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7/26/01 10:15 AM

## Learning and the Central Bank\*

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June 26, 2001

**Abstract:** It is well known that sunspot equilibria may arise under an interest rate operating procedure in which the central bank varies the nominal rate with movements in future inflation (a forward-looking Taylor rule). This paper demonstrates that these sunspot equilibria may be learnable in the sense of E-stability.

We thank John Carlson for interesting us in this topic. We have received helpful comments from Jim Bullard and Seppo Honkapohja. Any remaining errors are our own.

## I. Introduction.

The celebrated Taylor (1993) posits that central bank behavior can be described by a fairly simple rule linking nominal rate movements to movements in inflation and output. This seminal paper has spawned a large literature concerned with issues of stability: under what situations can a Taylor-rule formulation of monetary policy create real indeterminacy and thus sunspot<sup>1</sup> fluctuations in the model economy? See for example, Benhabib, Schmitt-Grohe and Uribe (1999), Bernanke and Woodford (1997), Carlstrom and Fuerst (2001a,2001b,2000a), Clarida, Gali and Gertler (2000), and Kerr and King (1996). As forcefully argued by Evans and Honkapohja (2001), sunspot equilibria are compelling only if they are not “fragile” to reasonable assumptions about “learning”. We follow Evans and Honkapohja (2001), and interpret “learning” as E-stability, so that an equilibrium is “fragile” if it is not E-stable. The issue raised in this paper is whether the sunspot equilibria induced by some Taylor-rules are E-stable.<sup>2,3</sup>

A robust result of the papers on indeterminacy is that sunspots are particularly likely in cases in which the central bank responds to forecasted inflation. We will thus focus on Taylor rules in which the central bank responds to forecasted inflation. Honkapohja and Mitra (2001) analyze the basic monetary model considered here, and conclude that the sunspot equilibria arising from a forward-looking monetary policy are

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<sup>1</sup> The term “sunspot” is in one sense misleading since these shocks are accommodated by monetary policy. But we use the term since the central bank introduces real indeterminacy by responding to forecasts which can be driven by sunspots.

<sup>2</sup> E-stability typically implies that least-squares learning converges to the rational expectations equilibrium, although there are some technical issues in the case of a continuum of equilibria (as is the case with the sunspot equilibria examined below). See Evans and Honkapohja (2001) for a detailed discussion.

<sup>3</sup> Woodford (1990) was the first to demonstrate the learnability of stationary sunspot equilibria in an overlapping generations model.

not E-stable.<sup>4</sup> They show that the only equilibria that are E-stable are the minimum state vector (msv) solutions where inflation depends only on fundamental shocks. McCallum (2001) concludes from this that only the msv solution is empirically relevant.

In this paper we consider two variants of the analysis in Honkapohja and Mitra (2001) and demonstrate the existence of E-stable sunspot equilibria. First, we consider a different timing scenario. A microfoundation of the model analyzed by Honkapohja and Mitra is that money balances at the end of goods market trading is what aids in transactions. Carlstrom and Fuerst (2001) refer to this as “cash-when-I’m-done” (CWID) timing. In a model with CWID timing Honkapohja and Mitra demonstrate that sunspot equilibria are not E-stable. But CWID is a peculiar timing convention. In contrast, suppose that cash balances held in advance of goods trading are the balances that aid in transactions, what Carlstrom and Fuerst (2001) call “cash-in-advance” (CIA) timing. One contribution of this paper is to demonstrate that in a model with CIA timing there exist E-stable sunspot equilibria.

Our second modeling variation is a different assumption on the nature of learning in the model. Honkapohja and Mitra (2001) examine a model in which there is symmetric learning by both the public and the central bank. That is, both the central bank and private sector have common expectations. This can be interpreted as the private sector learning, and the central bank operating off of private sector forecasts. In contrast, this paper examines a case in which the forecasts of the central bank and private sector differ, and coincide only in the long run. There are many possible differential learning scenarios. Here we take one extreme: We assume that only the central bank is subject to

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<sup>4</sup> However, Honkapohja and Mitra (2001) demonstrate that “resonant frequency” sunspot equilibria may be learnable under certain policy rules.

a learning process, while private sector expectations are always rational. This assumption is analogous to the assumption in Sargent's (1999) analysis of *The Conquest of American Inflation*. A second contribution of this paper is to demonstrate that in this case of central bank learning the sunspot equilibria are typically E-stable. In essence, central bank policy can lead the public to believe in sunspots.

The outline of the paper is as follows. In the next section we present the basic CWID monetary model and the results of Honkapohja and Mitra (2001). We then consider the CIA variant of this model. Here sunspots can be learnable. In section 3, we demonstrate that sunspots are typically learnable when there is asymmetric learning and it is the central bank doing the learning. Section 4 concludes.

## II. Symmetric Learning in a Sticky Price Model.

### A. Sunspots and Learnability in the CWID Model.

The analysis is conducted using the now-standard sticky price model that is given by the following two equations:

$$\pi_t = \lambda z_t + \beta E_t \pi_{t+1} + u_t \quad (1)$$

$$z_t = -[R_t - E_t \pi_{t+1}] + E_t z_{t+1} \quad (2)$$

where

$$u_t = \rho u_{t-1} + \varepsilon_t,$$

$\pi_t \equiv \frac{P_t}{P_{t-1}}$  denotes the inflation rate from time t-1 to time t,  $z_t$  denotes marginal cost,  $R_t$  is

the nominal interest rate, and  $u_t$  denotes a shock to the pricing equation.<sup>5</sup> All variables are expressed as log deviations from the non-stochastic steady-state. Below we find it convenient to assume  $\beta + \lambda > 1$ .

To close the model we need to specify the central bank reaction function. In what follows we assume a reaction function where the current nominal interest rate responds to expected inflation:

$$R_t = \tau E_t \pi_{t+1}, \quad (3)$$

where  $\tau > 0$  is the response of the nominal interest rate to movements in expected inflation. Under any such interest rate policy the money supply (not modeled) responds endogenously to satisfy the interest rate rule. It is this endogeneity of the money supply that leads to the possibility of real indeterminacy and sunspot equilibria. That is, there is real indeterminacy if different money growth rules support the interest rate target (3).<sup>6</sup> These are then associated with different real outcomes because of the sticky price assumption (1).

To proceed, use (1) to eliminate  $z_t$  from the system:

$$\pi_t - E_t \beta \pi_{t+1} - u_t = -\lambda [R_t - E_t \pi_{t+1}] + E_t \pi_{t+1} - E_t \beta \pi_{t+2} - \rho u_t. \quad (4)$$

Using (3) to eliminate the nominal rate, we have a second-order difference equation in  $\pi_t$ . For determinacy, we need both roots of the corresponding characteristic equation to be outside the unit circle. Straightforward calculations imply that there is real determinacy if

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<sup>5</sup> See Clarida, Gali, and Gertler (2000), and the references therein. Following Yun (1996), Carlstrom and Fuerst (2000b) demonstrate that with a linear production technology, the system can be written in the marginal cost form used above. In this case,  $\lambda$  represents the link between marginal cost and prices, while in the Clarida, Gali, and Gertler (2000) framework  $\lambda$  represents the link between output and prices. One can transform the current model by replacing our  $\lambda$  with Clarida et al.'s  $\lambda\sigma$ , where  $\sigma$  is the elasticity of intertemporal substitution.

and only if

$$1 < \tau < \frac{2(\beta+1)+\lambda}{\lambda}.$$

For reasonable calibrations ( $\beta = .99$ ,  $\lambda = .3$ ), the upper bound is quite high, about 14, so that the basic conclusion is that a  $\tau$  greater than unity will achieve determinacy. If there is determinacy, the equilibrium can be written as

$$\pi_t = \gamma_{\text{msv}} u_t$$

where  $\gamma_{\text{msv}}$  is unique and denotes the “minimum state vector” (msv) solution. If  $\tau$  lies outside the determinacy region, then we have two cases. For  $\tau < 1$  only one root of the characteristic equation given by (4) is explosive, while the other is in  $(0,1)$ . If

$\tau > \frac{2(\beta+1)+\lambda}{\lambda}$ , one root is explosive while the other is in  $(-1,0)$ . In either case we have

real indeterminacy and multiple equilibria. In particular there are sunspot equilibria given by

$$\pi_{t+1} = \alpha \pi_t + \gamma u_t + \sigma_1 \varepsilon_{t+1} + \sigma_2 s_{t+1} \quad (5)$$

where  $\alpha \in (-1,1)$  is unique,  $\gamma \neq \gamma_{\text{msv}}$  is unique,  $\sigma_1$  and  $\sigma_2$  are arbitrary,  $\varepsilon_{t+1}$  is the innovations in the  $u_t$  process, and  $s_{t+1}$  is an arbitrary iid, mean-zero sunspot shock. Note that although the msv solution uniquely determines the response of  $\pi_{t+1}$  to  $\varepsilon_{t+1}$ ,  $\sigma_1$  is arbitrary in the case of sunspot equilibria because both  $\varepsilon_{t+1}$  and  $s_{t+1}$  are white noise.

Are these sunspot equilibria learnable? Following the methodology outlined in Evans and Honkapohja (2001), posit the following perceived law of motion (PLM):

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<sup>6</sup> See Carlstrom and Fuerst (2000b) for a discussion.

$$\pi_t = a_1 \pi_{t-1} + b_1 u_{t-1} + c_1 \varepsilon_t + d_1 s_t. \quad (\text{PLM})$$

Notice that this PLM has the same form as the sunspot equilibria (5). Using this PLM scrolled forward to eliminate the forecasts in the equilibrium condition (4), we can then solve for the implied actual law of motion (ALM):

$$\pi_t = a_2 \pi_{t-1} + b_2 u_{t-1} + c_2 \varepsilon_t + d_2 s_t. \quad (\text{ALM})$$

By replacing all expectations with this common PLM, we are assuming symmetric learning between the public and the central bank.<sup>7</sup> We now have the mapping  $T(a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$ . The fixed points of this T-mapping are the rational expectations equilibria. An equilibrium is said to be E-stable if this mapping is stable evaluated at the equilibrium in question. Bullard and Mitra (2000) study the E-stability of the msv equilibrium.<sup>8</sup> Our focus is on sunspot equilibria.

It is straightforward to demonstrate that if agents know  $\pi_t$  when forecasting  $\pi_{t+1}$  and  $\pi_{t+2}$ , then the coefficient  $a_1$  maps into zero so that the sunspot equilibria are not E-stable. Hence, Honkapohja and Mitra (2001) extend the analysis by assuming that when forming expectations agents do not know  $\pi_t$ , so that time- $t$  forecasts are functions only of  $\pi_{t-1}$  and the exogenous shocks. As noted by Evans and Honkapohja (2001), this increases the chances for E-stability. One contribution of Honkapohja and Mitra (2001) is to demonstrate that even in this case the sunspot equilibria are still not E-stable so that sunspots are not learnable.<sup>9</sup>

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<sup>7</sup> In the next section we will consider a particular form of asymmetric learning in which only the central bank is learning. In this case we replace only the central bank's forecast with the PLM.

<sup>8</sup> It is important to note that our PLM does not include a constant term, while a constant term is central to the results in the Bullard-Mitra paper.

<sup>9</sup> However, Honkapohja and Mitra (2001) demonstrate that a different type of equilibria, "resonant frequency" sunspot equilibria, may be learnable under certain policy rules.

## B. Sunspots and Learnability in the CIA Model.

Before abandoning the possibility of E-stable sunspots in the case of symmetric learning, consider the alternative money-demand timing convention suggested by Carlstrom and Fuerst (2001). The Fisher equation given by (2) has as its microfoundations the assumption that money balances at the end of the period (after leaving the goods market) aid in transactions—what Carlstrom and Fuerst call “cash-when-I’m-done” timing (CWID). If we instead assume that cash available before entering the goods market aid in transactions—what Carlstrom and Fuerst call “cash-in-advance” timing (CIA), equation (2) becomes

$$z_t = -[R_{t+1} - E_t \pi_{t+1}] + E_t z_{t+1}. \quad (6)$$

As before we use (1) to eliminate  $z_t$  from the system.

$$\pi_t - E_t \beta \pi_{t+1} - u_t = -\lambda [R_{t+1} - E_t \pi_{t+1}] + E_t \pi_{t+1} - E_t \beta \pi_{t+2} - \rho u_t \quad (7)$$

In this case Carlstrom and Fuerst (2001) demonstrate that there is real indeterminacy under the forward-looking Taylor rule for all values of  $\tau$ . Are any of these sunspot equilibria E-stable? Yes, but only a few. We first characterize the indeterminacy, and then look at E-stability.

**Proposition 1:** Under the assumption of CIA timing there is real indeterminacy for all values of  $\tau$ . In particular:

- a. If  $\tau < 1$ , the equilibria are characterized by the AR(1) process

$$\pi_{t+1} = \alpha \pi_t + \mu_t + \sigma_1 \mathcal{E}_{t+1} + \sigma_2 S_{t+1} \quad \mathbf{AR(1)} \quad (8)$$



where  $0 < \alpha < 1$  is unique,  $\gamma$  is unique, and  $\sigma_1$  and  $\sigma_2$  are arbitrary.

b. If  $1 < \tau < \left[ \frac{(1 + \beta + \lambda)^2 - 4\beta}{4\lambda} \right] \equiv \tau^*$ , there are two stable real roots to the characteristic

equation, so that there are two distinct AR(1) processes of the form (8) where  $0 < \alpha < 1$  takes on one of these two values. There are also AR(2) equilibria characterized by

$$\pi_{t+1} = \frac{1 + \beta + \lambda}{\beta + \lambda\tau} \pi_t + \frac{-1}{\beta + \lambda\tau} \pi_{t-1} + \gamma u_t + \sigma_1 \varepsilon_{t+1} + \sigma_2 \delta_{t+1}. \quad \mathbf{AR(2)} \quad (9)$$

c. If  $\tau > \tau^*$ , the roots of the characteristic equation are complex with norm in  $(0,1)$  so that the equilibria are characterized by the AR(2) process (9).

*Proof:* Since questions of determinacy depend only upon deterministic dynamics, the proof focuses only on the AR coefficients without loss of generality. The characteristic equation of (7) is given by

$$h(e) = (\beta + \lambda\tau)e^2 - (1 + \beta + \lambda)e + 1.$$

We have  $h(0) > 0$ ,  $h'(0) < 0$ , and  $h(1) = \lambda(\tau - 1)$ . Hence, if  $\tau < 1$  there is one root in  $(0,1)$  and one outside  $(0,1)$ . Since there are no predetermined variables we have real indeterminacy.

Now suppose that  $\tau > 1$ . In this case we have  $h'(1) > 0$ . Hence, if the roots are real, they are both in  $(0,1)$ . These two roots are both possible AR(1) coefficients.

Alternatively, we can write this as the AR(2) in (9). The roots are real if and only if

$$(1 + \beta + \lambda)^2 > 4(\beta + \lambda\tau)$$

Solving this for  $\tau$  yields the  $\tau^*$  in the proposition. If the roots are complex, their norm is in  $(0,1)$  and the equilibria are then characterized by the AR(2). **QED**

In contrast to the CWID model in which there is indeterminacy only for very small or very large values of  $\tau$ , Proposition 1 implies that in the case of CIA timing real indeterminacy arises for all values of  $\tau$ . Note that the nature of the equilibria varies around  $\tau = 1$ . For  $\tau < 1$ , the sunspot equilibria are of the AR(1) form given by (8), while for  $\tau > 1$  there are sunspot equilibria of the AR(2) form given by (9).

We will now turn to E-stability of these equilibria. If we assume that  $\pi_t$  is known when generating forecasts the earlier discussion applies and the sunspot equilibria are not E-stable. Hence, we once again must restrict the information set by assuming that  $\pi_t$  is not known when generating forecasts.

**Proposition 2:** Assume CIA timing and that  $\pi_t$  is not observable for time- $t$  forecasting. For  $\tau < 1$  the AR(1) equilibria given by (8) are not E-stable. However, for

$$1 < \tau < \left[ \frac{(1 + \beta + \lambda)^2 - 2\beta}{2\lambda} \right]$$

the AR(2) equilibrium given by (9) are E-stable.

*Proof:* Let us first consider the AR(1) case. Suppose that the PLM is given by

$$\pi_t = a_1 \pi_{t-1} + b_1 u_{t-1} + c_1 \varepsilon_t + d_1 s_t.$$

Under the assumption that  $\pi_t$  is not observable for time- $t$  forecasting, we have

$$\begin{aligned} E_t \pi_{t+1} &= a_1^2 \pi_{t-1} + a_1 b_1 u_{t-1} + a_1 c_1 \varepsilon_t + a_1 d_1 s_t + b_1 \rho u_{t-1} + b_1 \varepsilon_t \\ E_t \pi_{t+2} &= a_1^3 \pi_{t-1} + a_1^2 b_1 u_{t-1} + a_1^2 c_1 \varepsilon_t + a_1^2 d_1 s_t + b_1 (a_1 + \rho)(\rho u_{t-1} + \varepsilon_t) \end{aligned}$$

Substituting this into (7) we have that the PLM maps into the ALM via:

$$T(a_1) = (1 + \beta + \lambda)a_1^2 - (\beta + \lambda\tau)a_1^3$$

$$T(b_1) = (1 + \beta + \lambda)(a_1 + \rho)b_1 - (\beta + \lambda\tau)[a_1^2 + \rho(a_1 + \rho)]b_1 + (1 - \rho)\rho$$

$$T(c_1) = (1 + \beta + \lambda)(a_1c_1 + b_1) - (\beta + \lambda\tau)[a_1^2c_1 + b_1(a_1 + \rho)] + (1 - \rho)$$

$$T(d_1) = (1 + \beta + \lambda)a_1d_1 - (\beta + \lambda\tau)a_1^2d_1$$

It is straightforward to demonstrate that at  $a_1 = \alpha$ ,  $T'(c_1) = T'(d_1) = 1$ , ie., there are no learning dynamics for the coefficients on the innovations. Following Evans and Honkapohja (2001), this implies that for E-stability of the sunspot equilibria we need focus only on the mappings of  $a_1$  and  $b_1$ . Since this system is diagonal, the E-stability condition is that  $T'(a_1) < 1$  and  $T'(g_1) < 1$ , evaluated at the sunspot equilibria. Consider  $a_1$  first:

$$T'(a_1) = 2(1 + \beta + \lambda)a_1 - 3(\beta + \lambda\tau)a_1^2$$

The AR(1) solution is  $\alpha$  such that  $T(\alpha) = \alpha$ . Using this fact we have that E-stability requires

$$\alpha > \frac{2}{1 + \beta + \lambda}$$

It is straightforward to show that only the larger of the two real roots satisfies this condition. If  $\tau < 1$ , the larger root is outside the unit circle so the AR(1) equilibria are not E-stable.<sup>10</sup>

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<sup>10</sup> If  $1 < \tau < \tau^*$ , the larger root is inside the unit circle so that  $a_1 = \alpha_{\text{high}}$  is E-stable. In this case, we must examine  $T'(b_1)$ :

$$T'(b_1) = 1 + (1 + \beta + \lambda)\rho - (\beta + \lambda\tau)\rho(a_1 + \rho)$$

For stability, we need this within the unit circle. Using  $a_1 = \alpha$  and  $T(\alpha) = \alpha$  we have

We now analyze the case where  $\tau > 1$  so that we have AR(2) equilibria. Let the PLM be given by

$$\pi_t = a_1 \pi_t + a_2 \pi_{t-2} + b_1 u_{t-1} + c_1 \varepsilon_t + d_1 s_t.$$

Using this PLM and the assumption that  $\pi_t$  is not part of the information set, we have the following T-mapping from PLM to ALM:

$$T(a_1) = (1 + \beta + \lambda)(a_1^2 + a_2) - (\beta + \lambda\tau)(a_1^2 + 2a_2 a_1)$$

$$T(a_2) = (1 + \beta + \lambda)a_1 a_2 - (\beta + \lambda\tau)(a_1^2 a_2 + a_2^2).$$

$$T(b_1) = [1 + \beta + \lambda - a_1(\beta + \lambda\tau)](a_1 + \rho)b_1 - (\beta + \lambda\tau)(a_2 + \rho^2)b_1 + \rho(1 - \rho)$$

$$T(c_1) = [1 + \beta + \lambda - a_1(\beta + \lambda\tau)](a_1 c_1 + b_1) - (\beta + \lambda\tau)(a_2 c_1 + b_1 \rho) + (1 - \rho)$$

$$T(d_1) = [1 + \beta + \lambda - a_1(\beta + \lambda\tau)]a_1 d_1 - (\beta + \lambda\tau)a_2 d_1$$

As before, our focus is on the system in  $a_1$ ,  $a_2$ , and  $b_1$ . Note first that this system of derivatives is once again block recursive. Evaluating the derivatives at the equilibrium values of

$$a_1 = \frac{1 + \beta + \lambda}{\beta + \lambda\tau}$$

$$a_2 = \frac{-1}{\beta + \lambda\tau}$$

we have  $T'(b_1) = -(\beta + \lambda\tau)\rho^2 < 1$ .<sup>11</sup> Hence, we need only examine the subsystem in  $a_1$  and  $a_2$ . The characteristic equation of this sub-matrix (evaluated at the AR(2) values) is

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$$(1 + \beta + \lambda)\rho - (\beta + \lambda\tau)\rho(a_1 + \rho) < 0$$

Note that if  $\rho = 0$ , we have instability, but in this case the sunspot equilibria would not depend upon  $u_{t-1}$ . If  $\rho > 0$ , then E-stability requires  $\alpha > \frac{1 + \beta + \lambda}{\beta + \lambda\tau} - \rho$ .

<sup>11</sup> As before we have  $T'(c_1) = T'(d_1) = 1$ , i.e., there are no learning dynamics in these coefficients.

$$g(e) = e^2 + \Delta e - \Delta, \text{ where } \Delta \equiv \frac{(1 + \beta + \lambda)^2}{\beta + \lambda\tau} - 4$$

Note that  $g(1) > 0$  and  $g(0) = -\Delta$ . Recall from Proposition 1 that the roots of  $h$  (the characteristic equation of (7)) are real when  $\Delta > 0$ . If  $\Delta > 0$ ,  $g(0) < 0$  so that the two roots of  $g$  are below unity and we have E-stability. If  $\Delta < 0$ , the roots of  $g$  are complex, and we need the real part to be less than unity. Expressing this condition in terms of  $\tau$  yields the expression in the proposition. **QED**

Proposition 2 implies that the AR(2) sunspot equilibria are learnable for an empirically relevant range. For example, with  $\beta = .99$ ,  $\lambda = .3$ , we have E-stability for  $1 < \tau < 5.44$ . This region includes the celebrated Taylor coefficient of 1.5.<sup>12</sup>

### III. Asymmetric Learning in a Sticky Price Model.

The former section made an extreme assumption: both the public and the central bank have common forecasts, both of which are rational only in the limit. In contrast, in this section we assume that the private sector's forecasts are rational but that the central

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<sup>12</sup> Curiously, this range gets arbitrarily large as the economy approaches a flexible price model ( $\lambda \rightarrow \infty$ ). Yet for an economy which is perfectly flexible (so that equilibrium is given by (6) with  $z_t = 0$ ) there is real indeterminacy but sunspots are never learnable (this is immediate given that  $\pi_t$  no longer enters into the system). This suggests there might be a problem in the above analysis. As in Honkapoinja and Mitra (2001), the above analysis assumes that when forming expectations agents do not know  $\pi_t$ . But actual inflation in the pricing equation (1) was assumed observable. Following Yun (1996) the microfoundations of this pricing equation are that firms who set prices in time- $t$  base their prices on the current price level and forecasts of future prices. If the current price level is assumed to be not observable, then we should also replace  $\pi_t$  in equation (1) with the expectation of  $\pi_t$  given current information. That is, in the analysis of E-stability, we should also replace  $\pi_t$  in equation (1) with its PLM. In this case we have an actual law of motion (ALM) solely in  $\pi_{t-1}$  so that the coefficient  $a_1$  maps into zero. (A similar argument holds in the case of the AR(2) equilibria.) Under this interpretation the sunspot equilibria are not E-stable. This criticism does not apply to the analysis in Section III as we assume that  $\pi_t$  is known when making forecasts.

bank uses a forecasting rule that is rational only in the limit. In this case it is much more likely for real indeterminacy to be learnable. If the central bank uses current inflation to forecast future inflation, and if the public knows that the central bank is doing so, then the AR(1) and AR(2) sunspot equilibria may be learnable. The central bank can lead the economy to indeterminacy.

## A. The CWID Model.

Let us begin with the case of CWID timing. The relevant equilibrium is given by

$$\pi_t - E_t \beta \pi_{t+1} - u_t = -\lambda [R_t - E_t \pi_{t+1}] + E_t \pi_{t+1} - E_t \beta \pi_{t+2} - \rho u_t \quad (10)$$

The sunspot equilibria are of the AR(1) form in (8). Since only the central bank is subject to learning we substitute the PLM only into the bank's forecast:

$$R_t = \tau E_t^{cb} \pi_{t+1} = \tau [a_1 \pi_t + b_1 u_t] \quad (11)$$

As will soon be evident, because of the dynamics of asymmetric learning, the sunspot equilibria can be E-stable even if the central bank observes  $\pi_t$  when forecasting  $\pi_{t+1}$ .

Without loss of generality, we thus proceed under this assumption. Notice that with asymmetric learning the forward rule with parameter  $\tau$  corresponds to (roughly) a current rule with parameter  $\tau a_1$ . Substituting (11) into (10), we have a second order system in  $\pi_t$ . This system is indeterminate, with one root in the unit circle. This root is the ALM.

Under this mapping, is the AR(1) coefficient ever E-stable? Yes.

**Proposition 3:** Assume CWID timing, and central bank learning. If  $\tau < 1$ , then there is real indeterminacy and the AR(1) equilibria of the form (8) are E-stable. If

$\tau > \frac{2(\beta+1)+\lambda}{\lambda}$ , then there is real indeterminacy but the AR(1) equilibria of the form (8)

are not E-stable

*Proof:* Substituting (11) into (10), we have the following system:

$$\pi_t(1 + \lambda \pi a_1) = (1 + \beta + \lambda)E_t \pi_{t+1} - E_t \beta \pi_{t+2} + (1 - \rho - \lambda \phi_1)u_t$$

In the neighborhood of the AR(1) equilibria,  $a_1 = \alpha$ , this system is subject to indeterminacy so that we can use the method of undetermined coefficients to solve it for the ALM:

$$\pi_t = T(a_1)\pi_{t-1} + T(b_1)u_{t-1}$$

where without loss of generality we ignore the sunspot coefficients. The mapping  $T(a_1)$  is given by the stable root (the smaller root) of the system:

$$T(a_1) = \frac{(1 + \beta + \lambda) - \sqrt{(\beta + \lambda)^2 + 2(\lambda - \beta) + 1 - 4\beta\lambda\tau a_1}}{2\beta}$$

so that

$$dT(a_1)/da_1 = \frac{\lambda\tau}{\sqrt{(\beta + \lambda)^2 + 2(\lambda - \beta) + 1 - 4\beta\lambda\tau a_1}}$$

For E-stability we need this to be less than one. Exploiting the fact that  $T(\alpha) = \alpha$ , where  $\alpha$  is the AR(1) solution, we can eliminate the square root and obtain:

$$dT(a_1)/da_1 = \frac{\lambda\tau}{1 + \beta + \lambda - 2\alpha\beta} \quad (12)$$

We can now consider the two cases:

**Case 1:**  $\tau > \frac{2(\beta+1)+\lambda}{\lambda}$ .

$$dT(a_1)/da_1 = \frac{\lambda\tau}{1+\beta+\lambda-2\alpha\beta} > \frac{1+\beta+\lambda+(\beta+1)}{1+\beta+\lambda-2\alpha\beta}$$

where the inequality follows from the restriction on  $\tau$ . Since the AR(1)  $\alpha < 0$  in this case, we have that  $dT(a_1)/da_1 > 1$ , so that the solution is not E-stable.

**Case 2:**  $\tau < 1$ .

Expression (12) is increasing in  $\alpha$ . Setting  $\alpha = 1$  we have

$$dT(a_1)/da_1 < \frac{\lambda\tau}{1+\lambda-\beta} < 1$$

where the last inequality follows from  $\tau < 1$ . Hence, we must proceed to the  $T(b_1)$

mapping:

$$T(b_1) = \frac{\lambda\bar{w}_1 - (1-\rho)}{(1+\beta+\lambda) - \beta(\alpha+\rho)}$$

For the case  $\tau < 1$  we have  $0 < \alpha < 1$ , so that  $T'(b_1) < 1$ . Hence, in the case of  $\tau < 1$  we have E-stability. **QED**

*Remark:* It is curious to note that the sunspots fail to be E-stable only when  $\tau$  is large so that the equilibria are oscillatory,  $\alpha < 0$ .

## B. The CIA Model.

In the case of CIA timing, the relevant equilibrium is given by

$$\pi_t - E_t\beta\pi_{t+1} = -\lambda[R_{t+1} - E_t\pi_{t+1}] + E_t\pi_{t+1} - E_t\beta\pi_{t+2} \quad (13)$$

As before, since only the central bank is subject to learning we only replace their



forecasts with the relevant PLM. Recall that in the CIA model there are two forms of indeterminacy, depending upon the size of  $\tau$ . For  $\tau < 1$ , we have AR(1) equilibria of the form in (8), so that we replace the interest rate with

$$R_{t+1} = \tau E_t^{cb} \pi_{t+2} = \tau [a_1 \pi_{t+1} + b_1 u_{t+1}]. \quad (14)$$

In the case of  $\tau > 1$ , we have indeterminacy of the AR(2) form given in (9), and replace the interest rate with

$$R_{t+1} = \tau E_t^{cb} \pi_{t+2} = \tau [a_1 \pi_{t+1} + a_2 \pi_t + b_1 u_{t+1}]. \quad (15)$$

We now state:

**Proposition 4:** Assume CIA timing, and asymmetric learning (central bank learning).

For  $\tau < 1$  the AR(1) equilibria in (8) are learnable if

$$\rho < \frac{2}{\beta + \lambda + 1}.$$

For  $\tau > 1$  the AR(2) equilibria in (9) are learnable for all values of  $\rho$ .

*Proof:*

**Case 1:  $\tau < 1$ .** Substitute (14) into (13). This system is indeterminate, with two positive roots, one in (0,1). This smaller root is the ALM and is given by  $T(a_1)$ :

$$T(a_1) = \frac{x - \sqrt{x^2 - 4/\beta}}{2} \text{ where } x = \frac{\lambda(1 - \tau a_1) + (1 + \beta)}{\beta} > 0.$$

E-stability is given by  $dT(a_1)/da_1 < 1$ .

$$\frac{dT(a_1)}{da_1} = \frac{dT(a_1)}{dx} \frac{dx}{da_1} = \left[ \frac{1}{2} + \frac{-x}{2\sqrt{x^2 - 4/\beta}} \right] \left( \frac{-\lambda\tau}{\beta} \right) = \frac{\lambda\tau\alpha}{\lambda(1 - \tau\alpha) + 1 + \beta - 2\alpha\beta}$$

where the last equality comes from exploiting  $T(\alpha) = \alpha$  to eliminate the square root. This

last term must be less than unity for E-stability. This implies there is E-stability if and only if

$$\alpha < \frac{\beta + \lambda + 1}{2(\beta + \lambda\tau)}. \quad (16)$$

For  $\tau < 1$  this is always satisfied as  $\alpha$  is the smaller root of the characteristic equation.

We now must turn to the  $b_1$  coefficient:

$$T(b_1) = \frac{\lambda\tau b_1 \rho - (1 - \rho)}{(1 + \beta + \lambda) - \alpha(\beta + \lambda\tau) - \beta\rho}$$

where we are evaluating this at  $a_1 = \alpha$ . For E-stability we need  $T'(b_1) < 1$ . Imposing this and using the fact that  $\alpha$  is the root of the characteristic equation, we have

$$T'(b_1) = \frac{\alpha\lambda\tau\rho}{1 - \alpha\beta\rho} < 1.$$

Combining this with (16), we have that the equilibria are learnable if and only if

$$\rho < \frac{2}{\beta + \lambda + 1}.$$

**Case 2:**  $\tau > 1$ . Substitute (15) into (13). This system is indeterminate and in the neighborhood of the candidate sunspot equilibria can be expressed as an AR(2). This AR(2) is our ALM:

$$\pi_{t+2} = \frac{1}{\beta} \left[ (1 + \beta + \lambda - \lambda\tau a_1)\pi_{t+1} - (1 + \lambda\tau a_2)\pi_t + (1/\rho)(1 - \rho - \rho\lambda\tau b_1)u_{t+1} \right]$$

We thus have the mapping

$$a_1 \rightarrow (1 + \beta + \lambda - \lambda\tau a_1)/\beta$$

$$a_2 \rightarrow -(1 + \lambda\tau a_2)/\beta.$$

$$b_1 \rightarrow (1 - \rho - \rho\lambda\phi_1) / \beta\rho$$

Inspection reveals that this is E-stable. **QED**

#### **IV. Conclusion.**

This paper has shown that the developing consensus that policy-induced sunspots are not learnable may be premature. This paper has considered two modifications to the typical model, either one of which leads to the learnability of sunspot equilibria. First, if we replace CWID money demand timing with the more intuitive CIA timing, then sunspots are learnable over a relevant range of the parameter space. Second, sunspots are learnable if the central bank is the one doing the learning.

There are several natural areas of further work. First, the Taylor rule examined depended only on expected inflation. Future work will consider the case of including a measure of output in the policy rule. Second, the sunspot equilibria arise because of the endogeneity of the supporting money supply process. What features of this money supply behavior lead to E-stability? Finally, in the case of asymmetric learning, the public was assumed to have rational expectations. Future work needs to investigate whether sunspots are learnable when both the public and the central bank learn but with potentially different learning rules.

While addressing whether sunspots are learnable we have left unanswered the question of how a particular sunspot is coordinated upon. While far from being a complete answer to this important question we note that if the monetary authority believes in a particular sunspot, rational expectations on the part of the public dictates that they too will believe in that sunspot. The central bank can lead us to real indeterminacy.

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