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Chaotic Interest Rate Rules*
(PRELIMINARY)

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Abstract

A growing empirical and theoretical literature argues in favor of specifying monetary policy in the form of Taylor-type interest rate feedback rules. That is, rules whereby the nominal interest rate is set as an increasing function of inflation with a slope greater than one around an intended inflation target. This paper shows that because of the zero bound on nominal interest rates, such rules can easily lead to chaotic dynamics. The result is obtained in the context of standard and well-calibrated monetary environments. Furthermore, the battery of fiscal policies that have recently been advocated for avoiding liquidity traps are shown to be unlikely to provide a remedy for the kind of dynamics characterized in this paper. JEL Classification Numbers: E52, E31, E63.
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1 Introduction

Monetary policy rules that set the nominal rate of interest as a function of the inflation rate can generate multiple equilibria and indeterminacies. (See for example Benhabib, Schmitt-Grohé and Uribe (2000), (2001a), (2001b) or Carlstrom and Fuerst (2001)). Such multiplicities can arise from the existence of alower bound on nominal interest rates and give rise to liquidity traps, or can be purely local, for axample giving rise to a continuum of equilibria converging to the targeted steady state or to a cycle around it. While the design of simple non-Ricardian monetary-fiscal regimes may eliminate the multiplicity of equilibria that arise from the existence of a lower bound on nominal rates and that converge to a liquity trap, it is harder to design policies that eliminate multiplicities that are local in nature, and which for example give rise to a contiuum of equilibria that converge to or remain in a neighborhood of the targeted inflation rate, whether inder active or passive monetary policy.

In this paper we show, in the context of the simplest flexible price economy where money is productive and calibrated to US data and in particular to standard money demand specifications, that Taylor rules can very easily give rise to cyclic and chaotic trajectories in interest rates and inflation. We then explore alternative specifications of the model and policies to investigate whether multiplicities persist or can be avoided.

2 The economic environment

2.1 Households

Consider an economy populated by a large number of infinitely-lived agents with preferences over streams of consumption and described by the following utility function:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \tag{1}$$

Agents have access to two financial assets: fiat money, M_t, and government bonds, B_t. Government bonds B_t, held between periods $t-1$ and t, pay the gross nominal interest rate R_t at the beginning of period t. Note that the nominal rate R_t has to be set at $t-1$. In addition, agents receive a stream of real income y_t and pay real lump-sum taxes τ_t. The budget constraint of

the representative household is then given by

$$M_{t+1} + B_{t+1} = M_t + B_t(1 + R_t) + p_t y\left(\frac{M_{t+1}}{p_t}\right) - p_t c_t + p_t \tau_t$$

where P_t denotes the price level in period t.[1] This budget constraint can be rearranged as:

$$c_t = \frac{M_t}{p_t} + \frac{B_t}{p_t}(1 + R_t) + F\left(\frac{M_{t+1}}{p_t}\right) + \tau_t - \frac{M_{t+1}}{p_t} - \frac{B_{t+1}}{p_t} \tag{2}$$

Let $a_t \equiv (M_t + B_t)/p_t = m_t + b_t$ and let the inflation rate be defined by $\pi_t = \frac{p_{t+1}}{p_t}$.To prevent Ponzi games, households are subject to a borrowing constraint of the form

$$\lim_{t\to\infty} \frac{a_t}{\prod_{j=0}^{t-1}(R_j/\pi_{j+1})} \geq 0 \tag{3}$$

We motivate a demand for money by assuming that real balances facilitate firms transactions as in Calvo (1979), Fischer (1974), and Taylor (1977). Specifically, we assume that output is the CES function of real balances and a constant endowment $\bar{y}$.[2]

$$y_{t+1} = F\left(\frac{M_{t+1}}{p_t}\right) = \left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{1}{\rho}} \tag{4}$$

Households choose sequences $\{c_t\}_{t=0}^{\infty}$ $\{M_t,\ B_t\}_{t=1}^{\infty}$ so as to maximize the utility function (1) subject to (2)-(4), given M_0 and B_0. The first-order optimality conditions for interior solutions are the constraints (2)-(4) holding with equality and the Euler equations with repect to bonds and money:

$$c_t^{-\sigma} = \frac{\beta\,(1 + R_{t+1})}{\pi_t}\, c_{t+1}^{-\sigma} \tag{5}$$

$$c_t^{-\sigma}\left(1 - F'\left(\frac{M_{t+1}}{p_t}\right)\right) = \frac{\beta}{\pi_t}\, c_{t+1}^{-\sigma} \tag{6}$$

[1] Note that net output here depends on end of the period money balances. Net output, money and principal plus interest from bonds can finance end of the period consumption and the purrchase of bonds and money holdings for the next period.

[2] Under this formulation one may view real balances either as directly productive or as decreasing the transaction costs of exchange and increasing net output. It is also possible to replace the endowment output $\bar{y}$ with a function increasing in labor supply, and add leisure to the utility function. The current formulation then would correspond to the case of an inelastic labor supply.

The implied money demand equation then is given by

$$\left(1 - F'\left(\frac{M_{t+1}}{p_t}\right)\right)^{-1} = \left(1 - a\left(\left(\frac{M_{t+1}}{p_t}\right)^{-1} y_{t+1}\right)^{1-\rho}\right)^{-1} = 1 + R_{t+1} \tag{7}$$

for which real balances are a decreasing function of the nominal rate of interest R.[3] Note that the interest elasticity of real balances in this formulation is $1-\rho$, which is the reason that we prefer it to a Cobb-Douglas specification with unitary interest elasticity. Finally the transversality condition for the agent, the flip side of the no-Ponzi condition is given by:

$$\lim_{t\to\infty} \frac{a_t}{\prod_{j=0}^{t-1}(R_j/\pi_{j+1})} \leq 0 \tag{8}$$

2.2 The monetary/fiscal regime

Following a growing recent empirical literature that has attempted to identify systematic components in monetary policy, we postulate that the government conducts monetary policy in term of an interest rate feedback rule of the form $R_t = \rho(\pi_{t-1})$. We impose three conditions of the feedback rule: First, in the spirit of Taylor (1993) we assume that monetary policy is active around a target rate of inflation π^*; that is, the interest elasticity of the feedback rule at π^* is greater than unity, or $\rho'(\pi^*)\pi^*/R^* > 1$. Second, we impose that the feedback rule satisfy the zero bound on nominal interest rates, $\rho(\pi) \geq 0$ for all π. Finally, we assume that the feedback rule is non-decreasing, $\rho'(\pi) \geq 0$ for all π. For analytical and computational purposes, we will focus on the following specific parameterization of the Taylor-type rule:

$$R_t = R^*\left(\frac{\pi_{t-1}}{\pi^*}\right)^{\frac{A}{R^*}}; \quad a\beta > 1, \tag{9}$$

where $1 + R \equiv \pi/\beta > 1$. Under this rule the nominal rate R_t on bonds B_t, set by the central bank in period $t-1$, depends (in the terminology of Carlstrom and Fuerst (2000)) on the forward-looking inflation rate between

[3] We note that an interior solution may require a lower bound on real balances $\left(\frac{M_{t+1}}{p_t}\right)$ so that $1 - a\left(\left(\frac{M_{t+1}}{p_t}\right)^{-1} y_{t+1}\right)^{1-\rho} \geq 0$. This becomes important below when we consider the dynamics of equilibrium trajectories in terms of $\left(\frac{M_{t+1}}{p_t}\right)$, since monetary policy is defined for non-negative nominal rates only.

$t-1$ and t.[4] Inverting this equation, we obtain:

$$\pi_t = \pi^* \left(\frac{R_{t+1}}{R^*} \right)^{\frac{R^*}{\lambda}} \tag{10}$$

Each period the government faces the budget constraint $M_t + B_t = M_{t-1} + (1 + R_{t-1}) B_{t-1} - P_t \tau_t$. This constraint can be written in real terms in the following form:

$$a_t = \frac{1 + R_{t-1}}{\pi_t} a_{t-1} - \frac{R_{t-1}}{\pi_t} m_{t-1} - \frac{\tau_t}{\pi_t} \tag{11}$$

This expression states that total government liabilities in period t are given by liabilities carried over from the previous period, including interest, minus total consolidated revenues. Consolidated government revenues, in turn, have two components, regular taxes and seignorage revenue. We assume that the fiscal regime consists of setting consolidated government revenues as a fraction of total government liabilities. Formally,

$$\frac{R_{t-1}}{\pi_t} m_{t-1} + \frac{\tau_t}{\pi_t} = \omega(\pi_t) a_{t-1}; \tag{12}$$

where $\omega(\cdot)$ is a non-decrerasing function of π. Combining the above two expression we obtain:

$$a_t = \left(\frac{R_{t-1}}{\pi_t} - \omega(\pi_t) \right) a_{t-1} \tag{13}$$

If $\omega(\cdot) > 0$ for all π, this expression implies that

$$\lim_{t \to \infty} \frac{a_t}{\prod_{j=0}^{t-1} (R_j / \pi_{j+1})} = 0. \tag{14}$$

Therefore, the assumed fiscal policy ensures government solvency and that the household's intertemporal borrowing constraint holds with equality under all circumstances. On the other hand, if $lim_{t \to \infty} \pi_t = \pi^p$ along a candidate equilibrium path and $\omega(\pi^p) < 0$, then such a path will not satisfy transversality conditions because the limiting value of discounted government liabilities held by the public does not converge to zero. Such a fiscal policy, with $\omega(\pi^p) < 0$, will be locally non-Ricardian.

[4] A number of authors have argued that in the post-Volker era, U.S. monetary policy is best described as incorporating a forward-looking component (Clarida et al., 1997).

2.3 Equilibrium

Combining equations (2) and (11) implies that the goods market clears at all times:

$$y_t = c_t \tag{15}$$

Combining (4), (7), (6), (10) and (15) yields

$$\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}} = \frac{\beta\left(1+R_{t+1}\right)}{\pi\left(\frac{R_{t+1}}{R^*}\right)^{\frac{R^*}{A}}}\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}} \tag{16}$$

If we use (7) to eliminate the nominal rate R_{t+1}, we obtain a difference equation in M:

$$\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}} \tag{17}$$

$$= \frac{\beta\left(1 - a\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t+1}}{p_t}\right)^{\rho-1}\right)^{-1}}{\pi^*\left(\frac{\left(1-a\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho}+(1-a)\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t+1}}{p_t}\right)^{\rho-1}\right)^{-1}-1}{R^*}\right)^{\frac{R^*}{A}}}\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

If we define $x_{t+1} = \left(\frac{M_{t+1}}{p_t}\right)^{\rho}$ then the above equation can be written as an explicit difference equation:

$$x_{t+2} \tag{18}$$

$$= a^{-1}\left(ax_{t+1} + (1-a)\,\bar{y}^{\rho}\right)\left(\frac{\beta\left(1 - a\left(ax_{t+1} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(x_{t+1}\right)^{\frac{\rho-1}{\rho}}\right)^{-1}}{\pi\left(\frac{\left(1-a\left(ax_{t+1}+(1-a)\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(x_{t+1}\right)^{\frac{\rho-1}{\rho}}\right)^{-1}-1}{R^*}\right)^{\frac{R^*}{A}}}\right)^{\frac{\rho}{\sigma}} - a^{-1}(1-a)\,\bar{y}^{\rho}$$

We are now ready to define an equilibrium real allocation.

Definition 1 *An equilibrium real allocation is a sequence* $\{\left(\frac{M_{t+1}}{p_t}\right), \pi_t, a_t\}_{t=1}^{\infty}$ *satisfying* $R_t > 0$, *(9), (13), (14) and (18), given* a_0.

Given a_0 and a pair of sequences $\{\left(\frac{M_{t+1}}{p_t}\right)_t, \pi_t\}_{t=1}^{\infty}$satisfying satisfying (16), if we also specify that $\omega(\cdot) > 0$, equation (13) gives rise to a sequence $\{a_t\}_{t=1}^{\infty}$ that satisfies the transversality condition (8). Thus, in such a case we can limit attention to the existence of sequences $\{\left(\frac{M_{t+1}}{p_t}\right), \pi_t\}_{t=1}^{\infty}$ satisfying (16) and (9). On the other hand, if $\omega(\pi) < 0$ for some π,some of the candidate equilibrium trajectories will fail to satisfy transversality, and may be ruled out.

2.4 Steady-state equilibria

Consider constant solutions $x_t = x^{ss} > 0$ to the difference equation (18). Because x_t is not predetermined in period t (i.e., x_t is a "jump" variable), such solutions represent equilibrium real allocations. The steady state solutions can be more easily constructed first in terms of the nominal rate $R_t = R^{ss}$ using equation (16). The steady state solutions π^{ss} and $R^{ss} = R^* \left(\frac{\pi}{\pi^*}\right)^{\frac{A}{R^*}}$ must satisfy

$$1 + R^* \left(\frac{\pi}{\pi^*}\right)^{\frac{A}{R^*}} = \beta^{-1}\pi \tag{19}$$

By construction one such pair is (π^*, R^*).There exists however another solution, (π^p, R^p) ,where π^p solves (19), with $\pi^p < \pi^*$ and $0 < R^p < R^*$ and $\pi^p \equiv \beta(1+R^p)$. It is straightforward to show that at the steady state equilibrium R^* monetary policy is active ($\rho'(\pi^*) > 1$), whereas at the steady-state equilibrium R^p monetary policy is passive ($\rho'(\pi^p) < 1$). The corresponding steady state values of x can be obtained by solving (18):

$$x^{ss} = \left(\left(\left(a^{-1}\left(1 + R^{ss}\right)^{-1}\right)^{-\frac{\rho}{1-\pi^{ss}}} - a\right)\left((1-a)\,\bar{y}^{-\rho}\right)\right)^{-\frac{1}{\rho}}$$

for $R^{ss} = R^p, R^*$, with $x^* < x^p$.

If we specify that $\omega(\cdot) > 0$, then (14) for all paths, and both steady states are equilibria. Since x is not predetermined, this already implies indeterminacy because either of the steady states can constitute an equilibrium. However as discussed in Benhabib, Schmitt-Grohé, and Uribe (2000), it is possible to rule out equilibria that converge to (π^p, R^p) by specifying $\omega(\cdot)$ such that $\omega(R^*) > 0$ and $\omega(R^p) < 0$.A simple policy that implements such

an $\omega(\cdot)$ is a policy that sets the growth of nominal government liabilities, $A = B + M$ to an approipriate constant rate k such that $R^{\rho} < k < R^{*}$.(See Woodford(1999) and Benhabib, Schmitt-Grohé, and Uribe (2000)). However, as we will show below, such a locally non-Ricardian policy may be ineffective if paths that diverge away from steady states converge to other periodic or non-periodic sets.

2.5 Non-steady state equilibria

2.5.1 Chaos

Let $z_t = ax_t + (1-a)\bar{y}^{\rho}$ This implies that $x_t = a^{-1}(z_t - (1-a)\bar{y}^{\rho}) \geq 0$, so the domain of z_t must be suitably restricted to maintain the non-negativity of real balances. Then equation (18) can be written as:

$$z_{t+1} = \left(\left(\frac{\beta R^{\frac{R}{A}}}{\pi} \right) \frac{\left(1 - a \left(\frac{az_t}{z_t - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1}}{\left(\left(1 - a \left(\frac{az_t}{z_t - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1} - 1 \right)^{\frac{R}{A}}} \right)^{\frac{\rho}{\sigma}} z_t \tag{20}$$

Let $q_t = \ell n\, z_t$, so that $z_t = e^{q_t}$,and let the natural log of the expression in brackets as $f(z_t) = f(e^{q_t})$. Then we have

$$\begin{aligned} q_{t+1} &= q_t + \frac{\rho}{\sigma} \ell n \left(\left(\frac{\beta R^{\frac{R}{A}}}{\pi} \right) \frac{\left(1 - a \left(\frac{ae^{q_t}}{e^{q_t} - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1}}{\left(\left(1 - a \left(\frac{ae^{q_t}}{e^{q_t} - (1-a)\bar{y}^{\rho}} \right)^{\frac{1-\rho}{\rho}} \right)^{-1} - 1 \right)^{\frac{R}{A}}} \right) \\ q_{t+1} &= q_t + \frac{\rho}{\sigma} \left(h\left(e^{q_t}\right) \right) \equiv q_t + \left(\frac{-\rho}{\sigma} \right) \left(f\left(q_t\right) \right) \end{aligned} \tag{21}$$

where we defined $h(e^{q_t}) \equiv -f(q_t)$.We can now apply the results of Yamaguti and Matano(1979), (and elaborated on in Ushiki, S., Yamaguti, M., and H. Matano(1980)), to show that (21) gives rise to chaotic equilibria. Let $\Delta \equiv \left(\frac{-\rho}{\sigma}\right)$.

Assumption 1 *Let $q_{t+1} = F_{\Delta}(q_t) = q_t + \Delta f(q_t)$ where f is continuous in R^1.Assume the differential equation $\dot{y} = f(y)$ has two stationary points, one of which is asymptotically stable.*

As pointed out in Yamaguti and Matano (1979), this assumption implies that after a linear transformation of variables, we have:

A) $f(0) = f(\bar{u}) = 0$ for some $\bar{u} > 0$.

B) $f(u) > 0$ *for* $0 < u < \bar{u}$.

C) $f(u) < 0$ *for* $\bar{u} < u < \kappa$ where the constant κ is possibly $+\infty$.

Note that the two steady states x^* and x^p in the transformed variables q_p and q^* will correspond to the zeros of $f(q_t)$. Then defining $u_t = q_t - q_p$ provides the appropriate linear transformation required above. Furthermore, the assumption used by Yamaguti and Matano (1979) is weaker than the one above, requiring $F_\Delta(q_t)$ to have *at least* two steady states, or $f(u)$ to have *at least* two zeros. Requiring exactly two steady states, as above, assures that $\kappa = +\infty$, since otherwise there would be some $\tilde{u} > \bar{u}$ such that $f(\tilde{u}) = 0$, implying the existence of a third steady state. We use the fact that our stronger assumption of two steady states implies $\kappa = +\infty$ in Theorem 2 below.

THEOREM 1(Yamaguti and Matano(1979)) : There exits a positive constant c_1 such that for any $\frac{\rho}{\sigma} > c_1$ the difference equation is chaotic in the sense of Li-Yorke (1975).

THEOREM 2(Yamaguti and Matano(1979)): Suppose the assumptions above hold and $\kappa = +\infty$. Then there exists another constant c_2, $0 < c_1 < c_2$, such that for any $0 \leq \frac{\rho}{\sigma} \leq c_2$ (where $\Delta \equiv \left(\frac{-\rho}{\sigma}\right)$), the map $F_\Delta : q_t \to q_{t+1}$ given by $q_{t+1} = q_t + \Delta f(q_t)$ has a finite interval $[0, \alpha_\Delta]$ such that F_Δ maps $[0, \alpha_\Delta]$ into itself, with $\alpha_\Delta > \bar{u}$. Moreover, if $c_1 < \frac{-\rho}{\sigma} \leq c_2$, the chaotic phenomenon in the sense of Li-Yorke (1975) occurs in this confinement interval.[5]

The theorems apply immediately to our transformed equation (21), since as discussed above, this equation has two steady states. However for an interior solution implied by the dynamics of (21) over the range $[0, \alpha_\Delta]$, we must check that real balances $\left(\frac{M_{t+1}}{p_t}\right)$ as well as the nominal rate, $R_{t+1} = \left(1 - F'\left(\frac{M_{t+1}}{p_t}\right)\right)^{-1} - 1$, remain non-negative. This implies that there is

[5] The value of c_2 can be computed as described in detail by Ushiki, S., Yamaguti, M., and H. Matano(1980). For our equation (21), the right hand side crosses the x-axis twice. We can redefine the origin to correspond to the lower zero, q_p, by transforming variables as $u = q - q_p$. Then c_2 will correspond to the highest value of Δ such that the maximum attained by the right-hand side is less than the higher zero of the right-hand side: in other words, c_2 will be the upper bound for Δ such that the right side of (21) maps an interval $[0, \alpha_\Delta]$ into itself.

a lower bound to $\left(\frac{M_{t+1}}{p_t}\right)$, and therefore that we must check whether the corresponding restrictions on the transforms of $\left(\frac{M_{t+1}}{p_t}\right)$, that is on x_t, z_t, or q_t hold along these trajectories. In the simulations below real balances as well as the nominal interest rate remain positive and interior along the computed equilibrium trajectories.

To match the interest elasticity of money in the data (see footnote 2.5.1) we choose $\rho = -9$, and we use σ as our parameter, so that $\Delta = \left(\frac{-\rho}{\sigma}\right)$. For $\sigma = 9$ so that $\Delta = 1$, the difference equation $q_{t+1} = q_t + \left(\frac{-\rho}{\sigma}\right)(f(q_t))$ given by (21) will have one stable steady state $\hat{q}$ at which $f'(\hat{q}) < 0$. As σ is reduced towards standard values, this steady state will become increasingly unstable and lead to chaotic dynamics. The simulations in the graphs below illustrate the results of Theorems 1 and 2, although these simulations are given for the equilibrium trajectories of real balances raise to the power ρ, that is by $x_t = \left(\frac{M_{t+1}}{p_t}\right)^{\rho}$ described by (18), rather than its transform $q_t = \ell n\left(ax_t + (1-a)\bar{y}^{\rho}\right)$ that is described by equation (21).

We start by calibrating the parameters of our economy. We set the target nominal rate of 6%, which for a quarterly calibration implies $R^* = 0.0015$, and the discount factor $\beta = (1.01/1.015) = 0.9951$. Since the associated target stationary inflation rate is given by $\pi^* = \beta(1+R^*)$, we have $\pi^* = 1.01$, or annual inflation target of 4%.[6] The Taylor coefficient on the interest rate at the active steady state (the slope of the Taylor rule) given by $A/\pi^* = 1.60/1.01 = 1.5842$. In our model the interest elasticity of money demand is given by $(1-\rho)^{-1}$. We set it to 0.1, which implies a value of $\rho = -9$. The coefficient a is a measure of the productive effect of money in the economy, and can be calibrated from the demand for money equation (7). We set $a = 0.000359$[7], and the constant endowment income $\bar{y} = 1$. Finally, the for the

[6] The target nominal annual rate of 6% corresponds to the average yield on 3-month T-bills over the period 1960:Q1 to 1998:Q3. The annual inflation rate of 4% matches the average growth rate of the U.S. GDP deflator during 1960:Q1-1998:Q3, which is 4.2%.

[7] The equilibrium condition (7) implies

$$\frac{R}{1+R} = a\left(\left(\frac{M_{t+1}}{p_t}\right)^{-1} y_{t+1}\right)^{\rho-1}$$

where $\frac{R}{1+R} \approx R$. Taking logs it is easy to show that

$$\frac{d\,\ell n\left(\frac{M_{t+1}}{p_t}\right)}{d\,\ell n\left(\frac{R}{1+R}\right)} = \frac{1}{\rho-1}$$

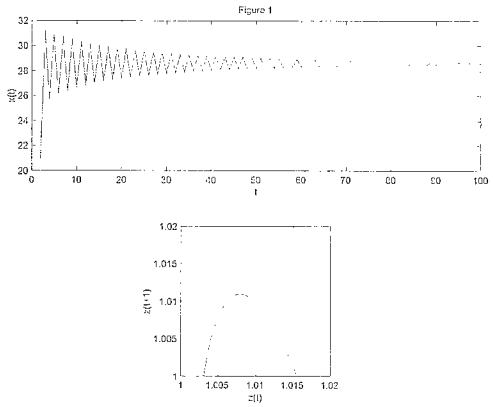

Figure 1:

intertemporal consumption elasticity σ we use the values of 2, 2.18, 2.5 and 2.75 to explore the dynamics parametrically.[8]

The upper Figure 1 below, where $\sigma = 2.75$, shows the solution to (18), for an initial value of $x(0) = 14$ which is close to the passive steady state. Note that x correponds to the value of real balances raised to the power ρ. Since $\rho < 0$, a higher value of x corresponds to lower real balances. The lower panel is a graph of (20), where z, which is a function of x,corresponds to the value of output raised to the power ρ. It is clear from the lower panel that the lower "passive" steady state with higher real balances is locally determinate, and that trajectories that start to its right must converge to the higher "active" steady state, which is locally stable and therefore indeterminate. The simulation in the upper panel illustrates the convergence to the active steady state.

In Figure 2, we set $\sigma = 2.5$,and it is clear from the lower panel that both steady states are now locally determinate: focussing on the local properties of the active steady state however will give a misleading result of a unique equilibrium. The trajectories now diverging from the active steady state converge to a cyclic equilibrium period 2 instead of the "active" steady

so that $\rho = -9$ seems reasonable. Given an annual velocity of around 5.8, or a quarterly velocity of 1.45, yields a value for $a = 0.0008597$.Some have argued that a is as high as 0.01 but we use a lower more conservative value. It is easier to obtain chaotic dynamics for higher values of a.

[8]These values correspond to values of L in the theorems above of around 3.6- 4.6.

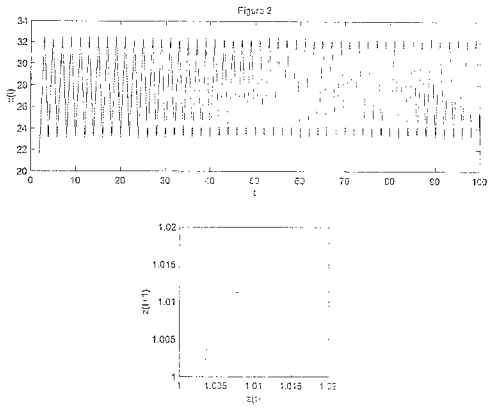

Figure 2:

state, as shown in the upper panel.

For $\sigma = 2.18$,the trajectoty is still cyclic, but of a period 4:

The transition to higher order stable cycles as we decrease σ illustrates the transition towards a chaotic equilibrium, as described by Sarkovskii's theorem. Finally in Figure 4, we set $\sigma = 2$,and the upper panel shows a chaotic equilibrium. The existence of period three cycles or which satisfy the conditions of the Li-Yorke theorem for topological chaos are east to show as well.

In certain cases, for which conditions are difficult to verify, the aperiodic chaotic equilibrium trajectories may exhibit ergodic chaos so that the $\lim_{t\to\infty} n^{-1} \sum_0^n x_t$ converges to a limit independently of the initial condition $x(0)$. This raises the possibility of eliminating chaotic trajectories as equilibria by designing the fiscal policy underlying the function $\omega(\cdot)$ appropriately , in analogy to the case where trajectories converging to the passive steady could be ruled out since they do not not respect government solvency or transversality conditions. In general however cyclic and aperiodic trajectories around the active steady state take values both above and below the steady state value, and it may not be possible to rule out such cyclic or chaotic trajectories as equilibria by the use of locally non-Ricardian fiscal policies. One possibility, along the lines discussed in Benhabib, Schmitt-Grohé and Uribe (2000) is to define a neighborhood of π^*, the inflation rate at the active steady state, in which the Taylor Rule is operative, and specify that for any π outside that neighborhood the policy rule becomes

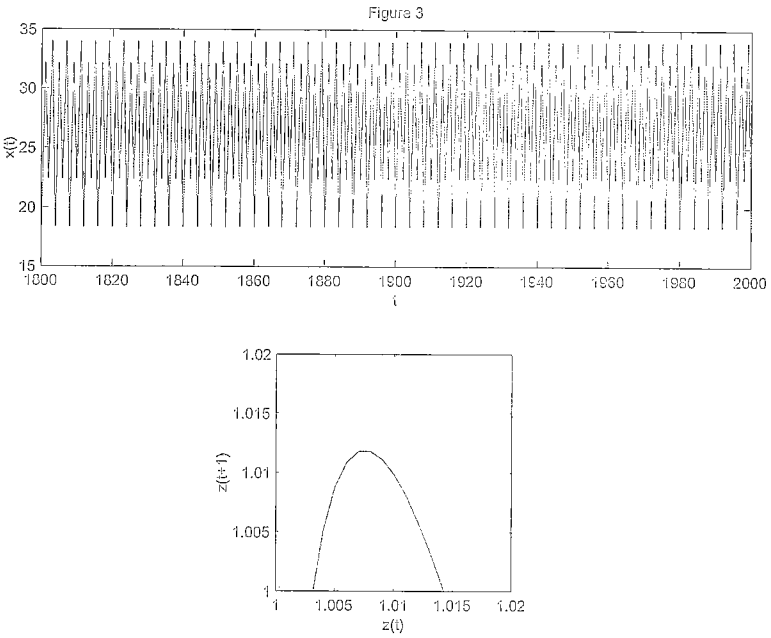

Figure 5:

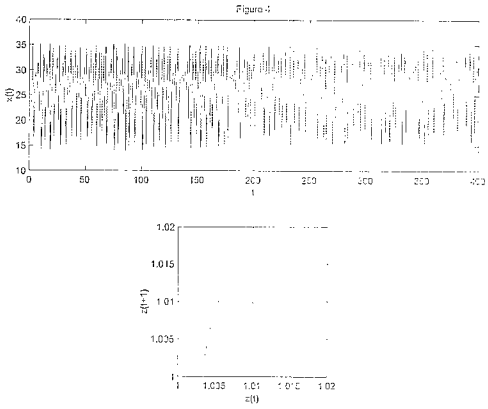

Figure 4:

non-Ricardian, for example it reverts to a constant money growth rule. This corresponds to the monetary-fiscal regime specified in equation (12), where $\omega(\pi)$ switches from positive to negative at some value of π below π^*. Any cycle or trajectory that strays out of the neigborhood of π^* where the Taylor rule is operative can then trigger a policy switch that sets it on a course that violates transversality condoitions and is therefore ruled out as a possible equilibrium.

2.6 Alternative Taylor Rules and Timing Conventions

2.6.1 A Linear Rule

It is important to realize however that the local properties of the "active" steady state do not depend on the particular Taylor rule that we adopted. Consider for example the linear Taylor rule:

$$R_t = R^* + A(\frac{\pi_{t-1} - \pi^*}{\pi^*})$$

which, at the active steady state, has the same slope $\left(\frac{A}{\pi^*}\right)$ as our previous non-linear Taylor rule given by (9) that respected the zero lower bound on the nominal rate. For $\sigma = 2.75$, for intial conditions in the neighborhood of the active steady state, this linear rule generates the same time-series picture as in the upper panel of Figure 1: the active steady state is indeterminate. For $\sigma = 2.65$ however, the active steady state is determinate, but now trajectories starting in a small neighborhood of it converge to a period 2 cycle, as in the upper panel of Figure 2. The amplitude of the cycle is zero at the bifurcation point $\hat{\sigma} \approx 2.709$, and grows as σ is decreased. Thus local indeterminacy, with convergence either to the active steady state or the cycle surrounding it, does not depend on the non-linear Taylor rule.

2.6.2 Backward Looking Rules

We have also explored alternative formulations with forward looking Taylor rules, and with money entering the utility function in various non-separable ways. In such cases another potentially serious problem emerges: the instantaneous or temporary equilibrium is often non-unique. Given the Taylor rule, money balances at time t do not uniquely determine money balances at time t+1, so that the dynamics of the model, calibrated by respecting the standard parameter values, are ambigously defined even before the usual indeterminacy considerations come into play. To see thgis consider a backward

looking rule given by

$$R_t = R^* \left(\frac{\pi_{t-2}}{\pi^*}\right)^{\frac{A}{R^*}}; \quad a\beta > 1, \tag{22}$$

which implies that

$$\pi_t = \pi^* \left(\frac{R_{t+2}}{R^*}\right)^{\frac{R^*}{A}} \tag{23}$$

Substituting into the Euler equation (16) we get

$$\left(a\left(\frac{M_{t+2}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}} \tag{24}$$

$$= \frac{\beta\left(1 - a\left(a\left(\frac{M_{t+1}}{p_t}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t+1}}{p_t}\right)^{\rho-1}\right)^{-1}}{\pi^*\left(\frac{\left(1-a\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho}+(1-a)\bar{y}^{\rho}\right)^{\frac{1-\rho}{\rho}}\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho-1}\right)^{-1}-1}{R^*}\right)^{\frac{R^*}{A}}}\left(a\left(\frac{M_{t+2}}{p_{t+1}}\right)^{\rho} + (1-a)\,\bar{y}^{\rho}\right)^{\frac{-\sigma}{\rho}}$$

which is a difference correspondence rather than a difference equation. There often is not a unique $\left(\frac{M_{t+2}}{p_{t+1}}\right)$ defined for a given $\left(\frac{M_{t+1}}{p_t}\right)$: instantaneous equilibrium is not well-defined. The equilibrium trajectory may jump from one branch of solutions to the next, creating a more severe problem than the standard difficulties associated with multiple equilibria and indeterminacy.

2.6.3 Timing Conventions

A similar problem emerges if the production function depends on the initial rather than end of period balances: $c_t = y_t = F\left(\frac{M_t}{p_t}\right) = F(m_t)$.[9] In such a case the money and bond markets are in equilibrium if $F_m(m_t) = R_t$.The Euler equation becomes:

$$(F(m_t))^{-\sigma}\,\pi_t = \beta\,(F(m_{t+1}))^{-\sigma}\,(1 + R_{t+1})$$

With a forward looking Taylor rule as in (??), this yields

$$(F(m_t))^{-\sigma}\,\pi\left(\frac{F_m(m_{t+1})}{Rq}\right)^{\frac{R}{A}} = \beta\,(F(m_{t+1}))^{-\sigma}\,(1 + F_m(m_{t+1}))$$

[9]For a general discussion of various timing assumptions and their impact on local determinacy properties, see Carlstrom and Fuerst (2001)

Again, whether we have a Cobb-Douglas or CES production function, m_t may not gernerate a unique m_{t+1}. If the Taylor Rule is backward looking as in (23) however, we obtain from the Euler equation:

$$\begin{aligned}
(F(m_t))^{-\sigma}\pi_t &= \beta(F(m_{t+1}))^{-\sigma}(1+R_{t+1}) \\
(F(m_t))^{-\sigma}\pi\left(\frac{F_m(m_{t+2})}{Rq}\right)^{\frac{R}{A}} &= \beta(F(m_{t+1}))^{-\sigma}(1+F_m(m_{t+1})) \\
F(m_{t+2}) &= \left((Rq)^{\frac{R}{A}}\pi^{-1}\beta(1+F_m(m_t))(F(m_t))^{\sigma}(F(m_{t+1}))^{-\sigma}\right)^{\frac{A}{R}}
\end{aligned}$$

The equilibrium dynamics are now defined by a difference equation of second order, and the solution can be complicated.

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