Zero bound on nominal interest rates and optimal monetary policy

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Abstract

What should a central bank do when faced with a weak aggregate demand even after reducing the short-term nominal interest rate to zero? To address this question, we solve a central bank's intertemporal loss minimization problem, in which the non-negativity constraint on nominal interest rates is explicitly considered. Given an adverse shock to aggregate demand, we compute the optimal path of short-term nominal interest rates under the assumption that the central bank has the ability to make a credible commitment about the future path of short-term nominal interest rates. We find that the optimal path is history dependent, in the sense that a zero interest rate policy should be continued for a while even after the economy returns to normal. By making such a commitment, the central bank is able to achieve higher expected inflation, lower long-term nominal interest rates, and weaker domestic currency in the adverse periods when the natural rate of interest significantly deviates from a normal level. We provide a numerical example to show that this channel of monetary policy transmission is quantitatively important.

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1 Introduction

When the short-term nominal interest rate is very close to zero, the substitutability between short-term bonds, or monetary policy instruments, and money becomes very high, which makes it extremely difficult for a central bank to implement further monetary easing. This phenomenon, which is called the zero bound on nominal interest rates or a liquidity trap, has been studied by many researchers including Keynes (1936). However, because a liquidity trap has not been observed in the real world until very recently, there was a tendency to view this phenomenon as a purely theoretical textbook problem.

This situation changed on February 12, 1999, when the Bank of Japan (BOJ) made an announcement of lowering overnight interest rates to be "as low as possible" to stimulate the Japanese economy, which was then believed to be at the edge of deflationary spiral. Following this announcement, the BOJ provided ample liquidity to the interbank money market until the uncollateralized overnight call rate reached zero (actually 2 basis points). These developments in Japanese monetary policy have alerted researchers and practitioners, inside and outside the country, that the same phenomenon could take place even in other industrialized countries where inflation has fallen to very low levels.

The BOJ's "zero interest rate policy" has revived the interest of researchers in the zero bound on nominal interest rates, and a number of studies have recently investigated this issue. Krugman (1998, 2000) argues that the natural rate of interest in Japan is negative, because of rapid aging and low productivity growth, so that the real interest rate, which is zero or slightly positive, is still higher than the natural rate of interest. Based on this diagnosis, he recommends the BOJ should raise the expected rate of inflation by announcing that it would never stick to price stability and instead would conduct "irresponsible" monetary policy in the future. Krugman (1998) reports an estimate that the Japanese economy needs four percent annual inflation for the next 15 years.

While Krugman proposes lowering the real interest rate by raising the expected rate of inflation, Woodford (1999d) and Reifschneider and Williams (2000) put less emphasis on expected inflation but more on long-term nominal interest rates. Woodford (1999d) points out that, even when the current overnight interest rate is close to zero, the long-term nominal interest rate could be well above zero if future overnight rates are expected to be well above zero. Expectations theory of the term structure of interest rates implies that, in this situation, a central bank could lower the long-term nominal interest rate by committing itself to an expansionary monetary policy in the future, thereby stimulating aggregate demand. Lowering the long-term nominal interest rate has also been a popular argument in the policy debate in Japan, where the 10-year JGB rate was well above two percent when the zero interest rate policy was initiated in February 1999.

In contrast to the above studies, McCallum (2000), Meltzer (2000), and Bernanke (2000) recommend that the BOJ lower the yen to increase net exports and stop deflation. An important issue in implementing this recommendation is how to realize a weaker yen. Given that the overnight interest rate has already reached zero, monetary policy is useless. Moreover, as is emphasized by Orphanides and Wieland (2000), foreign exchange intervention is not a reliable measure because of the uncertainty regarding its effectiveness. Taking these points into consideration, Svensson (1999, 2000) proposes to control market expectations about the future values of the yen. More specifically, Svensson (2000) recommends a central bank in a trap should declare an upward-sloping price-level target path, and announce that the home currency will be devalued and pegged at a lower level until the price-level target path has been reached.

An important element commonly contained in the above policy recommendations is that a central bank, when caught by a trap, should make a credible commitment to an expansionary monetary policy in the future. Under the assumption that the central bank's policy instrument is the short-term nominal interest rate rather than quantitative measures, as is currently observed in most industrialized countries, this means that a central bank must commit itself to keeping a zero interest rate policy for some time. More specifically, a central bank needs to specify and announce a contingency plan describing how long a zero interest rate policy would be continued, i.e., when and under what circumstances a zero interest rate policy should be terminated.

This was exactly the issue the BOJ policy board had been discussing since the introduction of the zero interest rate policy. At the early stage of the policy, there was a perception in money markets that such an unusual policy would not be continued for long. Reflecting this perception, implied forward interest rates for at least six months started to rise in early March, two months after the introduction of the policy, although implied forward rates for less than six months remained at very low levels.[1] This was clearly against the BOJ's expectation that the zero overnight call rate would spread to longer-term nominal interest rates.[2] Forced to make the bank's policy intention clearer, Governor Hayami announced on April 13, 1999 that the monetary policy board would keep the overnight interest rate at zero until "deflationary concerns are dispelled".[3] Some researchers and practitioners argue that this announcement has had the effects of lowering longer term interest rates by altering expectations of market participants.[4]

The objective of this paper is to characterize the contingency plan; in particular, we are interested in when and under what circumstances a central bank should terminate a zero interest rate policy. Returning to the BOJ example, we are interested in whether the condition "deflationary concerns are dispelled" was appropriate. For this purpose, we solve a central bank's intertemporal loss-minimization problem, in which the non-negativity constraint on nominal interest rates is explicitly considered. Given an adverse shock to the natural rate of interest, we compute the optimal path of short-term nominal interest rates under the assumption that a central bank has the ability to make a credible commitment about the future course of monetary policy.

Our main finding is that the optimal path is characterized by history dependence: a zero

[1]See Okina and Oda (2000) for more details on the movement of implied forward rates during this period.

[2]For example, at a press conference in mid-February, Governor Hayami of the BOJ stated an expectation that lowering the overnight call rate to zero would place downward pressure on longer-term interest rates.

[3]See Ueda (2000) for the BOJ's policy intention behind this announcement. Ueda identifies three transmission channels of the BOJ's commitment to the zero interest rate policy: (1) to minimize policy uncertainties; (2) to mitigate liquidity concern of financial institutions; and (3) to lower longer-term nominal interest rates.

[4]See, for example, Taylor (2000a).

interest rate policy should be continued for some time, even after the natural rate of interest returns to a normal level. By making such a commitment, a central bank is able to achieve higher expected inflation, lower long-term nominal interest rates, and weaker domestic currency in the adverse periods when the natural rate of interest significantly deviates from a normal level. This is as if a central bank "borrows" future monetary easing in the periods when current monetary easing is exhausted. Just like consumption smoothing, borrowing in this sense would improve the national welfare by reducing the variability of inflation and the output gap.[5]

The rest of the paper is organized as follows. Section 2 presents a central bank's intertemporal loss-minimization problem. Sections 3 and 4 characterize discretionary and commitment solutions to the problem. Section 5 gives a numerical example. Section 6 concludes the paper by comparing the BOJ's zero interest rate policy with the optimal commitment solution.

2 Central bank's optimization problem

In this section, we present a central bank's intertemporal optimization problem. The way we specify this problem is similar to those adopted in a number of studies, including Bernanke and Woodford (1997), Rotemberg and Woodford (1997, 1999), McCallum and Nelson (1999), Woodford (1999b, d), and Clarida et al. (1999), but differs from them in that we explicitly treat the non-negativity constraint on nominal interest rates. For example, Rotemberg and Woodford (1997, 1999) and Woodford (1999b) take account of the effects of the non-negativity

[5]While the focus of this paper is on how to *escape* from a liquidity trap, many researchers have discussed how to *avoid* a liquidity trap (Hicks (1967), Summers (1991), Fuhler and Madigan (1997), and Orphanides and Wieland (1998)). In particular, Summers (1991) argues that a central bank should set the target rate of inflation at a level higher than implied by "price stability," thereby making it possible for a central bank to create negative real interest rates enough to offset contractionary disturbances. An important aspect, not fully considered by the previous studies. is that the benefits of a higher target rate depend crucially on the central bank's ability to make a credible commitment about future monetary policy. For example, a central bank with a good commitment technology would be able to raise the expected rate of inflation whenever necessary, thereby reducing the real interest rate below zero. Thus, such a central bank might not need to set the target inflation rate at a higher level against a liquidity trap. However, a central bank without such strong commitment technology should accept the recommendation of Summers (1991). In this sense, the arguments about how to escape from a liquidity trap are closely related to the ones about how to avoid a trap. In the companion paper, Jung, Teranishi, and Watanabe (2001), we point out that setting a positive target inflation rate would indeed reduce the frequency of hitting the zero bound, but it would not necessarily improve the national welfare if a central bank has the ability to control the expected rate of inflation by making a credible commitment about future monetary policy.

constraint by introducing a constraint that the mean value of short-term nominal interest rates is greater than a prespecified positive number. Although this treatment approximates, in some sense, the effects of the non-negativity constraint, it does not allow the possibility that a zero interest rate policy might emerge as an optimal solution to the problem. By treating the non-negativity constraint in an explicit way, we are able to produce a zero interest rate policy as a solution to the optimization problem, thereby investigating several important aspects of the policy.[6]

2.1 Policy preferences

We assume that the central bank's policy instrument is the short-term nominal interest rate (e.g., the overnight call rate) rather than quantitative measures. The central bank chooses the path of the short-term nominal interest rates, starting from period 0, $\{i_0, i_1, \cdots\}$ to minimize

$$E_0 \sum_{t=0}^{\infty} \beta^t L_t, \tag{2.1}$$

where β is the discount factor and L is the loss function. Denoting the inflation rate by π_t and the output gap by x_t, the loss in a given period is given by

$$L_t = \pi_t^2 + \lambda x_t^2, \tag{2.2}$$

where λ is a positive parameter representing the weight assigned to output stability. We make the assumption throughout this paper that the target rate of inflation is equal to zero.[7] We also assume that the target level of the output gap is zero. Since the steady-state level of the output gap is also zero as shown below, the target coincides with the steady-state level. Thus, the central bank has no incentives to create a surprise inflation.

[6] On the other hand, a number of studies that conduct simulations of small macroeconomic models, including Fuhler and Madigan (1997), Orphanides and Wieland (1998), and Reifschneider and Williams (2000), treat the non-negativity constraint on interest rates explicitly in their simulation analyses, but do not solve an optimization problem: they just assume simple Taylor-type policy rules for setting the nominal interest rate with alternative inflation targets. An exception is Orphanides and Wieland (2000), who solve a dynamic optimization problem in the presence of the non-negativity constraint on nominal interest rates. However, market expectations about future monetary policy or future interest rates play no role in their backward-looking model. Therefore, it is impossible to investigate the expectational role of the central bank's commitment about future monetary policy, particularly about how long it should continue a zero interest rate policy.

[7] This assumption is relaxed in Jung, Teranishi, and Watanabe (2001), in which the target rate of inflation is endogenously determined in the process of optimization.

The loss function of this form has been widely used in the literature on monetary policy evaluation, but its theoretical foundation has seldom been discussed. An exception is Woodford (1999c), who shows that the quadratic loss function of this form can be regarded as an approximation to the utility function of the representative household in an economy with monopolistic competition and nominal price rigidities. In particular, Woodford (1999c) presents a theoretical justification for the inflation term in equation (2.2) by illustrating that instability of the general price level causes undesirable variation in the relative prices of goods whose prices are updated only infrequently. This results in unnecessary variations in the production of differentiated goods.

2.2 Economy

The economy outside the central bank is represented by two equations: an "IS curve" and an "AS curve."

$$x_t = E_t x_{t+1} - \sigma^{-1}\left[(i_t - E_t \pi_{t+1}) - r_t^n\right], \tag{2.3}$$

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \tag{2.4}$$

where i_t is the short-term nominal interest rate, r_t^n is the natural rate of interest, and σ and κ are positive parameters. The natural rate of interest is an exogenous variable that could deviate from its steady-state level, thereby giving rise to fluctuations in the output gap and the inflation rate.

Equation (2.3) states that the output gap in period t is determined by the expected value of the output gap in period $t+1$ and the deviation of the short-term real interest rate from the natural rate of interest in period t. Equation (2.3) can be iterated forward to obtain

$$x_t = -\sigma^{-1} \sum_{j=0}^{\infty} E_t \left[(i_{t+j} - \pi_{t+j+1}) - r_{t+j}^n\right]. \tag{2.5}$$

According to the expectations theory of the term structure of interest rates, the expression $\sum_{j=0}^{\infty} E_t \left[(i_{t+j} - \pi_{t+j+1}) - r_{t+j}^n\right]$ stands for the deviation of the long-term real interest rate from the corresponding natural rate of interest in period t, which implies that, given the path of the natural rate of interest, the output gap depends negatively on the long-term real interest rate.

To give a more concrete idea on the natural rate of interest, we interpret equation (2.3) as that which is obtained from the consumption Euler equation. Log-linearizing the Euler equation and imposing the market clearing condition yields

$$y_t - g_t = E_t(y_{t+1} - g_{t+1}) - \sigma^{-1}(i_t - E_t\pi_{t+1} - (1-\beta)/\beta),$$

where y_t is the logarithm of the real output, g_t is a disturbance that fluctuates independently of changes in the real interest rate, and the final term $(1-\beta)/\beta$ represents the discount rate. The above equation is rewritten to obtain

$$\begin{aligned} y_t - y_t^p &= E_t(y_{t+1} - y_{t+1}^p) - \sigma^{-1}(i_t - E_t\pi_{t+1} - (1-\beta)/\beta) \\ &+ E_t\left[(y_{t+1}^p - y_t^p) - (g_{t+1} - g_t)\right], \end{aligned}$$

where y_t^p is the natural rate of output or the potential output. Defining r_t^n as

$$r_t^n \equiv \sigma E_t\left[(y_{t+1}^p - y_t^p) - (g_{t+1} - g_t)\right] + (1-\beta)/\beta, \tag{2.6}$$

we immediately obtain equation (2.3). According to the above definition of r_t^n, variations in the natural rate of interest are caused by short-term factors such as changes in g_t, and long-term factors such as the growth rate of potential output. For example, as pointed out by Krugman (1998) in the context of the Japanese economy, a declining trend in the population of workers or a low growth rate of productivity reduces the potential growth rate, thereby having a negative effect on the natural rate of interest. At the same time, a temporary and autonomous decline in aggregate expenditure, which is triggered by, say, debt-overhang in the corporate sector, could also lead to a decline in the natural rate of interest.

Equation (2.4) is the so-called new-Keynesian Phillips curve,[8] which differs from the traditional Phillips curve in that current inflation depends on the expected rate of future inflation,

[8] Equation (2.4) can be interpreted as derived in a framework with staggered price-setting of Calvo (1983) type, as well as monopolistic competition among price-setting suppliers. For more on this, see Clarida, Gali and Gertler (1999) and Gali and Gertler (1999).

$E_t \pi_{t+1}$, rather than the expected rate of current inflation, $E_{t-1}\pi_t$. To see the implication of this difference, we integrate equation (2.4) forward to obtain

$$\pi_t = \kappa \sum_{j=0}^{\infty} E_t \beta^j x_{t+j}.$$

Current inflation depends entirely on the current and expected future values of the output gap, sharply contrasting with the traditional Phillips curve, in which the current inflation depends on the past values of the output gap.

The structure of the economy outside the central bank is extremely simple in that there is no inertia or lagged dependence in the rate of inflation and the output gap. Instead, these forward-looking variables are determined entirely by expected future economic conditions. Admittedly, this is an unrealistic aspect of the model, but the main focus of this paper is on the expectational channel of monetary policy transmission. In addition, our analysis in later sections investigates monetary policy inertia that is endogenously generated as a part of an optimal solution to the problem. For example, we see later that it is optimal for a central bank to continue a zero interest rate policy even after the economy returns to normal. The economic structure with no lags makes it easier for us to distinguish between endogenous and exogenous inertia.

Finally, we introduce the non-negativity constraint on short-term nominal interest rates,

$$i_t \geq 0. \tag{2.7}$$

The marginal utility of real money balances could be negative if the real balances held by a consumer exceeds the satiation level. If this applies to all consumers, the marginal utility of money balances at the aggregate level, and the nominal interest rate are negative. However, as pointed out by Woodford (1990), this possibility could be ruled out by assuming the existence of at least one consumer having zero cost of holding additional money balances, who would be able to earn a profit by borrowing an infinite amount of money at the negative interest rate and holding it at zero interest rate.

It should be emphasized that equation (2.7) differs from the treatment of the non-negativity constraint by Rotemberg and Woodford (1997, 1999) and Woodford (1999b), who adopt a constraint that the mean value of short-term nominal interest rates is no smaller than a prespecified positive level. These papers show that this constraint is equivalent to making a minor modification to the central bank's loss function by introducing the square of the deviation of the short-term nominal interest rate from its positive target level as an additional term, so that the structure of the quadratic linear optimization problem is not altered.[9] This treatment makes the analysis much simpler, but at the price of losing reality. It does not allow for the possibility that a zero interest rate policy emerges as an optimal solution. Therefore, it is impossible to address practical issues such as those faced by the BOJ policy board, including when and under what circumstances a zero interest rate policy should be terminated. Moreover, optimization under this simplified treatment predicts a symmetric response of a central bank to positive and negative demand shocks, despite the asymmetric property of the non-negativity constraint.

2.3 Adverse shock to the economy

We consider the situation that, in period 0, the economy is hit by a large-scale negative demand shock; the central bank responds to it by lowering the short-term nominal interest rate down to zero; but, aggregate demand is not strong enough to close the output gap. To be more specific, we assume that a large negative shock to the natural rate of interest, denoted by ϵ_0^n, occurs in period 0, so that the natural rate of interest takes a large negative value in period 0. The natural rate of interest is assumed to converge to its steady-state on and after period 1, but only gradually. That is,

$$r_t^n = \rho^t \epsilon_0^n + r_\infty^n \quad \text{for} \quad t = 0, \cdots \tag{2.8}$$

where r_∞^n is the steady-state value of the natural rate of interest, which is assumed to be non-negative, and ρ is a parameter satisfying $0 \leq \rho < 1$. We emphasize that the purpose of this paper is to investigate the optimal path of nominal interest rates, given that a one-time large negative shock has already occurred, so that the above assumption about the path of the natural rate of interest is sufficient to fulfill that purpose. If, instead, the focus were on

[9] In contrast, the way we treat the non-negativity constraint on nominal interest rates makes the optimization problem nonlinear.

the situation in which adverse shocks to the natural rate of interest occur rather frequently, then we would assume that the natural rate of interest obeys a stochastic process, which could take a large negative value in the future with a positive probability, rather than a deterministic process we are assuming here.

3 Optimization under discretion

The central bank minimizes equation (2.1) subject to (2.3), (2.4), and (2.7).[10] In this and the next sections, we characterize solutions to the central bank's optimization problem under two different assumptions about the way monetary policy is conducted: discretion in this section and commitment in the next section. In characterizing these solutions, we assume that the central bank is able to control not only the short-term nominal interest rate, but also the inflation rate and the output gap, including their future values. This assumption, which is adopted by Woodford (1999b) and Clarida et al. (1999), rules out the possibility of coordination failure in market expectations about future values of inflation and the output gap, thereby making it possible for us to avoid complicated issues such as the indeterminacy of an equilibrium. After characterizing this "cooperative solution", we proceed to discuss whether it could be implemented under the more realistic assumption that a unique control variable for the central bank is the short-term nominal interest rate.

[10] If we ignore the non-negativity constraint on nominal interest rates, the optimal solution to the problem is very simple. All that the central bank should do is to adjust the short-term nominal interest rate so that it always coincides with the natural rate of interest. By doing this, both the inflation rate and the output gap in each period become zero, thus the loss in each period is also zero. It is important to note that, since the loss is zero in each period, this is the first-best outcome. Thus, the solution under commitment always coincides with the one under discretion, so that time-inconsistency of Barro-Gordon type never arises. In addition, note that time-inconsistency that could arise in dynamic responses to shocks, which is pointed out by Woodford (1999b) and discussed further by Clarida et al. (1999), does not occur.

3.1 First-order conditions

We assume that the central bank reoptimizes in each period. The optimization problem is represented by a Lagrangean of the form[11]

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ L_t + 2\phi_{1t}[x_t - x_{t+1} + \sigma^{-1}(i_t - \pi_{t+1} - r_t^n)] + 2\phi_{2t}[\pi_t - \kappa x_t - \beta\pi_{t+1}] \right\}, \quad (3.1)$$

where ϕ_{1t} and ϕ_{2t} represent the Lagrange multipliers associated with the IS constraint (equation (2.3)) and the AS constraint (equation (2.4)), respectively. We differentiate the Lagrangean with respect to π_t, x_t, and i_t to obtain the first-order conditions

$$\pi_t + \phi_{2t} = 0 \quad (3.2)$$

$$\lambda x_t + \phi_{1t} - \kappa\phi_{2t} = 0 \quad (3.3)$$

$$i_t \phi_{1t} = 0 \quad (3.4)$$

$$i_t \geq 0 \quad (3.5)$$

$$\phi_{1t} \geq 0 \quad (3.6)$$

Equations (3.4), (3.5), and (3.6) are Kuhn-Tucker conditions regarding the non-negativity constraint on the nominal interest rate. Observe that $\partial\mathcal{L}/\partial i_t = 2\sigma^{-1}\beta^t\phi_{1t} \propto \phi_{1t}$. If the non-negativity constraint is not binding, $\partial\mathcal{L}/\partial i_t$ is equal to zero, so that ϕ_{1t} is zero also. On the other hand, if the constraint is binding, $\partial\mathcal{L}/\partial i_t$ is non-negative, ans so is ϕ_{1t}. The above conditions, together with the IS and AS equations, are the first-order conditions for loss minimization.

3.2 Steady-state values

The steady-state values of the endogenous variables, x_∞, π_∞, i_∞, $\phi_{1\infty}$, and $\phi_{2\infty}$ are calculated by substituting $x_t = x_{t+1} = x_\infty, \pi_t = \pi_{t+1} = \pi_\infty, i_t = i_\infty, \phi_{1t} = \phi_{1\infty}$, and $\phi_{2t} = \phi_{2\infty}$ into the first-order conditions. There are two sets of steady-state values; one is the interior solution

[11] Note that the expectation operator, E_t, does not appear here because the future path of the natural rate of interest is perfectly foreseen, as assumed in Section 2.3.

given by

$$x_\infty = 0; \quad \pi_\infty = 0; \quad i_\infty = r^n_\infty; \quad \phi_{1\infty} = 0; \quad \phi_{2\infty} = 0, \tag{3.7}$$

and the other is the corner solution given by

$$\begin{aligned} x_\infty &= -(1-\beta)\kappa^{-1} r^n_\infty; \quad \pi_\infty = -r^n_\infty < 0; \quad i_\infty = 0; \\ \phi_{1\infty} &= \left[\lambda(1-\beta)\kappa^{-1} + \kappa\right] r^n_\infty; \quad \phi_{2\infty} = r^n_\infty. \end{aligned} \tag{3.8}$$

It is important to note that the interior solution is the first-best outcome, in the sense that the value of the central bank's loss function, defined by (2.2), is equal to zero. The corner solution, or the so-called Friedman equilibrium, in which nominal interest rates are zero in each period, is inferior to the interior solution in terms of the central bank's preferences.[12] An important feature of the corner solution is the self-fulfilling deflationary spiral: people have an expectation of deflation and a negative output gap, which are indeed realized through a self-fulfilling mechanism; the short-term nominal interest rate is reduced to zero in response to the deterioration of the economy, but it is not sufficient to escape from a deflationary spiral.[13]

It is easy to see that the difference between the two solutions in terms of the central bank's preferences becomes larger with r^n_∞. It is assumed in this paper that r^n_∞ is sufficiently large, so that the dynamic path of the endogenous variables converging to the interior solution is superior to that converging to the corner solution. Thus, in this and the next sections, we focus our analysis on the dynamic path that converges to the interior solution.

3.3 Dynamic path

Given that the non-negativity constraint on nominal interest rates is not binding in the interior steady-state solution, and the assumption that the natural rate of interest converges monoton-

[12]Friedman (1969) argues that distortions due to the shoe-leather costs are proportional to the level of nominal interest rates, and that these costs are eliminated if nominal interest rates are equal to zero. The loss function defined by (2.2) ignores the existence of shoe-leather costs, while emphasizing the importance of stability in the inflation rate and the output gap, which results in the inferiority of the corner solution. See Woodford (1999c) for further details.

[13]Benhabib et al. (1999) emphasize the existence of multiple equilibria in a monetary economy with a zero bound on nominal interest rates. One of the equilibria, which is characterized by zero interest rates and self-fulfilling deflation, is similar to the corner solution in our model.

ically to its steady-state value, it is straightforward to guess that the non-negativity constraint is binding until some period, denoted by period T^d, but not binding afterwards.

To characterize the path of the endogenous variables for the periods on and after $T^d + 1$, we substitute $\phi_{1t} = 0$ into (3.2) and (3.3), eliminating ϕ_{2t}, to obtain

$$\lambda x_t + \kappa \pi_t = 0 \qquad \text{for} \quad t = T^d + 1, \cdots \tag{3.9}$$

which, together with the AS equation, yields a first-order difference equation of the form

$$\pi_{t+1} = \beta^{-1} \left(1 + \lambda^{-1}\kappa^2\right) \pi_t \qquad \text{for} \quad t = T^d + 1, \cdots \tag{3.10}$$

Since the coefficient of π_t on the right-hand side is greater than unity, this difference equation has a unique bounded solution, which is given by $\pi_t = 0$. Substituting $\pi_t = 0$ into equation (3.9) and the IS equation, we obtain a unique bounded solution for $t = T^d + 1, \cdots$, which is given by

$$z_t = 0 \qquad \text{for} \quad t = T^d + 1, \cdots \tag{3.11}$$

$$i_t = r_t^n \qquad \text{for} \quad t = T^d + 1, \cdots \tag{3.12}$$

where $z_t \equiv [\pi_t \; x_t]'$. Note that (3.12) implies that r_t^n must be non-negative for $t = T^d + 1, \cdots$.

For the periods during which a zero interest rate policy is adopted (i.e., $t = 0, \cdots, T^d$), we substitute $i_t = 0$ into the IS and AS equations to obtain

$$z_{t+1} = Q z_t - q r_t^n \qquad \text{for} \quad t = 0, \cdots, T^d \tag{3.13}$$

where

$$Q \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ -\sigma^{-1}\beta^{-1} & 1 + \sigma^{-1}\beta^{-1}\kappa \end{bmatrix},$$

and $q \equiv [0 \quad \sigma^{-1}]'$. Combined with the terminal conditions, $z_{T^d+1} = 0$, this difference equation has a unique bounded solution of the form

$$z_t = \sum_{k=t}^{T^d} Q^{-(k-t+1)} q \; r_k^n. \tag{3.14}$$

Finally, we need to confirm that ϕ_{1t} is positive for $t = 0, \cdots, T^d$. First, as we saw in (3.12), $r^n_{T^d+1}$ must be non-negative. Second, $r^n_{T^d}$ must be non-positive; otherwise, both x_{T^d} and π_{T^d} would be positive, so that ϕ_{1T^d} would be negative (recall that $\phi_{1t} = -\lambda x_t - \kappa\pi_t$), contradicting the Kuhn-Tucker condition. Combining the two, we see that the first-order conditions concerning ϕ_1 is satisfied if T^d is chosen so that $r^n_{T^d} < 0$ and $r^n_{T^d+1} \geq 0$. Note that the timing to terminate a zero interest rate policy is determined entirely by an exogenous factor, r^n, which is in sharp contrast with the case of the commitment solution, as we see in the next section.

3.4 Implementation

Given the cooperative solution characterized in the previous subsection, the next issue we address is how to implement it. As pointed out by McCallum (1981), the rational expectations equilibrium could be indeterminate under policy rules in which a policy instrument (such as the overnight call rate) is exogenously given. To illustrate this point in the context of our model, denote the cooperative solution characterized in the previous subsection by $\{i^d_t,\ \pi^d_t,\ x^d_t\}^\infty_{t=0}$, and suppose that the central bank sets the short-term interest rate in each period as

$$i_t = i^d_t \qquad \text{for} \quad t = 0, \cdots. \tag{3.15}$$

A system of equations now consists of (3.15), (2.3), and (2.4), which is rewritten as

$$z_{t+1} = Qz_t + q(i^d_t - r^n_t). \tag{3.16}$$

Since π and x are both non-predetermined variables, or jumping variables, this difference equation has a unique bounded solution if the matrix Q has two eigenvalues outside the unit circle, according to Proposition 1 in Blanchard and Kahn (1980). However, in this case, there is one eigenvalue outside the unit circle, so that there is an infinity of bounded solutions (see Proposition 3 in Blanchard and Kahn (1980)).

To cope with the problem of indeterminacy, the central bank needs to adopt a feedback policy rule in which a policy instrument depends on endogenous variables (McCallum, 1981).

As an example of such feedback rules, consider

$$i_t = max\left\{i_t^d + \theta_\pi(\pi_t - \pi_t^d) + \theta_x(x_t - x_t^d), 0\right\}, \tag{3.17}$$

where θ_π and θ_x are positive parameters representing the responsiveness of the short-term nominal interest rate to the deviations of π and x from the cooperative solution. It is important to note that equation (3.17) describes the "off-equilibrium path" of the short-term nominal interest rate, in that it specifies how the central bank behaves when the economy deviates from the cooperative solution.

Consider the system of equations consisting of (3.17), (2.3), and (2.4). Suppose there exists a path of z_t and i_t which differs from z_t^d and i_t^d but converges to the same interior steady-state. Since i_t converges to r_∞^n, which is positive, there must exist $T \in [0, \infty)$ such that $i_t \geq 0$ for $t \geq T+1$. Then, the system of equations can be rewritten as

$$z_{t+1} = Rz_t - q\theta' z_t^d + q(i_t^d - r_t^n), \qquad \text{for} \quad t \geq T+1 \tag{3.18}$$

where

$$R \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma^{-1}\left(\theta_\pi - \beta^{-1}\right) & 1 + \sigma^{-1}\left(\theta_x + \beta^{-1}\kappa\right) \end{bmatrix},$$

$z_t^d \equiv [\pi_t^d \; x_t^d]'$, and $\theta \equiv [\theta_\pi \theta_x]'$. The matrix R has now two eigenvalues outside the unit circle if and only if

$$\kappa(\theta_\pi - 1) + (1-\beta)\theta_x > 0.$$

In particular, a unique bounded solution obtains if $\theta_\pi > 1$ and $\theta_x > 0$. If these conditions are satisfied, the path of z_t and i_t is identical to that of z_t^d and i_t^d for $t \geq T+1$, which rules out the possibility that there exists a path of z_t and i_t that differs from z_t^d and i_t^d but converges to the same interior steady-state. Thus, a sufficient condition for the cooperative solution to be implemented is that the policy feedback rule satisfies these conditions. The condition $\theta_\pi > 1$, which is called the "Taylor principle", requires the central bank to raise (lower) the short-term nominal interest rate more than one-to-one when the rate of inflation is higher (lower) than the cooperative solution, thereby raising (lowering) the *real* interest rate.[14] It is important to note that the Taylor principle guarantees local, but not global, determinacy. To be more specific, some dynamic paths converging to the corner steady-state, which is given by (3.8), can be solutions to the system of equations given by (3.17), (2.3), and (2.4), even if the Taylor principle is satisfied. In this sense, we cannot rule out the possibility that the economy falls into a deflationary spiral equilibrium, which is investigated by Benhabib et al. (1999) in a related model.[15]

4 Optimization under commitment

A central bank that conducts monetary policy in a discretionary manner does not take into account the consequences of its policy on market expectations about future inflation and the output gap, thereby failing to achieve an optimal solution. In this section, we consider a different situation, in which the central bank solves an intertemporal optimization problem in period 0, taking account of the expectational channel of monetary policy, and commit itself to the computed optimal path.

4.1 First-order conditions

We differentiate the Lagrangean, equation (3.1), with respect to π_t, x_t, and i_t to obtain the first-order conditions

$$\pi_t - (\beta\sigma)^{-1}\phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0 \quad (4.1)$$

$$\lambda x_t + \phi_{1t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0 \quad (4.2)$$

$$i_t\phi_{1t} = 0 \quad (4.3)$$

$$i_t \geq 0 \quad (4.4)$$

$$\phi_{1t} \geq 0 \quad (4.5)$$

[14] Leeper (1991) shows that an "active" policy rule satisfying $\theta_\pi > 1$ guarantees the determinacy of equilibrium, while rational expectation equilibrium is indeterminate under "passive" policy rules in which θ_π is less than unity. Benhabib et al. (2000a) show that equilibrium could be determinate even under a passive policy, depending on the way in which money is assumed to enter preferences and technology.

[15] See Benhabib et al. (2000b) for discussion on how to avoid falling into a deflationary spiral equilibrium.

These first-order conditions differ from those obtained in the previous section in that lagged Lagrange multipliers, ϕ_{1t-1} and ϕ_{2t-1}, appear in the first two equations. For example, equation (4.1) captures not only a change in the loss in period t due to a marginal change in π_t, but also a change in the loss in period $t-1$ through changes in inflation expectation; changes in inflation expectation affect real interest rate in the IS equation, as well as current inflation through the AS equation, so that the Lagrange multipliers, ϕ_{1t-1} and ϕ_{2t-1}, appear there. The same expectational effects apply to a marginal change in the output gap, which is captured by (4.2).

4.2 Steady-state values

The same procedure as in the previous section allows us to specify the steady-state values of the endogenous variables. It is straightforward to see that the interior solution given by (3.7) satisfies the requirements of the steady-state values. On the other hand, the corner solution, in which the short-term nominal interest rate is zero, does not satisfy these requirements because

$$\phi_{1\infty} = -(\beta\sigma)\ r_{\infty}^{n} < 0 \tag{4.6}$$

is inconsistent with the Kuhn-Tucker conditions. Thus, the interior solution is a unique steady-state in the case of commitment.[16]

Since the first-order conditions contain lagged Lagrange multipliers, we need to specify the values of the two multipliers in period -1 in order to solve the optimization problem. We assume that the economy is in the interior steady-state before period 0, achieving zero inflation and zero output gap in each period. Since neither the IS nor the AS equations is binding before an adverse shock to the natural rate of interest occurs in period 0, the Lagrange multipliers in these periods should be equal to zero. More specifically, we assume that

$$\phi_{1-1} = 0; \quad \phi_{2-1} = 0. \tag{4.7}$$

[16] If we allow the possibility that the two Lagrange multipliers, ϕ_{1t} and ϕ_{2t} are *not* constant over time even in steady-state, the corner solution could be a steady-state. In this corner solution, the Lagrange multipliers are not bounded, while the other endogenous variables (π, x, and i) are bounded.

These two initial conditions, together with equations (4.1)-(4.5), (2.3), and (2.4), characterize the optimal solution under commitment.

4.3 Optimal dynamic path

The same argument as in the previous section allows us to guess that the non-negativity constraint on nominal interest rates is binding until some period, denoted by T^c, but not thereafter. To characterize the path of the endogenous variables for the periods on and after T^c+2, we substitute $\phi_{1T^c+1} = \phi_{1T^c+2} = \cdots = 0$ into (4.1) and (4.2) to obtain a system of difference equation of the form

$$\pi_t + \phi_{2t} - \phi_{2t-1} = 0 \quad \text{for} \quad t = T^c + 2, \cdots \tag{4.8}$$

$$\lambda x_t - \kappa \phi_{2t} = 0 \quad \text{for} \quad t = T^c + 2, \cdots \tag{4.9}$$

Eliminating x_t using the AS equation, we have a difference equation with respect to π_t and ϕ_{2t} of the form

$$\begin{bmatrix} \pi_{t+1} \\ \phi_{2t} \end{bmatrix} = \begin{bmatrix} \beta^{-1}(1+\kappa^2/\lambda) & -\beta^{-1}\kappa^2/\lambda \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ \phi_{2t-1} \end{bmatrix} \quad \text{for} \quad t = T^c + 2, \cdots \tag{4.10}$$

where ϕ_{2T^c+1} is given as an initial condition. This system of difference equation has one predetermined variable, ϕ_2, and one non-predetermined variable, π, and the two-by-two matrix on the right-hand side of the equation has two real eigenvalues, which are denoted by $|\mu_1| < 1$ and $|\mu_2| > 1$. Since the number of eigenvalues outside the unit circle is equal to the number of non-predetermined variables, there exists a unique bounded solution that converges to the interior steady-state, whose dynamic path is given by

$$\pi_t = -(\beta^{-1}\kappa^2/\lambda)^{-1}\left[\mu_1 - \beta^{-1}(1+\kappa^2/\lambda)\right]\phi_{2t-1}, \tag{4.11}$$

$$\phi_{2t} = \mu_1 \phi_{2t-1}. \tag{4.12}$$

We obtain an optimal path of x_t in the corresponding periods by substituting this solution into (4.2), and an optimal path of i_t in the corresponding periods by substituting into the IS equation.

Next, we characterize the path of the endogenous variables for $t = 0, \cdots, T^c + 1$. First, for $t = T^c + 1$, we substitute $\phi_{1T^c+1} = 0$ into (4.1) and (4.2) to obtain

$$\pi_t - (\beta\sigma)^{-1}\phi_{1t-1} + \phi_{2t} - \phi_{2t-1} = 0, \qquad \text{for} \quad t = T^c + 1 \tag{4.13}$$

$$\lambda x_t - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0. \qquad \text{for} \quad t = T^c + 1 \tag{4.14}$$

Together with (2.3) and (2.4), these two equations characterize an optimal path of the endogenous variables for $t = T^c + 1$. As for $t = 0, \cdots, T^c$, we substitute $i_t = 0$ into (2.3) to obtain

$$x_t - x_{t+1} - \sigma^{-1}(\pi_{t+1} + r_t^n) = 0 \tag{4.15}$$

This equation, and equations (4.1), (4.2), and (2.4) characterize an optimal path of the endogenous variables for $t = 0, \cdots, T^c$. Thus, a unique optimal solution for $t = 0, \cdots, T^c + 1$ is characterized by (4.13)-(4.15), two initial conditions, (4.7), and two terminal conditions concerning π_{T^c+2} and x_{T^c+2}, both of which are obtained by solving an optimization problem for the periods on and after $T^c + 2$. Note that, unlike the case of discretion, the path of the endogenous variables for $t = 0, \cdots, T^c + 1$ and that for $t = T^c + 2, \cdots$ are not mutually independent: we need the value of ϕ_{2T^c+1} as an initial condition in solving the difference equation for $t = T^c + 2, \cdots$, whereas we need the values of π_{T^c+2} and x_{T^c+2} as terminal conditions in solving the difference equation for $t = 0, \cdots, T^c + 1$.

Finally, we need to confirm that T^c is a finite number, and that ϕ_{1t} is positive for $t = 0, \cdots, T^c$, as postulated in the above discussion. We first eliminate ϕ_{2t} from equations (4.1) and (4.2) to obtain a second-order difference equation with respect to ϕ_{1t}.

$$\phi_{1t} - [1 + \beta^{-1} + \kappa(\beta\sigma)^{-1}]\phi_{1t-1} + \beta^{-1}\phi_{1t-2} = -\kappa\pi_t - \lambda x_t + \lambda x_{t-1}$$
$$\text{for} \quad t = 0, \cdots, T^c + 1, \tag{4.16}$$

where initial conditions are given by $\phi_{1-1} = \phi_{1-2} = 0$. A unique solution to this difference equation is given by

$$\phi_{1t} = -\kappa A(L)\pi_t - \lambda(1 - L)A(L)x_t, \tag{4.17}$$

where

$$A(L) \equiv \frac{1}{\eta_1 - \eta_2}\left(\frac{\eta_1}{1-\eta_1 L} - \frac{\eta_2}{1-\eta_2 L}\right),$$

and η_1 and η_2 are two real solutions to the characteristic equation associated with the difference equation (4.16), satisfying $\eta_1 > 1$ and $0 < \eta_2 < 1$. The right-hand side of (4.17) represents a marginal change in the sum of the central bank's loss up to period t when a zero interest rate policy is prolonged for one more period. Extending a zero interest rate policy increases x_t, which is followed by an increase in x_{t-1}, x_{t-2} and so on, as indicated by equation (2.3). The second term of the right-hand side of (4.17) captures the effect of these changes on the central bank's losses. Similarly, the effects of changes in inflation by extending a zero interest rate policy is captured by the first term of the right-hand side of (4.17). Note that an increase in x_t reduces losses in periods prior to t if t is smaller than T^d, so that extending a zero interest rate policy always increases welfare. On the other hand, if t is greater than T^d, an increase in x_t increases the losses in some periods prior to period t, while it reduces losses in earlier periods including period 0. A comparison of the marginal benefits in the latter periods and the marginal costs in the former periods determines the optimal timing to terminate a zero interest rate policy.

It is important to note that, since η_1 is greater than unity, ϕ_{1t} tends to explode once it becomes positive, unless the right-hand side of equation (4.16) takes sufficiently large negative values. In other words, inflation and the output gap must overshoot the steady-state values (i.e., zero) after they take negative values in period 0 and subsequent periods. This condition is satisfied if the central bank continues the zero interest rate policy for a sufficiently long period. By adopting such a policy, the output gap and inflation become sufficiently high, so that the expression on the right-hand side of (4.16) takes sufficiently large negative values to guarantee that ϕ_{1t} converges to zero within a finite period.

The same argument as in section 3.4 guarantees that the commitment solution characterized above can be implemented when the central bank follows a feedback policy rule of the form

$$i_t = max\left\{i_t^c + \theta_\pi(\pi_t - \pi_t^c) + \theta_x(x_t - x_t^c), 0\right\}, \tag{4.18}$$

where i_t^c, π_t^c, and x_t^c represent the commitment solution, and θ_π and θ_x are parameters satisfying $\theta_\pi > 1$ and $\theta_x > 0$.

4.4 Timing to terminate a zero interest rate policy

Equation (4.16) has several implications. First, given an adverse shock to the natural rate of interest, a zero interest rate policy is continued longer than for the case of discretion. To see this, recall that, in the case of discretion, both inflation and the output gap take negative values in periods when a zero interest rate policy is adopted (i.e., $t = 0, \cdots, T^d$). Thus, if a zero interest rate policy is terminated in period T^d, ϕ_{1t}, which is defined by (4.17), takes a large positive value at $t = T^d + 1$, indicating that a zero interest rate policy should be extended, or equivalently

$$0 \leq T^d \leq T^c < \infty. \tag{4.19}$$

Second, whether the short-term nominal interest rate is set at zero in period t depends not only on the current condition of the economy, which is represented by π_t and x_t, but also on the past values of the inflation rate and the output gap. For example, if inflation and the output gap take large negative values before period t, due to a negative shock to the natural rate of interest, then the short-term interest rate in period t should be set at zero, even if the natural rate of interest has already returned to a positive level in period t. This is in sharp contrast with the case of discretion, in which a zero interest rate policy is terminated as soon as the natural rate of interest becomes positive.

Put differently, the optimal path of the short-term nominal interest rates is characterized by history dependence, in the sense that a zero interest rate policy is continued for a while even after the natural rate of interest returns to an "above-water", or normal level. By making such a commitment, the central bank is able to achieve higher expected inflation and lower long-term nominal interest rates in the periods when the natural rate of interest significantly deviates from a steady-state level, thereby stimulating aggregate demand in these periods.[17] This is as if the central bank "borrows" future monetary easing in the periods when current monetary easing is already exhausted. Just like consumption smoothing, borrowing in this sense would improve the national welfare by reducing the variability of inflation and the output gap.[18] As emphasized by Woodford (1999b) and Clarida et al. (1999), history dependence in the commitment solution arises as a product of the central bank's ability to manipulate private sector expectations.

Third, the timing to terminate a zero interest rate policy depends more on the rate of inflation than the output gap. To see this, consider a limiting case in which $\kappa(\beta\sigma)^{-1}$ takes a value very close to zero. As we see later in our numerical example, this expression is actually close to zero as compared with other parameter values. When $\kappa(\beta\sigma)^{-1} \simeq 0$, equation (4.17) is approximated by

$$\phi_{1t} = -(1-\beta^{-1}L)^{-1}\left[\kappa(1-L)^{-1}\pi_t + \lambda x_t\right]. \tag{4.20}$$

The expression in the square brackets indicates that the past values of inflation have more influences on ϕ_{1t} as compared with those of the output gap. This is a simple reflection of the property of the new-Keynesian Phillips curve, (2.4), that the current inflation rate π_t is determined by the future inflation rates π_{t+j}, which are determined by the future values of the output gap x_{t+j}.

[17] Watanabe (2000) discusses the effects of this commitment in the context of a small open economy. This commitment creates an expectation of weaker currency, leading to the depreciation of the current spot rate.

[18] Consumption theories tell us that consumers are able to borrow money if (1) their future income is large enough to repay, or, put differently, the shortage of income is not long-lasting; and (2) they are able to make a credible commitment to repay in the future. In our model, the first condition corresponds to the assumption that an adverse shock to the natural rate of interest is temporary, or ρ in equation (2.8) is less than unity. If ρ were equal to unity, there would be no room for monetary policy smoothing. On the other hand, the second condition corresponds to the assumption that the central bank is able to make a credible commitment about the future course of monetary policy. An important issue related to this assumption is that the optimal solution under commitment is *not* time-consistent: the central bank has an incentive to terminate a zero interest rate policy once the natural rate of interest returns to an "above-water" level. In a related context, Taylor (2000b) points out that applying the idea of history dependent policy to events that are very rare may raise credibility problems. It should be emphasized that the unique source of time-inconsistency in our setting is the non-negativity constraint on nominal interest rates (see footnote 10).

Fourth, our commitment solution is closely related with, but different in an important respect from the augmented Taylor rule proposed by Reifschneider and Williams (2000), which is of the form[19]

$$i_t = max\left\{i_t^{Taylor} - Z_t, 0\right\}, \tag{4.21}$$

where Z_t is the cumulative sum of the deviation of the actual short-term rates from the standard unconstrained Taylor rule, which is given by $d_t = i_t - i_t^{Taylor}$. According to Reifschneider and Williams (2000), when the non-negativity constraint is binding in period t, d_t takes a positive value, and therefore Z_t increases. Zero interest is continued until a backlog of past deviations completely vanishes. In our setting in which an adverse shock to the economy decays monotonically over time, equation (4.21) implies that

$$\sum_{k=0}^{T^{ATR}+1} (i_k - i_k^{Taylor}) = 0, \tag{4.22}$$

where T^{ATR} is the final period of zero interest rate policy. This equation simply states that, during the periods of zero interest rate policy, the short-term nominal interest rate coincides *on average* with the unconstrained Taylor rule. Since the unconstrained interest rate rule in our setting is given by $i_t = r_t^n + \pi_{t+1}$, (4.22) can be further rewritten as

$$\sum_{k=0}^{T^{ATR}+1} [i_k - (r_k^n + \pi_{k+1})] = 0. \tag{4.23}$$

To make it possible to compare the augmented Taylor rule with our commitment solution, consider a limiting case of $\kappa \simeq 0$,[20] in which the condition to terminate a zero interest rate policy in our commitment solution, $\phi_{1T^c+1} = 0$, is simplified to

$$\sum_{k=0}^{T^c+1} \beta^{-(T^c+1-k)} x_k = 0. \tag{4.24}$$

[19]Their original definition is $i_t = max\left\{i_t^{Taylor} - \alpha Z_t, 0\right\}$, where α is a parameter satisfying $\alpha \in (0, 1]$. We discuss a simple case of $\alpha = 1$ in the text, but we could reach a similar conclusion as long as α is close to unity.

[20]Changes in the output gap have no effects on inflation if $\kappa = 0$. Then, according to the AS equation, the rate of inflation always equals to the steady-state value (namely, zero).

By substituting (2.5) into the above condition, we obtain

$$\sum_{k=0}^{T^c+1} \beta^{-(T^c+1-k)} D_k = 0, \tag{4.25}$$

where D_k is defined by $D_k \equiv \sum_{j=0}^{\infty} \left[(i_{k+j} - \pi_{k+j+1}) - r^n_{k+j} \right]$, representing the deviation in period t of the real long-term interest rate from the corresponding natural rate of interest. Note that our commitment solution requires the coincidence of the real *long*-term interest rate with the corresponding natural rate of interest on average during the periods of zero interest rate policy, while the augmented Taylor rule requires the coincidence of the real *short*-term interest rate with the natural rate of interest. Since the short-term nominal interest rate coincides with the unconstrained Taylor rule period by period on and after $T^{ATR} + 2$, (4.23) can be further rewritten as

$$D_0 = 0. \tag{4.26}$$

A simple comparison between (4.25) and (4.26) indicates the following. First, the augmented Taylor rule is a special case of our commitment solution, in which the discount factor β is very close to zero. In other words, the augmented Taylor rule is a good approximation of our commitment solution if the central bank does not care much about future values of its losses. This is because the augmented Taylor rule pays no attention to central bank's losses resulting from the adoption of a zero interest rate policy. Second, the augmented Taylor rule prescribes longer periods of zero interest rate policy than our commitment solution. To see this, recall that, in our commitment solution, D_t is positive for the earlier periods including $t = 0$ and turns to negative afterwards, so that (4.25) holds for the entire periods of zero interest rate policy. To change D_0 from positive to zero, we need a further extension of a zero interest rate policy, which means that T^{ATR} is greater than T^c.

$$0 \le T^d \le T^c \le T^{ATR} < \infty. \tag{4.27}$$

The augmented Taylor rule instructs a central bank to "borrow" future monetary easing to fully compensate the shortage of current monetary easing, while the discretionary solution prohibits borrowing. These two are extreme cases in the sense that the former exhibits very strong history dependence while the latter has no history dependence. What we learn from the discussion in this section is that these two extreme cases are inferior to an imtermediate case, the commitment solution.

5 Numerical example

In this section, we compute the optimal path of the short-term nominal interest rates, using the parameter values shown in Table 1. These parameter values are borrowed from Woodford (1999b),[21] except the steady-state value of the natural rate of interest, r^n_∞, which is calculated using equation (2.6) under the assumption that the growth rate of potential output is three percent per year. The parameter values are adjusted so that the length of a period in our model is interpreted as a quarter.

Figure 1 shows the responses of various variables to an adverse shock to the natural rate of interest in the case of discretion.[22] In the baseline case, shown in this figure, we assume that the initial shock to the natural rate of interest, ϵ^n_0 in equation (2.8), is equal to -0.10, which means a 40 percent decline in the annualized natural rate of interest. In addition, we assume that the persistence of the shock, which is represented by ρ in equation (2.8), is 0.5 per a quarter. The path of the natural rate of interest is shown at the bottom of Figure 1. In response to this shock, the short-term nominal interest rate is set to zero for the first four periods until period 3, and is positive in period 4 when the natural rate of interest turns positive. Inflation and the output gap take negative values for the first four periods during which a zero interest rate policy is adopted, and return to the steady-state value (i.e., zero) on and after period 4.

Figure 2 shows the responses of the same set of variables in the case of commitment.[23] An

[21]See Woodford (1999b) and Rotemberg and Woodford (1997) for details on the estimation of parameter values.

[22]We use MATLAB 5.3.1 for the calculations.

[23]A key part in computing the commitment solution is how to find the timing to terminate a zero interest rate policy, T^c. We search T^c as follows: (1) We set T^c at a sufficiently high value, say 50, under which ϕ_{1T^c} is supposed to be negative, and compute the path of the variables; (2) If ϕ_{1T^c} is negative, we try $T^c = 49$; and (3) We repeat this until ϕ_{1T^c} becomes non-negative.

important difference from the case of discretion is that a zero interest rate policy is continued longer, which is consistent with equation (4.19). A prolonged zero interest rate policy lowers the long-term interest rate during the first six periods. This reduces the central bank's losses in periods 0 to 3, during which the natural rate of interest is negative, by alleviating deflationary pressures, while it increases losses in periods 4 and 5 by creating positive inflation and output gap. It is also important to note that the cumulative sum of the deviation of the short-term *real* interest rate from the natural rate of interest during the periods of zero interest rate policy is negative, although significantly smaller as compared with the case of discretion. This implies that a zero interest rate policy would be extended further if we follow the augmented Taylor rule, proposed by Reifschneider and Williams (2000), which requires that the deviation should be zero, on average, during the periods of zero interest rate policy.

Table 2 compares T^d and T^c for various combinations of ρ and ϵ_0^n. If a shock to the natural rate of interest is small and non-persistent, the difference between the two is negligibly small (see, for example, the case of $\epsilon_0^n = -0.02$ and $\rho = 0$). However, given the value of ρ, the difference between the two becomes larger with the absolute value of ϵ_0^n. In the case of $\rho = 0$, for example, T^d and T^c coincide when $\epsilon_0^n = -0.02$, but the difference between the two emerges and increases as ϵ_0^n becomes larger in absolute value. On the other hand, given the size of the shock, a change in its persistence does not affect the difference between the two, although both T^d and T^c increases with ρ.

Finally, to check the sensitivity of the results, Table 3 computes T^c changing parameter values for β, λ, κ, and σ within a plausible range. Note that T^d is equal to 3, independent of these parameter values. As seen in the table, T^c indeed changes depending on parameter values, ranging from 3 to 6,[24] but exceeds T^d, which is equal to 3, in almost all cases. In this sense, the result we saw in the previous section that T^c is greater than T^d holds, not only qualitatively, but also quantitatively.

[24]The upper part of the table shows that T^c decreases as λ increases, whereas changes in β within a plausible range have no impact on T^c. Moreover, the lower part of the table shows that T^c decreases as κ increases while T^c increases as σ increases. These results are consistent with the qualitative results we saw in the previous section.

6 Conclusion

What should the central bank do when it faces a weak aggregate demand even after lowering the short-term nominal interest rate to zero? To address this question, we have solved a central bank's intertemporal loss-minimization problem, in which the non-negativity constraint on nominal interest rates is explicitly considered. Given an adverse shock to aggregate demand, we have computed the optimal path of short-term nominal interest rates under the assumption that the central bank has the ability to make a credible commitment about the future course of monetary policy. We have found that the optimal path is history dependent, in the sense that a zero interest rate policy is continued for a while even after the economy returns to normal.

There are some similarities between our commitment solution and the Bank of Japan's zero interest rate policy.[25] Governor Hayami's announcement, in April 1999, that the zero interest rate policy would be continued until "deflationary concerns are dispelled" has a flavor of policy commitment, as pointed out by many researchers, market participants, and policy makers (see, for example, Taylor (2000a) and Ueda(2000)). For example, Taylor (2000a) states that this announcement had the effects of lowering longer term interest rates by altering the market expectations about the future path of short-term interest rates. As stated by Ueda (2000), this announcement was a unique experience in the history of the BOJ which had been conducting monetary policy in a discretionary manner.

However, if we closely look at the BOJ policy, it differs from the optimal commitment solution in some important respects. First, according to the BOJ's announcement, the timing to terminate a zero interest rate policy depends solely on the rate of inflation, while our theoretical analysis indicates that it should depend on both the rate of inflation and the output gap.[26]

[25]See Watanabe (2000) for a detailed comparison between the BOJ's zero interest rate policy and the optimal commitment solution.

[26]In this respect, it should be emphasized that the meaning of "deflationary concerns are dispelled" was never clearly defined by BOJ officials. According to Ueda (2000), some board members state that "the zero interest rate will be kept until the risk of the rate of inflation falling down to a large negative number becomes small enough," and others say "inflation would fall to a lot when aggregate demand goes down sharply. Hence, we would wait until we are confident that domestic private demand is on a sustained recovery path." Interestingly, some of the board members link the timing to terminate the zero interest rate policy to aggregate demand, or the output gap, as clearly described here, but only to the extent that the output gap affects the rate of inflation.

Second, the BOJ's announcement lacks the element of history dependence. There were some arguments in the discussions at the policy board which had a weak flavor of history dependence,[27] but it was quite far from the consensus view at the board. History dependent policy in this context means that monetary easing is continued even after a negative shock to the economy has gone, or, put differently, even after deflationary concerns are gone. This is equivalent to creating an intentional delay of policy change. As shown in our analysis, such a policy lag is welfare-improving in the presence of a zero bound on nominal interest rates, but might not be easy to implement for a central bank with limited knowledge about the transmission mechanism of monetary policy.

[27]For example, during the discussions about the inflation forecast for the year 2000 at the monetary policy meeting on December 17, 1999, a member reported the result of an analytical study that "(1) the average inflation rate for 2000 would be nearly zero; (2) there was a relatively high probability that the rate would become negative; and (3) it was not very likely that the rate would rise" and concluded that "deflationary concern had not yet been dispelled." The argument of this member implies that the zero interest rate policy would be continued until the probability of deflation became negligible, or, put differently, until the average rate of inflation was well above zero, which is consistent with the optimal commitment solution.

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Table 1: Parameter values

$\lambda = 0.048/4^2$
$\beta = 0.990$
$\sigma = 0.157$
$\kappa = 0.024$
$r^n_\infty = 0.011$

Table 2: Various types of shock

T^d					
	$\epsilon^n_0 = -0.02$	-0.05	-0.10	-0.20	-0.30
$\rho = 0.7$	1	4	6	8	9
0.5	0	2	3	4	4
0.3	0	1	1	2	2
0.1	0	0	0	1	1
0.0	0	0	0	0	0
T^c					
	$\epsilon^n_0 = -0.02$	-0.05	-0.10	-0.20	-0.30
$\rho = 0.7$	2	6	9	11	13
0.5	1	3	5	7	8
0.3	0	2	3	5	6
0.1	0	1	2	4	5
0.0	0	1	2	3	4

Table 3: Sensitivity analysis

T^c					
	$\beta = 0.99$	0.97	0.95	0.93	0.91
$\lambda = 0.05/4^2$	5	5	5	5	5
$0.10/4^2$	5	5	5	5	5
$0.15/4^2$	4	4	4	4	4
$0.20/4^2$	4	4	4	4	4
$0.25/4^2$	3	3	3	3	3
T^c					
	$\kappa = 0.02$	0.04	0.06	0.08	0.10
$\sigma = 0.05$	3	4	4	4	4
0.10	5	5	4	4	4
0.15	5	5	5	5	5
0.20	5	5	5	5	5
0.25	6	5	5	5	5

Note: $\epsilon_0^n = -0.10$; $\rho = 0.5$.

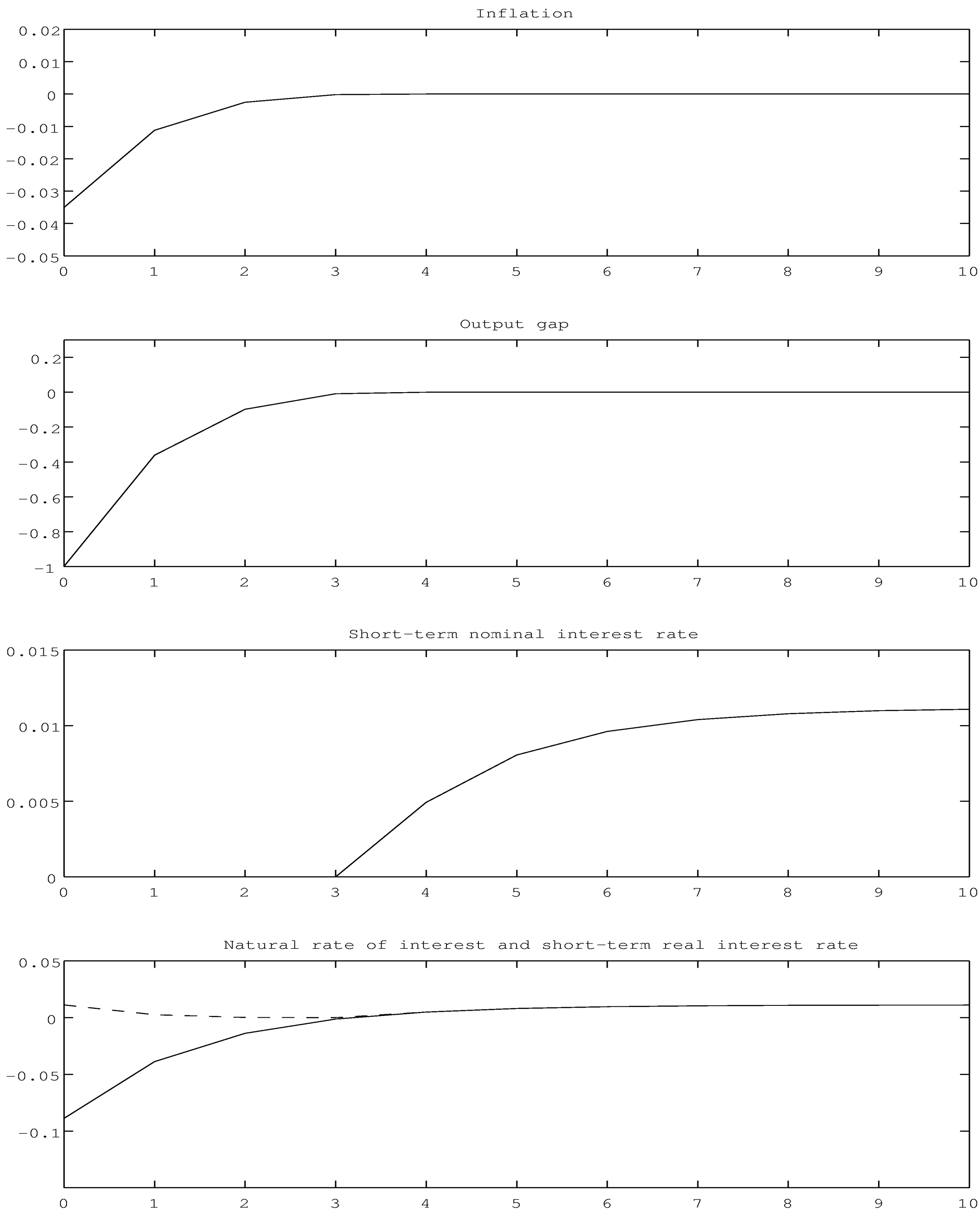

Figure 1: Optimal responses under discretion

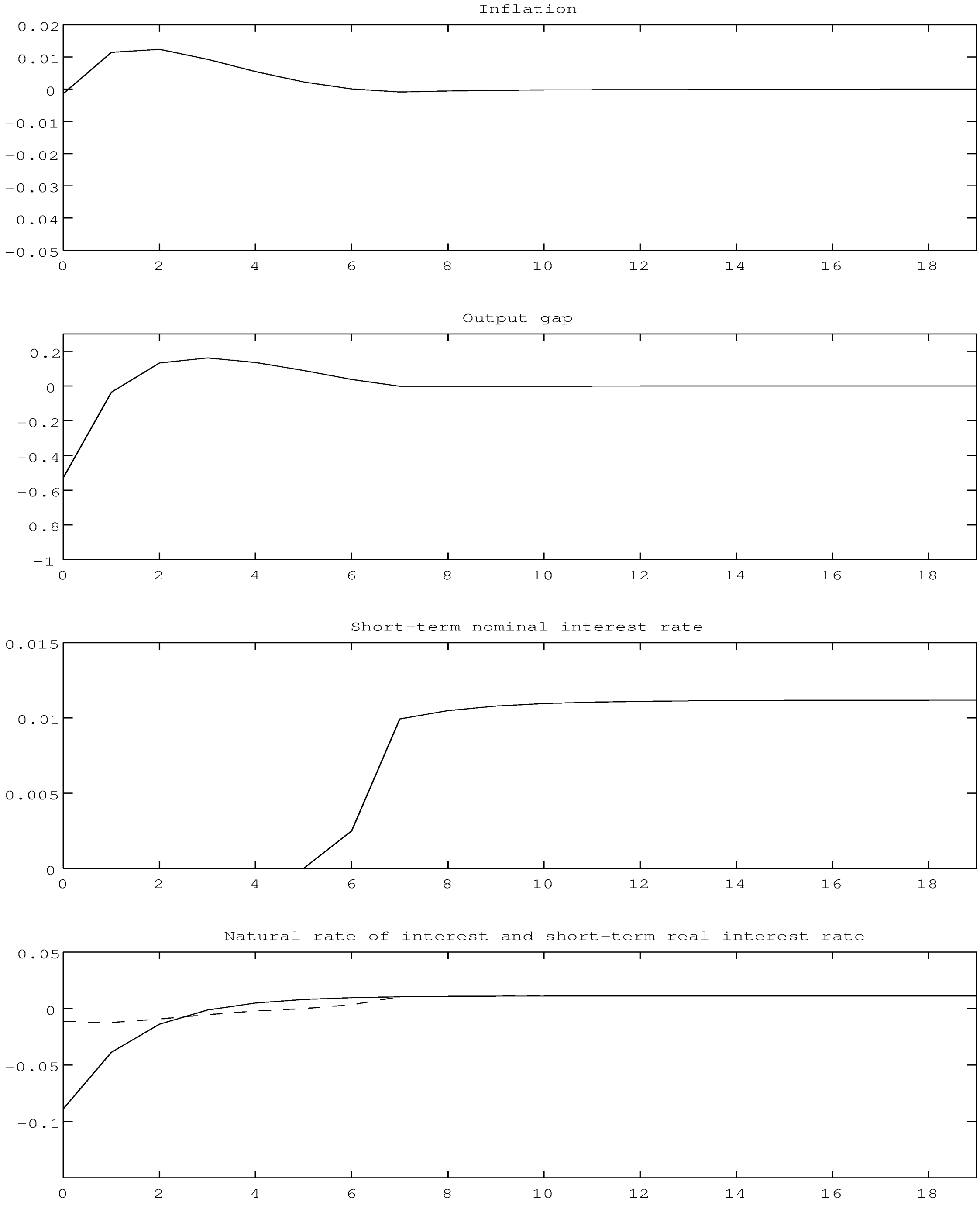

Figure 2: Optimal responses under commitment