Dynamic Derivative Strategies

Jun Liu and Jun Pan*

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Abstract

This paper studies the optimal investment strategy of an investor who can access not only the bond and the stock markets, but also the derivatives market. We consider the investment situation where, in addition to the usual diffusive price shocks, the stock market experiences sudden price jumps and stochastic volatility. The dynamic portfolio problem involving derivatives is solved in closed-form. Our results show that derivatives are important in providing access to the risk and return tradeoffs associated with the volatility and jump risks. Moreover, as a vehicle to the volatility risk, derivatives are used by non-myopic investors to exploit the time-varying opportunity set; and as a vehicle to the jump risk, derivatives are used by investors to dis-entangle their simultaneous exposure to the diffusive and jump risks in the stock market. In addition, derivatives investing also affects investors' stock position because of the interaction between the two markets. Finally, calibrating our model to the S&P 500 index and options markets, we find sizable portfolio improvement for taking advantage of derivatives.

*Liu is with the Anderson School at UCLA, jliu@anderson.ucla.edu. Pan is with the MIT Sloan School of Management, junpan@mit.edu. We benefited from discussions with Darrell Duffie, and comments from Harrison Hong, Leonid Kogan, Andrew Lo, Alex Shapiro, and seminar participants at Duke and NYU.

1 Introduction

"Derivatives trading is now the world's biggest business, with an estimated daily turnover of over US$2.5 trillion and an annual growth rate of around 14%."[1] Despite increasing popularity, derivatives largely retain their image as vehicles for either speculation or hedging. Rarely are derivatives regarded as part of an optimal portfolio decision, and, more often than not, they are excluded by academic studies on asset allocation.

In a complete market setting [e.g., Black and Scholes (1973) and Cox and Ross (1976)], such an exclusion can very well be justified by the fact that derivative securities are redundant. When the completeness of the market breaks down — either because of infrequent trading or by the presence of additional sources of uncertainty — it then becomes suboptimal to exclude derivatives. Among others, the spanning role of derivatives has been studied extensively by Ross (1976), Breeden and Litzenberger (1978), Arditti and John (1980), and Green and Jarrow (1987) in static settings, and, more recently, by Bakshi and Madan (2000) in a dynamic setting. In an economy with multiple goods, Breeden (1984) shows that Pareto optimality can be achieved with continuous trading of finite numbers of futures contracts. In an information context, Brennan and Cao (1996) analyze the role of derivatives in improving trading opportunities. In a buy-and-hold environment, Haugh and Lo (2001) use derivatives to mimic the dynamic trading strategy of the underlying stock. Using historical stock data, Merton, Scholes, and Gladstein (1978, 1982) investigate the return characteristics of various option strategies, and more recently, using historical option data, Coval and Shumway (2001) provide a direct examination of option returns.

While such studies clearly indicate that investors can potentially benefit from the additional investment opportunities associated with derivatives, they do not provide any guidance on what the optimal investment strategy should be. Our paper is motivated by this important question. Focusing on the two most important empirical risk factors in the stock market — stochastic volatility and price jumps, we examine the optimal strategy and the value of derivatives trading in the presence of such additional risk factors.[2]

Consider the situation that, in addition to diffusive price shocks, the stock market experiences sudden, adverse jumps in prices. While the diffusive price shocks can be controlled via dynamic trading, the risk of jumps literally takes the control out of investors' hands. Because of this, an investor who is otherwise attracted by the risk and return tradeoff associated with the diffusive price shock might invest significantly less in the stock market [Liu, Longstaff, and Pan (2002)]. By including derivatives in his portfolio, however, the investor is able to dis-entangle the two risk factors. In particular, he is free to choose his desired exposure to each risk factor according to the associated risk and return tradeoff. The risk factor that gives rise to stochastic volatility is another such example of derivative securities acting as a vehicle to the otherwise inaccessible risk factors. Moreover, empirical studies indicate that such risk factors — the jump and volatility risks – are indeed priced in options on the aggregate market [Pan (2001)], and there are additional gains from participating in

[1]From *Building the Global Market: A 4000 Year History of Derivatives* by Edward J. Swan.

[2]In a related article, Carr, Jin, and Madan (2001) consider the optimal portfolio problem in a pure-jump setting by including as many options as the jump states. Ahn, Boudoukh, Richardson, and Whitelaw (1999) considers the role of options in a portfolio Value-at-Risk setting.

the options market.[3]

To formally investigate the potential benefit of investing in the derivatives market and the optimal investment strategy including derivatives, we focus on the dynamic asset allocation problem [Merton (1971)] of a power-utility investor whose investment opportunity includes not only the usual riskless bond and risky stock, but also derivatives on the stock. We specialize in a model with three types of risk factors: the diffusive price shocks, price jumps, and volatility risks. The risk and return tradeoff associated with each risk factor is controlled by a pricing kernel, which is also used to price derivative securities in this economy. We focus on derivatives whose time-t price, $O_t = g(S_t, V_t)$, depends on the risky stock price S_t and its volatility V_t through a non-linear function g. Prominent examples of this class of derivatives include the European-style call and put options, which are among the most popular derivatives in organized exchanges, and can be priced analytically in our setting via transform analysis [Heston (1993); Duffie, Pan, and Singleton (2000)]. Although derivatives with more complicated payoff structure can be adopted, this class of derivatives provides the cleanest intuition possible.

We solve the dynamic asset allocation problem in closed form. Taking advantage of the analytic nature of our solutions, we further establish explicit links between the demands for the risky assets and their economic sources. This is illustrated in two examples, one focuses on the volatility risk, and the other on the jump risk.

Focusing first on the volatility risk, we find that the optimal portfolio weight on derivatives depends explicitly on the sensitivity of the chosen derivative to the stock volatility. This is quite intuitive because in this setting the demand for derivatives arises from the need to access the volatility risk. A derivative with more "volatility exposure per dollar" is more effective as a vehicle to the volatility risk. Hence a smaller portion of the investor's wealth needs to be invested in this particular derivative.

The need to access the volatility risk arises from two economically different sources, corresponding to the myopic and non-myopic behavior of an investor. Acting myopically, the investor participates in the derivatives market simply to take advantage of the risk and return tradeoff provided by the volatility risk. For example, if the volatility risk is not priced at all, he would find no "myopic" incentive to take on derivative positions. On the other hand, a negatively priced volatility risk induces his to sell volatility by writing options. Acting non-myopically, the investor holds derivatives to further exploit the time-varying nature of his investment opportunity, which, in our setting, is driven exclusively by the stochastic volatility. As the volatility becomes more persistent, this non-myopic demand for derivatives becomes more prominent, and it also changes sharply around the investment horizon close to the half life of the volatility.

Given that both the myopic and non-myopic demands for volatility risk are taken care of by derivatives positions, the "net" demand for stocks should simply depend on how the price risk is compensated. The non-trivial interaction between the stock and the derivative security, however, affects the total demand for stocks. For example, by writing a call option, the investor implicitly sells a fraction — the "delta" of the call option — of the underlying. Even for a derivative that is delta-neutral, the negative correlation between the volatility

[3] For example, Coval and Shumway (2001) report that selling "zero-beta" at-the-money straddles on the S&P 500 index produces average returns of approximately 3% per week.

and price shocks[4] implies that a short position on the volatility automatically involves long positions on the underlying stock. To maintain his optimal "net" position on stocks, an investor would have to unload the stock positions taken unintentionally through the derivatives position.

To assess the portfolio improvement for participating in the derivatives market, we compare the certainty equivalent wealth of two utility-maximizing investors with and without access to the derivatives market.[5] To further quantify the gain from taking advantage of derivatives, we calibrate the parameters of the stochastic volatility model to those reported by empirical studies on the S&P 500 index and option markets. Our results show that the improvement for including derivatives is driven mostly by the risk and return tradeoff associated with the additional volatility risk. At normal market condition and with a conservative estimate of the volatility-risk premium, the improvement in certainty equivalent wealth for an investor with relative risk aversion of three is about 14% per year, which becomes higher when the market becomes more volatile. To further assess the improvement when transaction costs render continuous trading infeasible in either market, we consider the situation where the investor rebalances his portfolio over finite time intervals. Comparing buy-and-hold strategies with and without derivatives, our analysis indicates that there is still gain for including derivatives. For example, for an investor who rebalances only once a year, the improvement is roughly 12% per year in certainty equivalent wealth.

One important issue not addressed in our example on the volatility risk is the possibility of sudden, large price jumps in the stock market. As discussed in Liu, Longstaff, and Pan (2002), the presence of such jump risks takes away the investor's ability to continuously trade his way out of a leveraged position so as to avoid negative wealth. Preparing for the worst case scenario, investors impose constraints on their stock holdings. In such an environment, our results show that derivatives are important for two reasons. First, they alleviate the investor from such self-imposed constraints, allowing them to choose their exposure more freely. Second, they provide direct access to the risk and return associated with the jump risk. Both functions require the derivative securities to be able to dis-entangle the jump risk from the diffusive price risk. A derivative security that is effective in this task is one with high sensitivity to large price movements, but low sensitivity to infinitesimal price movements. For example, a deep out-of-the-money put option is such an instrument if large negative price jumps are the concern, just as a deep out-of-the-money call option is for large positive price jumps. Finally, the more effective the derivative security is in providing separate exposure to diffusive and jump risks, the less is needed to be invested in it.

The rest of the paper is organized as follows. Section 2 describes the investment environment including the risky stock and the derivative securities. Section 3 formalizes the investment problem and provides the explicit solutions. Section 4 provides an extensive example on the role of derivatives in the presence of volatility risk, while Section 5 focuses on jump risks. Section 6 concludes the paper. Technical details are provided in the appendices.

[4]This empirical fact is typically referred to as the leverage effect [Black (1976)].

[5]Admittedly, this analysis is a partial-equilibrium one, and is relevant only for an investor who takes price dynamics and the market prices of risks as given. To quantify the welfare improvement of the society as a whole, a general-equilibrium analysis will be more informative. See, for example, the literature on financial innovation [Allen and Gale (1994)].

2 The Model

We assume that there are two types of risky assets — the risky stock and its derivatives. In addition, there is one riskless bond paying a constant rate of interest r. The price process S of the risky stock is assumed as follows

$$dS_t = \left(r + \eta V_t + \mu\left(\lambda - \lambda^Q\right) V_t\right) S_t\, dt + \sqrt{V}_t S_t\, dB_t + \mu S_{t-}\left(dN_t - \lambda V_t\, dt\right) \tag{1}$$

$$dV_t = \kappa(\bar{v} - V_t)\, dt + \sigma\sqrt{V}_t\left(\rho\, dB_t + \sqrt{1-\rho^2}\, dZ_t\right), \tag{2}$$

where B and Z are standard Brownian motions, and N is a pure-jump process. All three random shocks B, Z, and N are assumed to be independent.

In addition to the usual diffusive price shock B, this model incorporates two risk factors that are important in characterizing the aggregate stock market: stochastic volatility and price jumps. Specifically, the instantaneous variance process V is a stochastic process with long-run mean $\bar{v} > 0$, mean-reversion rate $\kappa > 0$, and volatility coefficient $\sigma \geq 0$. This formulation of stochastic volatility, due to Heston (1993), also allows the diffusive price shock B to enter the volatility dynamics via the constant coefficient $\rho \in (-1, 1)$, introducing correlations between the price and volatility shocks, a feature that is important in the data. The random arrival of jump events is dictated by the pure-jump process N with stochastic arrival intensity $\{\lambda V_t : t \geq 0\}$ for constant $\lambda \geq 0$. Intuitively, the conditional probability at time t of another jump before $t + \Delta t$ is, for some small Δt, approximately $\lambda V_t\, \Delta t$. This formulation, due to Bates (2000), has the intuitive interpretation that jumps are more likely to occur during volatile markets. Following Cox and Ross (1976), we adopt deterministic jump amplitudes. That is, conditional on a jump arrival, the stock price jumps by a constant multiple of μ: negative jumps if $-1 < \mu < 0$, and positive jumps if $\mu > 0$. This formulation, though simple, is capable of capturing the sudden and high-impact nature of jumps that cannot be produced by diffusions.

In the presence of the additional risk factors, the market is no longer complete with respect to the bond and the stock. Consequently, there are infinitely many candidate pricing kernels for the purpose of derivative pricing. We specialize in the following parametric form not only for its flexibility in separately pricing all three risk factors in the economy, but also for its analytical tractability,

$$d\pi_t = -\pi_t\left(r\, dt + \eta\sqrt{V_t}\, dB_t + \xi\sqrt{V_t}\, dZ_t\right) + \left(\frac{\lambda^Q}{\lambda} - 1\right)\pi_{t-}\left(dN_t - \lambda V_t\, dt\right), \tag{3}$$

where the constant coefficients η and ξ control the premiums for the diffusive price risk B and the additional volatility risk Z, respectively, and where $\lambda^Q \geq 0$ is a constant coefficient closely related to the premium for the jump-timing uncertainty.[6] For example, in the presence of adverse jump events ($\mu < 0$), setting $\lambda^Q > \lambda$ implies that whenever the stock price jumps down, the pricing kernel π jumps up, resulting in a positive jump-risk premium.[7] Finally, to verify that π is indeed a valid pricing kernel, which rules out arbitrage opportunities

[6]It should be noted that $\lambda^Q = 0$ if and only if $\lambda = 0$.

[7]Alternatively, one can show that λ^Q is in fact the counterpart of λ under the risk-neutral measure

involving the risky stock and the riskless bond, one can apply Ito's lemma and show that $\pi_t \exp(-rt)$ and $\pi_t S_t$ are local martingales.[8]

We now proceed to introduce derivative securities to this economy. For simplicity, we focus on derivatives whose time-t price depends on the underlying stock price S_t and the stock volatility V_t through $O_t^{(i)} = g^{(i)}(S_t, V_t)$, where $g^{(i)}$ is the pricing formula for the i-th derivative security $O^{(i)}$. Although derivatives with more complicated payoff structure can be adopted in our setting, this class of derivatives provides the cleanest intuition possible. Moreover, prominent examples of this class of derivatives include the European-style call and put options, which are among the most popular derivatives in organized exchanges. For example, if the i-th derivative security is a European-style call option with maturity τ_i and strike price K_i, then $g^{(i)} = c(S_t, V_t\,;\, K_i, \tau_i)$, where the explicit functional form of c can be derived via transform analysis [Heston (1993); Duffie, Pan, and Singleton (2000)], and is provided in Appendix A for the completeness of the paper.

Given that the i-th derivative security $O^{(i)}$ is priced by the pricing kernel π, $\{\pi_t\, O_t^{(i)} : 0 \le t \le \tau\}$ must be a martingale. Straightforward applications of the Ito's lemma then show that the option price should satisfy the following stochastic differential equation

$$dO_t^{(i)} = r\, O_t^{(i)}\, dt + \left(g_s^{(i)}\, S_t + \sigma\rho\, g_v^{(i)}\right)\left(\eta\, V_t\, dt + \sqrt{V_t}\, dB_t\right) + \sigma\sqrt{1-\rho^2}\, g_v^{(i)}\, \left(\xi\, V_t\, dt + \sqrt{V_t}\, dZ_t\right)$$
$$+ \Delta g^{(i)}\left(\left(\lambda - \lambda^Q\right) V_t\, dt + dN_t - \lambda V_t\, dt\right), \tag{4}$$

where $g_s^{(i)}$ and $g_v^{(i)}$ measure the sensitivity of the i-th derivative price to infinitesimal changes in the stock price and volatility, respectively, and where $\Delta g^{(i)}$ measures the change in derivative price for each jump in the underlying stock price. Specifically,

$$g_s^{(i)} = \left.\frac{\partial g^{(i)}(s,v)}{\partial s}\right|_{(S_t,V_t)}\,;\; g_v^{(i)} = \left.\frac{\partial g^{(i)}(s,v)}{\partial v}\right|_{(S_t,V_t)}\,;\; \Delta g^{(i)} = g^{(i)}\left((1+\mu)\, S_t, V_t\right) - g^{(i)}\left(S_t, V_t\right)\,. \tag{5}$$

For the purpose of understanding what each derivative security can offer in terms of providing exposure to risk factors, these sensitivity measures play a very important role. Specifically, g_s is associated with the diffusive price shock B, g_v with the additional volatility risk Z, and Δg with the jump risk.

Associated with each risk exposure is the risk premium, as controlled by η, ξ, and λ^Q, respectively. Take the underlying stock as an example. Its sensitivity measures are $g_s = 1$, $g_v = 0$, and $\Delta g = \mu S$. From (4), we can see that it provides exposures to the diffusive price shock B and the jump risk N, but none to the additional volatility risk Z. In return for such exposures, the risk premiums are ηV_t for the diffusive price shock B, and $\mu\,(\lambda - \lambda^Q)V_t$ for the jump risk. In comparison with the full scale of tradeoffs offered by this economy, this set of risk and return tradeoffs provided by the underlying stock is rather limited. In particular, linear positions on the underlying stock are not capable of exploiting the risk and return

defined by the pricing kernel π. When coupled with risk aversion, the arrival of adverse jump events ($\mu < 0$) is perceived to be more frequent. Hence $\lambda^Q > \lambda$ when $\mu < 0$, resulting in a positive jump-risk premium. See Naik and Lee (1990) for an example how such jump-risk premiums might arise in equilibrium.

[8]See, for example, Appendix B.2 of Pan (2000).

tradeoff associated with the volatility risk, nor are they capable of exploiting the tradeoff associated with the jump risk separately from that associated with the diffusive price shock.[9]

In this sense, introducing non-linear instruments such as derivatives is important for achieving the optimal risk and return tradeoff. In our setting, two additional non-redundant derivatives will be sufficient to complete the market, providing the desired exposure to all three risk factors.

3 The Investment Problem and the Solution

The investor starts with a positive wealth W_0. Given the opportunity to invest in the riskless asset, the risky stock and the derivative securities, he chooses, at each time t, $0 \le t \le T$, to invest a fraction ϕ_t of his wealth in the stock S_t, fractions $\psi_t^{(1)}$ and $\psi_t^{(2)}$ in the two derivative securities $O_t^{(1)}$ and $O_t^{(2)}$, respectively. The investment objective is to maximize the expected utility of his terminal wealth W_T,[10]

$$\max_{\phi_t,\, \psi_t,\, 0 \le t \le T} E\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right), \tag{6}$$

where $\gamma > 0$ is the relative risk-aversion coefficient of the investor, and where the wealth process satisfies the self-financing condition

$$\begin{aligned} dW_t = {} & r\, W_t\, dt + \theta_t^B\, W_t \left(\eta\, V_t\, dt + \sqrt{V_t}\, dB_t\right) + \theta_t^Z\, W_t \left(\xi\, V_t\, dt + \sqrt{V_t}\, dZ_t\right) \\ & + \theta_{t-}^N\, W_{t-}\, \mu \left(\left(\lambda - \lambda^Q\right) V_t\, dt + dN_t - \lambda V_t\, dt \right), \end{aligned} \tag{7}$$

where, for given portfolio weights ϕ_t and ψ_t on the stock and the derivatives, the θ's defined in (8) are effectively portfolio weights on the risk factors:

$$\begin{aligned} \theta_t^B &= \phi_t + \sum_{i=1}^{2} \psi_t^{(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho\, \frac{g_v^{(i)}}{O_t^{(i)}} \right); \; \theta_t^Z = \sigma\sqrt{1-\rho^2} \sum_{i=1}^{2} \psi_t^{(i)}\, \frac{g_v^{(i)}}{O_t^{(i)}}; \\ \theta_t^N &= \phi_t + \sum_{i=1}^{2} \psi_t^{(i)}\, \frac{\Delta g^{(i)}}{\mu\, O_t^{(i)}}. \end{aligned} \tag{8}$$

Specifically, by taking positions ϕ_t and ψ_t on the risky assets, the investor effectively invests θ^B on the diffusive price shock B, θ^Z on the additional volatility risk Z, and θ^N on the jump risk N. For example, a portfolio position ϕ_t on the risky stock provides equal exposures to both the diffusive and jump risks in stock prices. Similarly, a portfolio position ψ_t on the derivative security provides exposure to the volatility risk Z via a non-zero g_v, exposure to

[9]Empirically, there is strong evidence indicating that, at the aggregate market level, both the jump and the volatility risks are indeed priced. Moreover, the market price of jump risks differ qualitatively from that for diffusive risks. See, for example, Pan (2001), and references therein.

[10]Although the model could be extended to allow for intermediate consumption, we use this simpler specification to focus more directly on the intuition behind the results.

the diffusive price shock B via a non-zero g_s, and exposure to the jump risk via a non-zero Δg.

Except for adding derivative securities in the investor's opportunity set, the investment problem in (6) and (7) is the standard Merton (1971) problem. Before solving for this problem, we should point out that the maturities of the chosen derivatives do not have to match the investment horizon T. For example, it might be hard for an investor with a 10-year investment horizon to find an option with a matching maturity. He may choose to invest in options with much shorter time to expiration, say LEAPS, which typically expires in one or two years, and switch or roll over to other derivatives in the future. For the purpose of choosing the optimal portfolio weights at time t, what matters is his choice of derivative securities O_t at that time, not his future choice of derivatives. This is true as long as, at each point in time in the future, there exist non-redundant derivative securities to complete the market.

We now proceed to provide solutions to our investment problem with derivatives.[11] Following Merton (1971), we define the indirect utility function by

$$J(t,w,v) = \max_{\{\phi_s,\,\psi_s,\,t\le s\le T\}} E\left(\frac{W_T^{1-\gamma}}{1-\gamma} \,\middle|\, W_t = w, V_t = v \right), \tag{9}$$

which, by the principle of optimal stochastic control, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \max_{\phi_t\,,\,\psi_t} \Bigg\{ & J_t + W_t J_W \left(r_t + \theta^B \eta V_t + \theta_t^Z \xi V_t - \theta^N \mu \lambda^Q V_t \right) + \frac{1}{2} W_t^2 J_{WW} V_t \left(\left(\theta^B\right)^2 + \left(\theta^Z\right)^2 \right) \\ & \lambda V_t \, \Delta J + \kappa \left(\bar{v} - V_t \right) J_V + \frac{1}{2}\sigma^2 V_t J_{VV} + \sigma V_t W_t J_{WV} \left(\rho\, \theta^B + \sqrt{1-\rho^2}\, \theta_t^Z \right) \Bigg\} = 0\,, \end{aligned} \tag{10}$$

where $\Delta J = J(t, W_t(1+\theta^N\mu), V_t) - J(t, W_t, V_t)$ denotes the jump in the indirect utility function J for given jumps in the stock price, and where J_t, J_W, and J_V denote the derivatives of $J(t,W,V)$ with respect to t, W and V respectively, and similar notations for higher derivatives.

To solve the HJB equation, we notice that it depends explicitly on the portfolio weights θ^B, θ^Z, and θ^N, which, as defined in (8), are linear transformations of the portfolio weights ϕ and ψ on the risky assets. Taking advantage of this structure, we first solve the optimal positions on the risk factors B, Z, and N, and then transform them back via the linear relation (8) to the optimal positions on the risky assets. This transformation is feasible as long as the chosen derivatives are non-redundant in the following sense.

Definition: *At any time t, the derivative securities $O_t^{(1)}$ and $O_t^{(2)}$ are non-redundant if*

$$\mathcal{D}_t \neq 0 \quad where \quad \mathcal{D}_t = \left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \right) \frac{g_v^{(2)}}{O_t^{(2)}} - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}} \right) \frac{g_v^{(1)}}{O_t^{(1)}} \tag{11}$$

[11] Alternative to the stochastic control approach adopted here is the Martingale approach of Cox and Huang (1989). The results are the same. Details are available upon request.

Effectively, the non-redundancy condition in (11) guarantees market completeness with respect to the chosen derivative securities, the risky stock, and the riskless bond. Without access to derivatives, linear positions on the risky stock provide equal exposures to the diffusive and jump risks, and none to the volatility risk. To complete the market with respect to the volatility risk, we need to bring in a risky asset that is sensitive to changes in volatility: $g_v \neq 0$. To complete the market with respect to the jump risk, we need a risky asset with different sensitivities to the *infinitesimal* and *large* changes in stock prices: $g_s S_t / O_t \neq \Delta g / \mu O_t$. Moreover, (11) also ensures that the two chosen derivative securities are not identical in covering the two risk factors.

Proposition 1 *Assume that there are non-redundant derivatives available for trade at any time $t < T$. Then, for given wealth W_t and volatility V_t, the solution to the HJB equation is given by*

$$J(t, W_t, V_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left(\gamma\, h(T-t) + \gamma\, H(T-t)\; V_t\right), \tag{12}$$

where $h(\cdot)$ and $H(\cdot)$ are time-dependent coefficients that are independent of the state variables:

$$\begin{aligned} h(t) &= \frac{2\kappa\bar{v}}{\sigma^2} \ln\left(\frac{2\, k_2 \exp\left((k_1+k_2)\, t/2\right)}{2k_2 + (k_1+k_2)\left(\exp(k_2\, t) - 1\right)}\right) + \frac{1-\gamma}{\gamma} r\, t \\ H(t) &= \frac{\exp(k_2\, t) - 1}{2k_2 + (k_1+k_2)\left(\exp(k_2\, t) - 1\right)}\, \delta \end{aligned} \tag{13}$$

where

$$\begin{aligned} \delta &= \frac{1-\gamma}{\gamma^2}\left(\eta^2 + \xi^2\right) + 2\lambda^Q \left[\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1 - \frac{\lambda}{\lambda^Q}\right) - 1\right] \\ k_1 &= \kappa - \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2}\right)\sigma\,; \quad k_2 = \sqrt{k_1^2 - \delta\,\sigma^2} \end{aligned}$$

The optimal portfolio weights on the risk factors B, Z, and N are given by

$$\theta_t^{*\,B} = \frac{\eta}{\gamma} + \sigma\rho\, H(t)\,; \quad \theta_t^{*\,Z} = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2}\, H(t)\,; \quad \theta_t^{*\,N} = \frac{1}{\mu}\left(\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} - 1\right). \tag{14}$$

Transforming the θ^*'s to the optimal portfolio weights on the risky assets, ϕ_t^* for the stock and ψ_t^* for derivatives, we have

$$\begin{aligned} \phi_t^* &= \theta_t^{*B} - \sum_{i=1}^{2} \psi_t^{*\,(i)} \left(\frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho\, \frac{g_v^{(i)}}{O_t^{(i)}}\right) \\ \psi^{*\,(1)} &= \frac{1}{\mathcal{D}_t}\left[\frac{g_v^{(2)}}{O_t^{(2)}}\left(\theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z}\,\rho}{\sqrt{1-\rho^2}}\right) - \left(\frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}}\right)\frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}}\right] \\ \psi^{*\,(2)} &= \frac{1}{\mathcal{D}_t}\left[\left(\frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}}\right)\frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}} - \frac{g_v^{(1)}}{O_t^{(1)}}\left(\theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z}\,\rho}{\sqrt{1-\rho^2}}\right)\right]. \end{aligned} \tag{15}$$

Proof: *See Appendix.*

To further illustrate our results, we consider two examples in the next two sections, one on volatility risks and the other on jump risks.

4 Example I: Derivatives and Volatility Risk

This section focuses on the role of derivative securities as a vehicle to stochastic volatility. For this, we specialize in an economy with volatility risk but no jump risk. Specifically, we turn off the jump component in (1) and (2) by letting $\mu = 0$ and $\lambda = \lambda^Q = 0$.

In such a setting, only one derivative security with non-zero sensitivity to volatility risk is needed to help complete the market. Denoting this derivative security by O_t, we can readily use the result of Proposition 1 to derive the optimal portfolio weights:

$$\phi_t^* = \frac{\eta}{\gamma} - \frac{\xi\rho}{\gamma\sqrt{1-\rho^2}} - \psi_t^* \, \frac{g_s \, S_t}{O_t} \tag{16}$$

$$\psi_t^* = \left(\frac{\xi}{\gamma\sigma\sqrt{1-\rho^2}} + H\left(T-t\right) \right) \frac{O_t}{g_v} \, , \tag{17}$$

where ϕ_t^* and ψ_t^* denote the optimal positions on the risky stock and the derivative security, respectively, and where H is as defined in (13) with the simplifying restriction of no jumps.

4.1 The Demand for Derivatives

The optimal derivatives position ψ^* in (17) is inversely proportional to g_v/O_t, which measures the volatility exposure for each dollar invested in the derivative security. Intuitively, the demand for derivatives arises, in this setting, from the need to access the volatility risk. The more "volatility exposure per dollar" a derivative security provides, the more effective it is as a vehicle to the volatility risk. Hence a smaller portion of the investor's wealth needs to be invested in this derivative security.

The demand for derivatives — or the need for volatility exposures — arises for two economically different reasons. First, a myopic investor finds the derivative security attractive because, as a vehicle to the volatility risk, it could potentially expand his dimension of risk and return tradeoffs. This myopic demand for derivatives is reflected in the first term of ψ_t^*. For example, a negatively priced volatility risk ($\xi < 0$) makes short positions on volatility attractive, inducing investors to sell derivatives with positive "volatility exposure per dollar." Similarly, a positive volatility-risk premium ($\xi > 0$) induces opposite trading strategies. Moreover, the less risk-averse investor is more aggressive in taking advantage of the risk and return tradeoff through investing in derivatives.

Second, for an investor who acts non-myopically, there is benefit in derivative investments even when the myopic demand diminishes with a zero volatility risk-premium ($\xi = 0$). This non-myopic demand for derivatives is reflected in the second term of ψ_t^*. Without any loss of generality, let's consider an option whose volatility exposure is positive ($g_v > 0$). In our setting, the Sharpe ratio of the option return is driven exclusively by the stochastic volatility. In fact, it is proportional to the volatility. This implies that a higher realized option return at one instant is associated with a higher Sharpe ratio (better risk-return tradeoff) for the next-instant option return. In other words, a good outcome is more likely to be followed by another good outcome. By the same token, a bad outcome in the option return predicts a sequence of less attractive future risk-return tradeoffs. An investor with relative risk aversion $\gamma > 1$ is particularly averse to sequences of negative outcomes because his utility is unbounded from below. On the other hand, an investor with $\gamma < 1$ benefits from sequences of positive outcomes because his utility is unbounded from above. As a result, they act quite differently in response to this temporal uncertainty. The one with $\gamma > 1$ takes a short position on volatility so as to hedge against the temporal uncertainty, while the one with $\gamma < 1$ takes a long position on volatility so as to speculate on the temporal uncertainty. Indeed, it is easy to verify that $H(T-t)$, which is the driving force of this nonmyopic term, is strictly positive for investors with $\gamma < 1$, and strictly negative for investor with $\gamma > 1$, and zero for the log-utility investor.[12]

4.2 The Demand for Stock

Given that the volatility risk exposure is taken care of by the derivative holding, the "net" demand for stock should simply be linked to the risk and return tradeoff associated with the price risk. Focusing on the first term of ϕ_t^* in (16), this is indeed true. Specifically, it is proportional to the attractiveness of the stock and inversely proportional to the investor's risk aversion.

The interaction between the derivative security and its underlying stock, however, complicates the optimal demand for stocks. For example, by holding a call option, one effectively invests a fraction g_s — typically referred to as the "delta" of the option — on the underlying stock. The last term in ϕ^* is there to correct this "delta" effect. Even for a delta-neutral derivative security ($g_s = 0$), the negative correlation between the volatility and price shocks, typically referred to as the leverage effect [Black (1976)], implies that a short position on the volatility automatically involves long positions on the price shock, and, equivalently, the underlying stock. The second term in ϕ^* is there to correct this "correlation" effect.

4.3 Empirical Properties of the Optimal Strategies

To examine the empirical properties of our results, we fix a set of base-case parameters for our current model, using the results from the existing empirical studies.[13] Specifically, for the one-factor volatility risk, we set its long-run mean at $\bar{v} = (0.13)^2$, its rate of mean-reversion

[12] One way to show this is by taking advantage of the ordinary differential equation (B.4) for $H(\cdot)$ with the additional constraints of no jumps. Given the initial condition $H(0) = 0$, it is easy to see that the driving force for the sign of H is the constant term which has the same sign as $1 - \gamma$.

[13] The empirical properties of the Heston (1993) model have been extensively examined using either the time-series data on the S&P 500 index alone [Andersen, Benzoni, and Lund (2001); Eraker, Johannes, and Polson (2000)], or the joint time-series data on the S&P 500 index and options [Chernov and Ghysels (2000); Pan (2001)]. Because of different sample periods or/and empirical approaches in these studies, the exact model estimates may differ from one paper to another. Our chosen model parameters are in the generally agreed region, with the exception of those reported by Chernov and Ghysels (2000).

at $\kappa = 5$, and its volatility coefficient at $\sigma = 0.25$. The correlation between the price and volatility risks is set at $\rho = -0.40$.

Important for our analysis is how the risk factors are priced. Given the well established empirical property of the equity risk premium, calibrating the market price of the Brownian shocks B is straightforward. Specifically, setting $\eta = 4$ and coupling it with the base-case value of $\bar{v} = (0.13)^2$ for the long-run mean of volatility, we have an average equity risk premium of 6.76% per year.

The properties of the market price of the volatility risk, however, are not as well established. In part because that volatility is not a directly tradable asset, there is less consensus on reasonable values for market prices of volatility risk.[14] Empirically, however, there is strong support that volatility risk is indeed priced. For example, using the joint time-series data on the S&P 500 index and options, Chernov and Ghysels (2000), Pan (2001), Benzoni (1998), and Bakshi and Kapadia (2001) report that volatility risks are negatively priced. That is, short positions on volatility are compensated with a positive premium. Similarly, Coval and Shumway (2001) report large negative returns generated by positions that are long on volatility.

Given that the volatility risk at the aggregate level is generally related to the economic activity [Officer (1973); Schwert (1989)], it is quite plausible that it is priced. At an intuitive level, the negative volatility risk premium could be supported by the fact that the aggregate market volatility is typically high during recessions. A short position on volatility, which loses value when volatility becomes high during recessions, is therefore relatively more risky than a long position on volatility, requiring an additional risk premium.

Instead of calibrating the volatility-risk premium coefficient ξ to the existing empirical results, however, we will allow this coefficient to vary in our analysis so as to get a better understanding of how different levels and signs of the volatility risk premium could affect the optimal investment decision.

Using this set of base-case parameters, particularly the risk-and-return tradeoff implied by the data, we now proceed to provide some quantitative examples of optimal investments in the markets of S&P 500 index and options. To make the intuition as clean as possible, we focus on "delta-neutral" securities. Specifically, we consider the following "delta-neutral" straddle:

$$O_t = g\,(S_t, V_t\,;\, K, \tau) = c\,(S_t, V_t\,;\, K, \tau) + p\,(S_t, V_t\,;\, K, \tau)\ , \tag{18}$$

where c and p are pricing formulas for call and put options with the same strike price K and time to expiration τ. The explicit formulation of c and p is provided in Appendix A. For given stock price S_t, market volatility V_t, and time to expiration τ, the strike price K is selected so that the call option has a delta of 0.5, and, by put/call parity, the put option has a delta of -0.5, making the straddle delta neutral.[15]

[14] If it is believed that there are bounds on how attractive an investment opportunity should be — be it the Sharpe ratio of the investment opportunity or the maximum gain-loss ratio — constraints can be imposed on the pricing kernel, *a la* Cochrane and Saá-Requejo (2000) or Bernardo and Ledoit (2000), which in turn have a direct implication on the values for market prices of risk.

[15] Although "delta-neutral" positions can be constructed in numerous ways, we choose the "delta-neutral" straddle mainly because it is made of call and put options are typically very close to the money. In particular, we intentionally avoid deep out-of-the-money options in our quantitative examples because they are most

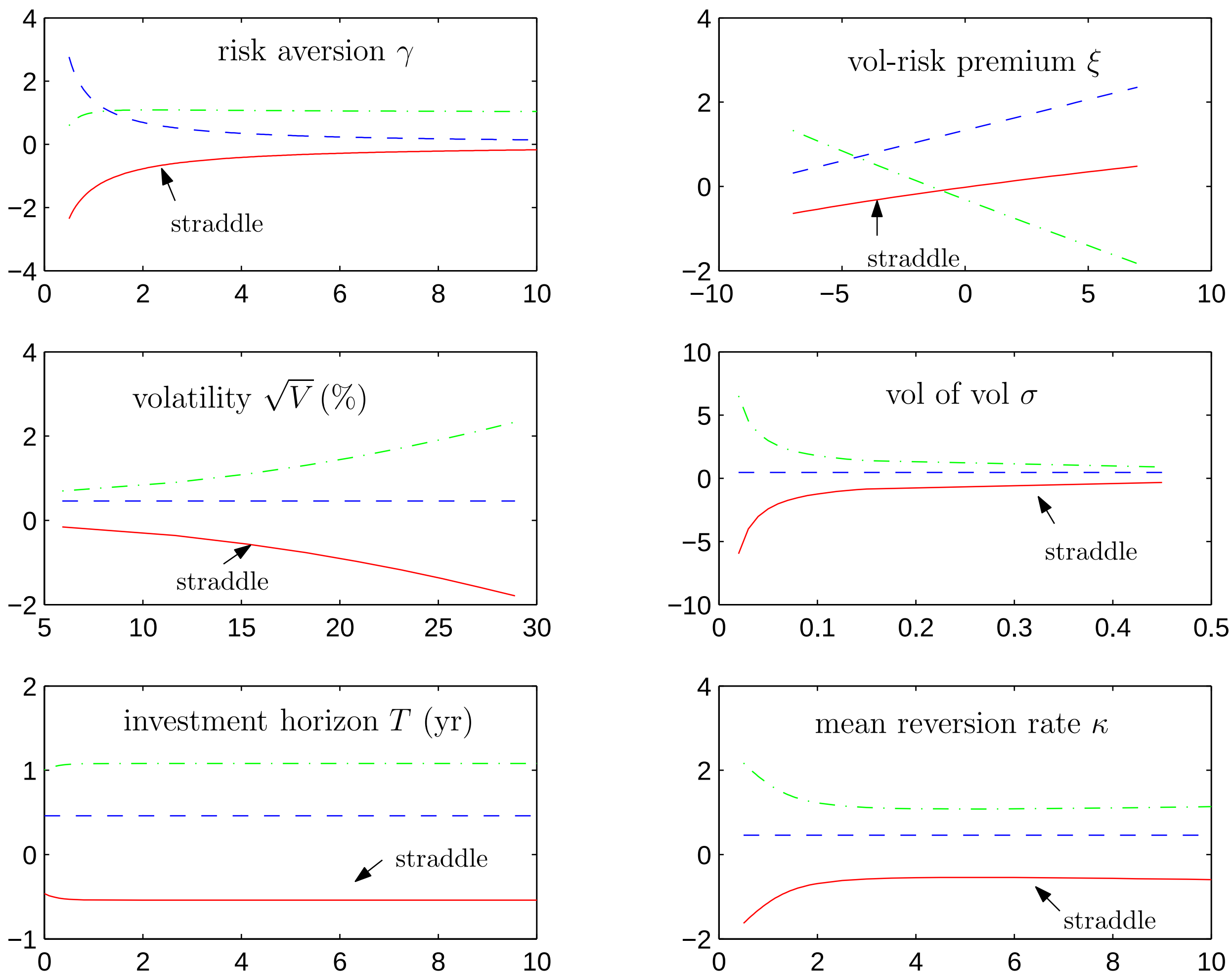

Figure 1: The optimal portfolio weights. The y-axes are the optimal weight ψ^* on the "delta-neutral" straddle (solid line), ϕ^* on the risky stock (dashed line), and $1 - \psi^* - \phi^*$ on the riskfree bank account (dashed-dot line). The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at $\xi = -6$. The base-case investor is the one with risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. The riskfree rate is fixed at $r = 5\%$, and the base-case market volatility is fixed at $\sqrt{V} = 15\%$.

Fixing riskfree rate at 5%, and picking a delta-neutral straddle with 0.1 year to expiration, Figure 1 provides optimal portfolio weights under different scenarios. The top-right panel examines the optimal portfolio allocation with varying volatility-risk premium. Qualitatively, this result is similar to our analysis in Section 4.1. Quantitatively, however, this result indicates that the demand for derivatives is driven mainly by the myopic component. In particular, when the volatility-risk premium is set to zero ($\xi = 0$), the non-myopic demand for straddles is only 2% of the total wealth for an investor with relative risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. In contrast, as we set $\xi = -6$, which is a conservative estimate for the volatility-risk premium, the optimal portfolio weight in the delta-neutral straddle increases to 54% for the same investor.

The quantitative effect of the non-myopic component can be best seen by varying the investment horizon (bottom left panel), or the volatility persistence (bottom right panel). Consider an investor with $\gamma = 3$, who would like to hedge against temporal uncertainty by taking short positions on volatility. The bottom left panel shows that as we increase his investment horizon, this intertemporal hedging demand increases. And, quite intuitively, the change is most noticeable around the region close to the half life of the volatility risk. Similarly, the bottom right panel shows that as we decrease the volatility persistency by increasing the mean-reversion rate κ, there is less benefit in taking advantage of the intertemporal persistence. Hence a reduction in the intertemporal hedging demand.

As the market becomes more volatile, the cost of straddle (O_t) increases, but the volatility sensitivity (g_v) of such straddles decreases. Effectively the delta-neutral straddles provide less "volatility exposure per dollar" as the market volatility increases. To achieve the optimal volatility exposure, more needs to be invested in the straddle. Hence the increase in $|\psi^*|$ with the market volatility $\sqrt{V}$. As the volatility of the volatility increases, the risk and return tradeoff on the volatility risk becomes less attractive. Hence the decrease in magnitude of the straddle position with increasing "vol of vol" σ. Finally, the optimal strategy with varying risk aversion γ is as expected: less risk-averse investors are more aggressive in their investment strategies.

4.4 Portfolio Improvement

Consider an investor with an initial wealth of W_0 and an investment horizon of T years. If he takes advantage of the derivatives market, his optimal expected utility is as provided in Proposition 1 (with the simplifying restriction of no jumps). For a given market volatility of V_0, his certainty equivalent wealth $\mathcal{W}^*$ is[16]

$$\mathcal{W}^* = W_0 \exp\left(\frac{\gamma}{1-\gamma}\Big[h\,(T) + H\,(T)\; V_0\Big]\right)\,, \tag{19}$$

where, again, the time-varying coefficients h and H are as defined in (13) with the simplifying constraint of no jumps. Alternatively, this investor might choose not to participate in

subject to concerns of option liquidity and jump risks, two important issues that are not accommodated formally in this section.

[16]It should be noted that the optimal expected utility is independent of the specific derivative contract chosen by the investor. This is quite intuitive, because, in our setting, the market is complete in the presence of the derivative security.

the derivatives market. Let $\mathcal{W}^{\text{no-op}}$ be the certainty equivalent wealth of such an investor who chooses not to invest in options. To quantify the portfolio improvement for including derivatives, we adopt the following measure[17]

$$\mathcal{R}^{\mathcal{W}} = \frac{\ln \mathcal{W}^* - \ln \mathcal{W}^{\text{no-op}}}{T} . \tag{20}$$

Effectively, $\mathcal{R}^{\mathcal{W}}$ measures the portfolio improvement in terms of the annualized, continuously compounded return in certainty equivalent wealth. The following Proposition summarizes the results.

Proposition 2 *For a power-utility investor with risk aversion coefficient $\gamma > 0$ and investment horizon T, the improvement for including derivatives is*

$$\mathcal{R}^{\mathcal{W}} = \frac{\gamma}{1-\gamma} \left(\frac{h(T) - h^{\textit{no-op}}(T)}{T} + \frac{H(T) - H^{\textit{no-op}}(T)}{T} V_0 \right) , \tag{21}$$

where V_0 is the initial market volatility, and $h^{\textit{no-op}}$ and $H^{\textit{no-op}}$ are defined in (C.6). For an investor with $\gamma \neq 1$, the portfolio improvement for including derivatives is strictly positive. For an investor with log utility, the improvement is strictly positive if $\xi \neq 0$, and zero otherwise.

Proof: See Appendix C.

Intuitively, there will be improvement for including derivatives if and only if the demand for derivatives is non-zero. For a myopic investor such as the one with log-utility, the demand for derivatives arises from the need to exploit the risk and return tradeoff provided by the volatility risk. When the volatility-risk premium is set to zero ($\xi = 0$), the myopic demand for derivatives diminishes, so does the benefit for including derivatives. There are, however, still non-myopic demands for derivatives. Hence the strict portfolio improvement for a nonmyopic investor.

To provide a quantitative assessment of the portfolio improvement, we again use the base-case parameters described in Section 4.3. The results are summarized in Figure 2. Focusing first on the top-right panel, we see that the portfolio improvement is very sensitive to how the volatility risk is priced. At normal market condition with a conservative estimate[18] of the volatility-risk premium $\xi = -6$, our results show that the portfolio improvement for including derivatives is about 14.2% per year in certainty equivalent wealth for an investor with risk aversion $\gamma = 3$. As the investor becomes less risk averse and more aggressive

[17] The indirect utility of the "no-option" investor can be derived using the results from Liu (1998). For the completeness of the paper, is provided in Appendix C.

[18] For example, Coval and Shumway (2001) report that zero-beta at-the-money straddle positions produce average losses of approximately 3% per week. This number roughly corresponds to $\xi = -12$. Using volatility-risk premium to explain the premium implicit in option prices, Pan (2001) reports a total volatility-risk premium that translates to $\xi = -23$. This level of volatility-risk premium, however, could be over-stated due to the absence of jump and jump-risk premium in the model. In fact, after introducing jumps and estimating jump-risk premium simultaneously with volatility-risk premium, Pan (2001) reports a volatility-risk premium that translates to $\xi = -10$.

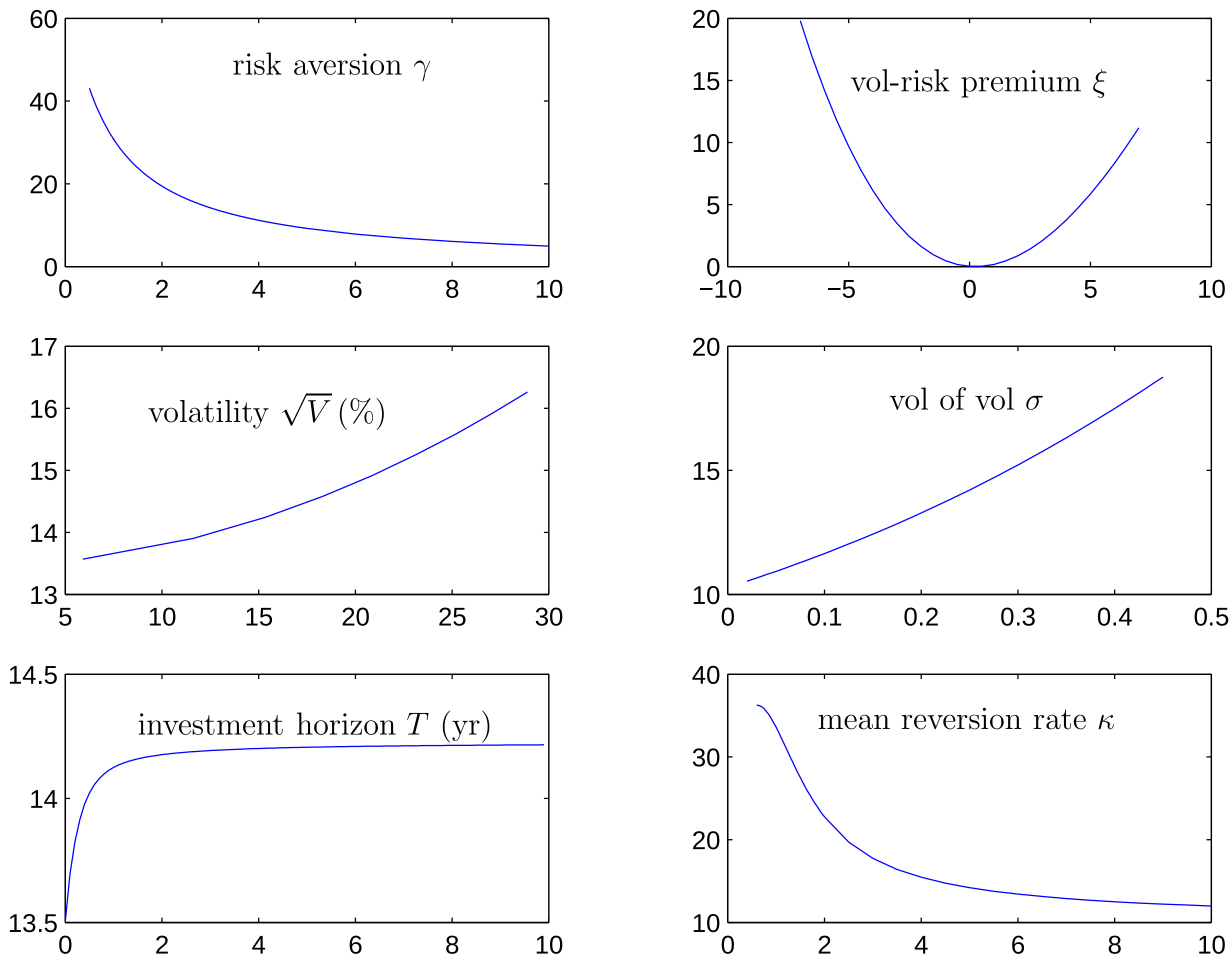

Figure 2: Portfolio improvement for including derivatives. The y-axes are the improvement measure $\mathcal{R}^W$, defined by (20) in terms of returns over certainty equivalent wealth. The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at $\xi = -6$. The base-case investor is the one with risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. The riskfree rate is fixed at $r = 5\%$, and the base-case market volatility is fixed at $\sqrt{V} = 15\%$.

in taking advantage of the derivatives market, the improvement for including derivatives becomes even higher (top left panel).

We can further evaluate the relative importance of the myopic and nonmyopic components of portfolio improvement by setting $\xi = 0$. The portfolio improvement from non-myopic trading of derivatives is as low as 0.02% per year. This is consistent with our earlier result: the demand for derivatives is driven mostly by the myopic component. The non-myopic component of the portfolio improvement is further examined in the bottom panels of Figure 2 as we vary the investment horizon and the persistence of volatility. Quite intuitively, as the investment horizon T increases, or, as the volatility shock becomes more persistent, the benefit of the derivative security as a hedge against temporal uncertainty becomes more pronounced. Hence the increase in portfolio improvement. Finally, from the middle two panels, we can also see that when the market volatility $\sqrt{V}$ increases, or when the volatility of volatility increases, there is more to be gained from investing in the derivatives market.

One natural question that arises from this analysis is how much of the improvement will remain when transaction cost renders dynamic trading infeasible in either the stock or the derivatives markets. To have a rough assessment of the portfolio improvement in such a situation, we consider an investor who, faced with non-zero transaction costs, decides to rebalance his portfolio over some fixed time intervals T. We assess his portfolio improvement by comparing buy-and-hold strategies *with* and *without* derivatives. We focus on his myopic behavior because, as shown in our earlier results, the portfolio improvement for including derivatives originates mainly from the myopic component.

When T is small, the n-th moment of the asset returns is proportional to the n-th power of T. Taylor-expanding the utility function up to the first power of T, one can show that, if he takes advantage of the derivatives market, his certainty equivalent wealth is

$$1 + \frac{1}{2\gamma}\left(\eta^2 + \xi^2\right) V_0 \, T \, .$$

On the other hand, if he decides to ignore the derivatives market and invest only in the bond and stock markets, his certainty equivalent wealth is

$$1 + \frac{1}{2\gamma}\, \eta^2 \, V_0 \, T \, .$$

This result implies that as long as the volatility risk is systematic, the portfolio improvement for including derivatives is strictly positive, even when non-zero transaction costs render continuous rebalancing infeasible. Calibrating this result to the same set of data, the portfolio improvement $\mathcal{R}^{\mathcal{W}}$ is 12% per year when rebalanced once a year, 12.7% per year when rebalanced twice a year, and 13.1% per year when rebalanced four times a year. Quite intuitively, as we rebalance more and more frequently, we reach the case of continuous rebalancing.[19]

5 Example II: Derivatives and Jumps

In contrast to the diffusive risks, which can be controlled via continuous trading, the sudden, high-impact nature of jump risks takes away the investor's ability to continuously trade his

[19] To be more precise, the myopic component of the continuous rebalancing case.

way out of a leveraged position to avoid negative wealth. As shown in Liu, Longstaff, and Pan (2002), self-imposed portfolio bounds arise in the presence of jump risks. That is, when being blindsided by things that they couldn't control, investors adopt investment strategies that prepare for the worst-case scenarios.

In this section, we examine the role of derivative securities in alleviating the constraint imposed by the jump risk. To be more concrete, we specialize in an economy with jump risk but no volatility risk. Specifically, we turn of the stochastic-volatility component in (1) and (2) by letting $V_0 = \bar{v}$ and $\sigma = 0$. That is, $V_t = \bar{v}$ at any time t. The resulting model contains two components: the Brownian price shock with constant volatility $\sqrt{\bar{v}}$, and the pure jump with Poison arrival $\lambda\bar{v}$ and deterministic jump size μ. In the absence of either risk factor, derivative securities are redundant since the market can be completed by dynamic trading of the stock and bond [Black and Scholes (1973) and Cox and Ross (1976)]. Their simultaneous presence is what makes derivative securities valuable. For example, an investor might be attracted by the risk and return tradeoff associated with the diffusive price risk. By taking a position on the risky stock, however, he is exposed to both the diffusive and jump risks. He can, nevertheless, use derivative securities to dis-entangle the two risk factors. For this to work, the derivative security must have different sensitivities to infinitesimal price movements dS and large price movements ΔS:

$$\frac{g_s S_t}{O_t} \neq \frac{\Delta g}{\mu O_t}$$

Out-of-the-money put and call options are examples of such derivative security with high sensitivity to large price movements, but low sensitivity to infinitesimal price movements.

Given the existence of such a derivative security, we can use the result of Proposition 1 to derive the optimal portfolio weight:

$$\phi_t^* = \frac{\eta}{\gamma} - \psi_t^* \frac{g_s S_t}{O_t} \tag{22}$$

$$\psi_t^* = \left(\frac{\Delta g}{\mu O_t} - \frac{g_s S_t}{O_t} \right)^{-1} \left(\frac{1}{\mu} \left[\left(\frac{\lambda}{\lambda^Q} \right)^{1/\gamma} - 1 \right] - \frac{\eta}{\gamma} \right) , \tag{23}$$

where ϕ_t^* and ψ_t^* are the optimal portfolio weights on the risky stock and the derivative security, respectively.

Focusing first on the optimal portfolio weight ϕ^* on the stock, the first term in (22) indicates the optimal investment in stocks depends on the investor's appetite for the diffusive price shock and the associated risk premium (controlled by η). In particular, with access to the derivative security, the optimal stock position ϕ^* is no longer subject to the portfolio bounds reported in Liu, Longstaff, and Pan (2002). Anticipating the optimal position ψ^* on the derivative security, the second term in (22) is to correct for the "delta" exposure introduced by the derivative security.

Focusing next on the optimal portfolio weight ψ^* on the derivative security. Evident in (23) is the role of derivative securities in separating the jump risk from the diffusive price risk. In particular, the demand for the derivative security is inversely proportional to its ability to dis-entangle the two. Quite intuitively, the more effective a derivative security is in providing the separate exposure, the less is needed to be invested in this derivative security. Also evident in (23) is the role of derivative securities in providing access to the risk and return tradeoff associated with the jump risk. Specifically, ψ^* is proportional to the disparity in the risk and return tradeoffs associated with the diffusive and jump risks. That is, if the jump risk is perceived to be better compensated than the diffusive price shocks, there will be more demand for derivatives.

6 Conclusion

In this paper, we studied the optimal investment strategy of an investor who can access not only the bond and the stock markets, but also the derivatives market. Our results demonstrate the importance of including derivative securities as an integrated part of the optimal portfolio decision. The analytical nature of our solutions also helps establish direct links between the demand for derivatives and their economic sources.

As a vehicle to the additional risk factors such as stochastic volatility and price jumps in the stock market, derivative securities play an important role in expanding the investor's dimension of risk and return tradeoffs. In addition, by providing access to the volatility risk, derivatives are used by non-myopic investors to take advantage of the time-varying nature of their opportunity set. Similarly, by providing acccess to the jump risk, derivatives are used by investors to dis-entangle their simultaneous exposure to the diffusive and jump risks in the stock market. Moreover, because of the non-trivial interaction between the derivatives market and its underlying stock market, derivatives positions further complicate the optimal portfolio position in the underlying stock.

Although our analysis focuses on volatility and jump risks, our intuition can be readily extended to other risk factors that are not accessible through linear positions on stocks. The risk factor that gives rise to stochastic predictor is such an example. If, in fact, there are derivatives providing access to such additional risk factors, then demands for the related derivatives will arise from the need to take advantage of the associated risk and return tradeoff, as well as the time-varying investment opportunity provided by such risk factors.

By focusing on the investment opportunity provided by derivative securities, this paper also raised an important question that has yet to be fully examined: What are the reasonable values for the market price of such additional risk factors? While empirically there is strong support indicating that these risk factors are indeed priced in the aggregate market, our theoretical understanding of this subject is still limited. In particular, while it is easy to include such risk factors in the pricing kernel (as we did in this paper), it remains an open question as to why they are in the pricing kernel,[20] and what types of restrictions are associated with their presence.[21] The importance of these questions naturally arises as we start to treat derivatives as an integrated part of the optimal portfolio decision.

[20]For example, in the setting of Campbell and Cochrane (1999), the time-varying risk aversion of an investor gives rise to stochastic volatility, which in turn finds its position in the pricing kernel.

[21]See, for example, Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000) for constraints on the pricing kernel via some intuitive criteria, and their impact on the market prices of the risk factors that affect derivatives pricing.

Appendices

A Option-Pricing Formulas

Option pricing for the stochastic-volatility model adopted in this paper is well established by Heston (1993). Using the notation established in Section 2, and letting $\kappa^* = \kappa - \sigma\left(\rho\,\eta + \sqrt{1-\rho}\,\xi\right)$ and $\bar{v}^* = \kappa\bar{v}/\kappa^*$ be the risk-neutral mean reversion rate and long-run mean, respectively, the time-t prices of European-style call and put options with time τ to expiration and striking at K are

$$C_t = c\left(S_t, V_t\,;\,K, \tau\right)\,; \quad P_t = p\left(S_t, V_t\,;\,K, \tau\right)\,, \tag{A.1}$$

where S_t is the spot price and V_t is the market volatility at time t, and where

$$c\left(S, V\,;\,K, \tau\right) = S\,\mathcal{P}_1 - e^{-r\,\tau} K\,\mathcal{P}_2\,,$$

and, by put/call parity, the put pricing formula is

$$p\left(S, V\,;\,K, \tau\right) = e^{-r\,\tau} K\,\left(1 - \mathcal{P}_2\right) - S\,\left(1 - \mathcal{P}_1\right)\,.$$

Very much like the case of Black and Scholes (1973), $\mathcal{P}_1$ measures the probability of the call option expiring in the money, while $\mathcal{P}_2$ is the adjusted probability of the same event. Specifically,

$$\begin{aligned}
\mathcal{P}_1 &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{du}{u}\,\mathrm{Im}\Big(e^{A(1-iu)+B(1-iu)\,V}\,e^{iu(\ln K - \ln S + r\tau)}\Big)\\
\mathcal{P}_2 &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{du}{u}\,\mathrm{Im}\Big(e^{A(-iu)+B(-iu)\,V}\,e^{iu(\ln K - \ln S + r\tau)}\Big)
\end{aligned} \tag{A.2}$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary component of a complex number, and where, for any $y \in \mathbb{C}$,

$$\begin{aligned}
B(y) &= -\,\frac{a\left(1 - \exp(-qt)\right)}{2q - (q+b)\left(1 - \exp(-qt)\right)}\\
A(y) &= -\,\frac{\kappa^*\bar{v}^*}{\sigma^2}\left((q+b)\,\tau + 2\ln\left[1 - \frac{q+b}{2q}\,\left(1 - e^{-q\tau}\right)\right]\right)
\end{aligned} \tag{A.3}$$

where $b = \sigma\rho y - \kappa^*$, $a = y(1-y) - 2\lambda^Q\left(\exp(y)(1+\mu) - 1 - y\mu\right)$ and $q = \sqrt{b^2 + a\sigma^2}$.

Connecting to the notation $O_t = g(S_t, V_t)$ adopted in Section 2, we can see that for a call option, g is simply c, while for a straddle, $g(S_t, V_t) = c\left(S_t, V_t\,;\,K, \tau\right) + p\left(S_t, V_t\,;\,K, \tau\right)$.

B A Proof of Proposition 1

The proof is a standard application of the stochastic control method. Suppose that the indirect utility function J exists, and is of the conjectured form in (12), then the first order condition of the HJB Equation (10) implies that the optimal portfolio weights ϕ^* and ψ^* are indeed as given by (16) and (17), respectively.

Substituting (12), (16), and (17) into the HJB equation (10), one can show that the conjectured form (12) for the indirect utility function J indeed satisfies the HJB equation (10) if the following ordinary differential equations are satisfied

$$\begin{aligned}\frac{dh(t)}{dt} &= \kappa\bar{v}\, H(t) + \frac{1-\gamma}{\gamma}\, r, \\ \frac{dH(t)}{dt} &= \left(-\kappa + \frac{1-\gamma}{\gamma}\left(\eta\rho + \xi\sqrt{1-\rho^2}\right)\sigma\right) H(t) + \frac{\sigma^2}{2} H(t)^2 + \frac{1-\gamma}{2\gamma^2}\left(\eta^2+\xi^2\right) \\ &\quad + \lambda^Q\left[\left(\frac{\lambda}{\lambda^Q}\right)^{1/\gamma} + \frac{1}{\gamma}\left(1-\frac{\lambda}{\lambda^Q}\right) - 1\right]\end{aligned} \tag{B.4}$$

Using the solutions provided in (13) for H and h, it is a straightforward calculation to verify that this is indeed true. ∎

C Appendix to Section 4

A "no-option" investor solves the same investment problem as that in (6) and (7) with the additional constraint that $\beta_t \equiv 0$. This problem is solved extensively in Liu (1998). For completeness of the paper, the following summarizes the results useful for our analysis of portfolio improvement in Section 4.4.

At any time t, the indirect utility of a "no-option" investor with a T-year investment horizon is

$$J^{\text{no-op}}\left(W_t, V_t, t\right) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp\left(\gamma\, h^{\text{no-op}}\left(T-t\right) + \gamma\, H^{\text{no-op}}\left(T-t\right)\, V_t\right)\,, \tag{C.5}$$

where $h^{\text{no-op}}(\cdot)$ and $H^{\text{no-op}}(\cdot)$ are time-dependent coefficients that are independent of the state variables:

$$\begin{aligned}h^{\text{no-op}}\left(t\right) &= \frac{2\kappa\bar{v}}{\sigma^2\left(\rho^2+\gamma(1-\rho^2)\right)}\ln\left(\frac{2k_2\exp\left(\left(k_1+k_2\right)t/2\right)}{2k_2+\left(k_1+k_2\right)\left(\exp(k_2\,t)-1\right)}\right) + \frac{1-\gamma}{\gamma}\, r\, t \\ H^{\text{no-op}}\left(t\right) &= \frac{\exp(k_2\,t)-1}{2k_2+\left(k_1+k_2\right)\left(\exp(k_2\,t)-1\right)}\,\frac{1-\gamma}{\gamma^2}\eta^2\end{aligned} \tag{C.6}$$

where

$$k_1 = \kappa - \frac{1-\gamma}{\gamma}\eta\sigma\rho\,; \quad k_2 = \sqrt{k_1^2 - \frac{1-\gamma}{\gamma^2}\eta^2\sigma^2\left(\rho^2+(1-\rho^2)\gamma\right)}\,. \tag{C.7}$$

The certainty equivalent wealth of such a "no-option" investor with initial wealth W_0 then becomes

$$\mathcal{W}^{\text{no-op}} = W_0\,\exp\left(\frac{\gamma}{1-\gamma}\left[h^{\text{no-op}}\left(T\right) + H^{\text{no-op}}\left(T\right)\, V_0\right]\right)\,. \tag{C.8}$$

The proof of Proposition 2 followings immediately from this result and that of Proposition 1.

Proof of Proposition 2: It is straightforward to verify that the portfolio improvement $\mathcal{R}^W$ is indeed of the form (21). To show that the improvement is strictly positive for investors with $\gamma \neq 1$, let $DH(t) = H(t) - H^{\text{no-op}}(t)$, and one can show that

$$DH(t) = \frac{1-\gamma}{2} \exp(-y(t)) \int_t^T \exp(-y(s)) \left(\frac{\xi}{\gamma} - \sqrt{1-\rho^2}\sigma \, H^{\text{no-op}}(s) \right)^2 ds \, ,$$

where

$$y(t) = \int_t^T \left[\kappa + \frac{1-\gamma}{\gamma} \left(\eta\rho + \xi\sqrt{1-\rho^2}\sigma \right) + \frac{\sigma^2}{2} \left(H(s) + H^{\text{no-op}}(s) \right) \right] ds$$

is finite for any $t \leq T$. Consequently, $DH(T)/(1-\gamma)$ is strictly positive. Moreover, it is straightforward to show that

$$\frac{Dh(t)}{1-\gamma} = \frac{h(t) - h^{\text{no-op}}(t)}{1-\gamma} = \kappa\bar{v} \int_0^t \frac{DH(s)}{1-\gamma} \, ds \, . \tag{C.9}$$

As a result, $Dh(T)/(1-\gamma)$ is also strictly positive, making $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$ for any $\gamma \neq 1$.

For the log-utility case, the intertemporal hedging demand is zero. That is, $H(t) = 0$ and $H^{\text{no-op}}(t) = 0$ for any t. One can show that

$$\lim_{\gamma \to 1} \frac{H^{\text{no-op}}(t)}{1-\gamma} = \frac{1-\exp(-\kappa t)}{2\kappa}\eta^2 \, ; \quad \lim_{\gamma \to 1} \frac{H(t)}{1-\gamma} = \frac{1-\exp(-\kappa t)}{2\kappa} \left(\eta^2 + \xi^2 \right)$$

Moreover, (C.9) also holds for the case of $\gamma = 1$, making $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$ when $\xi \neq 0$, and $\mathcal{W}^* = \mathcal{W}^{\text{no-op}}$ when $\xi = 0$. ▮

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