A Central-Planning Approach to Dynamic Incomplete-Market Equilibrium*

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Abstract

We show that a central planner with *two selves,* or two "pseudo welfare functions", are sufficient to deliver the market equilibrium that prevails among any (finite) number of heterogeneous individual agents acting competitively in an incomplete financial market. Furthermore, we are able to exhibit a *recursive formulation* of the two-central planner problem. In that formulation, every aspect of the economy can be derived one step at a time, by a process of backward induction as in dynamic programming.

Dynamic asset pricing increasingly considers models with incomplete markets and heterogeneity in an attempt to improve over the empirical performance of benchmark complete-market representative-agent models. Numerous authors have pointed out the difficulties faced when solving these models.[1]

In this paper, we aim to find a technique for computing an equilibrium in an incomplete financial market, that is less onerous than the fixed-point tâtonnement process. The tâtonnement process presents the major drawback that a stochastic process for securities prices must be postulated *ab initio* to start the procedure of obtaining optimal portfolios. The trial-and-error procedure would wander in a vast space of stochastic processes. It is a hopeless undertaking.

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[1]Early examples includes Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998) and Marcet and Singleton (1999). Levine and Zame (2001) show that market incompleteness is unimportant in the absence of aggregate risk. In the present paper we incorporate aggregate risk.

Direct calculation of equilibrium makes sense only in special cases in which the equilibrium has some properties that are known *a priori.*

Like Cuoco and He (1994), our line of attack of this problem is to use a representative-agent concept, where the representative agent utility is defined as a stochastically weighted average of individual utilities. In Cuoco and He, the way in which the representative agent is defined over time is derived separately on the basis of individual financial choices, based on the dual approach of He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991). These individual financial choices involve as many value functions (interpreted as each person's financial wealth) as there are individuals in the economy.[2]

Below, we show that a central planner with *two selves,* or two "pseudo welfare functions", are sufficient to deliver the market equilibrium that prevails among any (finite) number of heterogeneous individual agents. The first self solves for individual consumption decisions and individual-specific components of state prices, taking the economy-wide components of state prices as given. Simultaneously, the second self chooses individual consumption rules and equilibrium state prices (i.e. such that the aggregate resource restriction is satisfied) taking as given the individual-specific components of the state prices. In an equilibrium of this game, the two selves agree and the competitive equilibrium is found.

In a complete-market setting, competitive equilibrium with heterogeneous agents is typically obtained by virtue of the Pareto optimality of the competitive equilibrium. Solving a Planner problem, which is the sum of individual utilities weighted by Pareto weights, gives the equilibrium allocation so that one can price assets off the marginal rate of substitution of this constructed representative agent. This approach dates back to Negishi (1960) and was used in, for instance, Constantinides (1982) and Dumas (1989). The definition of the two pseudo welfare functions we propose is, however, not based on a claim that the competitive equilibrium in an incomplete financial market is constrained Pareto optimal. Indeed, Magill and Quinzii (1996, Chapter IV) have a simple counter-example showing that this claim is not true. Nonetheless, our approach is reminiscent of the work of Grossmann (1977) who shows that the market equilibrium has some welfare properties from the vantage point of a central planner who would act as several incompletely coordinated selves.[3]

Taking our method one step further, we are able to exhibit a *recursive formulation* of the two-central planner problem. In that formulation, every aspect of the economy can be derived one step at a time, by a process of backward induction. Dynamic programming is used but, in principle, the dynamic program involves two value functions which are solved for simultaneously. The only state variables needed to summarize the distribution of resources across individuals are the current values of the individual-specific components of the state prices.

[2] Barbachan (2001) extends Cuoco and He (1994) from the case of two individual agents to more than two.

[3] This was done in the context of a multi-good economy, whereas we employ here a multi-period economy. But it is well-known that analogous problems occur in both settings. In Grossmann (1977), the planner has as many selves as there are states of nature plus one, whereas our planner has only two selves.

Many economists were tempted to believe that no recursive formulation of an incomplete market equilibrium was possible.[4] Cuoco and He, however, write a system of partial differential equations – one for each individual – which is applicable to the exchange economy that we consider here. That large system of PDEs could be solved backward in a recursive fashion. We expect, however, that our two-planner algorithm is general enough to be applied later to more complex settings.

From the technical point of view, the recent paper by Harris and Laibson (2001) is also related to our work in that it shows that some decision problems plagued by time inconsistency – considered before them as being outside the reach of recursive techniques – can be formulated recursively provided that the decision maker is split into several selves which play a Nash game with each other.

Throughout this paper, we assume that the incomplete-market equilibrium exists. Hart (1975) has exhibited a well-known counter-example showing that equilibrium may fail to exist. It involves a situation in which the rank of the rate-of-return matrix drops in some states of nature. Fortunately, Duffie and Shafer (1986) have shown that this occurs for a negligible subset of economies.

At any rate, existence is not the topic of our paper. Our paper is useful only to calculate equilibria after someone has shown that they exist.

Not only do we take it for granted that competitive-market equilibrium exists but, for most of the paper, and for the entire theoretical part of the paper, *we take as given the variance-covariance matrix of equilibrium rates of return*,[5] for which we assume that it remains of constant rank N at all times with probability one. In a numerical illustration, however, we explain the way in which that matrix could be determined endogenously within an extended procedure, which would still be recursive. At this point, we cannot be sure that the extended procedure delivers the equilibrium when the variance-covariance matrix is endogenous. That issue is left for future research.

The balance of the paper is organized as follows. Section 1 describes the economy that we study. Section 2 reminds the reader of the dual formulation of the portfolio choice problem in He and Pearson (1991) and Karatzas et al. (1991), and gives the definition of the corresponding equilibrium. Section 3 presents a "simultaneous-game" formulation of the game played by the two selves of the central planner and shows that the extent to which the equilibrium of the game replicates the market equilibrium. Section 4 shows that the equi-

[4] See Judd, Kubler and Schmedders (2000) for the case of one (risky) asset. Kubler and Schmedders (2002) analyze the existence of recursive equilibria with minimal sufficient state spaces and construct a counter-example where the current exogenous state variables along with the wealth distribution across agents do not constitute sufficient state variables, even though the fundamentals of the economy are Markovian. Krebs (2001) shows the non-existence of recursive or Markov equilibria in infinite-horizon incomplete-market exchange economies without aggregate risk for general preferences (and with aggregate risk for homothetic preferences) when the wealth distribution is taken as a state variable of which decisions and value functions are to be continuous functions. In this paper however, we suggest a different set of state variables and obtain recursivity, but we do not claim continuity.

[5] More precisely, we take as given the diffusion matrix of securities prices.

librium in the Magill-Quinzii example is indeed a solution of the static game between the planners. Section 5 presents a recursive, or dynamic-game formulation of the same problem. This will be most useful for purposes of numerical implementation, since dynamic programming can be used. Section 6 presents an example of an actual numerical implementation that illustrates also how the variance-covariance of returns could be endogenized. Section 7 contains the conclusion.

1 The economy

1.1 Information and technical assumptions

The economy that we consider evolves over a finite interval $[0, T]$ of the real line. $(\Omega, \mathcal{F}, P)$ is a probability space endowed with a filtration $\mathbf{F}$. $w(\omega, t)$ is a K-dimensional Wiener $((\Omega, \mathcal{F}, P) \times [0, T] \to \mathbb{R}^K)$ relative to the given filtration where the components are independent of each other.

We define a filtration $\mathbf{F}^w$ which is the filtration generated by the Wiener w.

Definition 1 $\mathcal{L}^1$ *space: the set of adapted, measurable processes* b *such that for every* T*:*[6]

$$\int_0^T \|b\| \, dt < \infty \qquad \textit{with probability one} \tag{1}$$

Definition 2 $\mathcal{L}^2$ *is the space of adapted, measurable processes* b *such that for every* T*:*[7]

$$\int_0^T \|b\|^2 \, dt < \infty \qquad \textit{with probability one} \tag{2}$$

Assumption: In what follows, all processes for which an Itô stochastic integral is written are assumed to belong to the space $\mathcal{L}^2$. All processes for which an integral over time is written are assumed to belong to the space $\mathcal{L}^1$.

1.2 Individuals and endowments

We consider an exchange economy with one good. There is a large but finite number I of individuals who trade competitively in the financial market. They are indexed by i. They are endowed at time 0 with a stock of good $F_i(0)$ (initial wealth on hand or "financial wealth") and they receive over time a flow endowment $e_i(t)$ following a given Itô stochastic process which takes strictly positive values. That process belongs to $\mathcal{L}^1_+$. Their consumption process is denoted c_i and their utility functions are time additive: $E\left\{\int_0^T u_i(c_i(s), s)\, ds | \mathbf{F}^w_0\right\}$ with $u(.,.)$ satisfying the Inada conditions. As a result, the process c_i belongs to

[6] If a is a scalar, $\|a\|$ is absolute value. If a is $N \times 1$ dimensional, then a is in $\mathcal{L}^1$ if and only if each of its components is in $\mathcal{L}^1$.

[7] If a is matrix-valued, then $\|a\|^2 = tr\left(aa^\intercal\right)$.

$\mathcal{L}^1_+$. All individuals have the same information set, viz. the one provided by the filtration $\mathbf{F}^w$.

The process $[c_i(t) - e_i(t)]$ is called "net consumption". We are only interested in equilibria in which $\sum_i F_i(0) = 0$.

Definition 3 *The* aggregate resource restriction *is:*

$$\sum_i [c_i(t) - e_i(t)] = 0, \ \forall t \in [0, T], \ \textit{with probability 1} \tag{3}$$

1.3 Financial assets

There are $N+1$ securities one of which is instantaneously riskless. The N-dimensional Itô process for the "dividends" $\iota(s)$ is given. Individuals can choose to invest in these assets but, this being an exchange economy, the total net supply of each asset is equal to zero. Calling $[\alpha_i, \theta_i]$ the portfolio choice process of individual i, where $\alpha_i(t)$ is the number of units of the riskless asset and $\theta_i(t)$ is the vector containing the number of units of all the risky assets held by individual i at time t,

Definition 4 *The* market clearing condition *is:*

$$\begin{aligned} \sum_i \alpha_i(t) &= 0, \ \forall t \in [0, T], \ \textit{with probability 1} && (4) \\ \sum_i \theta_i(t) &= 0, \ \forall t \in [0, T], \textit{with probability 1} && (5) \end{aligned}$$

2 The static formulation: equilibrium

We now write down the formulation of an equilibrium in the financial market. In order to reach equilibrium, individuals have to choose their portfolios $[\alpha_i(t), \theta_i(t)]$. In order to choose their portfolio, they have to postulate a stochastic process for financial market prices. The central planning formulation, which comes later, presents the major advantage that there is no need to postulate such a stochastic process.

2.1 Financial market prices

The stochastic process for price is assumed to be an Itô process, denoted as follows:

$$\begin{aligned} B(t) &= B(0) e^{\int_0^t r(s) ds}; B(0) = 1 && (6) \\ S(t) + \int_0^t \iota(s) ds &= S(0) + \int_0^t \zeta(s) ds + \int_0^t \sigma(s) dw(s); S(0) = 1 && (7) \end{aligned}$$

$S(\omega, t)$ is a process in $\mathbb{R}_+^N$ $(N < K)$. At the individual level, the optimization problem to be solved is:

$$\sup_{c_i(s),\alpha_i(s),\theta_i(s)} E\left\{\int_0^T u_i\left(c_i\left(s\right), s\right) ds | \mathbf{F}_0^w\right\} \tag{8}$$

subject to:

$$\alpha_i\left(0\right) B\left(0\right) + \theta_i\left(0\right)^\intercal S\left(0\right) = F_i\left(0\right) \tag{9}$$

$$\begin{aligned} \alpha_i\left(t\right) B\left(t\right) + \theta_i\left(t\right)^\intercal S\left(t\right) + \int_0^t \left[c_i\left(s\right) - e_i\left(s\right)\right] ds = \\ \alpha_i\left(0\right) B\left(0\right) + \theta_i\left(0\right)^\intercal S\left(0\right) + \int_0^t \left[\alpha_i\left(s\right) B\left(s\right) r\left(s\right) + \theta_i\left(s\right)^\intercal \zeta\left(s\right)\right] ds \\ + \int_0^t \theta_i\left(s\right)^\intercal \sigma\left(s\right) dw\left(s\right); \forall t \in \left]0, T\right[\text{ with probability } 1 \end{aligned} \tag{10}$$

$$\alpha_i\left(T\right) B\left(T\right) + \theta_i\left(T\right)^\intercal S\left(T\right) = 0 \tag{11}$$

and subject to (6, 7). As the market is incomplete, the matrix σ has fewer rows than columns.

Definition 5 *A net-consumption plan $[c_i(t) - e_i(t)]$ is said to be marketable from $F_i(0)$ if there exist stochastic processes $[\alpha_i(t), \theta_i(t)]$ such that Equations (9, 10, 11) are satisfied with probability one.*

Obviously, the sum of two marketable plans is a marketable plan.

Lemma 6 *If $\sum_i F_i(0) = 0$ and the market clearing condition is satisfied, then the aggregate resource restriction is satisfied.*

Lemma 7 *If $\sum_i F_i(0) = 0$, the aggregate resource restriction is satisfied and $[c_i(t) - e_i(t)]$ is marketable from $F_i(0)$ for all i, then the market clearing condition is satisfied.*

2.2 Minimax Individual Consumption Choice

We define an adapted process κ in $\mathbb{R}^K$ such that:[8]

$$\sigma\left(t\right) \kappa\left(t\right) = \left[\zeta\left(t\right) - r\left(t\right) \times S\left(t\right)\right] \tag{13}$$

We define three scalar Itô processes ξ, η_i^κ and Z_i^{-1} where ν_i is an Itô process in $\mathcal{L}^2$, as yet unspecified:

$$\xi\left(0, t\right) \triangleq \exp\left\{-\int_0^t r\left(s\right) ds - \frac{1}{2}\int_0^t \left\|\kappa\left(s\right)\right\|^2 ds - \int_0^t \kappa\left(s\right)^\intercal dw\left(s\right)\right\} \tag{14}$$

[8] There are many processes satisfying that restriction. One such process is:

$$\kappa\left(t\right) \triangleq \sigma\left(t\right)^\intercal \left[\sigma\left(t\right) \sigma\left(t\right)^\intercal\right]^{-1} \left[\zeta\left(t\right) - r\left(t\right) \times S\left(t\right)\right] \tag{12}$$

In that case, κ is in the span of $\sigma^\intercal$: $\kappa(t) = \sigma(t)^\intercal x(t)$ where $x(t) = [\sigma(t)\sigma(t)^\intercal]^{-1}[\zeta(t) - r(t) \times S(t)]$. This is the process selected by He and Pearson.

$$\begin{aligned}
& \eta_i^\kappa (0,t) && (15) \\
\triangleq\ & \exp\left\{-\frac{1}{2}\int_0^t \|\nu_i(s)\|^2 ds - \int_0^t \kappa(s)^\intercal \nu_i(s)\,ds - \int_0^t \nu_i(s)^\intercal dw(s)\right\} \\
i\ =\ & 1,..I
\end{aligned}$$

$$Z_i^{-1}(t) \triangleq Z_i^{-1}(0)\,\xi(0,t)\,\eta_i^\kappa(0,t) \tag{16}$$

$Z_i^{-1}(0)$ will be given a meaning very shortly. These processes satisfy the following stochastic differential equations:

$$\frac{d\xi(t)}{\xi(t)} = -r(t)\,dt - \kappa(t)^\intercal dw(t)\,;\xi(0)=1 \tag{17}$$

$$\frac{d\eta_i^\kappa(t)}{\eta_i^\kappa(t)} = -\kappa(t)^\intercal \nu_i(t)\,dt - \nu_i(t)^\intercal dw(t)\,;\eta_i^\kappa(0)=1 \tag{18}$$

$$\frac{dZ_i^{-1}(t)}{Z_i^{-1}(t)} = -r(t)\,dt - [\kappa(t)+\nu_i(t)]^\intercal dw(t) \tag{19}$$

The restriction (13) on κ guarantees that:

$$E\left[S(t)\,\xi(0,t) + \int_0^t \iota(s)\,\xi(0,s)\,ds|\mathbf{F}_0^w\right] = S(0) = 1 \tag{20}$$

$$E\left[B(t)\,\xi(0,t)\,|\mathbf{F}_0^w\right] = B(0) = 1 \tag{21}$$

Lemma 8 *For as long as* $\nu_i \in \ker\sigma$ *(i.e.* $\sigma\nu_i = 0$*) and* $c_i(t) - e_i(t)$ *is marketable at all times with probability one, we have:*

$$E\left[\int_0^T [c_i(s) - e_i(s)]\,\xi(0,s)\,\eta_i^\kappa(0,s)\,ds|\mathbf{F}_0^w\right] = F_i(0) \tag{22}$$

Proof. This can be verified by direct application of Itô's lemma to (10) multiplied by (14) and (15). ■

One might reformulate the optimization problem as one of maximizing:

$$\sup_{c_i(s)} E\left\{\int_0^T u_i(c_i(s),s)\,ds|\mathbf{F}_0^w\right\} \tag{23}$$

subject to (22). However, that leaves the solution indeterminate for as long as we have not specified ν_i. A duality reasoning would show that the choice of ν_i

must be dictated by:[9]

$$\inf_{\nu_i(s)\in\ker\sigma(s)} \sup_{c_i(s)} E\left\{\int_0^T u_i\left(c_i\left(s\right),s\right)ds|\mathbf{F}_0^w\right\} \tag{26}$$

subject to (22). We call $Z_i^{-1}(0)$ the Lagrange multiplier of that constraint written at time 0. This is the main result of the dual approach of He and Pearson (1991) (Theorems 1, 2 and 3) and Karatzas et al. (1991), which is the extension to incomplete markets of the martingale methodology of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) and Pliska (1986).

The legitimacy of this procedure is established by the following lemma which shows that the solution of the dual problem is, indeed, the solution of the primal problem.

Lemma 9 *For any given ξ process, if a solution to problem (26) exists and technical conditions are satisfied, individual i optimizing (26) subject to (22) chooses net trades $[c_i(s)-e_i(s)]$ that are marketable from $F_i(0)$.*

Proof. See He and Pearson (1991) proof of Theorem 2, pages 292-295, which is applicable in the absence of an endowment stream. He and Pearson had selected a process κ in the span of $\sigma^\intercal$. That restriction is immaterial. If κ is not in the span, it can always be decomposed: $\kappa=\widehat{\kappa}+\widehat{\nu}$, where $\widehat{\kappa}$ is in the span of $\sigma^\intercal$ and $\widehat{\nu}$ is in the kernel of σ. The restriction $\sigma\nu=0$ is equivalent to the restriction $\sigma(\nu-\widehat{\nu})=0$. ■

The technical conditions just referred to in the Lemma are quite important. For the existence of a solution to the dual and primal problem and technical conditions guaranteeing the equivalence between the solution to the dual and the solution to the primal problem in the absence of intermediate endowments and consumption, see He and Pearson (1991) and Karatzas et al. (1991). Cuoco (1997) points out that the value of the optimal investment problem (23) subject to (22) does not define a convex function relative to ν_i (because ν_i appears as an exponent) so that minimization may not have a solution. He shows that technical conditions must be strengthened to generalize the result to the case in which individuals receive an endowment stream. Kramkov and Schachermayer (1999) provide conditions applicable when state prices are semimartingales. In our application, the economy-wide state price process $\xi(t)$ turns out to be continuous in equilibrium. As we shall see, however, one has to allow for off-equilibrium discontinuities in the state price process. A recent paper Hugonnier and Kramkov

[9] Starting at time t, the same sequence of decisions could have been obtained by solving the problem:

$$\inf_{\nu_i(s)\in\ker\sigma(s)} \sup_{c_i(s)} E\left\{\int_t^T u_i\left(c_i\left(s\right),s\right)ds|\mathbf{F}_t^w\right\} \tag{24}$$

subject to:

$$E\left[\int_t^T \left[c_i\left(s\right)-e_i\left(s\right)\right]\xi\left(t,s\right)\eta_i^\kappa\left(t,s\right)ds|\mathbf{F}_t^w\right] = F\left(t\right) \triangleq \alpha_i\left(t\right)B\left(t\right)+\theta_i\left(t\right)^\intercal S\left(t\right) \tag{25}$$

while the Lagrange multiplier of that constraint would have been equal to $1/Z_i(t)$.

(2002) provides technical conditions guaranteeing the validity of the dual approach for incomplete-market situations with random interim endowments when state prices are semi-martingales. To our knowledge, the case of intermediate consumption has not been studied.

All of these technical conditions constrain the choice of utility functions in some way. The most economically meaningful interpretation of these sufficient conditions is:

$$\limsup_{x\to\infty} \frac{x\frac{\partial}{\partial x}u_i(x,t)}{u_i(x,t)} < 1 \tag{27}$$

which, for the case of isoelastic utility x^γ/γ, imposes only that $\gamma < 1$ or that the investor should exhibit some degree of risk aversion.

Remark 10 *If we take the Lagrange multiplier at time 0, $Z_i^{-1}(0)$, as given, the time-0 problem (26) can be written equivalently:*

$$\begin{aligned} &\inf_{\nu_i(s)\in\ker\sigma(s)} \sup_{c_i(s)} E\left\{\int_0^T u_i(c_i(s),s)\,ds|\mathbf{F}_0^w\right\} \\ &-Z_i^{-1}(0)\,E\left[\int_0^T [c_i(s)-e_i(s)]\,\xi(0,s)\,\eta_i^\kappa(0,s)\,ds|\mathbf{F}_0^w\right] \end{aligned} \tag{28}$$

2.3 Market equilibrium

We continue to impose that $\sum_i F_i(0) = 0$.

Definition 11 *A competitive market equilibrium is a set of decision processes $\{\{c_i\},\{\alpha_i\},\{\theta_i\}\}$ and price processes $\{B,S\}$ such that, for each individual i, $\{c_i\},\{\alpha_i\}$ and $\{\theta_i\}$ are the optimizing argument of (8) subject to (9) through (11) and such that the market clearing conditions (4) and (5) hold.*

Remark 12 *Since the aggregate resource restriction holds, the set of equilibrium initial Lagrange multipliers $\{1/Z_i(0)\}$ is not just any element in $\mathbb{R}^I$. It is truly in a submanifold because we must have:*

$$\sum_i \left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(0),0\right) = \sum_i e_i(0) \tag{29}$$

where $\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}(\cdot,t)$ is the inverse marginal utility function of each individual with respect to consumption.[10]

Suppose that a competitive market equilibrium exists in which the initial Lagrange multipliers are equal to $\{1/Z_i(0)\}$ (satisfying (29)) and the diffusion matrix of traded asset prices is given by a $N\times K$ dimensional process σ. Then, we can define:

[10] $\frac{\partial}{\partial c_i}u_i(c_i(0),0) = Z_i^{-1}(0)$ is evidently the first-order condition with respect to consumption at time 0.

Definition 13 *A competitive market "sub-equilibrium" is a set of processes* $\{\{c_i\}, \{\nu_i\}, \xi\}$*, in which, for each individual* i*,* $\{c_i\}, \{\nu_i\}$ *are the optimizing arguments of (28), and which are such that the aggregate resource restriction holds.*

3 The static formulation: central planning

We suppose that a market equilibrium exists and is given, in which the initial Lagrange multipliers are equal to $\left\{Z_i^{-1}(0)\right\}$ (satisfying (29)) and the diffusion matrix of traded asset prices is given by a $N \times K$ dimensional process σ.[11]

Our goal is now to define a central-planning problem that generates a sub-equilibrium.

In the market setting, ξ has been implied from the behavior of market prices (6) and (7). In the central planning setting, however, ξ is just an adapted process to be determined with stochastic differential equation (17). In both contexts, ν_i is an adapted process, to be determined, which is in the kernel of σ and η_i^κ is defined by (18).

The central planner that achieves our goal has two selves which operate jointly in a Nash game with each other. The two selves solve two interdependent allocation problems with two different objective functions and constraints:

Problem 1:

$$\inf_{\{\nu_i \in \ker \sigma\}} \sup_{\{c_i(s)\}} \left\{ \sum_i E\left\{ \int_0^T u_i\left(c_i(s), s\right) ds | \mathbf{F}_0^w \right\} \right. \\ \left. - \sum_i Z_i^{-1}(0) E\left[\int_0^T \left[c_i(s) - e_i(s)\right] \xi(0,s) \eta_i^\kappa(0,s) ds | \mathbf{F}_0^w \right] \right\} \tag{30}$$

Problem 2:

$$\sup_{\{c_i\}} \left\{ \sum_i Z_i(0) E\left\{ \int_0^T \frac{1}{\eta_i^\kappa(0,s)} u_i\left(c_i(s), s\right) ds | \mathbf{F}_0^w \right\} \right. \\ \left. + \inf_{\xi \in \mathbb{R}^+} \left[-E\left[\int_0^T \sum_i \left[c_i(s) - e_i(s)\right] \xi(0,s) ds | \mathbf{F}_0^w \right] \right] \right\} \tag{31}$$

Self 1 makes sure that the budget constraints are satisfied. It takes ξ from Self 2 as given and makes exactly the same decisions as in the partial-equilibrium dual approach of He and Pearson and Karatzas et al. Self 2 acts very much like the central planner in a complete market problem or like an auctioneer; it makes sure that the aggregate resource restriction is satisfied at all times.[12] It takes

[11] In this section, therefore, σ is given. In Section 6, we illustrate how σ could be obtained from the recursive version of the central planning algorithm. This is done numerically only. We leave for future research the generalization of the theory for the case of endogenous σ.

[12] Although reminiscent of the 'auctioneer' algorithm of Lucas (1994) and Heaton and Lucas (1996), our auctioneer is different as he directly targets the aggregate resource restriction (by choosing aggregate state prices), rather than market clearing in financial markets (by searching for asset holdings), as in their approach.

the η_i^κ's from Self 1 as given in constructing his objective function. Observe that the decisions of one player serve to define the objective function of the other. The two selves could not be reduced to one since they face different objective functions and discount utility of consumption at different rates, but they agree on the consumption allocation. Indeed, the FOCs for consumption are the same in both cases:

$$\begin{aligned} \frac{\partial}{\partial c_i(s)} u_i(c_i(s), s) &= Z_i^{-1}(0)\, \xi(0,s)\, \eta_i^\kappa(0,s) \\ &\equiv Z_i^{-1}(s) \end{aligned} \tag{32}$$

Remark 14 *When markets are dynamically complete, i.e. when σ is a square matrix ($N = K$) of full rank, the kernel of σ is the singleton 0: there is a unique equivalent martingale measure. Problems 1 and 2 become respectively:*

$$\begin{aligned} &\sup_{\{c_i(s)\}} \left\{ \sum_i E\left\{ \int_0^T u_i(c_i(s), s)\, ds | \mathbf{F}_0^w \right\} \right. \\ &\left. - \sum_i Z_i^{-1}(0)\, E\left[\int_0^T [c_i(s) - e_i(s)]\, \xi(0,s)\, ds | \mathbf{F}_0^w \right] \right\} \end{aligned} \tag{33}$$

$$\begin{aligned} &\inf_{\xi(0,s)\in\mathbb{R}^+} \sup_{\{c_i(s)\}} \left\{ \sum_i Z_i(0)\, E\left\{ \int_0^T u_i(c_i(s), s)\, ds | \mathbf{F}_0^w \right\} \right. \\ &\left. + \left[-E\left[\int_0^T \sum_i [c_i(s) - e_i(s)]\, \xi(0,s)\, ds | \mathbf{F}_0^w \right] \right] \right\} \end{aligned} \tag{34}$$

so that central planning can be achieved by a planner with a single self. Indeed Planner 2 in this case needs no input from Planner 1.

Definition 15 *A Nash equilibrium of the above game is a set of decision processes $\{\{c_i\}, \{\nu_i\}\}$ that are optimal for Planner 1 (in particular, individually marketable) given the values $\{\xi\}$ of the decisions of Planner 2 and a set of decision processes $\{\{c_i\}, \xi\}$ that are optimal for Planner 2 (in particular, marketable in the aggregate and satisfying the budget constraint) given the values of the decisions $\{\nu_i\}$ of Planner 1.*

Theorem 16 *Suppose that a competitive market equilibrium exists in which the set of initial Lagrange multipliers is equal to $\left\{Z_i^{-1}(0)\right\}$ (satisfying the aggregate resource restriction (29) at time 0) and the diffusion matrix of traded asset prices is given by a $N \times K$ dimensional process σ. The Nash equilibrium of the above game is a market sub-equilibrium.*

Proof. *First, for any given ξ processes, Planner 1 chooses net trades $[c_i(s) - e_i(s)]$ that are marketable.*
Indeed, since Problem 1 is nothing but the sum taken over all individuals of the

individual problems of the form (28), lemma 9 above implies that individual net trades are marketable.
Second, for any given set of processes $\{\nu_i(s) \in \ker \sigma(s)\}$*, the choice of* ξ *by Planner 2 guarantees that the aggregate resource restriction is satisfied with probability one.* Indeed, the planner's objective function is nothing but a Lagrangian objective function incorporating that constraint. ■

Remark 17 *Given any solution* $\{\{c_i\}, \{\nu_i\}, \xi\}$ *of the above game, equivalently written as* $\{\{c_i\}, \{\eta_i^\kappa\}, \xi\}$, *define a process* $\widehat{\eta}$:

$$\widehat{\eta}^\kappa(0,t) \triangleq \exp\left\{-\frac{1}{2}\int_0^t \|\widehat{\nu}(s)\|^2 ds - \int_0^t \kappa(s)^\intercal \widehat{\nu}(s)\, ds - \int_0^t \widehat{\nu}(s)^\intercal dw(s)\right\}; i = 1, ..., I \tag{35}$$

where $\widehat{\nu}$ *is any adapted process satisfying the kernel constraint:* $\sigma\widehat{\nu} = 0$. *Then it can be checked readily that* $\{\{c_i\}, \{\eta_i^\kappa \widehat{\eta}^\kappa\}, \xi/\widehat{\eta}^\kappa\}$ *is another solution of the game. To understand this, observe that* κ *could have been reset at* $\kappa + \widehat{\nu}$ *where* $\widehat{\nu}$ *is an arbitrary element of the kernel of* σ. *Then the kernel condition would really be* $\sigma(\nu_i - \widehat{\nu}) = 0$. *But this last is equivalent to* $\sigma\nu_i = 0$.

Faced with this indeterminacy, one could impose the condition that κ be in the span of $\sigma^\intercal$or that, for the first individual, $\nu_1 = 0$. Either restriction pins down (or standardizes) κ, which otherwise would be indeterminate, with a cancelling indeterminacy in each of the ν_i's.
In terms of economics, the meaningful kernel condition is one that says only that *differences* in ν's between *any two* individuals should be in the kernel, not that each single ν_i should be in the kernel. This corresponds to the fact that we w*ant any pair of individuals to agree on the prices of traded securities.*

Therefore, if we impose spanning from now onwards, it is a normalization without loss of generality.

Proposition 18 *Since the Nash equilibrium of the game has the property that the aggregate resource restriction is satisfied at all times, we have, for an exchange economy:*

$$\sum_i \left[\frac{\partial}{\partial c_i} u_i\right]^{-1} \left(Z_i^{-1}(0)\,\xi(0,t)\,\eta_i^\kappa(0,t), t\right) = \sum_i e_i(t) \text{ with probability 1} \tag{36}$$

Applying Itô's calculus over time, it follows that the equilibrium choices of the r, ξ *and* $\{\nu_i\}$ *processes satisfy:*

$$-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2} u_i\left(\left[\frac{\partial}{\partial c_i} u_i\right]^{-1}\left(Z_i^{-1}(t), t\right), t\right)} Z_i^{-1}(t)\left[\kappa(t) + \nu_i(t)\right]^\intercal = \sum_i \sigma_i^e(t) \text{ with probability 1,} \tag{37}$$

where σ_i^e *is the diffusion row vector process of the endowment of individual* i, *as well as:*

$$\sum_i \frac{\partial}{\partial t}\left\{\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right)\right\} - \sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)} Z_i^{-1}(t)\, r(t)$$

$$+\frac{1}{2}\sum_i \frac{1}{\left[\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)\right]^3}$$

$$\times \frac{\partial^3}{\partial c_i^3}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)\left[Z_i^{-1}(t)\right]^2\left[\kappa(t)+\nu_i(t)\right]^\intercal\left[\kappa(t)+\nu_i(t)\right]$$

$$= \sum_i \mu_i^e(t) \text{ with probability 1} \tag{38}$$

where μ_i^e *is the drift process of the endowment of individual* i.

The solution of (37) for κ in the span ($\kappa = \sigma^\intercal x$ for some x) is:

$$\left[-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)} Z_i^{-1}(t)\right] x(t) = \left[\sum_i \sigma_i^e(t)\,\sigma(t)^\intercal\right]\left[\sigma(t)\,\sigma(t)^\intercal\right]^{-1} \tag{39}$$

$$\kappa(t)^\intercal = \frac{1}{\left[-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)} Z_i^{-1}(t)\right]}\left[\sum_i \sigma_i^e(t)\,\sigma(t)^\intercal\right]\left[\sigma(t)\,\sigma(t)^\intercal\right]^{-1}\sigma(t) \tag{40}$$

Residual restrictions on the corresponding equilibrium νs are:

$$-\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2}u_i\left(\left[\frac{\partial}{\partial c_i}u_i\right]^{-1}\left(Z_i^{-1}(t),t\right),t\right)} Z_i^{-1}(t)\,\nu_i(t)^\intercal = \left[\sum_i \sigma_i^e(t)\right]\left[I_K - \sigma(t)^\intercal\left[\sigma(t)\,\sigma(t)^\intercal\right]^{-1}\sigma(t)\right]. \tag{41}$$

In Section 5, we provide a recursive formulation of the same central planning problem. But, first, we look at an illustration of the static method.

4 The Magill-Quinzii example solved by the static central plan

In their textbook, Magill and Quinzii (1996, Chapter IV) construct an example that purports to show that an equilibrium in an incomplete market is not Pareto optimal, not even under the constraint of marketability of consumption plans. Indeed, we have been careful here not to give our algorithm involving two central planners any welfare interpretation.

We now use that same example to illustrate the way in which the two central planners would arrive at the incomplete market equilibrium.

Magill and Quinzii's example is set in a three-date (indexed by t), two-agent (indexed by i), two-state (indexed by s) environment. All uncertainty is resolved at $t = 1$ and both states have equal probability. The event-tree for the aggregate state $Y_{t,s}$ is:

$t=0$	$t=1$	$t=2$
	Y_{11}	Y_{21}
Y_0		
	Y_{12}	Y_{22}

The endowment processes e_i as a function of the aggregate state $Y_{t,s}$ are as follows for the two agents:

$$
\begin{aligned}
e_1 &= \left(e_1(Y_0), e_1(Y_{11}), e_1(Y_{12}), e_1(Y_{21}), e_1(Y_{22})\right) = (4,0,6,6,6) \\
e_2 &= \left(e_2(Y_0), e_2(Y_{11}), e_2(Y_{12}), e_2(Y_{21}), e_2(Y_{22})\right) = (9,8,0,8,8)
\end{aligned}
$$

If state Y_{11} occurs, agent 1 temporarily has a zero endowment, and similarly for agent 2 in state Y_{12}.

Agents have time-separable logarithmic utility, but are heterogeneous in terms of time preference:

$$
\begin{aligned}
E_0\left\{\sum_{t=0}^{2} u_i\left(c_i\left(Y_{t,s}\right), t\right)\right\} &= \log\left(c_i\left(Y_0\right)\right) + \beta_i\left[\frac{1}{2}\log\left(c_i\left(Y_{11}\right)\right) + \frac{1}{2}\log\left(c_i\left(Y_{12}\right)\right)\right] \\
&\quad + \beta_i^2\left[\frac{1}{2}\log\left(c_i\left(Y_{22}\right)\right) + \frac{1}{2}\log\left(c_i\left(Y_{22}\right)\right)\right]
\end{aligned}
$$

where the discount factors are given by $(\beta_1, \beta_2) = \left(\frac{1}{2}, \frac{1}{3}\right)$.

In each period, there is only one financial asset, a short-lived bond that permits lending and borrowing. Markets are incomplete, as there is no risky or state-contingent asset that would allow agents to hedge their endowment risk.

The spanning and kernel restrictions are now written with respect to the payoff matrix instead of the diffusion matrix. As there is no risky asset, the spanning condition implies that $\kappa\left(Y_{t,s}\right) = 0$. Also, the kernel restriction is vacuous, which means that ν_i can be chosen freely.

However, we also need to impose that $\eta_i\left(Y_{t,s}\right) = \exp\left(-\nu_i\left(Y_{t,s}\right)\right)$ be a martingale. This implies $\nu_i\left(Y_{t,s}\right) = 0$ for $t = 2$. For $t = 1$, we obtain

$$
\frac{1}{2} \times e^{-\nu(Y_{11})} + \frac{1}{2} \times e^{-\nu(Y_{12})} = 1
$$

It is straightforward to verify that the equilibrium consumption allocation described by Magill and Quinzii,

$$\begin{aligned} c_1 &= (c_1(Y_0), c_1(Y_{11}), c_1(Y_{12}), c_1(Y_{21}), c_1(Y_{22})) = (4, 0.8, 4.8, 2, 12) \\ c_2 &= (c_2(Y_0), c_2(Y_{11}), c_2(Y_{12}), c_2(Y_{21}), c_2(Y_{22})) = (9, 7.2, 1.2, 12, 2) \end{aligned}$$

along with the values

$$r = (r(Y_0), r(Y_{11}), r(Y_{12})) = (-38\%, 161\%, 161\%)$$

$$\begin{aligned} \nu_1 &= (\nu_1(Y_{11}), \nu_1(Y_{12}), \nu_1(Y_{21}), \nu_1(Y_{22})) = (-54\%, 125\%, 0, 0) \\ \nu_2 &= (\nu_2(Y_{11}), \nu_2(Y_{12}), \nu_2(Y_{21}), \nu_2(Y_{22})) = (125\%, -54\%, 0, 0) \end{aligned}$$

solves the following planning problems, given values for $\left\{Z_i^{-1}(0)\right\}$: $(Z_1(0), Z_2(0)) = (4, 9)$:

Problem 1:

$$\begin{aligned} &\inf_{\{\nu_i(Y_{t,s})\}} \sup_{\{c_i(Y_{t,s})\}} \left\{ \sum_{i=1}^{2} E_0 \left[\sum_{t=0}^{2} u_i(c_i(Y_{t,s}), t) \right] \right. \\ &\left. - \sum_{i=1}^{2} \frac{1}{Z_i(0)} E_0 \left[\sum_{t=0}^{2} [c_i(Y_{t,s}) - e_i(Y_{t,s})] \xi(Y_{t,s}) \eta_i(Y_{t,s}) \right] \right\} \end{aligned} \tag{42}$$

Problem 2:

$$\begin{aligned} &\inf_{\xi(Y_{t,s})} \sup_{\{c_i(Y_{t,s})\}} \left\{ \sum_{i=1}^{2} Z_i(0) E_0 \left[\sum_{t=0}^{2} \frac{1}{\eta_i(Y_{t,s})} u_i(c_i(Y_{t,s}), t) \right] \right. \\ &\left. + \left[-E_0 \left[\sum_{t=0}^{2} \sum_{i=1}^{2} [c_i(Y_{t,s}) - e_i(Y_{t,s})] \xi(Y_{t,s}) \right] \right] \right\} \end{aligned} \tag{43}$$

5 Recursive formulation

We now show that it is possible to develop a recursive (dynamic-programming) formulation of the static central-planner problem. This should be useful for numerical implementations.

For the purpose, adopt a Markovian setting:[13]

$$e_i(t) = e_i(0) + \int_0^t \mu_i^e(Y(s), s) ds + \int_0^t \sigma_i^e(Y(s), s) dw(s) \tag{44}$$

where Y is an Itô process in $\mathbb{R}^K$:

$$Y(t) = Y(0) + \int_0^t \mu(Y(s), s) ds + \int_0^t \rho(Y(s), s) dw(s) \tag{45}$$

[13] We assume that the functions μ_i^e, σ_i^e, μ and ρ satisfy growth and Lipschitz conditions.

The aim in the recursive setting is to derive the competitive market equilibrium one step at a time. We show that this can be done by having the two Planners play a dynamic game.

It is important in that context to keep two basic ideas in mind. First, even though the process ξ in equilibrium is a continuous process, since it solves Equation (36), it should nevertheless generally be conceived of as a jump process, or an instantaneous control, which accommodates the current values of $\{\eta_i\}$ to satisfy that equation.

Second, the timing of the game needs to be specified carefully (see Figure 1):

- The state variables of the game are $\{\eta_i(0,t)\}$. The two players arrive at time t with given values $\{\eta_i(0,t)\}$ for these state variables.

- At time t, they play simultaneously a Nash game in which Planner 1 chooses $\{\nu_i(t)\}$, Planner 2 chooses $\xi(0,t)$ to satisfy (36) and they both agree on the choice of $\{c_i(t)\}$. Note that the choice of $\xi(0,t)$ by Planner 2 only depends on the values of the state variables $\{\eta_i(0,t)\}$, and $\{e_i(t)\}$, not on the choice of $\{\nu_i(t)\}$ made by Planner 1. In other words, Planner 2 has a dominant strategy.

- As they move to time $t+dt$, a realization $dw(t)$ of the Wiener occurs. This leads to a realization of $\eta_i(t+dt)$. At that time Planner 2 instantaneously accommodates the aggregate resource constraint by adjusting $\xi(0,t+dt)$. That behavior can be anticipated at time t by Planner 1 *but he/she should hold it fixed when choosing his own actions.*

The game and equilibrium concept we have just described are not the standard dynamic (stochastic differential) game and corresponding Markov Perfect Equilibrium (henceforth MPE), as described, for instance, in Fudenberg and Tirole (1995), chapter 13. In an MPE, each player takes the decision *rule* of the other player as given. That means that, in making his/her current choices, each player takes into account the impact of his/her action on the other player's future actions via the the impact it has on the future values of the state variables. Were we to write a dynamic game of that type and examine the MPE, it would be a different equilibrium from the Nash equilibrium in the one-shot game we have proposed above.

Here, we are trying to find a dynamic game that replicates the static game we have described so far. In the static game each player took the entire process of the other player as given. The given process of the other player was not allowed to be contingent on this player's actions. That means that we must make sure that Planner 1 takes all future values of Planner 2's decisions as given. The dynamic game we are looking for is instead closer in spirit to an Open-Loop Equilibrium, albeit in a stochastic differential setting (so that the strategies must still be formulated as measurable functions of the relevant state variables). It is worth pointing out that an OLE is often deemed unattractive since it requires commitment of the players to their strategies. This is not an issue here, since

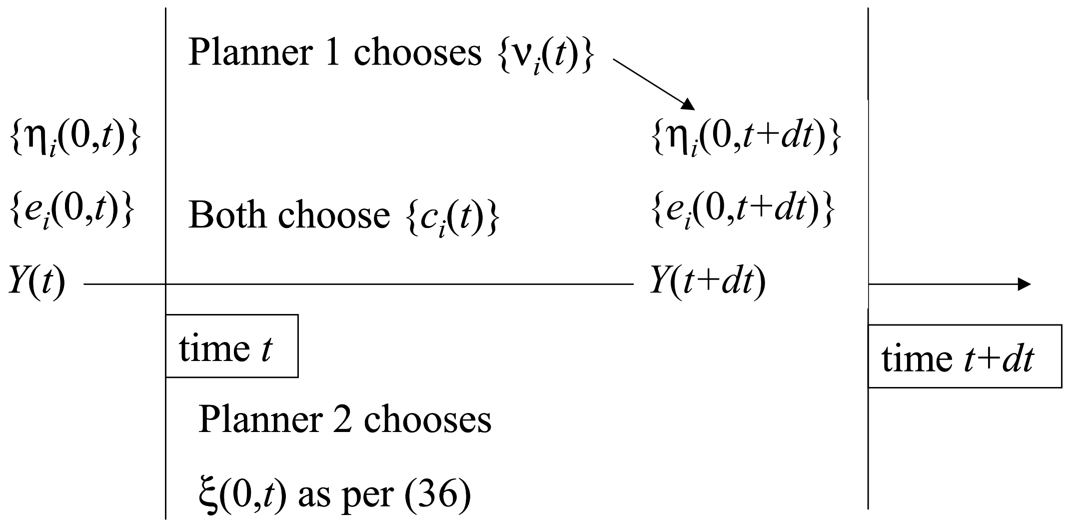

Figure 1: The timing of the dynamic game

the game and corresponding equilibrium are in any case artificial. It is a pure computational device which replicates the static game.It need not resemble what is done usually to represent human behavior in a setting that would be dynamic to start with. Characterizing the OLE will require some compensating term to eliminate from Planner 1's decision process the anticipated reaction of Planner 2.

In short, we have the following definition:

Definition 19 *A dynamic equilibrium of the game described in Section 3 is a set of admissible, measurable functions*

$$c_i^*\left(\{\eta_i\},\{e_i\},Y,t\right)$$
$$\nu_i^*\left(\{\eta_i\},\{e_i\},Y,t\right)$$

and

$$\xi^*\left(\{\eta_i\},\{e_i\},Y,t\right)$$

such that the decisions $\{\{c_i^*\},\{\nu_i^*\}\}$ *are optimal for Planner 1 given current and future* ***values*** $\{\xi^*\}$ *of the decisions of Planner 2 and given the current state variables* $\{\eta_i(t)\},\{e_i(t)\},Y(t),t$ *and such that the decisions* $\{\{c_i^*\},\xi^*\}$ *are optimal for Planner 2 given current and future* ***values*** $\{\nu_i^*\}$ *of the decisions of Planner 1 and given the current state variables* $\{\eta_i(t)\},\{e_i(t)\},Y(t),t,$ *where the processes* $(\{c_i^*\},\{\nu_i^*\},r^*,\kappa^*)$ *are defined by:*

$$c_{i,t}^* = c^*\left(\{\eta_i(0,t)\},\{e_i(t)\},Y(t),t\right) \tag{46}$$
$$\nu_{i,t}^* = \nu_i^*\left(\{\eta_i(0,t)\},\{e_i(t)\},Y(t),t\right) \tag{47}$$

and:

$$\xi_t^* = \xi^* \left(\left\{ \eta_i \left(0, t \right) \right\}, \left\{ e_i \left(t \right) \right\}, Y \left(t \right), t \right). \tag{48}$$

Theorem 20 *Suppose that a competitive market equilibrium exists in which the initial Lagrange multipliers are equal to* $\left\{ Z_i^{-1} (0) \right\}$ *and the diffusion matrix of traded asset prices is given by a* $N \times K$ *dimensional process* σ*. One equilibrium of the above dynamic game is a market sub-equilibrium relative to* $\left\{ Z_i^{-1} (0) \right\}$ *and* σ*.*

The task is now to prove this claim and to characterize the dynamic equilibrium. The remainder of this section is devoted to this. We define the value function(s) and construct the corresponding Hamilton-Jacobi-Bellman PDE(s). The first-order conditions are then shown to generate a competitive market sub-equilibrium.

5.1 Value functions

One could define an intertemporal value function for Planner 2 but, in the present exchange-economy setting, Planner 2 has no intertemporal decisions to make. Hence, that is really pointless. As we saw, Planner 2's decision $\xi (0, t)$ is dictated by Equation (36). The solution of that equation is denoted $\xi^* \left(\left\{ Z_i^{-1} (0) \eta_i \right\}, \sum_i e_i, t \right)$. This function does have MY as an argument and is homogeneous of degree -1 in $\left\{ Z_i^{-1} (0) \eta_i \right\}$.

Define the value function of Planner 1:

$$\begin{aligned} & J \left(\left\{ Z_i^{-1} (0) \eta_i \right\}, \left\{ e_i \right\}, Y, t \right) \\ & \triangleq \inf_{\left\{ \nu_i (s) \in \ker \sigma (s) \right\}} \sup_{\left\{ c_i (s) \right\}} \sum_i E \left\{ \int_t^T u_i \left(c_i \left(s \right), s \right) ds | \mathbf{F}_t^w \right\} \\ & -\xi^* \left(\left\{ Z_i^{-1} (0) \eta_i \right\}, \sum\nolimits_i e_i, t \right) \sum_i Z_i^{-1} (0) \eta_i E \left[\int_t^T \left[c_i (s) - e_i (s) \right] \xi^* (t, s) \eta_i (t, s) ds | \mathbf{F}_t^w \right] \end{aligned} \tag{49}$$

In this definition, $\left\{ Z_i^{-1} (0) \eta_i (0, t) \right\}$ is an argument because it is a state variable.

Planner 1 plays Nash and takes the value of the decision ξ^* of Planner 2 as a given. To accommodate that feature of the game, we also define a "compensated" value function $\widehat{J}$ of Planner 1 as follows:

$$\begin{aligned} & \widehat{J} \left(\left\{ Z_i^{-1} (0) \eta_i \xi \right\}, \left\{ e_i \right\}, Y, t \right) \\ & \triangleq \inf_{\left\{ \nu_i (s) \in \ker \sigma (s) \right\}} \sup_{\left\{ c_i (s) \right\}} \sum_i E \left\{ \int_t^T u_i \left(c_i \left(s \right), s \right) ds | \mathbf{F}_t^w \right\} \\ & -\xi \sum_i Z_i^{-1} (0) \eta_i E \left[\int_t^T \left[c_i (s) - e_i (s) \right] \xi^* (t, s) \eta_i (t, s) ds | \mathbf{F}_t^w \right] \end{aligned} \tag{50}$$

Assumption: We assume that J and $\widehat{J}$ are $C^{2,2,2,1}$.

Lemma 21 *The envelope theorem implies:*[14]

$$-\widehat{J}_i\left(\left\{Z_i^{-1}(0)\,\eta_i\xi\right\},\{e_i\},Y,t\right)=E\left[\int_t^T\left[c_i(s)-e_i(s)\right]\xi^*(t,s)\,\eta_i(t,s)\,ds|\mathbf{F}_t^w\right] \tag{51}$$

which can also be written:

$$-\widehat{J}_i\left(\left\{Z_i^{-1}(0)\,\eta_i\xi\right\},\{e_i\},Y,t\right)=E\left[\int_t^T\left[c_i(s)-e_i(s)\right]\xi^*(t,s)\,ds|\mathbf{F}_t^w\right] \tag{52}$$

so that the partial derivative $\widehat{J}_i$ is equal to (minus) the financial wealth of investor i. It follows from that observation that the function $\widehat{J}$ has the property that:

$$\sum_i\widehat{J}_i\left(\left\{Z_i^{-1}(0)\,\eta_i\xi\right\},\{e_i\},Y,t\right)=0;\forall\left\{Z_i^{-1}(0)\,\eta_i\xi\right\} \tag{53}$$

When equilibrium prevails the function J follows from the function $\widehat{J}$:

$$J\left(\left\{Z_i^{-1}(0)\,\eta_i\right\},\{e_i\},Y,t\right)\equiv\widehat{J}\left(\left\{Z_i^{-1}(0)\,\eta_i\xi^*\left(\left\{Z_i^{-1}(0)\,\eta_i\right\},\sum\nolimits_i e_i,t\right)\right\},\{e_i\},Y,t\right) \tag{54}$$

Given the functional forms, the functions J and $\widehat{J}$ have the homogeneity property that multiplying η by any real number λ and multiplying ξ by λ^{-1} leaves the value of the function $\widehat{J}$ unchanged, which implies that the function J is homogeneous of degree 0 in η. The reaction introduced by the function ξ^* causes the function J to be homogeneous of degree 0 in $\{\eta_i\}$.

5.2 Conditions of optimality

We need to incorporate a constraint that the individual-specific diffusion ν_i is in the kernel: $\sigma\nu_i=0$. We assign Lagrange multiplier $Z_i^{-1}(0)\,\xi^*\left(\left\{Z_i^{-1}(0)\,\eta_i\right\},\{e_i\},Y,t\right)\eta_i(0,t)\,\theta_i^\intercal$ to that last constraint. We look for the equilibrium for which κ is in the span of $\sigma^\intercal$.[15]

The Hamilton-Jacobi-Bellmann PDE for J is:

$$\begin{aligned}0=\sup_{\{\theta_i(t)\}}\inf_{\{\nu_i(t)\}}\Big|_{\xi_i^*}\sup_{\{c_i(t)\}}\sum_i\left[u_i(c_i,t)-Z_i^{-1}(0)\,\xi^*(\{\eta_i\},\{e_i\},Y,t)\,\eta_i(0,t)\,(c_i-e_i(t))\right]\\+\mathcal{D}^{\{\nu_i\}}J\left(\left\{Z_i^{-1}(0)\,\eta_i\right\},\{e_i\},Y,t\right)-\sum_i Z_i^{-1}(0)\,\xi^*\left(\left\{Z_i^{-1}(0)\,\eta_i\right\},\{e_i\},Y,t\right)\eta_i(0,t)\,\theta_i^\intercal\sigma\nu_i\end{aligned} \tag{55}$$

[14] $\widehat{J}_i\left(\left\{Z_i^{-1}(0)\,\eta_i\xi\right\},\{e_i\},Y,t\right)$ denotes the derivative of $\widehat{J}$ with respect to its ith argument among the first group of arguments.

[15] We impose that requirement later.

where the operator $\mathcal{D}^{\{\nu_i\}}$ is defined as:[16]

$$\begin{aligned}
\mathcal{D}^{\{\nu_i\}}J \triangleq \frac{\partial J}{\partial t} + \sum_i \frac{\partial J}{\partial e_i}\mu_i^e + \left(\frac{\partial J}{\partial Y}\right)^{\intercal}\mu \\
+\frac{1}{2}\sum_i\sum_j J_{ij}\nu_i^{\intercal}\nu_j Z_i^{-1}(0)\,\eta_i Z_j^{-1}(0)\,\eta_j \\
+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial e_i \partial e_j}\left(\sigma_i^e\right)\left(\sigma_j^e\right)^{\intercal} + \frac{1}{2}tr\left(\frac{\partial^2 J}{\partial Y \partial Y}\rho\rho^{\intercal}\right) + tr\left(\sum_i \frac{\partial^2 J}{\partial e_i \partial Y}\sigma_i^e\rho^{\intercal}\right) \\
-\sum_i\sum_j \frac{\partial J_i}{\partial e_j}\nu_i^{\intercal} Z_i^{-1}(0)\,\eta_i\left(\sigma_j^e\right)^{\intercal} - tr\left(\sum_i \frac{\partial J_i}{\partial Y}\nu_i^{\intercal} Z_i^{-1}(0)\,\eta_i\rho^{\intercal}\right)
\end{aligned} \tag{56}$$

Given that we plan to introduce a compensation when Planner 1 chooses ν_i, we use the equivalence (54) to rewrite the derivatives of J in terms of the compensated function $\widehat{J}$:[17]

$$J_i = \widehat{J}_i \times \xi^* + \left(\sum_k \widehat{J}_k Z_k^{-1}(0)\,\eta_k\right) \times \xi_i^* \tag{57}$$

$$\begin{aligned}
J_{ij} \quad = \quad & \widehat{J}_{ij} \times \left(\xi^*\right)^2 + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k\right] \times \xi_j^* \times \xi^* \\
& + \widehat{J}_i \times \xi_j^* + \left(\sum_k \widehat{J}_{kj} Z_k^{-1}(0)\,\eta_k\right) \times \xi^* \times \xi_i^* \\
& + \left(\sum_k\sum_{k'} \widehat{J}_{kk'} Z_k^{-1}(0)\,\eta_k Z_{k'}^{-1}(0)\,\eta_{k'}\right) \times \xi_i^* \times \xi_j^* \\
& + \widehat{J}_j \times \xi_i^* + \left(\sum_k \widehat{J}_k Z_k^{-1}(0)\,\eta_k\right) \times \xi_{ij}^*
\end{aligned} \tag{58}$$

$$\begin{aligned}
\frac{\partial J_i}{\partial e_j} \quad = \quad & \frac{\partial \widehat{J}_i}{\partial e_j} \times \xi^* + \left[\left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k\right) \times \xi^* + \widehat{J}_i + \left(\sum_k\sum_{k'} \widehat{J}_{kk'} Z_k^{-1}(0)\,\eta_k Z_{k'}^{-1}(0)\,\eta_{k'}\right) \times \xi_i^*\right] \\
& \times \frac{\partial \xi^*}{\partial e_j} + \left(\sum_k \frac{\partial \widehat{J}_k}{\partial e_j} Z_k^{-1}(0)\,\eta_k\right) \times \xi_i^* + \left(\sum_k \widehat{J}_k Z_k^{-1}(0)\,\eta_k\right) \times \frac{\partial \xi_i^*}{\partial e_j}
\end{aligned} \tag{59}$$

[16] $J_i\left(\left\{Z_i^{-1}(0)\,\eta_i\right\}, \{e_i\}, Y, t\right)$ denotes the derivative of J with respect to its ith argument among the first group of arguments.

[17] $\xi_i^*\left(\left\{Z_i^{-1}(0)\,\eta_i\right\}, \{e_i\}, t\right)$ denotes the derivative of ξ^* with respect to its ith argument among the first group of arguments.

$$\frac{\partial J_i}{\partial Y} = \frac{\partial \widehat{J}_i}{\partial Y} \times \xi^* + \left(\sum_k \frac{\partial \widehat{J}_k}{\partial Y} Z_k^{-1}(0)\,\eta_k \right) \times \xi_i^* \tag{60}$$

Given the simple behavior of Planner 2, there is also no problem in this exchange economy in calculating the derivatives of the function ξ^*. From (36), we have:

$$\xi_j^* = -\frac{\frac{1}{\frac{\partial^2}{\partial c_j^2} u_j\left(\left[\frac{\partial}{\partial c_j} u_j \right]^{-1} \left(Z_j^{-1}(0)\xi^* \eta_j, t \right), t \right)} \xi^*}{\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2} u_i\left(\left[\frac{\partial}{\partial c_i} u_i \right]^{-1} \left(Z_i^{-1}(0)\xi^* \eta_i, t \right), t \right)} Z_i^{-1}(0)\,\eta_i} \tag{61}$$

$$\frac{\partial \xi^*}{\partial e_j} = \frac{1}{\sum_i \frac{1}{\frac{\partial^2}{\partial c_i^2} u_i\left(\left[\frac{\partial}{\partial c_i} u_i \right]^{-1} \left(Z_i^{-1}(0)\xi^* \eta_i, t \right), t \right)} Z_i^{-1}(0)\,\eta_i} \tag{62}$$

In Equation (55), we wrote $\inf_{\{\nu_i(t)\}}\big|_{\xi_i^*}$ to signify that it is not literally the right-hand side of (55) that should be minimized. If it were, Planner 1 would be taking into account the impact of his/her choice of $\{\nu_i(t)\}$ on the future choices of ξ^* via its impact on the future values of state variables $\{\eta_i\}$. To prevent that, the first-order condition with respect to $\nu_i(t)$ involves the compensated value function $\widehat{J}$ defined above, keeping $\frac{\partial \xi^*}{\partial Z_i^{-1}} = 0$.

The four first-order conditions associated with the game are:

- Joint condition

$$\frac{\partial}{\partial c_i} u_i(c_i, t) = Z_i^{-1}(0)\,\xi^*\left(\left\{ Z_i^{-1}(0)\,\eta_i \right\}, \{e_i\}, t \right) \eta_i(0,t) \tag{63}$$

- Planner 2's condition:

$$\sum_i \left[c_i - e_i(t) \right] = 0 \tag{64}$$

- Planner 1's conditions:

$$\sum_j \left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*} \right] \times \frac{\xi_j^*}{\xi^*} \right] \nu_j^\intercal Z_j^{-1}(0)\,\xi^* \eta_j \tag{65}$$

$$-\sum_j \left[\frac{\partial \widehat{J}_i}{\partial e_j} + \left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*} \right) \frac{\partial \xi^*}{\partial e_j} \right] \sigma_j^e - \sum_k \left(\frac{\partial \widehat{J}_i}{\partial Y_k} \rho_k \right) = \theta_i^\intercal \sigma; \qquad 1 \times K \times I \tag{66}$$

$$\sigma \nu_i = 0; \qquad N \times 1 \times I \tag{67}$$

The first condition is the usual condition of optimality of consumption.

The interpretation (51) of the derivative $\widehat{J}_i$ as (minus) the financial wealth of individual i shows that the last two first-order conditions are, by construction, identical to He and Pearson's conditions (20, 21) but generalized to incorporate cross-derivatives across individuals. This shows that the $\theta_i^\intercal$s are interpretable as portfolios. Equation (53) above indicates that $\sum_i \theta_i^\intercal = 0$; financial markets clears.

The last three conditions are linear and can be organized into a large partitioned system which can be solved for all the $\theta_i^\intercal$s, and ν_is:

$$\theta_i^\intercal = -\left[\sum_j \left[\frac{\partial \widehat{J}_i}{\partial e_j} + \left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right)\frac{\partial \xi^*}{\partial e_j}\right]\sigma_j^e + \sum_k \left(\frac{\partial \widehat{J}_i}{\partial Y_k}\rho_k\right)\right]\sigma^\intercal \left(\sigma\sigma^\intercal\right)^{-1} \tag{68}$$

$$\begin{aligned} &\sum_j \left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right] \times \frac{\xi_j^*}{\xi^*}\right] \nu_j^\intercal Z_j^{-1}(0)\,\xi^* \eta_j \qquad (69) \\ = &\left[\sum_j \left[\frac{\partial \widehat{J}_i}{\partial e_j} + \left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right)\frac{\partial \xi^*}{\partial e_j}\right]\sigma_j^e + \sum_k \left(\frac{\partial \widehat{J}_i}{\partial Y_k}\rho_k\right)\right]\left\{I_K - \sigma^\intercal \left(\sigma\sigma^\intercal\right)^{-1}\sigma\right\} \end{aligned}$$

The matrix $\left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right] \times \frac{\xi_j^*}{\xi^*}\right]$ is singular because $\sum_i \widehat{J}_i \equiv 0$. There remains an indeterminacy in the values of the ν_is (although the portfolios are well determined). While we have chosen to look for the νs such that the corresponding κ is in the span of $\sigma^\intercal$ and while we have used that property (when writing Equation (55)), we have not yet imposed that requirement. That task is accomplished by appending Equations (41) to the above system. Delete one row in each of the matrices $\left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right] \times \frac{\xi_j^*}{\xi^*}\right]$ and $\left[\sum_j \left[\frac{\partial \widehat{J}_i}{\partial e_j} + \left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right)\frac{\partial \xi^*}{\partial e_j}\right]\sigma_j^e + \sum_k \left(\frac{\partial \widehat{J}_i}{\partial Y_k}\rho_k\right)\right]$ and replace them by the left-hand side and right-hand side of (41). The solution for ν is:

$$\begin{aligned} \left[\nu_j^\intercal Z_j^{-1}(0)\,\xi^* \eta_j\right] \;=\; &\left[\begin{array}{c} \left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right] \times \frac{\xi_j^*}{\xi^*}\right] \\ \left[-\frac{1}{\frac{\partial^2}{\partial c_j^2} u_j\left(\left[\frac{\partial}{\partial c_j} u_j\right]^{-1}\left(Z_j^{-1}(t), t\right), t\right)}\right] \end{array}\right]^{-1} \\ &\times \left[\begin{array}{c} \left[\sum_j \left[\frac{\partial \widehat{J}_i}{\partial e_j} + \left(\sum_k \widehat{J}_{ik} Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right)\frac{\partial \xi^*}{\partial e_j}\right]\sigma_j^e + \sum_k \left(\frac{\partial \widehat{J}_i}{\partial Y_k}\rho_k\right)\right] \\ \left[\sum_i \sigma_i^e(t)\right] \end{array}\right]\left\{I_K - \sigma^\intercal \left(\sigma\sigma^\intercal\right)^{-1}\right. \end{aligned}$$

6 Numerical implementation

We choose to illustrate our method on the example of the limited-participation equilibrium of Basak and Cuoco (1998). In this section, we adopt their notation. Even though we do not know yet how to handle the general case of limited participation – in which each investor is assigned a list of securities to which he/she has access, – we can handle the specific case situation analyzed by Basak and Cuoco, in which there is only one Wiener shock in the economy, one risky asset and one instantaneously riskless asset and just two (or two categories of) finite-lives agents. Agent 1 has access to both securities, whereas Agent 2 has access to the riskless security only. Basak and Cuoco calculate analytically the equilibrium for the case in which Agent 2 has logarithmic utility and receives no endowment. We show here how this can be generalized numerically to any power utility function.

In this setup, the risky security is effectively redundant since a group of identical agents (those of Category 1) are the only ones having access to it. No trading of it actually takes place at any time. In Basak and Cuoco, the security is nonetheless "held", but only because agents of Category 1 are endowed with it. The cash-flow process for the risky security, which we prefer to view as the process for the flow endowment of agents of Category 1 is:

$$\delta(t) = \delta(0) + \int_0^t \overline{\mu}\delta(s)\,ds + \int_0^t \overline{\sigma}\delta(s)\,dw(s) \tag{71}$$

The remainder of the agents' endowments is unconventional in the sense that they are not defined by exogenous cash flows. Instead, agents of the two categories share the total zero net supply of the riskless security, which yields an endogenous rate of interest. Agents of Category 1 are endowed with a short position in β shares of the bond and agents of Category 2 are endowed with a long position in the same β shares of the bond ($\beta \geqslant 0$). Agents of Category 1 are the only ones receiving a flow endowment, of the kind we consider in this article. On an average, Category 1 agents consume less than their endowment because they start out with a short position of the bond. This allows Category 2 agents to consume something out of Category 1's flow endowment δ.

Since the riskless security is tradable by all, the initial endowment of bonds only serves to specify the initial distribution of financial wealth. This is not important for our procedure since we are only interested in generating subequilibria, i.e., equilibria with given initial Lagrange multipliers, as opposed to equilibria with given initial wealth distribution.[18]

Concerning the diffusion matrix of security prices, we need to distinguish between securities to which both classes of investors have access and securities which are available to Category 1 agents exclusively. It is easiest to view this market as an incomplete market in which the only traded security is the instantaneously riskless one, the risky security being absent. In our approach, that pins down the diffusion matrix of traded securities: $\sigma = 0$. Note that here σ does not contain the volatility of the risky security. The risky security is redundant. Its volatility σ_1 can be determined separately from the determination of the equilibrium. We first determine the equilibrium by the central-planning approach and then show how to find σ_1 endogenously in our recursive numerical procedure. If we maintain that κ should be in the span of $\sigma^\intercal$, it follows that $\kappa = 0$, (as in the example of Magill and Quinzii).[19] Furthermore, there is no kernel restriction on the choice of ν_1 and ν_2.

[18] As Basak and Cuoco point out, the initial distribution of wealth, in their set up, also determines whether an equilibrium exists (β must be positive, but not so large that agents of Category 1 could never repay their initial short position in the bond).

[19] Here the notation differs from that of Basak and Cuoco. They call κ the price of risk in

6.1 Central planner solution

Planner 2's problem is easily solved. Assuming isoelastic utility functions ($e^{-\rho t}\frac{c^{\gamma_i}-1}{\gamma_i}$), it chooses $\xi^*\left(Z_1^{-1}(0)\,\eta_1, Z_2^{-1}(0)\,\eta_2, \delta, t\right)$ that solves:[20]

$$\sum_{i=1,2}\left[e^{\rho t}Z_i^{-1}(0)\,\xi\eta_i\right]^{\frac{1}{\gamma_i-1}} = \delta(t) \tag{73}$$

Let us now focus on Planner 1 who chooses ν_1 and ν_2. Under the present circumstances, the PDE for the value function $J\left(Z_1^{-1}(0)\,\eta_1, Z_2^{-1}(0)\,\eta_2, \delta, t\right)$ is:

$$\begin{aligned}0 = \sum_{i=1,2}&\left[\left(\frac{1}{\gamma_i}-1\right)e^{\frac{\rho t}{\gamma_i-1}}\left[Z_i^{-1}(0)\,\xi^*\left(Z_1^{-1}(0)\,\eta_1, Z_2^{-1}(0)\,\eta_2, \delta, t\right)\eta_i\right]^{\frac{\gamma_i}{\gamma_i-1}} - \frac{e^{-\rho t}}{\gamma_i}\right] \\ &+Z_1^{-1}(0)\,\xi^*\left(Z_1^{-1}(0)\,\eta_1, Z_2^{-1}(0)\,\eta_2, \delta, t\right)\eta_1\delta + \frac{\partial J}{\partial t} \qquad (74)\\ +\frac{\partial J}{\partial \delta}\overline{\mu}\delta + \frac{1}{2}\frac{\partial^2 J}{\partial \delta^2}\overline{\sigma}^2\delta^2 &+ \frac{1}{2}\sum_{i=1,2}\sum_{j=1,2}J_{ij}\nu_i Z_i^{-1}(0)\,\eta_i\nu_j Z_j^{-1}(0)\,\eta_j - \sum_{j=1,2}\frac{\partial J_i}{\partial \delta}\nu_i Z_i^{-1}(0)\,\eta_i\overline{\sigma}\delta\end{aligned}$$

the first-order conditions are:

$$\sum_j\left[\widehat{J}_{ij} + \left[\sum_k \widehat{J}_{ik}Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right]\times\frac{\xi_j^*}{\xi^*}\right]\nu_j Z_j^{-1}(0)\,\xi^*\eta_j = \left[\frac{\partial \widehat{J}_i}{\partial \delta} + \left(\sum_k \widehat{J}_{ik}Z_k^{-1}(0)\,\eta_k + \frac{\widehat{J}_i}{\xi^*}\right)\frac{\partial \xi^*}{\partial \delta}\right]\overline{\sigma}\delta; i = 1, \tag{75}$$

the market for the risky security. Our analog is denoted ν_1. Indeed, the risky security is priced by agents of Category 1 only. The common component ξ of state prices does not price the risky security. The riskless security is the only one the two categories of agents have to agree on.

[20]
$$e^{-\rho t}c_i^{\gamma_i-1} = Z_i^{-1}(0)\,\xi\eta_i \tag{72}$$

while the restriction (41) reads:[21]

$$-\sum_{i=1,2}\frac{\nu_i\left[e^{\rho t}Z_i^{-1}(0)\,\xi^*\eta_i\right]^{\frac{1}{\gamma_i-1}}}{(\gamma_i-1)}=\overline{\sigma}\delta. \tag{81}$$

Given the fact that $\widehat{J}_1+\widehat{J}_2\equiv 0$, it is possible to write $\widehat{J}$ as a function of the difference in the Z^{-1}s. Furthermore, for purposes of numerical implementation, it is somewhat more convenient to work in terms of an undiscounted value function and undiscounted arguments. Let us, therefore, introduce a function H: $e^{-\rho t}H\left(\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\xi,\delta,t\right)\triangleq\widehat{J}\left(Z_1^{-1}(0)\,\eta_1\xi,Z_2^{-1}(0)\,\eta_2\xi,\delta,t\right)$.[22] Let H' denote the derivative of the function H with respect to its first argument.

[21] In the special case of Basak and Cuoco in which investor 2 (receiving no endowment) is a log investor ($\gamma_2=0$), we can anticipate part of the solution of the problem. From (51) and the first-order conditions for consumption of a log investor, it follows that:

$$\widehat{J}_2\left(Z_1^{-1}(0)\,\eta_1\xi,Z_2^{-1}(0)\,\eta_2\xi^*,\delta,t\right)=-\frac{1}{Z_2^{-1}(0)\,\eta_2e^{\rho t}\xi}\frac{1}{\rho}\left[e^{-\rho(T-t)}-1\right] \tag{76}$$

so that:

$$\left[\widehat{J}_{21}+\left[\sum_{k=1,2}\widehat{J}_{2k}Z_k^{-1}(0)\,\eta_k+\frac{\widehat{J}_2}{\xi^*}\right]\times\frac{\xi_1^*}{\xi^*}\right]Z_1^{-1}(0)\,\xi^*\eta_1=\left[\widehat{J}_{22}Z_2^{-1}(0)\,\eta_2+\frac{\widehat{J}_2}{\xi^*}\right]\times\frac{\xi_1^*}{\xi^*}Z_1^{-1}(0)\,\xi^*\eta_1 \tag{77}$$

$$=0 \tag{78}$$

$$\left[\frac{\partial\widehat{J}_2}{\partial\delta}+\left(\sum_{k=1,2}\widehat{J}_{ik}Z_k^{-1}(0)\,\eta_k+\frac{\widehat{J}_2}{\xi^*}\right)\times\frac{\partial\xi^*}{\partial\delta}\right]=\frac{\partial\widehat{J}_2}{\partial\delta}+\left(\widehat{J}_{22}Z_2^{-1}(0)\,\eta_2+\frac{\widehat{J}_2}{\xi^*}\right)\times\frac{\partial\xi^*}{\partial\delta}=0 \tag{79}$$

Substituting into (70) gives: $\nu_2=0$. As a result of the restriction (81):

$$\nu_1=\overline{\sigma}\,(1-\gamma_1)\,\frac{\delta}{c_1}. \tag{80}$$

[22] The stochastic differential equation for $\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\xi^*$ is:

$$\begin{aligned}
d\left[\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\xi^*\right]&=\Big\{\rho\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\xi^*\\
&+\frac{1}{2}\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\left[\frac{\partial\xi^*}{\partial\delta}\overline{\mu}\delta+\xi_{11}^*Z_1^{-1}(0)\,\eta_1\nu_1^2\right.\\
&+2\xi_{12}^*Z_1^{-1}(0)\,\eta_1\nu_1\xi_2^*Z_2^{-1}(0)\,\eta_2\nu_2+\xi_{22}^*\left(\xi_2^*Z_2^{-1}(0)\,\eta_2\nu_2\right)\\
&\left.+\frac{\partial^2\xi^*}{\partial\delta^2}(\overline{\sigma}\delta)^2-2\frac{\partial\xi_1^*}{\partial\delta}\overline{\sigma}\delta Z_1^{-1}(0)\,\eta_1\nu_1-2\frac{\partial\xi_2^*}{\partial\delta}\overline{\sigma}\delta Z_2^{-1}(0)\,\eta_2\nu_2\right]\\
&-\left(Z_1^{-1}(0)\,\eta_1\nu_1-Z_2^{-1}(0)\,\eta_2\nu_2\right)e^{\rho t}\left(-\xi_1^*Z_1^{-1}(0)\,\eta_1\nu_1-\xi_2^*Z_2^{-1}(0)\,\eta_2\nu_2+\frac{\partial\xi^*}{\partial\delta}\overline{\sigma}\delta\right)\Big\}\,dt\\
&+\Big[-\left(Z_1^{-1}(0)\,\eta_1\nu_1-Z_2^{-1}(0)\,\eta_2\nu_2\right)e^{\rho t}\xi^*\\
&+\left(Z_1^{-1}(0)\,\eta_1-Z_2^{-1}(0)\,\eta_2\right)e^{\rho t}\left(-\xi_1^*Z_1^{-1}(0)\,\eta_1\nu_1-\xi_2^*Z_2^{-1}(0)\,\eta_2\nu_2+\frac{\partial\xi^*}{\partial\delta}\overline{\sigma}\delta\right)\Big]\,dw
\end{aligned} \tag{82}$$

Then the PDE is:

$$
\begin{aligned}
0 = \sum_{i=1,2} & \left[\left(\frac{1}{\gamma_i} - 1 \right) \left[e^{\rho t} Z_i^{-1}(0) \xi^* \eta_i \right]^{\frac{\gamma_i}{\gamma_i - 1}} - \frac{1}{\gamma_i} \right] \\
& + Z_1^{-1}(0) e^{\rho t} \xi^* \eta_1 \delta - \rho H + \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \delta} \overline{\mu} \delta + \frac{1}{2} \frac{\partial^2 H}{\partial \delta^2} \overline{\sigma}^2 \delta^2 \\
+ H' \times & \left\{ \rho \left(Z_1^{-1}(0) \eta_1 - Z_2^{-1}(0) \eta_2 \right) e^{\rho t} \xi^* + \frac{1}{2} \left(Z_1^{-1}(0) \eta_1 - Z_2^{-1}(0) \eta_2 \right) \left[e^{\rho t} \frac{\partial \xi^*}{\partial \delta} \overline{\mu} \delta + e^{\rho t} \xi_{11}^* \left(Z_1^{-1}(0) \eta_1 \nu_1 \right)^2 \right. \right. \\
& + 2 e^{\rho t} \xi_{12}^* Z_1^{-1}(0) \eta_1 \nu_1 Z_2^{-1}(0) \eta_2 \nu_2 + e^{\rho t} \xi_{22}^* \left(Z_2^{-1}(0) \eta_2 \nu_2 \right)^2 \\
& \left. + e^{\rho t} \frac{\partial^2 \xi^*}{\partial \delta^2} (\overline{\sigma} \delta)^2 - 2 e^{\rho t} \frac{\partial \xi_1^*}{\partial \delta} \overline{\sigma} \delta Z_1^{-1}(0) \eta_1 \nu_1 - 2 e^{\rho t} \frac{\partial \xi_2^*}{\partial \delta} \overline{\sigma} \delta Z_2^{-1}(0) \eta_2 \nu_2 \right] \\
& \left. - \left(Z_1^{-1}(0) \eta_1 \nu_1 - Z_2^{-1}(0) \eta_2 \nu_2 \right) \left(-e^{\rho t} \xi_1^* Z_1^{-1}(0) \eta_1 \nu_1 - e^{\rho t} \xi_2^* Z_2^{-1}(0) \eta_2 \nu_2 + e^{\rho t} \frac{\partial \xi^*}{\partial \delta} \overline{\sigma} \delta \right) \right\} \\
& + \frac{1}{2} H'' \times \left[- \left(Z_1^{-1}(0) \eta_1 \nu_1 - Z_2^{-1}(0) \eta_2 \nu_2 \right) e^{\rho t} \xi^* \right. \\
& \left. + \left(Z_1^{-1}(0) \eta_1 - Z_2^{-1}(0) \eta_2 \right) \left(-e^{\rho t} \xi_1^* Z_1^{-1}(0) \eta_1 \nu_1 - e^{\rho t} \xi_2^* Z_2^{-1}(0) \eta_2 \nu_2 + e^{\rho t} \frac{\partial \xi^*}{\partial \delta} \overline{\sigma} \delta \right) \right]^2 \\
& + \frac{\partial H'}{\partial \delta} \left[- \left(Z_1^{-1}(0) \eta_1 \nu_1 - Z_2^{-1}(0) \eta_2 \nu_2 \right) e^{\rho t} \xi^* \right. \\
& \left. + \left(Z_1^{-1}(0) \eta_1 - Z_2^{-1}(0) \eta_2 \right) \left(-e^{\rho t} \xi_1^* Z_1^{-1}(0) \eta_1 \nu_1 - e^{\rho t} \xi_2^* Z_2^{-1}(0) \eta_2 \nu_2 + e^{\rho t} \frac{\partial \xi^*}{\partial \delta} \overline{\sigma} \delta \right) \right] \overline{\sigma} \delta
\end{aligned} \tag{83}
$$

while the system of equations for ν is:

$$
\begin{bmatrix}
\begin{array}{c} \left\{ H'' + \left[H'' \times \left(Z_1^{-1}(0) \eta_1 \right. \right. \right. \\ \left. \left. - Z_2^{-1}(0) \eta_2 \right) + \frac{H'}{e^{\rho t} \xi^*} \right] \times \left. \frac{\xi_1^*}{\xi^*} \right\} \\ \times Z_1^{-1}(0) e^{\rho t} \xi^* \eta_1 \end{array} & \begin{array}{c} \left\{ -H'' + \left[H'' \times \left(Z_1^{-1}(0) \eta_1 \right. \right. \right. \\ \left. \left. - Z_2^{-1}(0) \eta_2 \right) + \frac{H'}{e^{\rho t} \xi^*} \right] \times \left. \frac{\xi_2^*}{\xi^*} \right\} \\ \times Z_2^{-1}(0) e^{\rho t} \xi^* \eta_2 \end{array} \\
\frac{\left[e^{\rho t} Z_1^{-1}(0) \xi^* \eta_1 \right]^{\frac{1}{\gamma_1 - 1}}}{(1 - \gamma_1)} & \frac{\left[e^{\rho t} Z_2^{-1}(0) \xi^* \eta_2 \right]^{\frac{1}{\gamma_2 - 1}}}{(1 - \gamma_2)}
\end{bmatrix}
\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}
= \begin{bmatrix} \frac{\partial H'}{\partial \delta} + \left[H'' \times \left(Z_1^{-1}(0) \eta_1 - Z_2^{-1}(0) \eta_2 \right) + \frac{H'}{e^{\rho t} \xi^*} \right] e^{\rho t} \frac{\partial \xi^*}{\partial \delta} \\ 1 \end{bmatrix} \overline{\sigma} \delta \tag{84}
$$

TO BE COMPLETED

6.2 Endogenizing the diffusion matrix of all securities

Once a solution has been obtained for $\xi^*(\{\eta_i\}, \{e_i\}, Y, t)$ and $\nu_i^*(\{\eta_i\}, \{e_i\}, Y, t)$, the entire process ξ is known.

Define the price of a risky security paying a cash flow stream ι by:[23]

$$S(t) = \frac{1}{\xi(0,t)} E\left[\int_t^T \iota(s)\,\xi(0,s)\,ds|\mathbf{F}_t^w\right] \tag{85}$$

If the cash flow stream is specified as $\iota(Y,t)$, then by the Feynmann-Kac formula, the process S can be obtained as a function $S(\{\eta_i\},\{e_i\},Y,t)$ satisfying the partial differential equation:

$$0 = \xi^*(\{\eta_i\},\{e_i\},Y,t)\,\iota(Y,t) + \mathcal{D}^{\{\nu_i\}}\{\xi^*(\{\eta_i\},\{e_i\},Y,t) \times S(\{\eta_i\},\{e_i\},Y,t)\} \tag{86}$$

where the operator $\mathcal{D}^{\{\nu_i\}}$ is defined by (56). This partial differential equation involves neither κ nor r explicitly but the application of the operator $\mathcal{D}^{\{\nu_i\}}$ to $\xi^*(\{\eta_i\},\{e_i\},Y,t)$ generates them indirectly, as we have seen in Equations (38) and following.

When this equation is solved, Itô's lemma provides the diffusion matrix of all traded securities.

It is important to realize that the PDE (86) can be solved recursively alongside the PDE (55) for the function J. In general, the diffusion matrix of traded securities is needed to impose the kernel condition on ν. We just showed how that matrix can be made available at each time step, as needed.

Our numerical illustration is special in that the volatility of the risky security is not needed to write the kernel condition and the PDE for the value function J. However, the general case could also be handled recursively by pricing numerically and recursively the risky security, which is defined by Basak and Cuoco as the security that gives title to the endowment stream δ.

7 Conclusion

We present a methodology for solving the competitive equilibria of economies with dynamically incomplete markets and heterogeneous agents. The nature of the algorithmic device we propose, a central planner with two selves, provides new insights regarding the fundamental difference between economies with and without complete financial markets. The first central Planner essentially solves for individual consumptions, portfolios and investor-specific components of state prices in the sense of He and Pearson, given economy-wide state prices. In other words, he solves a partial equilibrium problem. Simultaneously, the second Planner chooses equilibrium state prices to satisfy the aggregate resource restriction, given the individual-specific choices of Planner 1. Planner 2 acts like an (intertemporal) auctioneer. It is crucial that Planner 2 internalizes the investor-specific components of the state prices of Planner 1 in his choices. This makes the two Planners agree on consumption and generates equilibrium.

Our analysis is reminiscent of the work of Grossman (1977) who studied equilibria in multi-good economies with incomplete markets. He analyzed the

[23] Here the definition of one share of the security is based on the dividend flow whereas in (7), we had imposed $S(0) = 1$. This is just a different normalization.

welfare properties of these equilibria by introducing the notion of a central planner with incomplete coordination. Instead of exploring welfare properties, we pursue a similar construction in order to solve for the competitive equilibrium in a multi-period economy.

In a Markovian setting, we establish a recursive formulation of the two-central planner problem. The equilibrium can, therefore, be constructed one time step at a time, using standard dynamic programming techniques.

We believe our methodology has numerous interesting applications in dynamic asset pricing and the analysis of risk-sharing, beyond the confines of the standard complete market paradigm. We plan to pursue these applications in future research. In future work we also aim to extend our methodology to handle the general case of limited participation, where asset markets are incomplete in different ways for different individuals.

Appendix: the derivatives of ξ^* in the numerical example

$$e^{\rho t}\xi_j^* = -\frac{\left(\gamma_j - 1\right)^{-1}\left[e^{\rho t}Z_j^{-j}\left(0\right)\xi\eta_j\right]^{-\frac{\gamma_j-2}{\gamma_j-1}}e^{\rho t}\xi^*}{\sum_{i=1,2}\left(\gamma_i - 1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i} \tag{87}$$

$$e^{\rho t}\frac{\partial\xi^*}{\partial\delta} = \frac{1}{\sum_{i=1,2}\left(\gamma_i - 1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i} \tag{88}$$

$$\begin{aligned}
e^{\rho t}\xi_{jk}^* = &-\frac{1}{\sum_{i=1,2}\left(\gamma_i - 1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i} \\
&\times\left[-\frac{\gamma_j-2}{\left(\gamma_j-1\right)^2}\left[e^{\rho t}Z_j^{-j}\left(0\right)\xi\eta_j\right]^{-\frac{\gamma_j-2}{\gamma_j-1}-1}\right. \\
&\left.\times\left(e^{\rho t}\xi\mathbf{1}_{j=k} + e^{\rho t}Z_j^{-j}\left(0\right)\xi_k^*\eta_j\right)e^{\rho t}\xi^* + \left(\gamma_j-1\right)^{-1}\left[e^{\rho t}Z_j^{-j}\left(0\right)\xi\eta_j\right]^{-\frac{\gamma_j-2}{\gamma_j-j}}e^{\rho t}\xi_k^*\right] \\
&+\frac{\left(\gamma_j - 1\right)^{-1}\left[e^{\rho t}Z_j^{-j}\left(0\right)\xi\eta_j\right]^{-\frac{\gamma_j-2}{\gamma_j-1}}e^{\rho t}\xi^*}{\left[\sum_{i=1,2}\left(\gamma_i - 1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i\right]^2} \\
&\times\left[-\sum_{i=1,2}\left(\frac{\gamma_i-2}{\left(\gamma_i-1\right)^2}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}-1}\right.\right. \\
&\left.\left.\times\left(e^{\rho t}\xi^*\mathbf{1}_{i=k} + e^{\rho t}Z_i^{-1}\left(0\right)\xi_k^*\eta_i\right)Z_i^{-1}\left(0\right)\eta_i\right) + \left(\gamma_k-1\right)^{-1}\left[e^{\rho t}Z_k^{-1}\left(0\right)\xi\eta_k\right]^{-\frac{\gamma_k-2}{\gamma_k-1}}\right]
\end{aligned} \tag{89}$$

$$e^{\rho t}\frac{\partial \xi_k^*}{\partial \delta} = -\frac{1}{\left[\sum_{i=1,2}\left(\gamma_i-1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i\right]^2}$$

$$\times\left[-\sum_{i=1,2}\left(\frac{\gamma_i-2}{\left(\gamma_i-1\right)^2}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}-1}\right.\right.$$

$$\left.\left.\times\left(e^{\rho t}\xi^*\mathbf{1}_{i=k}+e^{\rho t}Z_i^{-1}\left(0\right)\xi_k^*\eta_i\right)Z_i^{-1}\left(0\right)\eta_i\right)+\left(\gamma_k-1\right)^{-1}\left[e^{\rho t}Z_k^{-1}\left(0\right)\xi\eta_k\right]^{-\frac{\gamma_k-2}{\gamma_k-1}}\right] \tag{90}$$

$$e^{\rho t}\frac{\partial^2 \xi^*}{\partial \delta^2} = -\frac{1}{\left[\sum_{i=1,2}\left(\gamma_i-1\right)^{-1}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}}Z_i^{-1}\left(0\right)\eta_i\right]^2}$$

$$\times\left[-\sum_{i=1,2}\frac{\gamma_i-2}{\left(\gamma_i-1\right)^2}\left[e^{\rho t}Z_i^{-1}\left(0\right)\xi\eta_i\right]^{-\frac{\gamma_i-2}{\gamma_i-1}-1}\right.$$

$$\left.\times\left(e^{\rho t}Z_i^{-1}\left(0\right)\frac{\partial \xi^*}{\partial \delta}\eta_i\right)Z_i^{-1}\left(0\right)\eta_i\right] \tag{91}$$

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