of information created by the recent advances in information technology can feed the illusion of knowledge of investors and cause them to trade more aggressively and perform worse in their trading.

We illustrate the effect of the amount of information in the signals, i_s, on trading costs K using the parameter values from our calibration exercise:

$$r = 5\%,\ \phi = 2.77,\ \lambda = 0,\ \theta = 0.1,\ i_D = 2.0,\ c = 0.1. \tag{66}$$

As shown by Figure 4, the total trading costs K increases with i_s. In this case, the increased amount of information causes more variation in the difference of agents' beliefs, and inducing higher frequency of trading.[22] Therefore, as a result of overconfidence, the trading performance of agents can deteriorate as more information becomes available.

9 Can the price of a subsidiary be larger than its parent firm?

The existence of the option value component in asset prices can potentially create violations to the law of one price. In this section, we use an example to illustrate this type of situation. We want, in particular, to illustrate a situation where the price of a subsidiary can be larger than its parent company.

There are two firms, indexed by 1 and 2. For simplicity, we assume the dividend processes of both assets follow the process in equation (1) with the same parameter σ_D, but with independent innovations, and with different fundamental variables f_1 and f_2 respectively. The fundamental variables f_1 and f_2 are unobservable and both follow the linear mean-reverting process in equation (2) with the same parameters λ, $\bar{f}$ and σ_f. To illustrate our point, we consider a very special case in which the innovations in the processes of f_1 and f_2 are perfectly negatively correlated. Discussions on more general cases are left for future research.

There is another firm (firm 3), and the dividend flow of firm 3 is exactly the sum of the dividend flows of firms 1 and 2. In this sense, firms 1 and 2 are both subsidiaries of firm 3, and the fundamental variable of firm 3 is the sum of that of firms 1 and 2: $f_3 = f_1 + f_2$. Since the innovations of f_1 and f_2 are perfectly negatively correlated, f_3 is a constant determined by initial conditions.

Shares of these three firms are traded by the two groups of agents described in Section 2. Since the fundamental variables of firms 1 and 2 fluctuate and are unobservable, these agents try to infer their values. According to our earlier discussion, overconfidence can

[22]This result may only hold under certain conditions.

generate heterogeneous beliefs among agents in different groups. As a result, an option component exists in the prices of the shares of firm 1 and firm 2. Since innovations to the fundamental variables f_1 and f_2 are perfectly negatively correlated, the beliefs of agents about these two assets are also perfectly negatively correlated, i.e., when $\hat{f}_1^A$ ($\hat{f}_1^B$) moves up by certain amount, $\hat{f}_2^A$ ($\hat{f}_2^B$) moves down by the same amount. To simplify our discussion, let us consider the case without trading costs. In this case, agents with higher beliefs hold the asset. Therefore, when agents in group A are holding firm 1, agents in group B must be holding firm 2. Consequently, the option components in the prices of these two firms are always the same. Therefore, the prices of firms 1 and 2 can be expressed as

$$p_1 = \frac{\bar{f}}{r} + \frac{\hat{f}_1 - \bar{f}}{r+\lambda} + q(x), \quad p_2 = \frac{\bar{f}}{r} + \frac{\hat{f}_2 - \bar{f}}{r+\lambda} + q(x), \tag{67}$$

where $x = \hat{f}_1^{\bar{o}} - \hat{f}_1^{o} = \hat{f}_2^{\bar{o}} - \hat{f}_2^{o} < 0$.

Since agents in both groups know that the fundamental variable of firm 3 is a constant, there are no heterogeneous beliefs about f_3. Therefore, there is no option component or bubble in the price of firm 3. The price of firm 3 can be expressed as

$$p_3 = \frac{2\bar{f}}{r} + \frac{\hat{f}_1 + \hat{f}_2 - 2\bar{f}}{r+\lambda}. \tag{68}$$

According to our calibration exercise in Section 6.5, for reasonable parameter values, the magnitude of the option component in the prices of assets 1 and 2 can be eight multiples of their fundamental volatility, which could be large enough to cause the value of either firm 1 or firm 2 to exceed that of firm 3, even though all prices are nonnegative.

From this simple example, we can see that the speculative bubble can cause the price of a subsidiary to be larger than its parent firm.[23] Although highly stylized, this analysis may help clarify the episodes such as 3Com's equity carve-out of Palm and its subsequent spinoff.[24] In early 2000, for a period of over two months the total market capitalization of 3Com was significantly less than the market value of its holding in Palm, a subsidiary

[23]Duffie, Garleanu, and Pedersen (2001) provide another mechanism to explain this phenomenon based on the lending fee that the asset owner can expect to collect.

[24]The missing link is that we have not demonstrated that the divergence of beliefs on the combined entity was smaller than the divergence of beliefs on the Palm spinoff.

of 3Com. Similar situations also happened in other carveout cases studied in Lamont and Thaler (2001), Mitchell, Pulvino and Stafford (2001), Schill and Zhou (2000), and Ofek and Richardson (2001). Interestingly, according to Lamont and Thaler (2001), the turnover rate of the subsidiaries' stocks is on average about six times higher than that of the parent firms' stocks. Our model also predicts that, in this context, trading in the subsidiary would be much higher than trading in the parent company, because of the much higher fluctuation in beliefs about the value of the subsidiary.

This example also illustrates the fact that the diversification of a firm reduces the bubble component in the firm's stock price because diversification reduces the fundamental uncertainty of the firm and therefore reducing the potential disagreements among investors. This result provides a possible explanation to the diversification discount puzzle - the fact that the stock of a diversified firm appears to trade at a discount when compared to the stock of a similar undiversified firm.[25]

10 Bubbles and corporate strategies

Due to the short-sale constraints, arbitrageurs or other rational investors cannot take advantage of the bubbles studied in our model. On the other hand, firm managers may be able to profit by adopting strategies that increase share prices. As we showed in Section 6.5 above, for reasonable parameter values, the bubble increases whenever we increase the information content of the signals, i_s, or decrease the information content of the dividend process, i_D. In this section, we briefly discuss how one may, in principle, use the model in this paper to rationalize the recent popularity of certain corporate strategies.

10.1 IPO underpricing

The underpricing of a firm's initial public offering (IPO) has been severe and puzzling. As reviewed by Ritter (2002), the average first day return of an IPO is about 10 to 15 percent. For the recent internet stock IPOs, it is common to see first day returns of 50% or even more than 100%. This means in some cases hundreds of millions of dollars left on the table.

[25] See Lang and Stultz (1994), Burger and Ofek (1995), and Servaes (1996).

Rajan and Servaes (1997), and Aggarwal, Krigman, and Womack (2001) show that higher initial returns on an IPO leads to more analysts and media coverage. Since investors may disagree about the precision of information provided by analysts and media, the increase in these information could increase the option component of the stock. Therefore, the IPO unerpricing could be a strategy used by firm managers to boost the price of their stocks.

Firm managers, who typically hold residual shares, can get greater payoffs from subsequent sales of their own personal shares after the lock-up period. Our model thus predicts that underpricing is more likely to occur when managers hold a larger share of the firm. This agrees with the empirical results in Aggarwal, Krigman, and Womack (2001) who show that managerial share and option holdings are positively related to first day IPO underpricing.

If underpricing occurs because of the mechanism we propose, a larger underpricing should be associated with a larger trading volume. In fact, Reese (2000) finds that the higher initial IPO returns is associated with larger trading volume for more than three years after issuance.

10.2 Name changes

In the recent internet stock bubble, many firms changed their names to a "dotcom" name. Cooper, Dimitrov, and Rau (2001) use a sample of 147 firms that changed their names to a dotcom name between June 1998 and July 1999, to document abnormal returns on the order of 53 percent in the five days around the announcement date. Lee (2001) also documents that the average trading volume rises twelve fold on the announcement date in a sample of 114 firms that change their names to dotcom between January 1995 and June 1999, even though these name changes were not accompanied by any changes in strategy.

If, as it seems likely, the name change increased the attention of the analyst and investor communities during the period when internet stocks were "hot", the name change produced an increase in the amount of information available about the company. The numerical illustration in Section 6.5 shows that this name change could increase the value of the stock.

10.3 Spinoffs and carve outs

In Section 9, we argued that our model may explain the large number of spin offs and carve outs that occurred during the internet bubble. In the model, another possible source of increased valuation following a breakup is an increase in the amount of information that is generated about the parts resulted from the breakup. Gilson et al. (2001) document that breakups are accompanied by an increase in analyst coverage. However, they also show that consensus among analysts forecasts increases, which should decrease stock prices.

10.4 Earnings management

Investigating a sample of 100,000 quarterly earnings reports from 1974-1996, Degeorge, Patel, and Zeckhauser (1999) argue that firms make great efforts to match analyst forecasts or just beat them by one penny. They also argue that managers have the ability to smooth earnings and obscure the information revealed by earnings reports. As stated by Degeorge, Patel, and Zeckhauser (1999), "within generally accepted accounting principles (GAAP), firm managers have considerable flexibility in the choice of inventory methods, allowance for bad debt, expensing of research and development, recognition of sales not yet shipped, estimation of pension liabilities, capitalization of leases and marketing expenses, delay in maintenance expenditures, and so on". In fact, D'Avolio, Gildor and Shleifer (2001) argue that the quality of information available to investors has deteriorated over time.

Although in our model the firm's production of information is summarized by the dividend flow, in reality firms produce much more detailed information in their earnings report. Since we expect investors to agree on information revealed by the firm, we interpret these information as information contained in the dividend process. Hence, a decrease in the information content of the earnings report corresponds to a decrease in i_D which, as we argued above, may lead to a higher price of the stock.

11 Conclusion

In this paper, we provide a simple model to study bubbles generated from speculative trading among agents with heterogeneous beliefs. Heterogeneous beliefs come from the overconfidence of agents. With a short-sale constraint, an asset owner has an option to sell the asset to other agents when they have more optimistic beliefs. Agents value this option, and consequently pay prices that exceed their own valuation of future dividends, because they believe that in the future they will find a buyer willing to pay even more. By solving for the optimal exercise strategy of the asset owner, we are able to solve analytically for the equilibrium. This allows us to discuss the magnitude of the bubble, trading frequency, asset return volatility and the predictability of returns in the equilibrium. These properties are consistent with the recent internet stock bubble. In particular, our model shows that a small trading tax may be effective in reducing speculative trading, but it may not be very effective in reducing price volatility or the size of the bubble. Through a simple example, we also illustrate that the bubble can cause the price of a subsidiary to be larger than its parent firm, a strong violation of the law of one price. In addition, our model allows us to discuss certain strategies that firm managers can adopt to exploit bubbles. These strategies include IPO underpricing or name changes. The observation of these strategies in the real world strengthens the case for our model.

A Some Proofs

A.1 Proof to Proposition 1

The process of g^A can be derived from the conditional beliefs $\hat{f}^A$ and $\hat{f}^B$ in equations (6) and (10):

$$dg^A = d\hat{f}^B - d\hat{f}^A = -\left[\lambda + (1+\phi^2)\frac{\gamma}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2}\right] g^A dt + \frac{(\phi^2-1)\gamma}{\sigma_s^2}(ds^B - ds^A). \quad \text{(A1)}$$

The difference of beliefs g^A mean-reverts with a parameter of

$$\rho = \lambda + (1+\phi^2)\frac{\gamma}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2}. \quad \text{(A2)}$$

In the mind of agents in group A,

$$ds^A = \hat{f}_A dt + \frac{\sigma_s}{\phi} dW_A^A, \quad \text{(A3)}$$

$$ds^B = \hat{f}_A dt + \sigma_s dW_B^A, \quad \text{(A4)}$$

according to equations (7) and (8). Therefore,

$$dg^A = -\rho g^A dt + \frac{(\phi^2-1)\gamma}{\sigma_s^2}\left(\sigma_s dW_B^A - \frac{\sigma_s}{\phi} dW_A^A\right). \quad \text{(A5)}$$

We can simplify the notation to

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A \quad \text{(A6)}$$

with

$$\sigma_g = (\phi^2-1)\sqrt{1+1/\phi^2}\,\frac{\gamma}{\sigma_s}, \quad \text{(A7)}$$

$$dW_g^A = \frac{1}{\sigma_s\sqrt{1+1/\phi^2}}\left(\sigma_s dW_B^A - \frac{\sigma_s}{\phi} dW_A^A\right). \quad \text{(A8)}$$

It is easy to verify that W_g^A is independent to the innovations to $\hat{f}^A$ in the mind of agents in group A.

Similar derivation can be done for the difference of beliefs g^B in the mind of agents in group B.

A.2 Proof to Proposition 2

Let $v(y)$ be a solution to the following differential equation

$$yv''(y) + (1/2 - y)v'(y) - \frac{r}{2\rho}v(y) = 0. \tag{A9}$$

It is straightforward to verify that

$$q(x) = v\left(\frac{\rho}{\sigma_g^2}x^2\right) \tag{A10}$$

satisfies the equation we need to solve:

$$\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) = rq(x). \tag{A11}$$

According to Chapter 13 of Abramowitz and Stegum (1964), the general solution of equation (A9) is

$$v(y) = \alpha M\left(\frac{r}{2\rho}, \frac{1}{2}, y\right) + \beta U\left(\frac{r}{2\rho}, \frac{1}{2}, y\right). \tag{A12}$$

$M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$ are Kummer functions defined as

$$M(a,b,y) = 1 + \frac{ay}{b} + \frac{(a)_2 y^2}{(b)_2 2!} + \cdots + \frac{(a)_n y^n}{(b)_n n!} + \cdots \tag{A13}$$

where

$$(a)_n = a(a+1)(a+2)...(a+n-1), \quad (a)_0 = 1, \tag{A14}$$

and

$$U(a,b,y) = \frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,y)}{\Gamma(1+a-b)\Gamma(b)} - y^{1-b}\frac{M(1+a-b, 2-b, y)}{\Gamma(a)\Gamma(2-b)}\right\}. \tag{A15}$$

For our purpose, the following asymptotic properties of these functions are useful:

$$M_y(a,b,y) > 0, \quad \forall y > 0 \tag{A16}$$

$$M(a,b,y) \to +\infty, \quad U(a,b,y) \to 0, \quad \text{as } y \to +\infty. \tag{A17}$$

Due to the non-uniqueness of the transformation in equation (A10), we need to discuss the solution for two separate regions $x \leq 0$ and $x > 0$. In order to guarantee that $u(x)$ is positive and increasing for $x < 0$, α coefficient in equation (A12) must be zero. Therefore,

$$u(x) = \beta_1 U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) \quad \text{if} \;\; x \leq 0. \tag{A18}$$

In the other region, the solution can be expressed as

$$u(x) = \alpha_2 M\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) + \beta_2 U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) \quad \text{if} \quad x > 0. \tag{A19}$$

The solution must have continuous value and first order derivative at the point $x = 0$. From the definition of the two Kummer functions, we can derive

$$\begin{array}{llll} x \to 0-, & u(x) \to \frac{\beta_1\pi}{\Gamma\left(\frac{1}{2}+\frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}, & u'(x) \to \frac{\beta_1\pi\sqrt{\rho}}{\sigma_g\Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)} \\ x \to 0+, & u(x) \to \alpha_2 + \frac{\beta_2\pi}{\Gamma\left(\frac{1}{2}+\frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}, & u'(x) \to -\frac{\beta_2\pi\sqrt{\rho}}{\sigma_g\Gamma\left(\frac{r}{2\rho}\right)\Gamma\left(\frac{3}{2}\right)} \end{array} \tag{A20}$$

By matching the values and first order derivatives of $u(x)$ from the two sides of $x = 0$, we have

$$\beta_2 = -\beta_1, \qquad \alpha_2 = \frac{2\beta_1\pi}{\Gamma\left(\frac{1}{2}+\frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{A21}$$

The function value at $x = 0$ is

$$q(0) = \frac{\beta_1\pi}{\Gamma\left(\frac{1}{2}+\frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{A22}$$

A.3 Proof to Lemma 1

$h(x)$ is solution to the

$$\alpha h'' - xh' - \beta h = 0, \tag{A23}$$

where $\alpha = \frac{\sigma_g^2}{2\rho} > 0$ and $\beta = \frac{r}{\rho} > 0$, that is positive and increasing in $(-\infty, 0)$.

If $x^* \in R$ with $h(x^*) > 0$ and $h'(x^*) = 0$ then $h''(x^*) = \beta h(x^*)/\alpha > 0$. Hence h has no local maximum while it is positive and as a consequence it is always positive and has no local maxima. In particular h is monotonically increasing. Since $h' > 0$ for $x \le 0$ and $h'' \ge 0$ for $x \ge 0$, $h'(x) > 0$ for all x. From the solution constructed in Proposition 2, $\lim_{x\to-\infty} h(x) = 0$.

Note any solution to the differential equation is infinitely differentiable. Next, we show that h is convex. For $x > 0$, $h''(x) = xh'(x)/\alpha + \beta h(x)/\alpha > 0$. To prove that h is also convex for $x < 0$, let us assume that there exists $x^* < 0$ such that $h''(x^*) \le 0$. Then

$$h'''(x^*) = x^*h''(x^*)/\alpha + (\beta+1)h'(x^*)/\alpha > 0. \tag{A24}$$

This directly implies that $h''(x) < 0$ for $x < x^*$. Then $\lim_{x\to-\infty} h'(x) = \infty$. In this situation the boundary condition $h(-\infty) = 0$ can not be satisfied. In this way, we get a contradiction.

Let $v(x) = h'(x)$. $v(x)$ is positive and increasing from the properties that we have proved for $h(x)$. v also satisfies the following equation:

$$\alpha v''(x) - xv'(x) - (\beta + 1)v(x) = 0. \tag{A25}$$

This equation is very similar to the one satisfied by $h(x)$. By repeating the same proof for h, one can show that $v(x)$ is also convex and $\lim_{x\to-\infty} v(x) = 0$.

Actually, one can show that any higher order derivative of $h(x)$ is positive, increasing and convex.

A.4 Proof to Theorem 1

Let

$$l(k) = [k - c(r + \lambda)](h'(k) + h'(-k)) - h(k) + h(-k). \tag{A26}$$

We first show that there exists a unique k^* that solves $l(k) = 0$.

If $c = 0$, $l(0) = 0$, and

$$l'(k) \;\; = \;\; x[h''(k) - h''(-k)] > 0, \;\; \text{for all}\, k \neq 0. \tag{A27}$$

Therefore $k^* = 0$ is the only root to $l(k) = 0$.

If $c > 0$, then

$$l(k) < 0, \text{ for all}\, k \in [0, c(r + \lambda)]. \tag{A28}$$

Also, since h'' and h''' are increasing (Lemma 1),

$$\begin{aligned} l'(k) &= [k - c(r + \lambda)][h''(k) - h''(-k)] > 0, \; \forall k > c(r + \lambda), \qquad \text{(A29)} \\ l''(k) &= h''(k) - h''(-k) + [k - c(r + \lambda)][h'''(k) - h'''(-k)] > 0, \; \forall k > c(r + \lambda) \qquad \text{(A30)} \end{aligned}$$

Therefore $l(k) = 0$ has a unique solution $k^* > c(r + \lambda)$.

A.5 Proof to Theorem 2

First we show that q satisfies equation (33). Using equation (44), we have

$$q(-x) = \begin{cases} \frac{b}{h(-k^*)}h(-x) & \text{for} \quad x > -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}h(x) - c & \text{for} \quad x \leq -k^*. \end{cases} \tag{A31}$$

We must establish that

$$U(x) = q(x) - \frac{x}{r+\lambda} - q(-x) + c \geq 0, \quad \forall x. \tag{A32}$$

A simple calculation shows that

$$U(x) = \begin{cases} 2c & \text{for} \quad x < -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}[h(x) - h(-x)] + c & \text{for} \quad -k^* \leq x \leq k^* \\ 0 & \text{for} \quad x > k^* \end{cases} \tag{A33}$$

Thus,

$$U''(x) = \frac{b}{h(-k^*)}[h''(x) - h''(-x)], \ -k^* \leq x \leq k^*. \tag{A34}$$

From lemma 1 we know for $U''(x) > 0$ for $0 < x < k^*$, and $U''(x) < 0$ for $-k^* < x < 0$. Since $U'(k^*) = 0$, $U'(x) < 0$ for $0 < x < k^*$. On the other hand, $U'(-k^*) = 0$, so $U'(x) < 0$ for $-k^* < x < 0$. Therefore $U(x)$ is monotonically decreasing for $-k^* < x < k^*$. Since $U(-k^*) = 2c > 0$ and $U(k^*) = 0$, $U(x) > 0$ for $-k^* < x < k^*$.

We now show that equation (34) holds. By construction, equation (34) holds in the region $x \leq k^*$. Therefore we only need to show for $x \geq k^*$,

$$\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - r q(x) \leq 0. \tag{A35}$$

In this region, $q(x) = \frac{x}{r+\lambda} + \frac{b}{h(-k^*)}h(-x) - c$, thus $q'(x) = \frac{1}{r+\lambda} - \frac{b}{h(-k^*)}h'(-x)$ and $q''(x) = \frac{b}{h(-k^*)}h''(-x)$. Hence,

$$\begin{aligned} & \frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - r q(x) \\ = \ & \frac{b}{h(-k^*)}\left[\frac{1}{2}\sigma_g^2 h''(-x) + \rho x h'(-x) - r h(-x)\right] - \frac{r+\rho}{r+\lambda}x + rc \\ = \ & -\frac{r+\rho}{r+\lambda}x + rc \leq -(r+\rho)c + rc = -\rho c < 0 \end{aligned} \tag{A36}$$

where the inequality comes from the fact that $x \geq k^* > (r+\lambda)c$ from Theorem 1.

Also q has an increasing derivative in $(-\infty, k^*)$ and has a derivative bounded in absolute value by $\frac{1}{r+\lambda}$ in (k^*, ∞). Hence q' is bounded.

If τ is any stopping time, the version of Ito's lemma for twice differentiable functions with absolutely continuous first derivatives (*e.g.* Revuz and Yor (1999), Chapter VI) implies that

$$\mathrm{e}^{-r\tau} q(g^o_\tau) = q(g^o_0) + \int_0^\tau \left[\frac{1}{2}\sigma_g^2 q''(g^o_s) - \rho g^o_s q'(g^o_s) - r q(g^o_s)\right] ds + \int_0^\tau \sigma_g q'(g^o_s) dW_s \quad \text{(A37)}$$

Equation (34) states that the first integral is non positive, while the bound on q' guarantees that the second integral is a Martingale. Using equation (33) we obtain,

$$\mathrm{E}^o\left\{\mathrm{e}^{-r\tau}\left[\frac{g^o_\tau}{r+\lambda} + q(-g^o_\tau) - c\right]\right\} \leq \mathrm{E}^o\left[\mathrm{e}^{-r\tau} q(g^o_\tau)\right] \leq q(g^o_0). \quad \text{(A38)}$$

This shows that no policy can yield more than $q(x)$.

Now consider the stopping time $\tau = \inf\{t : g^o_t \geq k^*\}$. Such τ is finite with probability one, and g^o_s is in the continuation region for $0 \leq s < \tau$. Hence using exactly the same reasoning as above, but recalling that in the continuation region (34) holds with equality we obtain that

$$q(g^o) = \mathrm{E}^o\left\{\mathrm{e}^{-r\tau}\left[\frac{g^o_\tau}{r+\lambda} + q(-g^o_\tau) - c\right]\right\}. \quad \text{(A39)}$$

A.6 Proof to Proposition 3

Since $q(x) = \frac{1}{2(r+\lambda)}\frac{h(x)}{h'(0)}$, the volatility of $q(g^o_t)$ is given by $\frac{1}{2(r+\lambda)}\frac{h'(g^o_t)}{h'(0)}$ multiplied by the volatility of g^o_t. From the proof to proposition 1,

$$dg^o_t = -\left[\lambda + (1+\phi^2)\frac{\gamma}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2}\right] g^o_t dt + \frac{(\phi^2-1)\gamma}{\sigma_s^2}(ds^{\bar{o}} - ds^o). \quad \text{(A40)}$$

We need to determine the volatility of this process from the perspective of an objective econometrician. From equations (3) and (4) the volatility of $s^{\bar{o}} - s^o$ is $\sqrt{2}\sigma_s$ in an objective measure. Hence the volatility of g^o is $\frac{\sqrt{2}(\phi^2-1)\gamma}{\sigma_s}$.

A.7 Proof to Lemma 2

It is obvious that b increases with α. We can directly show that $\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)}$ decreases with β by plotting it. Therefore, b decreases with β.

The option value component is $q(x) = b\frac{h(x)}{h(0)}$ where $h(x)$ is a positive and increasing solution to

$$\alpha h''(x) - xh'(x) - \beta h(x) = 0, \quad h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{A41}$$

$q(x)$ is not effected by letting $h(0) = 1$.

Assume $\tilde{\alpha} > \alpha$, let $\tilde{h}(x)$ satisfy the following differential equation

$$\tilde{\alpha}\tilde{h}''(x) - x\tilde{h}'(x) - \beta\tilde{h}(x) = 0, \quad \tilde{h}(-\infty) = 0, \quad \tilde{h}(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{A42}$$

We can show $\tilde{h}(x) > h(x)$ for all $x < 0$. Let

$$f(x) = \tilde{h}(x) - h(x). \tag{A43}$$

Then $f(-\infty) = f(0) = 0$ using Lemma 1. $f(x)$ has no local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, from the equations satisfied by $\tilde{h}(x)$ and $h(x)$, we have

$$[\tilde{\alpha}\tilde{h}''(x) - \alpha h''(x)] - x[\tilde{h}'(x) - h'(x)] - \beta[\tilde{h}(x) - h(x)] = 0. \tag{A44}$$

This equation implies that

$$\tilde{\alpha}\tilde{h}''(x^*) < \alpha h''(x^*). \tag{A45}$$

Since $\tilde{\alpha} > \alpha$, this further implies that $\tilde{h}''(x^*) < h''(x^*)$. This is equivalent to $f''(x^*) < 0$, which contradicts with x^* being a local minimum. Therefore, $f(x)$ cannot have a local minimum with its value less than zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x \in (-\infty, 0)$. Therefore, $\tilde{h}(x) > h(x)$ for all $x < 0$. This directly implies that the option value component $q(x)$ increases with α for all $x < 0$.

Assume $\bar{\beta} > \beta$, let $\bar{h}(x)$ satisfy the following differential equation

$$\alpha\bar{h}''(x) - x\bar{h}'(x) - \bar{\beta}\bar{h}(x) = 0, \quad \bar{h}(-\infty) = 0, \quad \bar{h}(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \tag{A46}$$

We can show $\bar{h}(x) < h(x)$ for all $x < 0$. Again let

$$f(x) = \bar{h}(x) - h(x). \tag{A47}$$

We first establish that $f(x)$ has no local maximum x^* with $f(x^*) > 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \leq 0$. On the other hand, from the equations satisfied by $\bar{h}(x)$ and $h(x)$, we have

$$\alpha[\bar{h}''(x) - h''(x)] - x[\bar{h}'(x) - h'(x)] - [\bar{\beta}\bar{h}(x) - \beta h(x)] = 0. \tag{A48}$$

This equation implies that

$$\bar{\beta}\bar{h}(x^*) < \beta h(x^*). \tag{A49}$$

Since $\bar{\beta} > \beta$, this further implies that $\bar{h}(x^*) < h(x^*)$. This is equivalent to $f(x^*) < 0$, which contradicts with $f(x^*) > 0$. Therefore, $f(x)$ cannot have a local maximum above zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay below zero for $x < 0$. This directly implies that $\bar{h}(x) < h(x)$ for all $x < 0$, and $q(x)$ decreases with β for all $x < 0$.

A.8 Proof to Proposition 4

By substituting the price function in equation (30) into the excess return, we have

$$dQ = \frac{d\hat{f}^o}{r+\lambda} + dq(x) + dD - \left[\bar{f} + \frac{r}{r+\lambda}(\hat{f}^o - \bar{f}) + rq(x)\right] dt. \tag{A50}$$

Without losing generality, we assume the asset owner is from group A, therefore $\hat{f}^o = \hat{f}^A$. From the perspective of the econometrician,

$$\begin{aligned}
dD &= \hat{f}^R dt + \sigma_D dW_D^R, && \text{(A51)}\\
d\hat{f}^A &= -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi^2\gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt) + \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt) \\
&= -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi^2\gamma}{\sigma_s^2}(\hat{f}^R dt + \sigma_s dW_A^R - \hat{f}^A dt) \\
&\quad + \frac{\gamma}{\sigma_s^2}(\hat{f}^R dt + \sigma_s dW_B^R - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(\hat{f}^R dt + \sigma_D dW_D^R - \hat{f}^A dt) \\
&= -\lambda(\hat{f}^A - \bar{f})dt - \gamma\left(\frac{1+\phi^2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)(\hat{f}^A - \hat{f}^R)dt \\
&\quad + \frac{\phi^2\gamma}{\sigma_s}dW_A^R + \frac{\gamma}{\sigma_s}dW_B^R + \frac{\gamma}{\sigma_D}dW_D^R. && \text{(A52)}
\end{aligned}$$

Therefore, we can derive the expected return as

$$
\begin{aligned}
\mathrm{E}^R[dQ] &= -\frac{\lambda}{r+\lambda}(\hat{f}^A - \bar{f})dt - \frac{\gamma}{r+\lambda}\left(\frac{1+\phi^2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)(\hat{f}^A - \hat{f}^R)dt \\
&\quad + \mathrm{E}^R[dq(x)] + \hat{f}^R dt - \left[\bar{f} + \frac{r}{r+\lambda}(\hat{f}^A - \bar{f}) + rq(x)\right] dt \\
&= -\left[1 + \frac{\gamma}{r+\lambda}\left(\frac{1+\phi^2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)\right](\hat{f}^A - \hat{f}^R)dt + \mathrm{E}^R[dq(x)] - rq(x)dt. \quad \text{(A53)}
\end{aligned}
$$

Since the difference of beliefs x follows the process in equation (25) from the perspective of the econometrician, $\mathrm{E}^R[dq(x)] = -\rho x q'(x)dt + \frac{1}{2}\sigma'^2_g q''(x)dt$. We also have $\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - rq(x) = 0$,

$$
\begin{aligned}
\mathrm{E}^R[dq(x)] - rq(x)dt &= \left[-\rho x q'(x) + \frac{1}{2}\sigma'^2_g q''(x) - rq(x)\right] dt \\
&= \frac{1}{2}\left(\sigma'^2_g - \sigma_g^2\right) q''(x)dt \quad \text{(A54)}
\end{aligned}
$$

Therefore,

$$
\mathrm{E}^R[dQ] = -\left[1 + \frac{\gamma}{r+\lambda}\left(\frac{1+\phi^2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)\right](\hat{f}^A - \hat{f}^R)dt + \frac{1}{2}\left(\sigma'^2_g - \sigma_g^2\right) q''(x)dt. \quad \text{(A55)}
$$

A.9 Proof to Proposition 5

Let

$$
l(k, c) = [k - c(r+\lambda)](h'(k) + h'(-k)) - h(k) + h(-k). \quad \text{(A56)}
$$

$k^*(c)$ is the root of $l(k, c) = 0$. If $c > 0$

$$
\frac{dk^*}{dc} = \frac{(r+\lambda)}{[k^* - c(r+\lambda)]} \frac{[h'(k^*) + h'(-k^*)]}{[h''(k^*) - h''(-k^*)]} > 0. \quad \text{(A57)}
$$

Hence $k^*(c)$ is differentiable in $(0, \infty)$. Now suppose $c_n \to 0$. The sequence $k^*(c_n)$ is bounded and every limit point $\bar{k}^*$ must solve $l(\bar{k}^*, 0) = 0$. Hence $\bar{k}^* = 0$ and the function $k^*(c)$ is continuous. Hence $\frac{dk^*}{dc}$ approaches ∞ as $c \to 0$. The claims on b and $q(x)$ follow from equations equations (43) and (44), and Lemma 1. The derivative of $\eta(x)$ with respect to c is

$$
\begin{aligned}
\frac{d\eta(x)}{dc} &= \frac{\sqrt{2}(\phi^2 - 1)\gamma}{\sigma_s(r+\lambda)} \frac{h'(x)(h''(k^*) - h''(-k^*))}{(h'(k^*) + h'(-k^*))^2}\left(-\frac{dk^*}{dc}\right) \\
&= -\frac{\sqrt{2}(\phi^2 - 1)\gamma h'(x)}{\sigma_s[k^* - c(r+\lambda)](h'(k^*) + h'(-k^*))} < 0. \quad \text{(A58)}
\end{aligned}
$$

Therefore, $\eta(x)$ decreases with c. However, note that $\frac{d\eta(x)}{dc}$ is finite as $c \to 0$ although $\frac{dk^*}{dc} \to \infty$ as $c \to 0$.

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