LOSING MONEY ON ARBITRAGES:

Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities

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ABSTRACT

In theory, an investor can make infinite profits by taking unlimited positions in an arbitrage. In reality, however, investors must satisfy margin requirements which completely change the economics of arbitrage. We derive the optimal investment policy for a risk-averse investor in a market where there are arbitrage opportunities. We show that it is often optimal to underinvest in the arbitrage by taking a smaller position than margin constraints allow. In some cases, it is actually optimal for an investor to walk away from a pure arbitrage opportunity. Even when the optimal policy is followed, the arbitrage strategy may underperform the riskless asset or have an unimpressive Sharpe ratio. Furthermore, the arbitrage portfolio typically experiences losses at some point before the final convergence date. These results have important implications for the role of arbitrageurs in financial markets.

So there's an arbitrage. So what? This desk has lost a lot of money on arbitrages. Arbitrages aren't particularly great trades.

— Treasury bond trader at a major Wall Street investment bank.

1. INTRODUCTION

One of the foundational principles of financial economics is that arbitrages cannot exist in security markets. The reasoning is that if arbitrages did exist, then investors could attain infinite wealth by taking unlimited positions in the arbitrage. Economic theory implies that an arbitrage is an investment opportunity that is literally too good to be true.

In actual financial markets, however, investors may not be able to attain infinite wealth even if pure arbitrage opportunities exist. Recall that the textbook strategy for exploiting an arbitrage requires taking offsetting long and short positions and holding them until convergence. An investor who takes a short position, however, is required to post collateral as margin. This margin requirement drives an important wedge between textbook arbitrage strategies and strategies that are actually feasible. For example, consider an investor who implements an arbitrage strategy to exploit an arbitrage opportunity. If the arbitrage were then to widen rather than narrow, the investor would experience mark-to-market losses on the position. If the losses were severe enough, the investor might not have sufficient collateral to meet margin calls, and then be forced to liquidate some or all of the position at a loss before it had converged to its theoretical no-arbitrage value. Some recent examples of this include the following:

When spreads widened in a disorganized, tumbling market, gains on short positions weren't enough to offset losses on long ones. Lenders demanded more collateral, forcing the funds either to abandon the arbitrage plays or to raise money for the margin calls by selling other holdings at fire sale prices. Long-Term Capital responded to the crisis by shedding marginal deals, such as bets on the direction of interest rates, at losses.

— Business Week, September 21, 1998.

A Minneapolis-based hedge fund that specializes in fixed-income arbitrage, Eagle Global Value Fund, was compelled to begin liquidating its holdings Friday in the latest trouble for the hedge-fund business. The move was made after the fund failed to meet margin calls, or demands to put up more collateral.

— The Wall Street Journal, October 12, 1998.

Because an investor might be forced to liquidate part of an arbitrage position prior to convergence, even the simplest strategies to exploit arbitrages could actually result in losses when investors face margin constraints, a lesson painfully learned recently by many highly-leveraged hedge funds. This inherent risk in taking arbitrage positions is also discussed in recent papers by Shleifer and Vishny (1997) and Loewenstein and Willard (2000a).

If arbitrages are actually risky investments from the perspective of an investor or hedge-fund manager facing margin constraints, then a number of interesting economic issues arise. For example, what is the optimal investment strategy when markets have arbitrage opportunities? Similarly, how do arbitrages compare with other investments in terms of their risk and return characteristics?

To address these issues, this paper studies a continuous-time model in which there are explicit arbitrage opportunities. To capture the spirit of standard textbook examples, we model the arbitrage opportunity as a security whose price follows a process that converges to zero at some specified future time. In this setting, an investor could make arbitrage profits with certainty if he could hold the position until convergence at maturity. In the short run, however, the arbitrage may widen and force the investor to liquidate positions at a loss. Thus, there is no guarantee that the investor can hold the position until it converges.

The results are surprising. We find that it is often optimal for the investor to underinvest in the arbitrage opportunity. Specifically, the investor often will not take the largest arbitrage position allowed by the margin constraint; the margin constraint frequently is not binding. This contrasts with the popular view that an investor should take the largest position possible in any arbitrage opportunity. In fact, we demonstrate that there are actually circumstances in which an investor will walk away from an arbitrage opportunity. In particular, an investor may turn down an arbitrage strategy with a payoff that dominates the payoff on a riskless asset in order to follow a strategy that may underperform the riskless asset.

Even when the investor follows the optimal investment strategy, the returns from investing in the arbitrage may not be as attractive as those from conventional assets. For example, we demonstrate that the investor can experience substantial losses on his portfolio prior to the convergence date of the arbitrage. In some cases, these losses can be more than 75 percent of the value of the portfolio. For some parameter values, it is also possible for the investor to have a loss even after the arbitrage has converged at its maturity date. In this situation, the investor is worse off than if he had invested only in the riskless asset. We show that the return distributions from following the optimal strategy are highly skewed towards negative values during the early stages of the arbitrage, and that the arbitrage portfolio is usually worth less that its initial value at some point during the life of the arbitrage. Finally, we find that the Sharpe ratio from investing in the arbitrage generally only averages about two in our numerical examples, which is not significantly better than for many types of traditional investments.

Our results demonstrate that experiencing large losses during the early stages of an arbitrage strategy is almost a hallmark of the optimal strategy. From this perspective, the real problem during the hedge fund crisis of 1998 may not have been that arbitrage funds used too much leverage or that they were speculating, but rather that many market participants had unrealistic expectations about how arbitrage strategies should perform over time.

These results also have important implications for the role of arbitrageurs in financial markets. Standard economic theory takes as given the notion that arbitrages cannot exist in the markets since if they did, arbitrageurs would immediately buy and sell the cheap and rich securities until the prices came back into line. Our analysis calls this simplistic view into question since it is not clear that an investor would actually choose to take a position in a specific arbitrage. If investors found it optimal to take only a very limited position in an arbitrage opportunity, or to avoid taking any position at all, then there is no reason why the arbitrage could not persist or even become wider. Given that margin requirements are a fact of life in all financial markets, these results suggest that many theoretical valuation arguments based on the absence of arbitrage principles may need to be reexamined.[1]

Our research complements an important recent literature focusing on whether arbitrage opportunities can exist in equilibrium. Key examples of this literature include Basak and Croitoru (2000) and Loewenstein and Willard (2000a, 2000b, 2000c). These papers demonstrate that arbitrage or mispricing can be sustained in general equilibrium when financial markets have frictions or imperfections. In the models studied in these papers, however, the arbitrageur always takes the maximum possible position allowed by the financial market constraints. This paper contributes to the literature by demonstrating that when the real-world feature of margin constraints is introduced, arbitrages become risky and agents may actually choose to take smaller positions that allowed by constraints. Our results underscore the importance of the papers by Basak and Croitoru and Loewenstein and Willard and suggest that their analysis could be extended to provide even richer general equilibrium implications for financial markets.

[1]Important recent papers addressing derivatives valuation without using no-arbitrage arguments include Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000).

Other important related work includes Brennan and Schwartz (1988, 1990), De Long, Shleifer, Summers, and Waldmann (1990), Duffie (1990), Dumas (1992), Tuckman and Vila (1992), Delgado and Dumas (1994), Yadev and Pope (1994), Chen (1995), Dow and Gorton (1997), Detemple and Murthy (1997), Zigrand (1997), Willard and Dybvig (1999), and Xiong (1999). In addition, our paper complements and extends the literature on margin constraints in financial markets. Important examples of this literature include Heath and Jarrow (1987), Hindy (1995), and Cuoco and Liu (1999). Finally, our results corroborate Shleifer and Vishny (1997) who show that arbitrage can be risky when there are margin constraints. Unlike Shleifer and Vishny, however, we explicitly study the optimal portfolio strategy for an investor in a market with arbitrage opportunities.

The remainder of this paper is organized as follows. Section 2 presents the dynamic portfolio choice problem in markets with arbitrage opportunities. Section 3 discusses the optimal portfolio strategy. Section 4 examines the return distributions resulting from following the optimal strategy. Section 5 considers several alternative strategies. Section 6 summarizes the results and makes concluding remarks.

2. THE DYNAMIC PORTFOLIO CHOICE PROBLEM

In this section, we describe the continuous-time framework and explain how we model arbitrage opportunities. We then solve for the optimal portfolio strategy and the investor's derived utility of wealth function. To make the intuition as clear as possible, we focus on the simplest version of the model.[2] Extensions of the basic model are discussed later.

We model a simple two-investment financial market in which trading takes place continuously in time. The first investment is a riskless asset with value R_t which earns a constant rate of interest r. The dynamics of the riskless asset are given by

$$dR = rRdt, \tag{1}$$

where $R_0 = 1$. Solving this equation for the value of the riskless asset gives $R_t = e^{rt}$.

[2]Basak and Croitoru (2000) and Loewenstein and Willard (2000a, 2000b, 2000c) focus on the important issue of whether arbitrage or mispricing can exist in general equilibrium. Motivated by their results as well as by practitioner claims and empirical evidence that arbitrages do exist, we focus on how an agent optimally exploits arbitrages in financial markets. In doing this, we use a partial equilibrium framework to highlight the implications for the portfolio choice problem. In this sense, our results both complement those of Basak and Croitoru and Loewenstein and Willard and suggest ways in which their general equilibrium models could be extended to allow richer effects.

The second investment is an arbitrage opportunity with value A_t, where $0 \leq t \leq T$. Intuitively, A_t can be thought of as the value of a portfolio that converges to zero at time T. As an example of this type of portfolio, consider the case where there is a violation of the put-call parity relation for European options on non-dividend-paying stock. Standard option-pricing theory such as Merton (1973) shows that a portfolio consisting of a long call option with strike K and maturity T, a short put option with the same strike and maturity, a short position in the underlying stock, and an initial investment of Ke^{-rT} in the riskless asset must be worth zero at time T. This convergence to zero at time T is a simple consequence of the contractual cash flows from the options at their expiration date, and does not depend on assumptions about the distributions of returns or the rationality of the options market at time T. If the value of this portfolio is non-zero at some time $t < T$, then this portfolio becomes a clear example of a pure arbitrage opportunity. If the value of the portfolio is negative (positive), the investor makes arbitrage profits by taking a long (short) position in this portfolio. Note that the arbitrage portfolio requires taking both long and short positions in some securities (see Longstaff (1995)). Other examples of arbitrage portfolios which mathematically must converge to zero at some future time T include taking offsetting long and short positions in two Treasury STRIPS or zero-coupon bonds with identical maturity dates (see Daves and Ehrhardt (1993) and Grinblatt and Longstaff (2000)) or in a Treasury bill and an off-the-run Treasury bond after it pays its final coupon (see Amihud and Mendelson (1991) and Kamara (1994)). Similarly, the difference between a stock index futures price and the price implied by standard cost-of-carry arguments must converge to zero at the expiration date of the contract (see Brennan and Schwartz (1988, 1990) and Duffie (1990)).

To capture the classical notion of an arbitrage as a portfolio with a value converging to zero at some future point in time T, we assume that the dynamics of A follow the Brownian-bridge process

$$dA = \frac{-\alpha A}{T - t} dt + \sigma \; dZ, \tag{2}$$

where α and σ are positive constants, $0 \leq t \leq T$, and Z is a standard Brownian motion.[3] Note that as $t \to T$, the drift of this process approaches $+\infty$ when $A_t < 0$, and $-\infty$ when $A_t > 0$. Thus, as $t \to T$, the mean reversion of the process towards zero become stronger and stronger, forcing A_T to converge to zero with probability one.[4]

[3]The Brownian bridge process has been applied to financial market prices by Ball and Torous (1983), Brennan and Schwartz (1988, 1990), Duffie (1990), and Cheng (1991). Loewenstein and Willard (2000a) study the viability of a Brownian bridge as a return process.

[4]Rather than converging to zero at time T, the arbitrage process could be generalized to converge to some other fixed value by a simple modification of the drift term.

The parameter α governs the speed at which the arbitrage opportunity converges to zero. The parameter σ represents the volatility of the arbitrage and determines the distribution of possible arbitrage opportunities. Solving this stochastic differential equation results in the following expression for A_s, where $0 \leq t \leq s \leq T$,

$$A_s = \left(\frac{T-s}{T-t}\right)^{\alpha} A_t + \sigma \int_t^s \left(\frac{T-s}{T-\tau}\right)^{\alpha} dZ_\tau. \tag{3}$$

It is easily seen that A_s is normally distributed for $s < T$. We denote the expected value of A_s conditional on the value of A_t by M_s, where

$$M_s = \left(\frac{T-s}{T-t}\right)^{\alpha} A_t. \tag{4}$$

Similarly, the conditional variance of A_s, which we denote V_s^2, is given by

$$V_s^2 = \frac{\sigma^2(T-t)}{1-2\alpha}\left[\left(\frac{T-s}{T-t}\right)^{2\alpha} - \left(\frac{T-s}{T-t}\right)\right], \tag{5}$$

for $\alpha \neq 1/2$, and by

$$V_s^2 = \sigma^2(T-s)\ln\left(\frac{T-t}{T-s}\right), \tag{6}$$

for $\alpha = 1/2$. As $s \to T$, both M_s and V_s^2 converge to zero.

The Brownian-bridge process allows A_t to take on both positive and negative values. When A_t is positive, the investor receives a positive cash flow of A_t by investing in a negative number of units of the arbitrage. When A_t is negative, the investor receives a positive cash flow of $-A_t$ by investing in a positive number of units of the arbitrage. It is important to observe that by taking a position in the arbitrage and receiving a positive cash flow at time t, the investor simultaneously creates a liability, since the investor would need to pay the same amount to immediately unwind the arbitrage position. It is easily shown that $|A(t)|$ can exceed any fixed value with strictly positive probability.[5] Thus, this specification implies that there is always a

The qualitative results, however, are unaffected by the specific number to which the arbitrage converges since the investor can always modify his final payoff by a constant by trading the riskless asset.

[5]This follows from the distribution of the maximum of a Brownian bridge. See Karatzas and Shreve (1991).

risk that the arbitrage can widen further before its final convergence date. This risk plays both a major role in this model as well as in actual financial markets.

> *More losses arose in Salomon Smith Barney's U.S. arbitrage group. Its traders had placed \$1 billion bets on the London Interbank Offered Rate, or Libor, on the expectation that the spread between that rate and U.S. Treasury yields would narrow. Instead, it ballooned, leading to aftertax losses of \$120 million.*
>
> — The Wall Street Journal, September 22, 1998.

As shown by Cheng (1991), the unboundedness of the drift of the arbitrage process as $t \to T$ implies that the usual sufficient Novikov condition for the existence of an equivalent martingale measure is not satisfied. In our simple two-investment financial market, it is straightforward to demonstrate directly that no equivalent martingale measure is possible.

Proposition 1. Non-Existence of an Equivalent Martingale Measure.

The Brownian bridge describing the dynamics of A_t in this financial market does not admit the existence of an equivalent martingale measure.

Proof of Proposition 1. *See Appendix.*

Since there is no equivalent martingale measure in this market, arbitrage is possible.[6] Thus, the process for A_t represents an arbitrage in a fundamental sense.[7]

Let N_t and P_t denote the number of units of the arbitrage and the riskless asset held by the investor. The amount invested in the riskless asset can be viewed as the balance in the investor's margin account. The balance in the interest-accruing

[6]Because the riskless rate is constant in this financial market, the corollary on pg. 123 of Duffie (1996) implies the existence of an (approximate) arbitrage. In the next section, we provide an explicit example of a strategy that generates arbitrage profits.

[7]Loewenstein and Willard (2000a, 2000b, 2000c) provide excellent discussions of the various definitions of arbitrage that are relevant in markets with different types of constraints.

margin account represents collateral available to the investor's creditors. Recall that in financial markets, collateral is often posted in the form of interest-bearing securities.

The investor's wealth at time t is given by

$$W_t = N_t A_t + P_t R_t. \tag{7}$$

Following standard portfolio choice theory, we assume that the investor follows a self-financing strategy.[8] Applying the self-financing condition results in the following dynamics for W_t,

$$\begin{aligned} dW &= NdA + rPRdt \\ &= NdA + r(W - NA)dt \\ &= \left(rW - \left(r + \frac{\alpha}{T-t} \right) NA \right) dt + \sigma N dZ \end{aligned} \quad . \tag{8}$$

This equation, along with the dynamics of A_t in equation (2), implies that W_t and A_t follow a joint Markov process. Thus, the state of economy is completely specified by the current values of the state variables W_t and A_t. From equation (8), W_T can also be expressed as

$$W_T = W_t \exp\left(\int_t^T \left(r - \left(r + \frac{\alpha}{T-s} \right) \frac{NA}{W} - \frac{\sigma^2}{2} \frac{N^2}{W^2} \right) ds + \sigma \int_t^T \frac{N}{W} dZ \right). \tag{9}$$

Harrison and Kreps (1979) and Harrison and Pliska (1991) show that restrictions on trading strategies are necessary to rule out unrealistic arbitrages arising from doubling strategies. Dybvig and Huang (1988) and Cox and Huang (1989) demonstrate that requiring admissible trading strategies to satisfy the non-negative wealth condition $W_t > 0$ for all t, $0 \leq t \leq T$, eliminates these types of unrealistic trading strategies.

In actual financial markets, however, even stronger restrictions on trading strategies are imposed though the standard requirement that investors hold collateral in margin accounts as protection against the risk of their short positions. Specifically, whenever an investor generates a liability by either shorting an asset or borrowing

[8]This rules out the possibility of later capital injections into the investor's portfolio. This is without much loss of generality, however, since the investor's initial wealth can be viewed as inclusive of the value of contingent capital injections.

funds (which is the same as shorting the riskless asset), financial institutions virtually always require collateral exceeding the amount of the liability as protection against mark-to-market losses. The amount by which the value of collateral exceeds the liability is often referred to as the 'haircut,' or the investor's equity in the position. The size of the required 'haircut' is typically a function of the type of collateral and risk of the short position. For example, an investor can initially borrow up to 50 percent of percent of the value of stock.[9] Large institutional investors may be able to borrow as much as 99 percent of the value of Treasury bonds through the repo market. In some cases, nearly 100 percent financing of Treasury bills is possible. The key point, however, is that no matter what the leverage, extra collateral is still required to protect against the risk of market movements in which the liability is no longer fully secured.[10] If the trade goes against the investor and generates losses, or if the value of the collateral itself falls, investors may be forced to either provide additional collateral in response to a margin call or be liquidated.

> *Creditors of hedge funds, convinced the funds wouldn't get back all the money they had put into Russia, issued demands for more collateral, known as margin calls. The funds had to raise capital to meet the calls, but they couldn't do so by selling Russian securities with those markets paralyzed. So they began selling other assets, including U.S. stocks.*
>
> — The Wall Street Journal, September 22, 1998.

As we show later, requiring margins against losses from following arbitrage strategies actually has the potential to cause those losses to occur; the fear that trading losses may occur can essentially become a self-fulfilling prophesy through the margining mechanism.[11]

[9]The level of margin on stock positions is governed by Federal Reserve regulations. Specifically, Federal Reserve regulations G, T, and U place constraints on the amount of leverage that financial institutions can offer their clients.

[10]Long-Term Capital appears to be one of the few exceptions to the rule since they were apparently able to avoid 'haircuts' on a number of their positions such as swap contracts. Dunbar (2000) argues, however, that when the positions began to go against Long-Term Capital, their counterparties imposed effective 'haircuts' by being more aggressive in the way they marked the collateral to market. In this sense, Long-Term Capital essentially faced 'contingent haircuts'.

[11]Loewenstein and Willard (2000a, 2000b, 2000c) consider models in which agents can obtain credit and may actually have liabilities in excess of the value of their assets.

When an investor invests in N_t units of the arbitrage, the investor receives an immediate cash flow of $-N_t A_t$ and has a current mark-to-market liability of the same amount. For example, imagine that two bonds with identical future cash flows have prices of 100 and 101. Define the arbitrage as a long position in the first and a short position in the second. Thus, $A_t = 100 - 101 = -1$, and taking a position in 10 units of the arbitrage generates an immediate cash flow of $-N_t A_t = 10$ and a liability of $N_t A_t = 10$. To capture the economics of the margining system in a simple way, we assume that the investor is required to hold liquid securities in the amount of the liability plus a margin of λ per unit of the arbitrage held, where λ is a non-negative constant.[12] Thus, we require that

$$P_t R_t \;\geq\; \mid N_t A_t \mid \;+\; \lambda \;\mid N_t \mid . \tag{10}$$

Assuming $N_t A_t \leq 0$ (which we show later to be optimal), the margin constraint can be expressed as a simple wealth constraint,

$$W_t \;\geq\; \lambda \;\mid N_t \mid . \tag{11}$$

Since λ is non-negative, satisfying this constraint generally satisfies the less-restrictive condition that $W_t > 0$ for all t, $0 \leq t \leq T$. Note that the form of this constraint parallels actual market practice in that the required margin is directly related to the size of the position and that λ can be interpreted as a percentage of the notional amount of the short leg of the arbitrage.[13]

Intuitively, the margin requirement insures that the counterparty taking the other side of the arbitrage position has collateral at least equal to the amount owed by the investor. Thus, if $\lambda = 1$, the investor in the above example who invested in $N_t = 10$ units of the arbitrage would have a long position of 1,000 in the first bond, and a short position of 1,010 in the second bond, implying $N_t A_T = -10$. The investor would need to have collateral of $P_t R_t = 20$ to cover the net liability of $\mid N_t A_t \mid = 10$ generated by the arbitrage and to post the additional $\lambda |N_t| = 10$ margin required.

[12] By requiring that margin only be held against the net value of the arbitrage rather than against the gross value of the short position, we are making the conservative assumption that both legs of the arbitrage are with the same counterparty. If not, the investor could face much higher total collateral requirements. Gross margin constraints could easily be modeled within this framework by setting λ to a larger value.

[13] It does not make sense to have the margin constraint depend on the value of A_t since the position still would be risky even when $A_t = 0$. Furthermore, from the dynamics in equation (2), the instantaneous volatility of the arbitrage process which governs the short-term mark-to-market risk of an arbitrage position does not depend on the level of A_t.

The parameter λ can also be thought of as the 'haircut' required for the short position. In this example, the investor has a liability of 1,010 and needs total collateral of 1,020 consisting of a long bond position with value 1,000 and 20 of the riskless asset. The margin requirement of 10 represents the excess of the collateral over the investor's short position. Note that the investor in this example is leveraged nearly 50 to 1. In general, since the arbitrage A_t is defined as the net of the long and short positions, the actual leverage inherent in a unit of the arbitrage could be almost arbitrarily large from the perspective of the balance sheet of the investor.

Another way of thinking about the margining mechanism is from the perspective of the cash generated. If entering into an arbitrage position were a source of funds rather than a use of funds, then whenever the investor experienced losses on a trading position, the investor could generate additional funds by simply doubling up on the arbitrage strategy. By requiring that investors meet margin requirements, financial markets insure that the moral hazard problem of an investor doubling up a losing position does not occur.

It is important to observe that requiring collateral as margin is fundamentally different from short-selling restrictions. In this framework, investors can take arbitrarily large short positions as long as they can post the required collateral. In fact, as we show later, the optimal portfolio strategy has the property that the portfolio weight for the arbitrage can take on any negative value. Thus, there is no limitation on the fraction of the portfolio invested in a short position in the arbitrage.[14] Margin constraints are also fundamentally different from transactions costs. Intuitively, this is because investors receive all of the interest, dividends, and appreciation on the securities held as collateral in margin accounts. Thus, the investor incurs no direct economic costs or losses from holding securities in a margin account. Finally, margin constraints differ from position limits such as those imposed in futures markets. This is because there is no specific upper bound on the number of units of the arbitrage that can be held by the investor; the margin constraint is based only on the investor's wealth, not on the absolute size of the position.

The investor is endowed with strictly positive initial wealth W_0 and has a finite investment horizon T corresponding to the date at which the arbitrage converges to zero. Note that the investor in this model could be viewed either as an individual agent, or as a hedge fund manager. To simplify the exposition, we assume that the investor only consumes at time T, although this assumption can be relaxed without affecting the basic results. In particular, the investor dynamically chooses a portfolio N_t to maximize an expected utility function defined over the logarithm of his terminal wealth W_T,

[14]This differs from Basak and Croitoru (2000) who place explicit upper and lower bounds on the fraction of wealth that can be invested in an arbitrage.

$$E_t\,[\,\ln W_T\,]\,. \tag{12}$$

We use this simple preference structure to focus more directly on the intuition of how the arbitrage opportunity affects the portfolio problem.

Define the derived utility of wealth function $J(W, A, t)$ by the following expression

$$J(W, A, t) = \max_N E_t\,[\,\ln W_T\,]\,, \tag{13}$$

subject to the budget constraint in equation (9) and where N is a member of the set of admissible strategies satisfying the margin constraint. Because the problem and margin constraint are homogeneous in W_t, we demonstrate in the following proposition that N_t must be of the form $F_t W_t$, where F is a function of A and t only. Substituting this into equation (9) implies

$$\begin{aligned} J(W, A, t) &= \ln W_t + \max_F E_t\left[\int_t^T r - \left(r + \frac{\alpha}{T-s}\right) FA - \frac{\sigma^2}{2} F^2 ds\right] \\ &= \ln W_t + r(T-t) - \min_F E_t\left[\int_t^T \left(r + \frac{\alpha}{T-s}\right) FA + \frac{\sigma^2}{2} F^2 ds\right]. \end{aligned} \tag{14}$$

Because of the quadratic form of the integrand in F and the fact that the dynamics of A are independent of F, the optimal portfolio strategy can be determined in closed form by a state-by-state minimization.

Proposition 2. The Optimal Arbitrage Position.

The optimal portfolio strategy for the investor is

$$N_t = \begin{cases} \frac{1}{\lambda} W_t, & \text{if} \quad A_t < -\frac{1}{\lambda} \frac{\sigma^2}{(r + \frac{\alpha}{T-t})}, \\ -\frac{r + \frac{\alpha}{T-t}}{\sigma^2} A_t W_t, & \text{if} \quad |\, A_t \,| \leq \frac{1}{\lambda} \frac{\sigma^2}{(r + \frac{\alpha}{T-t})}, \\ -\frac{1}{\lambda} W_t, & \text{if} \quad A_t > \frac{1}{\lambda} \frac{\sigma^2}{(r + \frac{\alpha}{T-t})}. \end{cases} \tag{15}$$

Proof of Proposition 2. *See Appendix.*

This optimal portfolio strategy has many interesting features which are discussed in the next section. Substituting the optimal portfolio strategy into equation (14) and evaluating the expectations gives the following result.

Proposition 3. The Derived Utility of Wealth.

The derived utility of wealth $J(W, A, t)$ is given by the following expression

$$J(W, A, t) = \ln W_t + r(T - t) - \int_t^T H(A, s)\ ds, \tag{16}$$

where $H(A, s) =$

$$\begin{aligned}
&\frac{\sigma^2}{2\lambda^2} - \left(r + \frac{\alpha}{T-s}\right)\frac{M_s}{\lambda} \\
&+ \left(\frac{\sigma^2}{2\lambda^2} + \left(r + \frac{\alpha}{T-s}\right)\frac{M_s}{\lambda} + \left(r + \frac{\alpha}{T-s}\right)^2 \frac{M_s^2 + V_s^2}{2\sigma^2}\right)\ \Phi\left(-\frac{L_s + M_s}{V_s}\right) \\
&- \left(\frac{\sigma^2}{2\lambda^2} - \left(r + \frac{\alpha}{T-s}\right)\frac{M_s}{\lambda} + \left(r + \frac{\alpha}{T-s}\right)^2 \frac{M_s^2 + V_s^2}{2\sigma^2}\right)\ \Phi\left(\frac{L_s - M_s}{V_s}\right) \\
&- \frac{1}{\sqrt{2\pi}}\left(\left(r + \frac{\alpha}{T-s}\right)\frac{V_s}{\lambda} - \left(r + \frac{\alpha}{T-s}\right)^2 \frac{V_s(L_s - M_s)}{2\sigma^2}\right)\ \exp\left(-\frac{(L_s + M_s)^2}{2V_s^2}\right) \\
&- \frac{1}{\sqrt{2\pi}}\left(\left(r + \frac{\alpha}{T-s}\right)\frac{V_s}{\lambda} - \left(r + \frac{\alpha}{T-s}\right)^2 \frac{V_s(L_s + M_s)}{2\sigma^2}\right)\ \exp\left(-\frac{(L_s - M_s)^2}{2V_s^2}\right),
\end{aligned}$$

and

$$L_s = \frac{\sigma^2}{\lambda\left(r + \frac{\alpha}{T-s}\right)},$$

and where $\Phi(\cdot)$ is the cumulative standard normal distribution function. Furthermore, if $\mid A_t \mid < \infty$, $W_t < \infty$, and $\lambda > 0$, then $J(W, A, t) < \infty$ for all t, $0 \le t \le T$.

Proof of Proposition 3. *See Appendix.*

The result that the derived utility of wealth is finite depends critically on the condition that $\lambda > 0$. If $\lambda = 0$, then it is easily shown that the strategy

$$N_t = -\frac{r + \frac{\alpha}{T-t}}{\sigma^2} A_t W_t, \tag{17}$$

implies that $E[\ \ln W_T\ ] = \infty$. Thus, the margin constraint fundamentally changes the economics of the arbitrage opportunity in this financial market.

3. THE OPTIMAL PORTFOLIO STRATEGY

In this section, we examine in more detail the optimal portfolio strategy. We focus first on the properties of the optimal strategy and then present numerical examples illustrating their implications for trading behavior.

Several key properties of the optimal strategy are immediately apparent from Proposition 2. First, the optimal strategy always requires taking a position in the arbitrage opposite in sign from the value of the arbitrage A_t. Thus, when A_t is negative, the investor optimally invests in a positive number of units of the arbitrage. Since A_t is negative, this generates an immediate cash inflow to the investor. Similarly, when A_t is positive, the optimal strategy is to invest in a negative number of units of the arbitrage. This again results in an immediate cash inflow to the investor. Since there are margin requirements, however, the investor must keep liquid assets at least equal in amount to the cash inflow plus $\lambda|N_t|$. Thus, while the optimal strategy generates cash, the investor is constrained in the way this cash can be used.

Since the investor faces margin constraints, it is perhaps not surprising that the investor only takes a finite position in the arbitrage. For example, if $A_t < 0$, the maximum value of N_t that the margin restriction allows is $\frac{1}{\lambda} W_t$. Thus, if $W_t = 100$ and $\lambda = 1$, the maximum number of units of the arbitrage the investor can hold is $N_t = 100$, independent of how large the arbitrage opportunity A_t becomes. Note, however, that this does not limit the leverage that the investor can utilize in his portfolio. In particular, since the portfolio weight for the arbitrage is $N_t A_t / W_t$, the maximum portfolio weight for the arbitrage in this example is $-100 A_t / 100 = -A_t$, which is unbounded. Thus, while the number of units of the arbitrage that can be held is bounded for a given W_t, the portfolio weight invested in the arbitrage is not. This is the sense in which margin constraints differ from short-selling constraints.

What is surprising, however, is that the investor often finds it optimal to take a smaller position in the arbitrage opportunity than the margin restrictions allow. For example, when

$$-\frac{1}{\lambda}\frac{\sigma^2}{\left(r+\frac{\alpha}{T-t}\right)} < A_t < \frac{1}{\lambda}\frac{\sigma^2}{\left(r+\frac{\alpha}{T-t}\right)}, \tag{18}$$

the optimal N_t is less in absolute value than the maximum number of units of the arbitrage that could be held while satisfying the margin constraint. In fact, when $| A_t |$ is close to zero, the optimal N_t may only be a small fraction of the maximum allowable number of units of the arbitrage.

To illustrate this, we simulate how often an investor following the optimal strategy will reach the margin constraint. In particular, we simulate paths of A_t and report in Table 1 the percentage of paths where the bounds shown in equation (18) are exceeded for different values of $t < T$, and for various values of the parameters. The lower the margin requirement λ, the less frequently the margin constraint is binding. Similarly, the riskier the arbitrage as measured by σ^2, the less frequently the investor finds it optimal to take the maximum position. As the speed of convergence α increases, the investor takes a more aggressive position and the margin constraint is more likely to be binding.

The intuition for why the investor does not always take the largest possible position in the arbitrage is directly related to the risk of the arbitrage widening. When A_t differs only slightly from zero, it is almost as likely that the arbitrage will widen as narrow, since the drift is close to zero. Furthermore, when A_t is close to zero, the potential loss from the arbitrage widening can be much larger than the possible gain from the arbitrage converging, at least in the near term. Specifically, the investor can realize a small gain per unit of the arbitrage if it converges to zero over the next short interval, but can experience a large loss if it widens to several times its current value. If the investor suffers large losses in the early stages, he clearly has less wealth to exploit arbitrages at a later stage. By being too aggressive with small arbitrages, the investor risks finding himself in a state of the world where there is a large arbitrage, but his ability to exploit the arbitrage is severely reduced because of losses suffered as the arbitrage widened.

Since the bounds in equation (18) converge to zero at rate $(T-t)$ while the standard deviation of A_t converges to zero at rate $(T-t)^{1/2}$, the probability that the margin constraint becomes binding becomes one as $t \to T$. This convergence, however, typically takes place at nearly the last instant before time T; Table 1 shows that even when $t = .999$ and $T = 1$, the probability that the constraint is reached is clearly less than one.

From Proposition 2 it is clear that the optimal N_t is continuous even at the boundary where the margin constraint becomes binding. Thus, there are no abrupt changes in the size of the arbitrage position when the boundary is reached. Over time, however, the absolute value of N_t tends to decreases after the boundary is reached. To see the intuition for this, consider the case where the boundary is just reached and the value of the arbitrage then moves back toward zero. The constraint is no longer binding and the size of the arbitrage position is reduced since the arbitrage is no longer as large. On the other hand, if the boundary is reached and the arbitrage widens, the investor then suffers a decrease in his wealth. Because of this decline in wealth, the constraint $W_t \geq \lambda \mid N_t \mid$ can only be satisfied by reducing the absolute value of N_t in this self-financing framework. Thus, the investor must partially liquidate his position in the arbitrage at a loss. Either way, the tendency is for the size of the arbitrage position to be reduced once the boundary is reached.

Since the optimal strategy involves taking a position in the arbitrage opposite in sign to A_t, the portfolio weight for the arbitrage position,

$$w_t = \frac{N_t A_t}{W_t}, \tag{19}$$

is less than or equal to zero. To give a sense of the distribution of portfolio weights that results from following the optimal portfolio, Table 2 provides summary statistics for the percentage portfolio weights for different values of t and of the parameters.

The optimal portfolio strategy can be highly leveraged even when there are margin constraints. For example, when $A_0 = 0$ and $\lambda = 1$, the investor may optimally leverage his portfolio by a factor of almost four by taking a short position in the arbitrage. The leverage factor is typically even higher when there is an initial arbitrage at time zero. Again, this leverage measure does not map directly into a traditional debt-equity ratio since the arbitrage is expressed in terms of the net value of the long and short legs, not in terms of the absolute size or notional amount of the long and short legs. Note also that this leverage factor is determined largely by the parameter σ; for larger values of σ than those in Table 2, much higher leverage factors would be obtained. Furthermore, there is considerable variability in the possible portfolio weights as evidenced by the standard deviations. The means are typically well below the median, indicating that the distribution of portfolio weights is skewed toward large negative values. Table 2 also shows that the distribution of portfolio weights converges to zero at the final horizon date of the arbitrage. In particular, the portfolio weights and their standard deviations at time $t = .999$ are typically an order of magnitude smaller in absolute value than for the other values of t.

4. THE RETURNS FROM ARBITRAGE

In this section, we examine the wealth distributions obtained from following the optimal investment strategy in a market with arbitrage opportunities. As before, we first present general results about the distribution of returns and then provide numerical examples.

The investor's wealth at time t, when the investor follows the optimal strategy, can be expressed as

$$W_t = W_0 e^{rt} \exp\left(\int_0^t -\left(r + \frac{\alpha}{T-s}\right) FA - \frac{\sigma^2}{2} F^2 ds + \sigma \int_0^t F dZ\right). \tag{20}$$

It can be shown that the investor's wealth is strictly positive for all t, $0 \leq t \leq T$. Thus, the optimal investment strategy satisfies the positive wealth constraint of Dybvig and Huang (1988). Because of the boundedness of F, it follows from Proposition 2 that W_t is finite with probability one. Thus, an investor following the optimal investment strategy in this market faces a well-defined wealth distribution with support $(0, \infty)$ for any time t.

In specific cases, the range of possible returns that can be obtained from investing optimally in the arbitrage opportunity can be narrowed. The following proposition gives sufficient conditions for the optimal investment portfolio to dominate the riskless asset at time T.

Proposition 4. Dominance of the Optimal Strategy.

If $0 < \alpha \leq 1$, *then* $W_T \geq W_0 e^{rT}$ *a.s. when the optimal strategy is followed.*

Proof of Proposition 4. *See Appendix.*

When the condition $0 < \alpha \leq 1$ is satisfied, this proposition implies that an investor who follows the optimal investment strategy cannot achieve a lower return than the riskless rate. In this situation, the investment portfolio clearly dominates the riskless asset and becomes an arbitrage in the classic textbook sense. Thus, following the optimal strategy can lead to a pure arbitrage at time T. Conversely, however, the fact that there is a pure arbitrage at time T does not imply that the investor takes the maximum possible position in the arbitrage at any time $t < T$. As shown earlier, the investor often finds it optimal to take a position smaller than would be allowed by the margin requirements.

In general, however, the returns obtained from investing in the arbitrage do not dominate those from the riskless asset. To illustrate this, Table 3 provides summary statistics for the wealth distributions at different horizons obtained from following the optimal portfolio strategy. In each of these examples, the initial wealth of the investor is assumed to be 100. Table 3 confirms the dominance result that when $\alpha = 1$, the optimal arbitrage portfolio ends up doing better than the riskless asset at time T; the final value of the arbitrage portfolio exceeds the $100e^{.06} = 106.18$ value of a portfolio fully invested in the riskless asset. When $\alpha > 1$, however, Table 3 shows that the final value of the arbitrage portfolio can be less than the value of the riskless portfolio, and can even be less than the initial value of the investor's wealth. Thus, when $\alpha > 1$, the arbitrage portfolio is no longer even an arbitrage in the classic sense.

This latter result is particularly interesting given that it is actually possible to find an investment strategy that dominates the riskless asset even when $\alpha > 1$. Specifically, let D_t denote the portfolio strategy given by

$$D_t = \begin{cases} \frac{1}{\lambda} W_t, & \text{if} \quad A_t < -\frac{\sigma^2}{2r\lambda}, \\ 0 & \text{if} \quad \mid A_t \mid \leq \frac{\sigma^2}{2r\lambda}, \\ -\frac{1}{\lambda} W_t, & \text{if} \quad A_t > \frac{\sigma^2}{2r\lambda}. \end{cases} \tag{21}$$

We denote this strategy the barrier strategy since it is zero until the arbitrage reaches a specific level. Because $\mid D_t \mid$ is always less than or equal to the margin constraint, this portfolio strategy is always feasible. The following result is easily shown:

Proposition 5. Dominance of the Barrier Strategy.

If $\alpha > 0$, then $W_T \geq W_0 e^{rT}$ a.s. when the barrier strategy is followed.

Proof of Proposition 5. *See Appendix.*

This result implies that by following the barrier strategy, the investor can achieve a wealth distribution that dominates that available from the riskless asset, and hence, is again a pure arbitrage. Even though this strategy is available to the investor when $\alpha > 1$, the investor finds it optimal to walk away from this pure arbitrage opportunity. Surprisingly, the optimal strategy is to invest in a way that runs the risk of underperforming the riskless asset even though there is a strategy available that guarantees the investor's return cannot be less than the riskless asset.

Another interesting feature relates to the shape of the distribution of investment returns. During the early stages of the investment horizon, the mean value of the portfolio is often substantially lower than the median value, suggesting a distribution that is highly skewed towards lower values. As the final convergence date approaches, the distribution typically becomes skewed toward higher values and the mean exceeds the median. To illustrate this, Figures 1 and 2 graph the distribution of values for the optimal arbitrage portfolio at times $t = .250$, $t = .500$, $t = .750$, and $t = 1.000$. The distribution at $t = .250$ is highly skewed towards the left. This is also true for $t = .500$. At $t = .750$, however, the nature of the distribution begins to change and a more symmetrical pattern appears. At the final maturity date T, the distribution now becomes highly skewed towards higher values and values less than the riskless asset value of 106.18 disappear since $\alpha = 1$ in these graphical examples.

The value of the optimal arbitrage portfolio is highly variable over time. Starting from an initial value of 100, the arbitrage portfolio can actually lose more than 75 percent of its value by $t = .250$. Analyzing these particular paths reveals that the investor takes a large position in the arbitrage portfolio at an early date, but then loses significant amounts as the arbitrage continues to widen. When the investor reaches the margin constraint, the investor is forced to begin to unwind his position at a loss in order to satisfy the margin constraint as the arbitrage widens further. Although the arbitrage ultimately converges to zero at time T, the investor is unable to fully participate at later stages since his wealth is now much lower. Thus, investors who experience large losses early during the life of the arbitrage end up with lower returns at time T. This can be seen Figure 3 which graphs the final value of the portfolio at time T against the minimum value of the portfolio during the life of the arbitrage. This demonstrates clearly that losses that occur early during the life of the strategy due to the widening of the arbitrage do not entirely "come back" later on as the arbitrage ultimately converges to zero.

The mean values of the arbitrage portfolios display an interesting pattern. Initially, they tend to be somewhat larger than the value of the riskless portfolio. Over time, however, the means grow rapidly and ultimately far exceed the value of the riskless portfolio. The farther the initial value of the arbitrage is from zero, the higher the final expected value of the optimal portfolio. This is intuitive, since when $A_0 \neq 0$, the investor immediately has the opportunity to invest in an arbitrage. Table 3 also shows that the means are decreasing functions of the leverage parameter λ. This follows since a lower value of λ places fewer constraints on the investor's ability to exploit arbitrage opportunities. The distribution of returns is typically shifted towards higher values when the value of α increases. Intuitively, a higher value of the speed of mean reversion implies that an arbitrage tends to converge more rapidly. On the other hand, the investor finds it optimal to take a larger position in the arbitrage for any given value of A_t. Because of this latter effect, there can be paths where the investor does worse that would be the case for a smaller value of α. In particular, when α is greater than one, the investor can actually end with less wealth at time T than he had at time zero. The expected return typically increases with the value of σ.

The standard deviation of the value of the arbitrage portfolio demonstrates that uncertainty about the ultimate value of the portfolio is not resolved evenly over time. At early stages of the life of the arbitrage, the standard deviation of the value of the optimal portfolio is fairly small. As the final convergence date T is approached, however, the standard deviation of the value of the arbitrage portfolio grows rapidly.

A detailed analysis of the returns from following the optimal strategy reveals that returns have three primary sources. First, the investor benefits by investing directly in an arbitrage which then eventually converges. The more frequently there is an arbitrage which then converges, the higher the value of the portfolio at the final maturity.

Second, the final value of the portfolio is adversely affected by reaching the margin constraint. This can be seen in Figure 4 which graphs the final value of the portfolio against the percentage of times that the margin constraint is binding along a path. There is a strong negative relation between the final value of the portfolio and the frequency with which the margin constraint is binding. Intuitively, this is because when the margin constraint is binding and there is an increase in the size of the arbitrage, the investor is forced to reduce his position at a loss rather than more aggressively exploiting the wider arbitrage. Figure 5 plots the final value of the portfolio against the number of units of the arbitrage that the investor must unwind in order to satisfy the margin constraints. The larger the number of units in absolute terms, the greater the total impact on the investor; the final value of the portfolio is clearly less as the number of units of the arbitrage unwound in order to meet the margin requirement increases.

Given these two effects, the investor does best when the value of the arbitrage tends to return frequently to zero and stays away from larger values which would then cause the margin constraint to be binding more frequently. This surprising implication is illustrated in Figure 6 which plots the final value of the portfolio against the average value of the arbitrage during its life. When the arbitrage is initially zero, the highest final value of the portfolio tends to be for those paths for which the average value of the arbitrage is close to zero. Similarly, when $A_0 = 1$, the highest final values of the arbitrage portfolio tend to occur for paths where the arbitrage returns quickly to the neighborhood of zero, resulting in average values of A_t of between zero and one. Thus, the highest returns occur along paths where there is a steady flow of small arbitrages which converge rapidly, and where large widenings in the value of A_t do not occur. This is consistent with the well-known Wall Street description of the business of relative value or arbitrage investing as "picking up nickels in front of a steamroller."

> *"Myron once told me they (LTCM) are sucking up nickels from all over the world," says Merton Miller, a University of Chicago business professor and himself a Nobel Prize winner in economics. "But because they are so leveraged, that amounts to a lot of money."*
>
> — The Wall Street Journal, November 16, 1998.

The third source of returns is more subtle. Because of the margin constraint, the investor is forced to place any cash generated by taking a position in the arbitrage into the riskless asset. Since the arbitrage is a source of cash, the balance invested in the margin account tends to be larger when the investor takes a position in the arbitrage. Over time, the excess funds in the riskless asset generate additional returns from the accrual of interest. Thus, the investor receives a subtle but direct benefit from the collateral requirement imposed by the margin constraint. The interest from the margin account can represent an important source of returns in many situations.

Although the eventual value of the optimal portfolio is on average much higher than the riskless asset, the intermediate values of the portfolio typically reflect losses at some point during the life of the arbitrage. Table 4 reports pathwise statistics from following the optimal portfolio. These pathwise statistics indicate that for a very high percentage of paths, the minimum value of the optimal portfolio is actually less than its initial value of 100. Specifically, the percentage of paths for which there is an actual capital loss at some point during the life of the arbitrage is typically in excess of 96 percent. Note that this is also true for the case where $\alpha = 1$ which guarantees that the final value of the arbitrage is strictly greater than the riskless asset. Clearly, the probability of the value of the portfolio dropping below the value of the riskless asset at some point is even higher than the probability of dropping below 100; Table 4 shows that the probability of underperforming the riskless asset at some point during the investment horizon is typically greater than 97 percent.

These results have many interesting implications for performance expectations for hedge funds investing in arbitrage opportunities. These results indicate that experiencing capital losses prior to the final horizon is part of the inherent nature of investments in arbitrage opportunities in markets with margin constraints. Thus, there is a definite "darkest before dawn" nature to arbitrage investments. This contrasts dramatically with the widely-held view that investors in arbitrage opportunities should never experience significant losses. An immediately corollary of this widely-held view is that arbitrage funds can experience losses only if they are not really investing in arbitrage opportunities but speculating in conventional types of investments. Our analysis, however, demonstrates that this common wisdom is flawed; losses during the early stages of an arbitrage opportunity are almost inevitable for an investor pursuing an optimal investment strategy in the arbitrage.[15]

[15]These results also have interesting implications for the willingness of an investor to inject capital into a hedge fund which has experienced significant losses. The expected return from acquiring the assets of a distressed fund optimally investing in arbitrage opportunities may far exceed that from the initial investment. This may have been one of the reasons why Long-Term Capital Management was able to so quickly receive a capital injection of $3.65 billion from a consortium of 16 financial institutions in exchange for most of the equity in the fund. Certainly, there is no evidence that any of the financial institutions experienced losses on the additional capital contributed

In general, the maximum value of the portfolio is obtained when the arbitrage converges to its final value of zero at time T. It is interesting to note, however, that there are paths for which the maximum is obtained earlier than at time T. Although the optimal strategy involves taking a position in the arbitrage whenever $A_t \neq 0$, this shows that there can be cases where the investor will wish that he had terminated his arbitrage position before the final convergence date.

Table 4 also reports the average times at which the minimum and maximum values are attained. The average time at which the minimum is attained ranges from .05 to .45 years for these examples. This is again consistent with the notion that the performance of an arbitrage position is generally worse during its early stages. The average time at which the maximum is attained is always very close to the final convergence date of the arbitrage.

Finally, Table 4 shows that the average minimum ranges from about 60 to 98 for an portfolio initially worth 100. Thus, an investor following an optimal strategy can expect to be down as much as 40 percent at some point for some parameter values. This again contrasts with the common view that true arbitrage positions should never show losses. The average maximum values are generally very similar to the expected portfolio values at time T shown in Table 3.

One popular measure of the attractiveness of a portfolio's return is the traditional Sharpe ratio, and the performance of hedge funds is often compared in terms of the Sharpe ratio. To make our analysis of Sharpe ratios compatible with the ratios typically reported by the financial industry, we do the following. For each simulated path, we compute the sample standard deviation of changes in the value of the portfolio and annualize the estimate. We do this for horizons of .250, .500, .750, and 1.000 years. We then take the excess return of the portfolio at the same horizons over the value of the riskless asset R_t and annualize the excess returns. We then divide the annualized excess return by the annualized standard deviation to obtain the estimated Sharpe ratio. Since the resulting Sharpe ratio is computed for each path, it is consistent with Sharpe ratios reported by the financial industry which are also computed from a single realized path. We repeat this process for 10,000 paths and provide summary statistics for the resulting distribution of Sharpe ratios at the various horizons. These summary statistics are reported in Table 5. Figures 7 and 8 also graph the distribution of Sharpe ratios for selected values of the parameters.

The Sharpe ratios for investing in the arbitrage are quite variable. This is particularly true at the early stages. At the convergence date, however, the average Sharpe ratio is roughly about two for all of the examples shown in Table 5. Curiously, this is about the same as the average Sharpe ratio of 2.39 for the relative-value hedge funds reported as of March 23, 2000 by the website HedgeFund.net which tracks the

during the bailout.

performance of approximately 1,000 hedge funds.[16] Figures 8 and 9 show that most of the Sharpe ratios at the final convergence date are between zero and four. Thus, even when there is a pure arbitrage available in the market, there is no guarantee than a hedge fund following the optimal investment strategy will have a Sharpe ratio even as large as that for the S&P 500.[17]

5. ALTERNATIVE STRATEGIES

In earlier sections, we derived the optimal strategy from the perspective of an investor with logarithmic preferences and explored its implications for return distributions. To provide additional insight, however, it is also interesting to study the performance of a number of alternative strategies that are not optimal, but may still have some intuitive justification.

5.1 The Maximal Strategy.

We consider first a simple strategy in which the investor always takes the largest possible position allowed by the margin constraint whenever A_t deviates from zero. In particular, when $A_t > 0$, the investor holds $N_t = -W_t/\lambda$ units of the arbitrage, and $N_t = W_t/\lambda$ when $A_t < 0$. This strategy, although suboptimal, loosely reflects the common belief that an investor faced with an arbitrage should plunge by taking the maximal allowable position.

To examine the performance of this strategy, we again simulate paths of the arbitrage and the resulting value of the arbitrage portfolio. This allows us to directly compare the results to those obtained by following the optimal strategy. Summary statistics for the return distributions obtained by following the maximal strategy are reported in Table 6.

The return distributions obtained by following the maximal strategy are very different from those obtained by following the optimal strategy. In general, the expected returns are higher at the final maturity date for the maximal strategy than for the optimal strategy. We conjecture that the maximal strategy may be close to the optimal strategy for a risk-neutral investor. Although the expected returns are higher for the maximal strategy, the standard deviations for the final value of the portfolio are all substantially higher than for the optimal strategy. It is for this reason that the

[16]On March 23, 2000, HedgeFund.net reports that the average Sharpe ratio for convertible arbitrage hedge funds is 2.32, for fixed-income arbitrage funds is 2.09, for options-arbitrage hedge funds is 2.91, and for risk-arbitrage hedge funds is 2.23.

[17]Empirical evidence about the return performance and trading strategies of hedge funds is presented in recent papers by Fung and Hsieh (1997, 1998) and Ackermann, McEnally, and Ravenscraft (1999).

maximal strategy fails to be optimal for an investor with logarithmic preferences; the higher expected return for the maximal strategy does not offset the higher standard deviation of the returns. This is also clear from the Sharpe ratios for the maximal strategy which are all lower than for the optimal strategy.

In addition to the higher standard deviations, Table 6 also illustrates that the final value of the portfolio following the maximal strategy can be less than the riskless rate for any value of α. Thus, the maximal strategy never generates a final value that is an arbitrage in the sense that it dominates the riskless asset.

5.2 The Barrier Strategy.

In contrast to the maximal strategy, we demonstrated earlier that the barrier strategy always generates a final value that dominates the riskless asset. Despite this, the barrier strategy is not optimal for an investor with logarithmic preferences.

To examine the return distributions produced by the barrier strategy, we again simulate paths of the arbitrage and the resulting value of the portfolio and report the results in Table 7. The expected return from following the barrier strategy is usually well below the expected return from following the optimal strategy. Table 7 also demonstrates that the final value of the portfolio is always greater than or equal to the value of the riskless asset for any value of α. On the other hand, the standard deviations of the returns are also much lower than for the optimal strategy. Intuitively, this is because the barrier strategy is a much less aggressive strategy than the optimal strategy. Ultimately, however, the decrease in the expected return more than offsets the decrease in the standard deviation, and the average Sharpe ratio for this strategy is less than for the optimal strategy.

6. CONCLUSION

We examine the optimal investment policy of a risk-averse investor in a market where there are arbitrage opportunities. The model includes the realistic feature that investors must hold collateral as margin against the risk of their short positions. We find that the optimal policy often results in the investor underinvesting in the arbitrage by taking a smaller position than would be allowed by the margin constraint. We also show that the investor may find it optimal to walk away from a strategy that is a pure arbitrage in the sense that its returns dominate those of the riskless asset. Even when the optimal policy is followed, the returns from the arbitrage strategy may not be much more attractive than those obtained from traditional investments. For example, the optimal strategy may generate losses even at the convergence date of the arbitrage, or may have a Sharpe ratio less that those available by directly investing in the stock market.

Our results also have implications for the current debate about hedge-fund leverage and the role of arbitrageurs in financial markets. Hedge funds such as Long-Term

Capital Management have been criticized for employing too much leverage in their trading strategies. This analysis shows that optimal leverage is determined by the volatility and speed of convergence of the arbitrage as well as by the nature of the margin requirements. Long-Term Capital Management had substantial investments in fixed-income spread trades, which historically have displayed rapid convergence. In addition, Long-Term Capital Management was successful in negotiating some of the lowest margin requirements on Wall Street. Given the combination of these two factors, it is possible that the optimal leverage structure of Long-Term Capital Management may have been much higher than for other market participants.

There are many possible extensions of this analysis. For example, alternative preference structures or objective functions could be used in solving the investor's or hedge-fund manager's problem. A simple perturbation argument, however, suggests that our basic results hold for preference structures sufficiently close to logarithmic, and are not an artifact of the myopic nature of logarithmic preferences. In fact, we conjecture that our results are true for virtually all risk averse preferences. Furthermore, using a simple binomial tree example, we can show that our results hold in a discrete-time setting and are not an artifact of continuous-time modeling. In addition, it would be useful to introduce additional risky assets and arbitrage opportunities into the investment opportunity set. The primary message of this paper, however, is that when the real-world feature of margin constraints is introduced, the economics of arbitrage become fundamentally different. In particular, arbitrages become risky investments and the issue of whether there would be sufficient demand from investors to completely eliminate arbitrage opportunities becomes relevant.

APPENDIX

Proof of Proposition 1. Since $A_T = 0$ *a.s.* under the objective measure, $A_T = 0$ *a.s.*, and hence $e^{-rT}A_T = 0$ *a.s.* under any equivalent measure. If there were an equivalent martingale measure, then by definition, the deflated price process $e^{-rt}A_t$ should be a martingale under the equivalent martingale measure,

$$e^{-rt}A_t = \mathrm{E}_t\left[e^{-rT}A_T\right] = 0, \quad a.s.$$

which implies $A_t = 0$ *a.s.*. Since the probability of the event $\{A_t \neq 0\}$, $t < T$, for a Brownian bridge is strictly positive under the objective measure, and therefore, under any equivalent measure, this is a contradiction.

∎

Proof of Proposition 2. We first prove that because of the homogeneity of the problem, the optimal portfolio strategy N_t must be of the form F_tW_t where F_t is a function of t and A_t only. The agent's optimization problem is

$$\max_N \mathrm{E}_t\left[\ln W_T\right] \tag{A1}$$

subject to the constraints

$$\begin{aligned} dW &= \left(rW - \left(r + \frac{\alpha}{T-t}\right)NA\right)dt + \sigma N dZ, \\ W_t &\geq \lambda|N_t|, \end{aligned} \tag{A2}$$

where $W_t > 0$ for all t, $0 \leq t \leq T$. Since $W_t > 0$, $F_t = \frac{N_t}{W_t}$ is well-defined and the original optimization problem is equivalent to the optimization problem

$$\max_F \mathrm{E}_t\left[\ln W_T\right]$$

subject to the constraints

$$\begin{aligned} dW &= \left(r - \left(r + \frac{\alpha}{T-t}\right)FA\right)W dt + \sigma F W dZ, \\ |F_t| &\leq \frac{1}{\lambda}. \end{aligned} \tag{A3}$$

By an application of Itô's lemma,

$$\ln W_T = \ln W_t + r(T-t) - \int_t^T \left(\left(r + \frac{\alpha}{T-s}\right)FA + \frac{\sigma^2}{2}F^2\right)ds + \sigma F dZ,$$

and the optimal F solves the following problem

$$\min_F \mathrm{E}_t \left[\int_t^T \left(\left(r + \frac{\alpha}{T-t} \right) FA + \frac{\sigma^2}{2} F^2 \right) ds \right], \tag{A4}$$

subject to the constraint

$$|F_s| \leq \frac{1}{\lambda}.$$

However, since W_t does not appear in (A4) or the constraint, the optimal control F can only depend on A_t and t. Hence N_t is of the form $F_t W_t$.

Turning now to the optimal F, note that a realization of a path of A_t does not depend on the control F. Thus, minimizing the integral in (A4) path-wise for A_t clearly minimizes the conditional expectation in (A4). Given a path of A_t, the problem

$$\min_F \int_t^T \left(\left(r + \frac{\alpha}{T-s} \right) FA + \frac{\sigma^2}{2} F^2 \right) ds, \tag{A5}$$

subject to the constraint

$$|F_s| \leq \frac{1}{\lambda},$$

can then be solved using standard calculus of variation techniques (see for example, Kamien and Schwartz (1991)). Given the quadratic form of the integrand in equation (A5), it is now easily shown that the optimal portfolio strategy N_t is given by

$$N_t = \begin{cases} \frac{1}{\lambda} W_t, & \text{if} \quad A_t < -\frac{1}{\lambda} \frac{\sigma^2}{r + \frac{\alpha}{T-t}}. \\ -\frac{r+\frac{\alpha}{T-t}}{\sigma^2} A_t W_t, & \text{if} \quad |A_t| \leq \frac{1}{\lambda} \frac{\sigma^2}{r+\frac{\alpha}{T-t}}, \\ -\frac{1}{\lambda} W_t, & \text{if} \quad A_t > \frac{1}{\lambda} \frac{\sigma^2}{r+\frac{\alpha}{T-t}}, \end{cases}$$

■

Proof of Proposition 3. We first derive some basic results for normal random variables which will be used in the proof.

Lemma. If x is a normal random variable with mean M and standard deviation V, and L is a constant, then the following formulas are true

$$\textbf{A)} \qquad \mathrm{E}\left[x^2 1_{\{|x| \leq L\}} \right] = -\frac{V^2}{\sqrt{2\pi}} \left(\frac{L-M}{V} e^{-\frac{(L+M)^2}{2V^2}} + \frac{L+M}{V} e^{-\frac{(L-M)^2}{2V^2}} \right)$$
$$+ (V^2 + M^2) \left(\Phi\left(\frac{L-M}{V} \right) - \Phi\left(-\frac{L+M}{V} \right) \right),$$

$$\textbf{B)} \qquad \mathrm{E}\left[|x|1_{\{|x|>L\}}\right] = \frac{V}{\sqrt{2\pi}}\left(e^{-\frac{(L-M)^2}{2V^2}} + e^{-\frac{(L+M)^2}{2V^2}}\right)$$
$$+M\left(1 - \Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right)\right),$$
$$\textbf{C)} \qquad \mathrm{E}\left[1_{\{|x|>L\}}\right] = 1 - \Phi\left(\frac{L-M}{V}\right) + \Phi\left(-\frac{L+M}{V}\right). \qquad \text{(A6)}$$

Proof of the Lemma. Let z denote the standard normal random variable $z = \frac{X-M}{V}$.

A) We have

$$\begin{aligned}
\mathrm{E}\left[x^2 1_{\{|x|\le L\}}\right] &= V^2 \mathrm{E}\left[\left(\frac{x-M}{V} + \frac{M}{V}\right)^2 1_{\{|\frac{x-M}{V}+\frac{M}{V}|\le\frac{L}{V}\}}\right] \\
&= V^2 \mathrm{E}\left[\left(z + \frac{M}{V}\right)^2 1_{\{|z+\frac{M}{V}|\le\frac{L}{V}\}}\right] \\
&= V^2 \mathrm{E}\left[\left(z^2 + 2z\frac{M}{V} + \frac{M^2}{V^2}\right) 1_{\{-\frac{L+M}{V}\le z\le\frac{L-M}{V}\}}\right]. \qquad \text{(A7)}
\end{aligned}$$

The three terms in the above equation can be evaluated as

$$\mathrm{E}\left[z^2 1_{\{-\frac{L+M}{V}\le z\le\frac{L-M}{V}\}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\frac{L+M}{V}}^{\frac{L-M}{V}} z^2 e^{-z^2/2} dz$$
$$= \frac{1}{\sqrt{2\pi}}\left(-ze^{-z^2/2}\Big|_{-\frac{L+M}{V}}^{\frac{L-M}{V}} + \int_{-\frac{L+M}{V}}^{\frac{L-M}{V}} e^{-z^2/2} dz\right)$$
$$= -\frac{1}{\sqrt{2\pi}}\left(\frac{L+M}{V}e^{-\frac{(L+M)^2}{2V^2}} + \frac{L-M}{V}e^{-\frac{(L-M)^2}{2V^2}}\right)$$
$$+\Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right), \qquad \text{(A8)}$$

$$\mathrm{E}\left[z 1_{\{-\frac{L+M}{V}\le z\le\frac{L-M}{V}\}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\frac{L+M}{V}}^{\frac{L-M}{V}} z e^{-z^2/2} dz$$
$$= \frac{1}{\sqrt{2\pi}}\left(-e^{-z^2/2}\Big|_{-\frac{L+M}{V}}^{\frac{L-M}{V}}\right)$$
$$= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(L+M)^2}{2V^2}} - e^{-\frac{(L-M)^2}{2V^2}}\right), \qquad \text{(A9)}$$

$$\mathrm{E}\left[1_{\{-\frac{L+M}{V}\le z\le\frac{L-M}{V}\}}\right] = \Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right). \qquad \text{(A10)}$$

Combine the above three expressions, we obtain

$$\mathrm{E}\left[x^2 1_{\{|x|\le L\}}\right] = V^2 \mathrm{E}\left[\left(z^2 + 2z\frac{M}{V} + \frac{M^2}{V^2}\right) 1_{\{-\frac{L+M}{V}\le z\le\frac{L-M}{V}\}}\right]$$

$$
\begin{aligned}
&= -\frac{V^2}{\sqrt{2\pi}}\left(\frac{L+M}{V}e^{-\frac{(L+M)^2}{2V^2}} + \frac{L-M}{V}e^{-\frac{(L-M)^2}{2V^2}}\right) \\
&+V^2\left(\Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right)\right) \\
&+2\frac{M}{V}\frac{V^2}{\sqrt{2\pi}}\left(e^{-\frac{(L+M)^2}{2V^2}} - e^{-\frac{(L-M)^2}{2V^2}}\right) \\
&+M^2\left(\Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right)\right) \\
&= -\frac{V^2}{\sqrt{2\pi}}\left(\frac{L-M}{V}e^{-\frac{(L+M)^2}{2V^2}} + \frac{L+M}{V}e^{-\frac{(L-M)^2}{2V^2}}\right) \\
&+(V^2+M^2)\left(\Phi\left(\frac{L-M}{V}\right) - \Phi\left(-\frac{L+M}{V}\right)\right). \qquad \text{(A11)}
\end{aligned}
$$

B) Next, we compute

$$
\begin{aligned}
&\mathrm{E}\left[|x|1_{|x|>L}\right] \\
&= \mathrm{E}\left[x1_{x>L} - x1_{x<-L}\right] \\
&= V\mathrm{E}\left[\frac{x-M+M}{V}1_{\frac{x-M}{V}>\frac{L-M}{V}} - \frac{x-M+M}{V}1_{\frac{x-M}{V}<-\frac{L+M}{V}}\right] \\
&= V\mathrm{E}\left[\left(z+\frac{M}{V}\right)1_{z>\frac{L-M}{V}} - \left(z+\frac{M}{V}\right)1_{z<-\frac{L+M}{V}}\right]. \qquad \text{(A12)}
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathrm{E}\left[z1_{\left\{z>\frac{L-M}{V}\right\}}\right] &= \frac{1}{\sqrt{2\pi}}\int_{\frac{L-M}{V}}^{\infty} ze^{-z^2/2}dz \\
&= \frac{1}{\sqrt{2\pi}}\left(-e^{-z^2/2}|_{\frac{L-M}{V}}^{\infty}\right) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{(L-M)^2}{2V^2}} \qquad \text{(A13)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}\left[z1_{\left\{z<-\frac{L+M}{V}\right\}}\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{L+M}{V}} ze^{-z^2/2}dz \\
&= \frac{1}{\sqrt{2\pi}}\left(-e^{-z^2/2}|_{-\infty}^{-\frac{L+M}{V}}\right) \\
&= -\frac{1}{\sqrt{2\pi}}e^{-\frac{(L+M)^2}{2V^2}}, \qquad \text{(A14)}
\end{aligned}
$$

so

$$
\mathrm{E}\left[|x|1_{|x|>L}\right]
$$

$$= V\mathrm{E}\left[\left(z+\frac{M}{V}\right)1_{z>\frac{L-M}{V}} - \left(z+\frac{M}{V}\right)1_{z<-\frac{L+M}{V}}\right]$$
$$= \frac{V}{\sqrt{2\pi}}\left(e^{-\frac{(L-M)^2}{2V^2}} + e^{-\frac{(L+M)^2}{2V^2}}\right)$$
$$+M\left(1-\Phi\left(\frac{L-M}{V}\right)-\Phi\left(-\frac{L+M}{V}\right)\right). \quad \text{(A15)}$$

C) Finally,

$$\mathrm{E}\left[1_{|x|>L}\right]$$
$$= V\mathrm{E}\left[1_{z>\frac{L-M}{V}} + 1_{z<-\frac{L+M}{V}}\right]$$
$$= 1-\Phi\left(\frac{L-M}{V}\right)+\Phi\left(-\frac{L+M}{V}\right).$$

We now turn to the proof of Proposition 3. The value function $J(W, A, t)$ is determined by

$$\int_t^T\left\{\left(r+\frac{\alpha}{T-s}\right)\mathrm{E}_t\left[FA\right]+\frac{\sigma^2}{2}\mathrm{E}_t\left[F^2\right]\right\}ds=\int_t^T Hds.$$

Substituting the definition of F_t, we have

$$\begin{aligned} H &= -\frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}\mathrm{E}\left[A_s^2 1_{\{|A_s|\leq\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right]-\frac{r+\frac{\alpha}{T-s}}{\lambda}\mathrm{E}_t\left[|A_s|1_{\{|A_s|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] \\ &+ \frac{1}{2}\frac{\sigma^2}{\lambda^2}\mathrm{E}\left[1_{\{|A_s|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right]. \end{aligned} \quad \text{(A16)}$$

Let $M = M_s$, $V = V_s$, and $L = L_s$ in the Lemma, we immediately have

$$\begin{aligned} \mathrm{E}_t\left[A_s^2 1_{\{|A_t|\leq\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] &= -\frac{V_s^2}{\sqrt{2\pi}}\left(\frac{L_s-M_s}{V_s}e^{-\frac{(L_s+M_s)^2}{2V_s^2}}+\frac{L_s+M_s}{V_s}e^{-\frac{(L_s-M_s)^2}{2V_s^2}}\right) \\ &+(V_s^2+M_s^2)\left(\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right), \\ \mathrm{E}_t\left[|A_s|1_{\{|A_t|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] &= \frac{V_s}{\sqrt{2\pi}}\left(e^{-\frac{(L_s-M_s)^2}{2V_s^2}}+e^{-\frac{(L_s+M_s)^2}{2V_s^2}}\right) \\ &+M_s\left(1-\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right), \\ \mathrm{E}_t\left[1_{\{|A_s|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] &= 1-\Phi\left(\frac{L_s-M_s}{V_s}\right)+\Phi\left(-\frac{L_s+M_s}{V_s}\right). \end{aligned} \quad \text{(A17)}$$

We now have

$$
\begin{aligned}
H &= -\frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}\mathrm{E}_t\left[A_s^2 1_{\{|A_s|\le\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] - \frac{r+\frac{\alpha}{T-s}}{\lambda}\mathrm{E}_t\left[|A_s|1_{\{|A_s|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] \\
&+ \frac{1}{2}\frac{\sigma^2}{\lambda^2}\mathrm{E}_t\left[1_{\{|A_s|>\frac{1}{\lambda}\frac{\sigma^2}{r+\frac{\alpha}{T-s}}\}}\right] \\
&= -\frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}\left\{(V_s^2+M_s^2)\left(\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right)\right. \\
&- \left.\frac{V_s^2}{\sqrt{2\pi}}\left(\frac{L_s-M_s}{V_s}e^{-\frac{(L_s+M_s)^2}{2V_s^2}}+\frac{L_s+M_s}{V_s}e^{-\frac{(L_s-M_s)^2}{2V_s^2}}\right)\right\} \\
&- \frac{r+\frac{\alpha}{T-s}}{\lambda}\left\{\frac{V_s}{\sqrt{2\pi}}\left(e^{-\frac{(L_s-M_s)^2}{2V_s^2}}+e^{-\frac{(L_s+M_s)^2}{2V_s^2}}\right)\right. \\
&\quad \left.+M_s\left(1-\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right)\right\} \\
&+ \frac{1}{2}\frac{\sigma^2}{\lambda^2}\left(1-\Phi\left(\frac{L_s-M_s}{V_s}\right)+\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right). \qquad \text{(A18)}
\end{aligned}
$$

Rearranging the above terms gives the expression of Proposition 3.

We next prove the finiteness of the value function. We have

$$
J = \ln W_t + r(T-t) - \int_t^T H(A,s)ds. \qquad \text{(A19)}
$$

Since the integrand $H(A,s)$ is a differentiable function of s at all $t \le s < T$, we only need to study its behavior at s near T to show the finiteness of the integral.

Note that the function $\Phi(x)$ and $e^{-\frac{x^2}{2}}$ are both bounded from above by 1, so we have

$$
\begin{aligned}
H(A,s) &= \frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}(V_s^2+M_s^2)\left(\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right) \\
&+ \frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}\frac{V_s^2}{\sqrt{2\pi}}\left(\frac{L_s-M_s}{V_s}e^{-\frac{(L_s+M_s)^2}{2V_s^2}}+\frac{L_s+M_s}{V_s}e^{-\frac{(L_s-M_s)^2}{2V_s^2}}\right) \\
&+ \frac{r+\frac{\alpha}{T-s}}{\lambda}\left\{\frac{2V_s}{\sqrt{2\pi}}+M_s\right\}+\frac{\sigma^2}{\lambda^2}. \qquad \text{(A20)}
\end{aligned}
$$

The last two terms are of order

$$
\begin{aligned}
&\frac{r+\frac{\alpha}{T-s}}{\lambda}\left\{\frac{2V_s}{\sqrt{2\pi}}+M_s\right\}+\frac{\sigma^2}{\lambda^2} \\
&= O((T-s)^0)+O\left(\frac{1}{T-s}\left((T-s)^{\min(\alpha,1/2)}O(\ln(T-s))+(T-s)^{\alpha}\right)\right),
\end{aligned}
$$

which will lead to a finite integration over s as long as $\alpha > 0$.

We now consider the integration over the first two terms in H. Note first, when there is no margin constraint, $\lambda = 0$, then $L_s \to +\infty$. In this case, the second term in H is zero, while the first term becomes (note that $\Phi(+\infty) = 1$ and $\Phi(-\infty) = 0$)

$$\frac{1}{2}\frac{\left(r + \frac{\alpha}{T-s}\right)^2}{\sigma^2}(V_s^2 + M_s^2).$$

When $s \to T$, depending on α, this term diverges at least at the order of $(T-s)^{-1}$ so the integration over this term is infinite and thus the utility will be infinite. Therefore the margin constraint is important for yielding a unique optimal solution with finite utility.

Case 1. If $0 < \alpha < 1/2$, then

$$\begin{aligned} M_s &= \left(\frac{T-s}{T-t}\right)^\alpha A_t, \\ V_s &= \frac{\sigma}{\sqrt{T-t}}\left(\frac{T-s}{T-t}\right)^\alpha, \\ L_s &= \frac{\sigma^2}{\lambda}(T-s). \end{aligned} \quad \text{(A21)}$$

As $s \to T$, we have

$$\begin{aligned} \frac{L_s + M_s}{V_s} &\to C, \\ \frac{L_s - M_s}{V_s} &\to -C, \end{aligned} \quad \text{(A22)}$$

where C is a constant. So we have

$$\left(\Phi\left(\frac{L_s - M_s}{V_s}\right) - \Phi\left(-\frac{L_s + M_s}{V_s}\right)\right) \sim 2\Phi(-C)\Phi'(-C)\frac{L_s}{V_s} = O((T-s)^{1-\alpha}),$$

and

$$\frac{L_s - M_s}{V_s}e^{-\frac{(L_s+M_s)^2}{2V_s^2}} + \frac{L_s + M_s}{V_s}e^{-\frac{(L_s-M_s)^2}{2V_s^2}} = e^{-\frac{M_s^2}{2V_s^2}}\frac{L_s}{V_s} = O((T-s)^{1-\alpha}).$$

Using the above two equations, we see that the integration over the first two terms in H is finite.

Now, for $x \to -\infty$, we have

$$\Phi(x) = O\left(e^{-\frac{x^2}{2}}x^2\right),$$

so the first term in H is also of order $x^n e^{-x^2/2}$ and goes to 0 as well.

Case 2. If $\alpha > 1/2$, we have, for $s \to T$,

$$\begin{aligned} M_s &= \left(\frac{T-s}{T-t}\right)^\alpha A_t, \\ V_s &= \sigma\sqrt{\frac{T-s}{2\alpha-1}}, \\ L_s &= \frac{\sigma^2}{\lambda}(T-s). \end{aligned} \tag{A23}$$

From the above equations, it is clear that

$$\begin{aligned} \frac{L_s+M_s}{V_s} &= \frac{\frac{\sigma^2}{\lambda}(T-s)+\left(\frac{T-s}{T-t}\right)^\alpha A_t}{\sigma\sqrt{\frac{T-s}{2\alpha-1}}} = O\left((T-s)^{\min(1/2,\alpha-1/2)}\right) \to 0, \\ \frac{L_s-M_s}{V_s} &= \frac{\frac{\sigma^2}{\lambda}(T-s)-\left(\frac{T-s}{T-t}\right)^\alpha A_t}{\sigma\sqrt{\frac{T-s}{2\alpha-1}}} = O\left((T-s)^{\min(1/2,\alpha-1/2)}\right) \to 0. \end{aligned}$$

Therefore, as $s \to T$, the second term in H goes as

$$\begin{aligned} & \frac{1}{(T-s)^2}(T-s)O\left((T-s)^{\min(1/2,\alpha-1/2)}\right) \\ =\ & O\left((T-s)^{\min(-1/2,\alpha-1/2-1)}\right), \end{aligned} \tag{A24}$$

so the integration is well-defined.

For the first term, note that $\Phi(x)-\Phi(-x)=O(x)$ for $x \approx 0$, so we have

$$\begin{aligned} & \frac{1}{2}\frac{\left(r+\frac{\alpha}{T-s}\right)^2}{\sigma^2}(V_s^2+M_s^2)\left(\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right) \\ & = O\left(\frac{1}{(T-s)^2}(T-s)(T-s)^{\min(1/2,\alpha-1/2)}\right) \\ & = O\left((T-s)^{\min(-1/2,\alpha-1/2-1)}\right), \end{aligned} \tag{A25}$$

hence the integration is well-defined.

Case 3. $\alpha = 1/2$. In this case,

$$\begin{aligned} \frac{L_s+M_s}{V_s} &= C_1 + O\left((T-s)^{1/2}\right), \\ \frac{L_s-M_s}{V_s} &= C_1 - O\left((T-s)^{1/2}\right), \end{aligned} \tag{A26}$$

where C_1 is a constant. Therefore

$$\begin{aligned} & \left(\Phi\left(\frac{L_s-M_s}{V_s}\right)-\Phi\left(-\frac{L_s+M_s}{V_s}\right)\right) = O\left((T-s)^{1/2}\right), \\ & \left(\frac{L_s-M_s}{V_s}e^{-\frac{(L_s+M_s)^2}{2V_s^2}}+\frac{L_s+M_s}{V_s}e^{-\frac{(L_s-M_s)^2}{2V_s^2}}\right) = O\left((T-s)^{1/2}\right), \end{aligned}$$

so again, the integration of first and second terms in H is well defined.

$\blacksquare$

Proof of Proposition 4. We now prove the dominance of the optimal strategy when $\alpha \leq 1$. Note that

$$W_T = W_0 \exp(rT) \exp\left(\int_0^T -\left(\left(r + \frac{\alpha}{T-t}\right) FA + \frac{\sigma^2}{2} F^2\right) dt + \sigma F dZ\right),$$

the dominance of the optimal strategy

$$W_T \geq W_0 e^{rT} \quad a.s.$$

is equivalent to the positivity of

$$h^+ = \int_0^T -\left(\left(r + \frac{\alpha}{T-t}\right) FA + \frac{\sigma^2}{2} F^2\right) dt + \sigma F dZ \quad a.s..$$

Using the equation

$$dA = -\frac{\alpha A}{T-t} dt + \sigma dZ, \tag{A27}$$

we have

$$h^+ = \int_0^T \left(-rFAdt - \frac{\sigma^2}{2} F^2\right) dt + FdA. \tag{A28}$$

The function F is given by

$$\begin{aligned} F &= -\frac{r + \frac{\alpha}{T-t}}{\sigma^2} A_t 1_{\left\{|A_t| \leq \frac{1}{\lambda} \frac{\sigma^2}{r + \frac{\alpha}{T-t}}\right\}} - \frac{\text{sign}(A_t)}{\lambda} 1_{\left\{|A_t| \geq \frac{1}{\lambda} \frac{\sigma^2}{r + \frac{\alpha}{T-t}}\right\}} \\ &= -\frac{r + \frac{\alpha}{T-t}}{\sigma^2} A_t \chi_t^< - \frac{\text{sign}(A_t)}{\lambda} \chi_t^>, \end{aligned} \tag{A29}$$

where

$$\begin{aligned} \chi_t^< &= 1_{\left\{|A_t| \leq \frac{1}{\lambda} \frac{\sigma^2}{r + \frac{\alpha}{T-t}}\right\}}, \\ \chi_t^> &= 1_{\left\{|A_t| \geq \frac{1}{\lambda} \frac{\sigma^2}{r + \frac{\alpha}{T-t}}\right\}}. \end{aligned} \tag{A30}$$

Let us Define

$$g = -\frac{r + \frac{\alpha}{T-t}}{2\sigma^2} A_t^2 \chi_t^> + \left(\frac{\sigma^2}{2\lambda^2} \frac{1}{r + \frac{\alpha}{T-t}} - \frac{|A_t|}{\lambda}\right) \chi_t^<. \tag{A31}$$

Noting that $g_A = F$, we have

$$dg = g_A dA + \left(\frac{1}{2} g_{AA}\sigma^2 + g_t\right) dt = F dA + \left(\frac{\sigma^2}{2} F_A + g_t\right) dt.$$

One can show that

$$\begin{aligned} g_t &= -\frac{\frac{\alpha}{(T-t)^2}}{2\sigma^2} A^2 \chi^< - \frac{\sigma^2}{2\lambda^2} \frac{\frac{\alpha}{(T-t)^2}}{\left(r + \frac{\alpha}{T-t}\right)^2} \chi^> \\ &- \left(\frac{r + \frac{\alpha}{T-t}}{2\sigma^2} A_t^2 + \left(\frac{\sigma^2}{2\lambda^2} \frac{1}{r + \frac{\alpha}{T-t}} - \frac{|A_t|}{\lambda}\right)\right) \delta\left(\frac{\sigma^2}{\lambda} \frac{1}{r + \frac{\alpha}{T-t}} - |A_t|\right) \\ &\times \frac{\sigma^2 \frac{\alpha}{(T-t)^2}}{\lambda (r + \frac{\alpha}{T-t})}, \end{aligned}$$

where $\delta(\cdot)$ is the Dirac-δ function. Note that $\delta\left(\frac{\sigma^2}{\lambda}\frac{1}{r+\frac{\alpha}{T-t}} - |A_t|\right)$ has support only when $|A_t| = \frac{\sigma^2}{\lambda}\frac{1}{r+\frac{\alpha}{T-t}}$, at which $\left(\frac{r+\frac{\alpha}{T-t}}{2\sigma^2} A_t^2 + \left(\frac{\sigma^2}{2\lambda^2}\frac{1}{r+\frac{\alpha}{T-t}} - \frac{|A_t|}{\lambda}\right)\right) = 0$, it immediately follows that

$$\left(\frac{r + \frac{\alpha}{T-t}}{2\sigma^2} A_t^2 + \left(\frac{\sigma^2}{2\lambda^2} \frac{1}{r + \frac{\alpha}{T-t}} - \frac{|A_t|}{\lambda}\right)\right) \delta\left(\frac{\sigma^2}{\lambda} \frac{1}{r + \frac{\alpha}{T-t}} - |A_t|\right) = 0,$$

which in turn implies

$$g_t = -\frac{\frac{\alpha}{(T-t)^2}}{2\sigma^2} A^2 \chi^< - \frac{\sigma^2}{2\lambda^2} \frac{\frac{\alpha}{(T-t)^2}}{\left(r + \frac{\alpha}{T-t}\right)^2} \chi^>. \tag{A32}$$

Therefore we have

$$\begin{aligned} dh^+ &= -\left(rFA + \frac{\sigma^2}{2} F^2\right) dt + F dA \\ &= -\left(rFA + \frac{\sigma^2}{2} F^2\right) dt + dg - \left(\frac{\sigma^2}{2} F_A + g_t\right) dt \\ &= -\left(g_t + rFA + \frac{\sigma^2}{2} F^2 + \frac{\sigma^2}{2} F_A\right) dt + dg. \end{aligned} \tag{A33}$$

Using the expressions for g_t, F, and noting that

$$\begin{aligned} F_A &= -\frac{r + \frac{\alpha}{T-t}}{\sigma^2} \chi_t^<, \\ F^2 &= \left(\frac{r + \frac{\alpha}{T-t}}{\sigma^2} A_t\right)^2 \chi^< + \frac{1}{\lambda^2} \chi^>, \end{aligned}$$

we obtain

$$
\begin{aligned}
&-(g_t + rFA + \frac{\sigma^2}{2}F^2 + \frac{\sigma^2}{2}F_A) \\
&= \frac{\frac{\alpha}{(T-t)^2}}{2\sigma^2}A^2\chi^< + \frac{\sigma^2}{2\lambda^2}\frac{\frac{\alpha}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2}\chi^> + r\frac{r+\frac{\alpha}{T-t}}{\sigma^2}A^2\chi^< + r\frac{1}{\lambda}|A|\chi^> \\
&-\frac{\sigma^2}{2}\left(\left(\frac{r+\frac{\alpha}{T-t}}{\sigma^2}A_t\right)^2\chi^< + \frac{1}{\lambda^2}\chi^>\right) + \frac{1}{2}\left(r+\frac{\alpha}{T-t}\right)\chi^< \\
&= \chi^<\left(\frac{\frac{\alpha}{(T-t)^2}}{2\sigma^2}A^2 + r\frac{r+\frac{\alpha}{T-t}}{\sigma^2}A^2 - \frac{\left(r+\frac{\alpha}{T-t}\right)^2A^2}{2\sigma^2} + \frac{1}{2}\left(r+\frac{\alpha}{T-t}\right)\right) \\
&+\chi^>\left(-\frac{\sigma^2}{2\lambda^2} + \frac{\sigma^2}{2\lambda^2}\frac{\frac{\alpha}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2} + r\frac{1}{\lambda}|A|\right) \\
&= \chi^<\left(\frac{r^2+\frac{\alpha-\alpha^2}{(T-t)^2}}{2\sigma^2}A^2 + \frac{1}{2}\left(r+\frac{\alpha}{T-t}\right)\right) \\
+ \quad & \chi^>\left(r\frac{1}{\lambda}|A| + \frac{\sigma^2}{2\lambda^2}\frac{\frac{\alpha}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2} - \frac{\sigma^2}{2\lambda^2}\right). \qquad \text{(A34)}
\end{aligned}
$$

The first term $\chi^<\left(\frac{r^2+\frac{\alpha-\alpha^2}{(T-t)^2}}{2\sigma^2}A^2 + \frac{1}{2}\left(r+\frac{\alpha}{T-t}\right)\right)$ is greater than 0 if $\alpha \leq 1$. For the second term, we have

$$
\begin{aligned}
& \chi^>\left(r\frac{1}{\lambda}|A| + \frac{\sigma^2}{2\lambda^2}\frac{\frac{\alpha}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2} - \frac{\sigma^2}{2\lambda^2}\right) \\
\geq \quad & \chi^>\left(r\frac{\sigma^2}{\lambda\left(r+\frac{\alpha}{T-t}\right)} + \frac{\sigma^2}{2\lambda^2}\frac{\frac{\alpha}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2} - \frac{\sigma^2}{2\lambda^2}\right) \\
= \quad & \chi^>\frac{\sigma^2}{2\lambda}\frac{r^2+\frac{\alpha-\alpha^2}{(T-t)^2}}{\left(r+\frac{\alpha}{T-t}\right)^2} \geq 0,
\end{aligned}
$$

which is non-negative as long as $\alpha \leq 1$.

∎

Proof of Proposition 5. The delay strategy is specified as

$$D_t = \frac{W_t}{\lambda}\left(-1_{\{A_t > \frac{\sigma^2}{2r\lambda}\}} + 1_{\{A_t \leq -\frac{\sigma^2}{2r\lambda}\}}\right).$$

The wealth follows the following equation

$$W_t = W_0 \exp(rt) \exp\left(\int_0^t (r\frac{1}{\lambda}|A| - \frac{1\sigma^2}{2\lambda^2})\chi dt - \frac{1}{\lambda}\left(1_{\{A_t > \frac{\sigma^2}{2r\lambda}\}} - 1_{\{A_t \leq \frac{\sigma^2}{2r\lambda}\}}\right) dA\right).$$

where $\chi_t \equiv 1_{\{|A_t| > \frac{\sigma^2}{2r\lambda}\}}$. Using the formulas (Equation (7.7) and (7.8) of Karatzas and Schreve (1991)) for semimartingale local times, we have

$$-\int_0^T 1_{\{A_t > \frac{\sigma^2}{2r\lambda}\}} dA_t = \left(A_0 - \frac{\sigma^2}{2r\lambda}\right)^+ - \left(A_T - \frac{\sigma^2}{2r\lambda}\right)^+ + L_T\left(\frac{\sigma^2}{2r\lambda}\right),$$

$$-\int_0^T 1_{\{A_t \leq -\frac{\sigma^2}{2r\lambda}\}} dA_t = \left(A_0 + \frac{\sigma^2}{2r\lambda}\right)^- - \left(A_T + \frac{\sigma^2}{2r\lambda}\right)^- + L_T\left(-\frac{\sigma^2}{2r\lambda}\right),$$

where $L_T(a) \geq 0$ denotes the local time of A_t over $[0, T]$ of level a. Since

$$\left(A_T - \frac{\sigma^2}{2r\lambda}\right)^+ = \left(-\frac{\sigma^2}{2r\lambda}\right)^+ = 0$$

and

$$\left(A_T + \frac{\sigma^2}{2r\lambda}\right)^- = \left(\frac{\sigma^2}{2r\lambda}\right)^- = 0,$$

we have

$$\begin{aligned} W_T &= W_0 \exp(rT) \exp\left(\int_0^T (r\frac{1}{\lambda}|A| - \frac{1\sigma^2}{2\lambda^2})\chi dt\right) \\ &\quad \exp\left(\left(A_0 - \frac{\sigma^2}{2r\lambda}\right)^+ + L_T\left(\frac{\sigma^2}{2r\lambda}\right) + \left(A_0 + \frac{\sigma^2}{2r\lambda}\right)^- + L_T\left(-\frac{\sigma^2}{2r\lambda}\right)\right). \end{aligned}$$

Note that $(r\frac{1}{\lambda}|A_t| - \frac{1\sigma^2}{2\lambda^2})\chi_t$ and local times are always non-negative, we conclude that

$$W_T \geq W_0 e^{rT} \quad a.s..$$

∎

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Table 1

Percentage of Times that Margin Constraint is Binding. This table reports the percentage of 10,000 simulated paths for which the margin constraint is binding at the indicated horizons. The initial value of the arbitrage is set equal to zero, the final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. The riskless rate is 6 percent.

A_0	λ	α	σ	$t = .250$	$t = .500$	$t = .750$	$t = .999$
0	1	1	1	9.44	32.40	56.58	97.32
0	1	1	2	.09	5.17	25.26	94.70
0	1	2	1	33.18	51.82	66.26	98.14
0	1	2	2	5.11	19.40	38.03	95.93
0	10	1	1	86.18	92.41	95.83	99.77
0	10	1	2	73.18	84.93	91.37	99.50
0	10	2	1	91.86	94.86	96.51	99.86
0	10	2	2	84.23	90.05	93.05	99.59
1	1	1	1	53.27	53.43	62.20	97.28
1	1	1	2	.66	7.72	26.47	94.47
1	1	2	1	70.77	59.94	66.72	98.13
1	1	2	2	11.78	21.68	37.98	95.93
1	10	1	1	96.96	95.51	95.98	99.78
1	10	1	2	81.82	85.57	90.68	99.47
1	10	2	1	97.49	95.31	96.80	99.86
1	10	2	2	88.22	90.52	93.16	99.59

Table 2

Summary Statistics for the Percentage Portfolio Weights Invested in the Arbitrage. This table reports summary statistics for the percentage portfolio weights for the indicated horizons based on 10,000 simulated paths. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. The riskless rate is 6 percent.

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
0	1	1	1	.250	-202.65	-23.20	-11.65	.00	23.81
				.500	-196.48	-34.09	-23.26	.00	33.07
				.750	-187.47	-32.50	-29.13	.00	27.40
				.999	-15.16	-2.60	-2.21	.00	1.97
0	1	1	2	.250	-405.30	-25.61	-11.65	.00	36.29
				.500	-392.95	-48.17	-23.26	.00	61.11
				.750	-374.95	-56.07	-34.45	.00	57.60
				.999	-30.33	-5.19	-4.42	.00	3.94
0	1	2	1	.250	-179.28	-25.91	-17.72	.00	25.23
				.500	-150.16	-28.35	-25.56	.00	24.36
				.750	-115.83	-21.98	-19.15	.00	17.78
				.999	-7.83	-1.54	-1.30	.00	1.17
0	1	2	2	.250	-358.55	-36.69	-17.72	.00	46.53
				.500	-300.32	-47.03	-26.53	.00	50.58
				.750	-231.66	-40.13	-29.56	.00	37.61
				.999	-15.67	-3.08	-2.61	.00	2.34
0	10	1	1	.250	-20.27	-3.40	-2.89	.00	2.62
				.500	-19.65	-3.96	-3.36	.00	2.99
				.750	-18.75	-3.43	-2.91	.00	2.58
				.999	-1.52	-.26	-.22	.00	.20
0	10	1	2	.250	-40.53	-6.70	-5.78	.00	5.33
				.500	-39.30	-7.88	-6.72	.00	6.02
				.750	-37.50	-6.85	-5.83	.00	5.18
				.999	-3.03	-.52	-.44	.00	.39
0	10	2	1	.250	-17.93	-3.00	-2.55	.00	2.29
				.500	-15.02	-3.03	-2.56	.00	2.27
				.750	-11.58	-2.27	-1.92	.00	1.71
				.999	-.78	-.15	-.13	.00	.12
0	10	2	2	.250	-35.86	-5.96	-5.10	.00	4.60
				.500	-30.03	-6.05	-5.11	.00	4.56
				.750	-23.17	-4.53	-3.83	.00	3.43
				.999	-1.57	-.31	-.26	.00	.23

Table 2 Continued

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
1	1	1	1	.250	-241.25	-70.80	-75.11	.00	45.64
				.500	-246.48	-54.13	-52.36	.00	43.00
				.750	-201.68	-38.29	-34.01	.00	31.04
				.999	-15.06	-2.60	-2.21	.00	1.97
1	1	1	2	.250	-407.51	-44.89	-23.56	.00	56.06
				.500	-442.95	-58.52	-29.55	.00	70.22
				.750	-378.36	-59.20	-37.54	.00	59.86
				.999	-30.23	-5.20	-4.41	.00	3.94
1	1	2	1	.250	-201.96	-56.70	-56.65	.00	36.19
				.500	-174.46	-34.78	-31.78	.00	28.36
				.750	-115.76	-22.49	-19.49	.00	18.20
				.999	-7.83	-1.54	-1.30	.00	1.17
1	1	2	2	.250	-347.68	-54.13	-29.91	.00	60.25
				.500	-323.93	-50.68	-30.03	.00	53.30
				.750	-225.41	-40.41	-29.74	.00	37.84
				.999	-15.67	-3.08	-2.61	.00	2.34
1	10	1	1	.250	-24.13	-7.63	-7.51	.00	4.04
				.500	-24.65	-5.79	-5.24	.00	3.96
				.750	-20.17	-3.98	-3.40	.00	2.96
				.999	-1.51	-.26	-.22	.00	.20
1	10	1	2	.250	-40.75	-9.22	-8.22	.00	6.66
				.500	-44.30	-8.82	-7.58	.00	6.71
				.750	-37.84	-7.72	-6.08	.00	5.40
				.999	-3.02	-.52	-.44	.00	.39
1	10	2	1	.250	-20.20	-5.85	-5.67	.00	3.41
				.500	-17.45	-3.64	-3.18	.00	2.68
				.750	-11.58	-2.32	-1.95	.00	1.75
				.999	-.78	-.15	-.13	.00	.12
1	10	2	2	.250	-34.77	-7.59	-6.62	.00	5.54
				.500	-32.39	-6.35	-5.44	.00	4.79
				.750	-22.54	-4.55	-3.84	.00	3.45
				.999	-1.57	-.31	-.26	.00	.23

Table 3

Summary Statistics for the Value of the Optimal Portfolio. The table reports summary statistics for the value of the optimal portfolio at the indicated horizons based on 10,000 simulated paths. The initial value of the portfolio is 100. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. If the initial wealth of 100 was invested in the riskless asset only, its value in one year would be 106.18.

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
0	1	1	1	.250	20.63	105.27	111.33	119.64	15.39
				.500	20.79	118.32	128.70	151.15	28.82
				.750	23.78	144.88	149.04	220.03	44.69
				1.000	113.53	278.85	266.45	763.86	88.83
0	1	1	2	.250	8.30	105.38	111.38	119.64	15.45
				.500	6.67	120.39	131.62	151.15	30.13
				.750	5.69	158.30	176.55	223.09	53.44
				1.000	131.45	482.73	453.52	1413.09	186.92
0	1	2	1	.250	22.02	110.98	118.60	139.39	22.99
				.500	28.58	135.44	137.98	214.04	40.09
				.750	39.94	183.14	176.55	428.22	65.85
				1.000	98.83	420.94	387.01	1457.41	179.84
0	1	2	2	.250	5.94	112.72	122.03	139.39	24.70
				.500	7.87	147.22	162.17	214.16	48.42
				.750	11.79	228.79	230.84	430.42	97.04
				1.000	90.10	995.53	873.80	5488.36	583.91
0	10	1	1	.250	85.71	102.33	102.47	114.63	4.27
				.500	85.39	105.50	105.29	128.65	5.85
				.750	90.06	109.93	109.48	138.61	6.83
				1.000	106.78	120.27	119.32	157.13	7.36
0	10	1	2	.250	71.99	103.04	103.76	118.31	7.59
				.500	71.97	107.76	107.64	141.92	10.86
				.750	77.92	115.04	114.30	165.52	13.17
				1.000	107.45	135.31	133.22	221.21	15.29
0	10	2	1	.250	87.72	102.95	102.92	117.66	4.26
				.500	89.02	107.15	106.82	130.83	5.60
				.750	95.94	112.86	112.20	143.55	6.66
				1.000	107.11	125.42	124.59	163.00	7.99
0	10	2	2	.250	75.68	104.31	104.38	128.91	8.06
				.500	77.49	111.14	110.48	159.90	11.01
				.750	87.72	121.24	119.87	184.65	13.66
				1.000	107.97	147.20	144.87	236.91	18.01

Table 3 Continued

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
1	1	1	1	.250	22.37	126.10	123.85	201.28	43.63
				.500	19.43	155.30	156.46	252.65	58.47
				.750	28.49	199.18	195.50	366.68	76.16
				1.000	184.98	390.99	366.18	1049.28	141.19
1	1	1	2	.250	9.36	111.29	119.89	136.56	24.49
				.500	4.35	131.71	145.59	172.88	38.60
				.750	5.53	176.57	197.01	251.04	63.33
				1.000	139.82	541.30	512.30	1664.83	214.64
1	1	2	1	.250	33.05	150.73	142.36	278.51	52.08
				.500	39.56	201.94	197.95	414.53	72.05
				.750	65.25	278.91	261.88	739.87	109.86
				1.000	182.21	640.59	580.94	2669.69	287.75
1	1	2	2	.250	7.38	129.38	142.73	179.41	40.08
				.500	6.85	174.35	189.56	273.14	65.14
				.750	12.45	272.57	270.88	547.72	123.60
				1.000	88.47	1183.89	1029.33	6326.61	717.29
1	10	1	1	.250	88.25	104.21	104.08	119.45	4.46
				.500	89.23	108.70	108.42	130.27	5.53
				.750	94.83	114.08	113.41	143.87	6.08
				1.000	117.10	125.20	123.87	160.32	6.16
1	10	1	2	.250	74.55	104.34	104.24	126.03	8.58
				.500	72.96	109.95	109.42	147.84	11.48
				.750	78.96	117.78	116.61	177.71	13.41
				1.000	115.72	138.73	136.19	215.37	15.32
1	10	2	1	.250	91.77	106.17	106.02	122.64	4.04
				.500	95.80	111.87	111.44	136.12	4.94
				.750	103.13	118.17	117.35	145.75	5.86
				1.000	117.20	131.35	130.31	165.08	7.15
1	10	2	2	.250	79.12	106.51	106.17	135.19	8.22
				.500	82.06	114.32	113.41	167.62	10.94
				.750	91.73	124.83	123.23	195.80	13.42
				1.000	115.82	151.57	149.19	243.08	17.70

Table 4

Pathwise Summary Statistics for the Value of a Portfolio following the Optimal Strategy. This table reports summary statistics taken over 10,000 paths for the value of a portfolio where the optimal strategy is followed. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. The riskless rate is 6 percent. Percent < 100 is the percentage of paths for which the minimum value of the portfolio was below 100. Percent $< R_t$ is the percentage of paths for which the return on the portfolio was less than the riskless asset at some point. Percent Max $> W_T$ is the percentage of paths for which the maximum value of the portfolio occurred prior to T. The values of Average t_{Min} and Average t_{Max} are the average of the times at which the minimum and maximum values of the arbitrage occurred. The values Average Min. and Average Max. are the average minimum and maximum values of the portfolio.

A_0	λ	α	σ	Percent < 100	Percent $< R_t$	Percent Max $> W_T$	Average t_{Min}	Average t_{Max}	Average Min.	Average Max.
0	1	1	1	99.89	100.00	.17	.3605	.9999	75.34	278.85
0	1	1	2	99.91	100.00	.49	.4366	.9999	67.09	482.73
0	1	2	1	99.94	99.98	.57	.2532	.9997	75.81	420.94
0	1	2	2	99.96	99.99	1.74	.3170	.9985	65.64	995.60
0	10	1	1	99.50	99.86	.00	.1722	1.0000	96.85	120.27
0	10	1	2	99.71	99.89	.00	.2271	1.0000	93.13	135.31
0	10	2	1	99.38	99.72	.00	.1269	1.0000	97.27	125.42
0	10	2	2	99.69	99.84	.01	.1625	1.0000	94.00	147.20
1	1	1	1	98.60	98.66	.14	.1980	.9999	72.17	390.99
1	1	1	2	99.18	99.33	.50	.3538	.9999	65.27	541.30
1	1	2	1	97.86	97.98	.53	.1182	.9999	80.04	640.59
1	1	2	2	98.77	98.83	1.80	.2272	.9980	63.67	1183.97
1	10	1	1	97.64	98.49	.00	.1036	1.0000	97.67	125.20
1	10	1	2	98.63	99.00	.00	.1877	1.0000	93.32	138.73
1	10	2	1	96.55	97.42	.00	.0547	1.0000	98.42	131.35
1	10	2	2	98.26	98.65	.01	.1273	1.0000	94.89	151.57

Table 5

Summary Statistics for the Annualized Sharpe Ratio. The table reports summary statistics for the distribution of annualized Sharpe ratios based on 10,000 simulated paths. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The Sharpe ratios are computed pathwise from the annualized mean and standard deviations of changes in the value of the optimal portfolio. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process.

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
0	1	1	1	.250	-8.95	1.05	1.24	8.03	1.97
				.500	-4.00	.85	1.00	5.98	1.38
				.750	-2.51	.86	.85	5.13	1.03
				1.000	.12	1.66	1.60	5.13	.67
0	1	1	2	.250	-9.05	1.08	1.24	8.03	1.94
				.500	-3.98	.95	1.09	5.98	1.32
				.750	-2.81	.99	1.10	5.13	1.06
				1.000	.34	1.82	1.76	4.79	.63
0	1	2	1	.250	-7.70	1.07	1.20	7.99	2.00
				.500	-2.97	.95	.92	5.94	1.26
				.750	-1.64	1.06	1.01	5.02	.89
				1.000	-.18	2.19	2.18	5.17	.70
0	1	2	2	.250	-7.72	1.23	1.37	7.99	1.91
				.500	-3.96	1.12	1.23	5.94	1.29
				.750	-2.10	1.15	1.12	5.02	.98
				1.000	-.18	2.19	2.17	5.17	.70
0	10	1	1	.250	-7.26	.44	.42	7.75	1.94
				.500	-3.76	.52	.46	6.02	1.25
				.750	-2.10	.71	.65	4.57	.92
				1.000	.06	1.36	1.28	4.63	.68
0	10	1	2	.250	-7.85	.54	.53	8.02	2.01
				.500	-3.59	.55	.50	5.85	1.27
				.750	-2.11	.72	.67	4.52	.92
				1.000	.07	1.40	1.33	4.90	.68
0	10	2	1	.250	-6.06	.62	.58	7.86	1.81
				.500	-2.96	.82	.76	5.56	1.13
				.750	-1.22	1.06	.99	4.61	.84
				1.000	.09	1.79	1.74	4.81	.69
0	10	2	2	.250	-6.25	.66	.63	7.72	1.85
				.500	-2.87	.82	.76	5.79	1.14
				.750	-1.28	1.06	.99	4.55	.84
				1.000	.09	1.84	1.78	4.88	.70

Table 5 Continued

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.
1	1	1	1	.250	-6.36	.99	.86	8.14	2.06
				.500	-3.10	1.05	.99	6.91	1.33
				.750	-1.88	1.11	1.05	4.72	.95
				1.000	.68	1.83	1.75	4.68	.61
1	1	1	2	.250	-6.22	1.14	1.29	8.11	1.96
				.500	-3.91	1.06	1.20	5.48	1.33
				.750	-2.86	1.07	1.16	4.79	1.05
				1.000	.45	1.86	1.79	4.60	.62
1	1	2	1	.250	-4.45	1.56	1.35	8.11	1.80
				.500	-1.83	1.50	1.42	6.71	1.11
				.750	-.68	1.46	1.38	4.68	.80
				1.000	.60	2.29	2.25	5.27	.65
1	1	2	2	.250	-6.11	1.38	1.57	8.27	1.92
				.500	-3.07	1.26	1.32	5.58	1.28
				.750	-1.98	1.24	1.20	4.96	.97
				1.000	-.17	2.22	2.20	5.17	.70
1	10	1	1	.250	-5.63	1.04	1.01	7.75	1.76
				.500	-2.89	1.08	1.03	5.51	1.07
				.750	-1.37	1.19	1.11	4.91	.76
				1.000	1.00	1.74	1.63	4.64	.53
1	10	1	2	.250	-6.19	.64	.55	8.07	1.90
				.500	-3.44	.71	.64	5.70	1.21
				.750	-1.98	.85	.78	4.86	.87
				1.000	.46	1.49	1.40	4.58	.64
1	10	2	1	.250	-4.07	1.79	1.75	8.77	1.56
				.500	-1.47	1.67	1.59	6.18	.92
				.750	-.20	1.66	1.58	4.73	.69
				1.000	1.02	2.24	2.17	4.79	.58
1	10	2	2	.250	-5.03	1.00	.91	7.92	1.70
				.500	-2.30	1.08	1.00	6.14	1.05
				.750	-.93	1.23	1.16	5.15	.78
				1.000	.47	1.96	1.90	4.78	.65

Table 6

Summary Statistics for the Value of a Portfolio following the Maximal Strategy. The table reports summary statistics for the value of a portfolio where the maximal strategy is followed. The initial value of the portfolio is 100. The summary statistics are based on 10,000 simulated paths. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. The Sharpe ratio the average of the pathwise Sharpe ratios for the individual paths. If the initial wealth of 100 was invested in the riskless asset only, its value in one year would be 106.18.

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.	Sharpe Ratio
0	10	1	1	.250	87.06	102.36	102.12	122.96	4.81	.29
				.500	84.75	105.55	105.02	137.05	6.46	.45
				.750	89.22	109.98	109.19	143.58	7.51	.65
				1.000	106.08	120.33	119.14	158.81	8.15	1.30
0	10	1	2	.250	74.48	103.18	102.51	148.54	9.74	.24
				.500	69.34	107.98	106.50	181.40	13.33	.40
				.750	75.54	115.30	113.14	195.58	16.01	.60
				1.000	104.92	135.61	132.36	235.12	18.81	1.27
0	10	2	1	.250	88.54	102.96	102.74	122.86	4.57	.53
				.500	88.79	107.16	106.66	132.18	5.97	.76
				.750	95.19	112.87	112.06	147.51	7.08	1.02
				1.000	106.50	125.44	124.44	169.93	8.46	1.74
0	10	2	2	.250	77.03	104.36	103.73	148.33	9.31	.48
				.500	76.11	111.23	109.85	168.72	12.52	.71
				.750	85.99	121.36	119.16	206.46	15.48	.97
				1.000	105.74	147.37	144.37	269.23	20.26	1.74
1	10	1	1	.250	88.25	104.22	104.08	121.81	4.49	1.02
				.500	89.23	108.69	108.39	135.93	5.61	1.06
				.750	94.83	114.08	113.27	146.62	6.27	1.16
				1.000	117.10	125.20	123.72	164.66	6.42	1.72
1	10	1	2	.250	74.55	104.35	103.83	149.73	9.25	.48
				.500	72.96	110.06	108.52	185.71	12.75	.60
				.750	78.96	117.94	115.62	206.36	15.02	.77
				1.000	115.59	138.92	135.18	245.11	17.49	1.41
1	10	2	1	.250	91.77	106.16	106.02	125.71	4.06	1.78
				.500	95.80	111.88	111.40	136.59	5.04	1.64
				.750	103.13	118.17	117.26	146.16	6.02	1.64
				1.000	117.20	131.37	130.30	166.53	7.37	2.22
1	10	2	2	.250	79.12	106.52	105.89	147.60	8.67	.91
				.500	82.06	114.34	112.72	180.41	11.77	1.00
				.750	91.73	124.80	122.52	223.29	14.53	1.16
				1.000	115.82	151.54	148.57	256.57	19.07	1.89

Table 7

Summary Statistics for the Value of a Portfolio following the Barrier Strategy. The table reports summary statistics for the value of a portfolio where the barrier strategy is followed. The initial value of the portfolio is 100. The summary statistics are based on 10,000 simulated paths. The final convergence date for the arbitrage is one year, and the simulation uses 10,000 discretization points per year in modeling the arbitrage process. The initial value of the arbitrage is A_0. The parameter λ represents the margin requirement. The parameters α and σ represent the speed of convergence and the volatility of the arbitrage process. The Sharpe ratio is the average of the pathwise Sharpe ratios for the individual paths. If the initial wealth of 100 was invested in the riskless asset only, its value in one year would be 106.18.

A_0	λ	α	σ	t	Min.	Mean	Median	Max.	Std. Dev.	Sharpe Ratio
0	10	1	1	.250	90.23	101.57	101.51	106.81	.61	.15
				.500	93.34	103.44	103.05	113.82	1.31	.35
				.750	94.72	105.61	104.60	124.06	1.85	.51
				1.000	106.18	107.45	106.18	127.94	2.05	.52
0	10	1	2	.250	96.94	101.51	101.51	103.88	.05	.00
				.500	99.38	103.05	103.05	111.76	.21	.01
				.750	104.60	104.63	104.60	117.63	.33	.01
				1.000	106.18	106.21	106.18	119.41	.35	.01
0	10	2	1	.250	93.25	101.57	101.51	107.20	.44	.12
				.500	98.12	103.38	103.05	112.02	.98	.30
				.750	102.07	105.14	104.60	116.50	1.27	.35
				1.000	106.18	106.74	106.18	118.41	1.30	.31
0	10	2	2	.250	100.58	101.51	101.51	101.51	.01	.00
				.500	103.05	103.05	103.05	105.95	.03	.00
				.750	104.64	104.60	104.60	107.55	.03	.00
				1.000	106.18	106.18	106.18	109.18	.03	.00
1	10	1	1	.250	88.25	103.50	104.00	112.63	2.96	1.51
				.500	89.23	106.70	106.63	118.65	2.97	1.41
				.750	95.67	109.67	109.17	125.71	2.78	1.35
				1.000	107.97	111.74	111.14	128.59	2.86	1.24
1	10	1	2	.250	94.67	101.51	101.51	106.59	.13	.01
				.500	93.00	103.08	103.05	115.96	.48	.02
				.750	104.60	104.67	104.60	117.71	.62	.02
				1.000	107.97	110.74	110.16	125.52	2.31	02
1	10	2	1	.250	91.77	104.35	104.31	113.46	2.02	2.14
				.500	95.80	107.12	106.64	119.99	2.05	1.79
				.750	105.05	109.08	108.51	122.98	2.27	1.54
				1.000	107.97	110.74	110.16	125.52	2.31	1.35
1	10	2	2	.250	101.51	101.51	101.51	104.77	.05	.00
				.500	103.05	103.05	103.05	110.72	.12	.00
				.750	104.60	104.61	104.60	112.39	.12	.00
				1.000	106.18	106.19	106.18	114.09	.12	.00

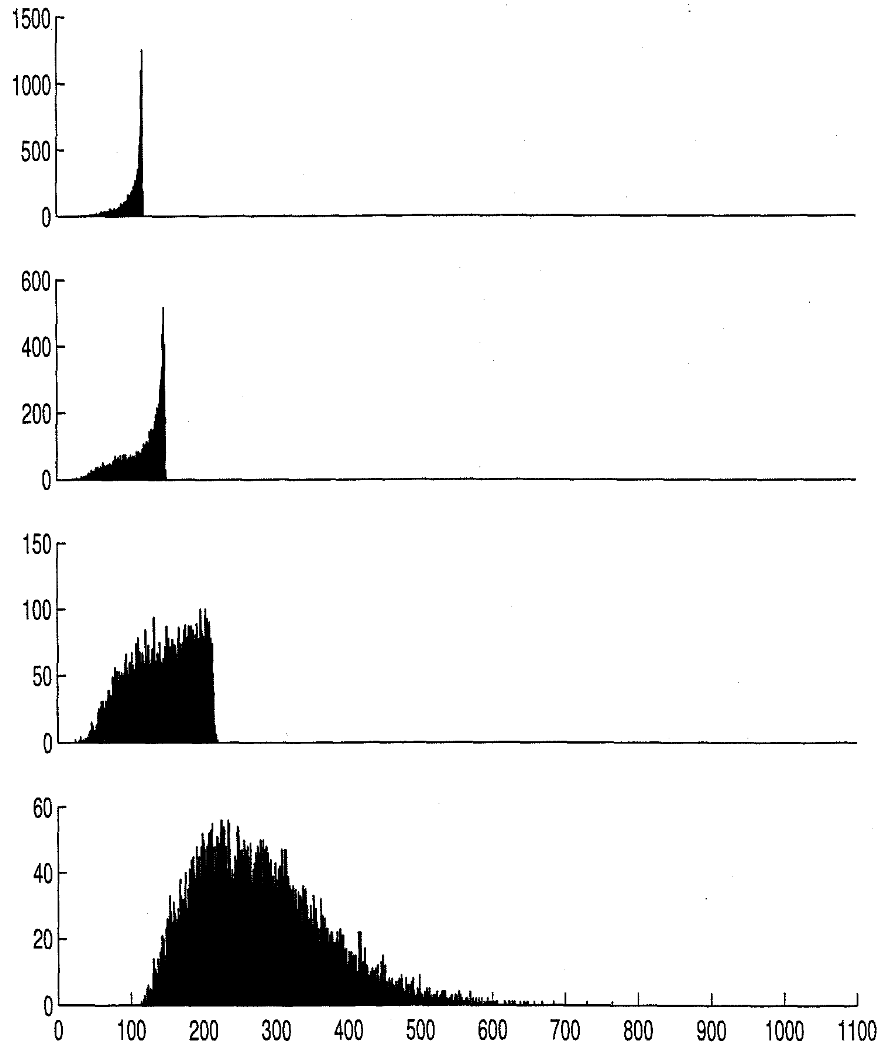

Figure 1. The Distribution of Wealth. From top down, the graph shows the distribution of the value of the optimal portfolio at time t=0.25, 0.50, 0.75, and 1.00. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

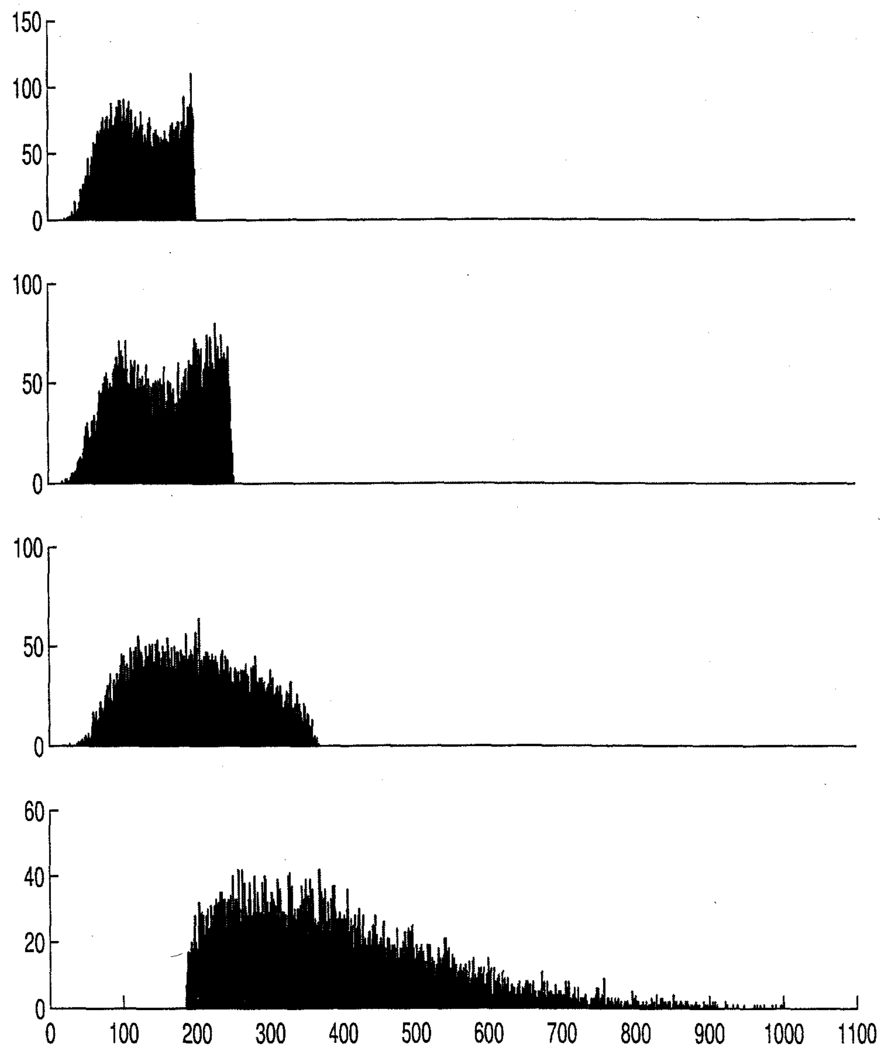

Figure 2. The Distribution of Wealth. From top down, the graph shows the distribution of the value of the optimal portfolio at time t=0.25, 0.50, 0.75, and 1.00. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 1. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

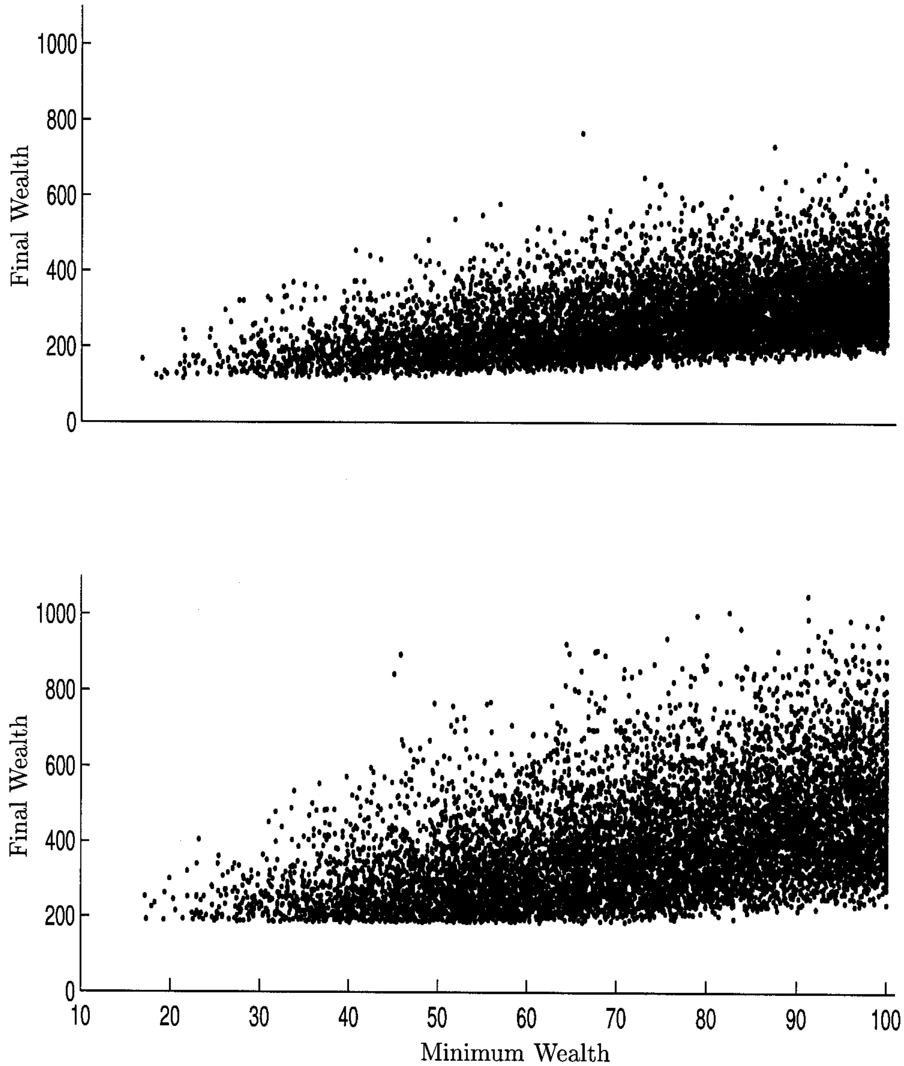

Figure 3. Graph of Wealth versus the Minimum Value of Wealth. The graph shows the relation between the final value and the minimum value of the optimal portfolio. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0 and 1 respectively in the the top and bottom graph. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

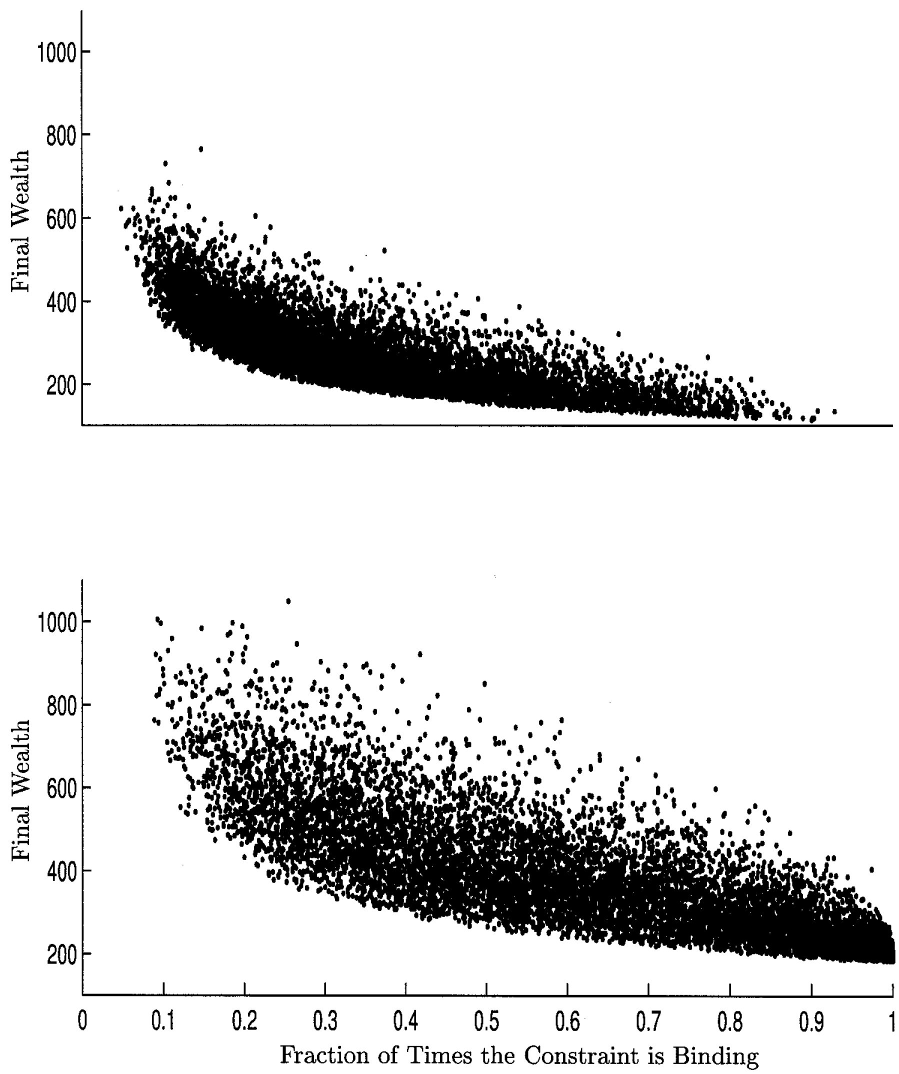

Figure 4. Graph of Wealth versus the Fraction of Times the Constraint is Binding. The graph shows the relation between the final value of the optimal portfolio and the fraction of times that the margin constraint is binding. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0 and 1 respectively in the the top and bottom graph. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

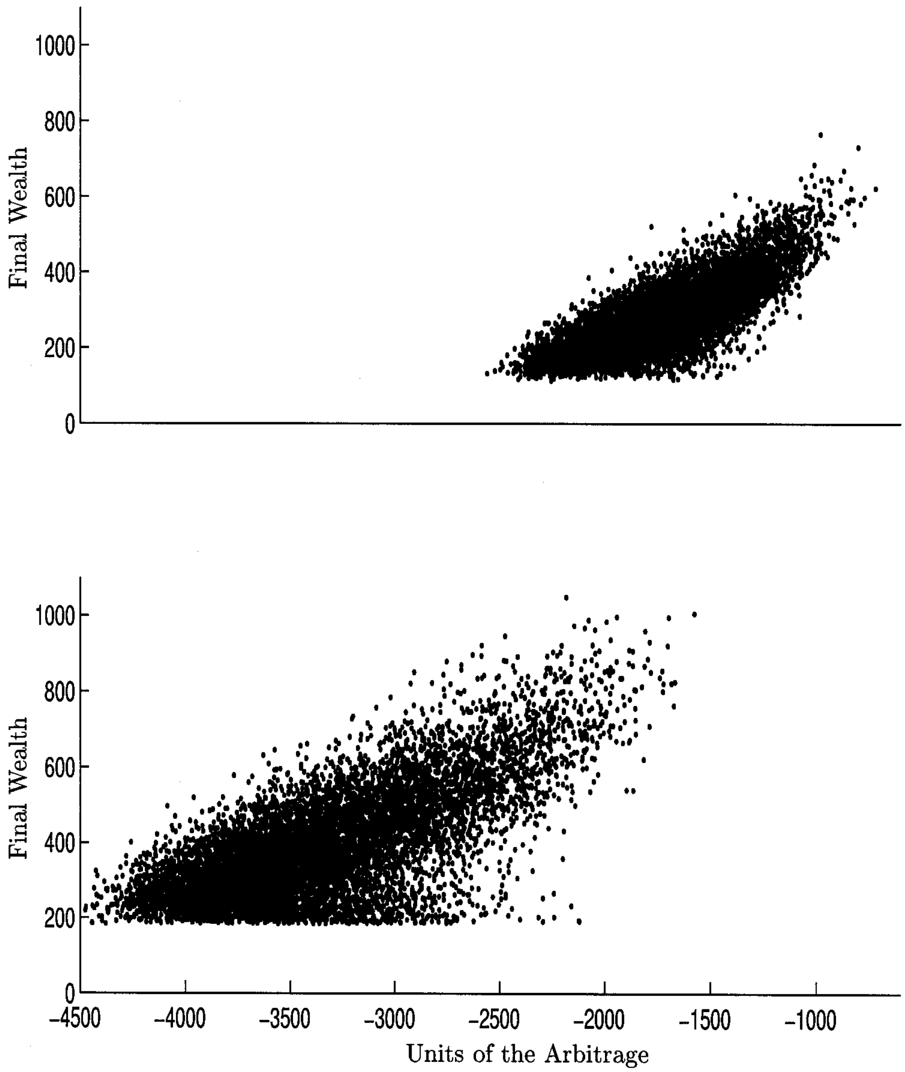

Figure 5. Graph of Wealth versus the Number of Units of the Arbitrage Liquidated. The graph shows the relation between the final value of the optimal portfolio and the total number of units of the arbitrage liquidated because of margin calls. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0 and 1 respectively in the the top and bottom graph. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

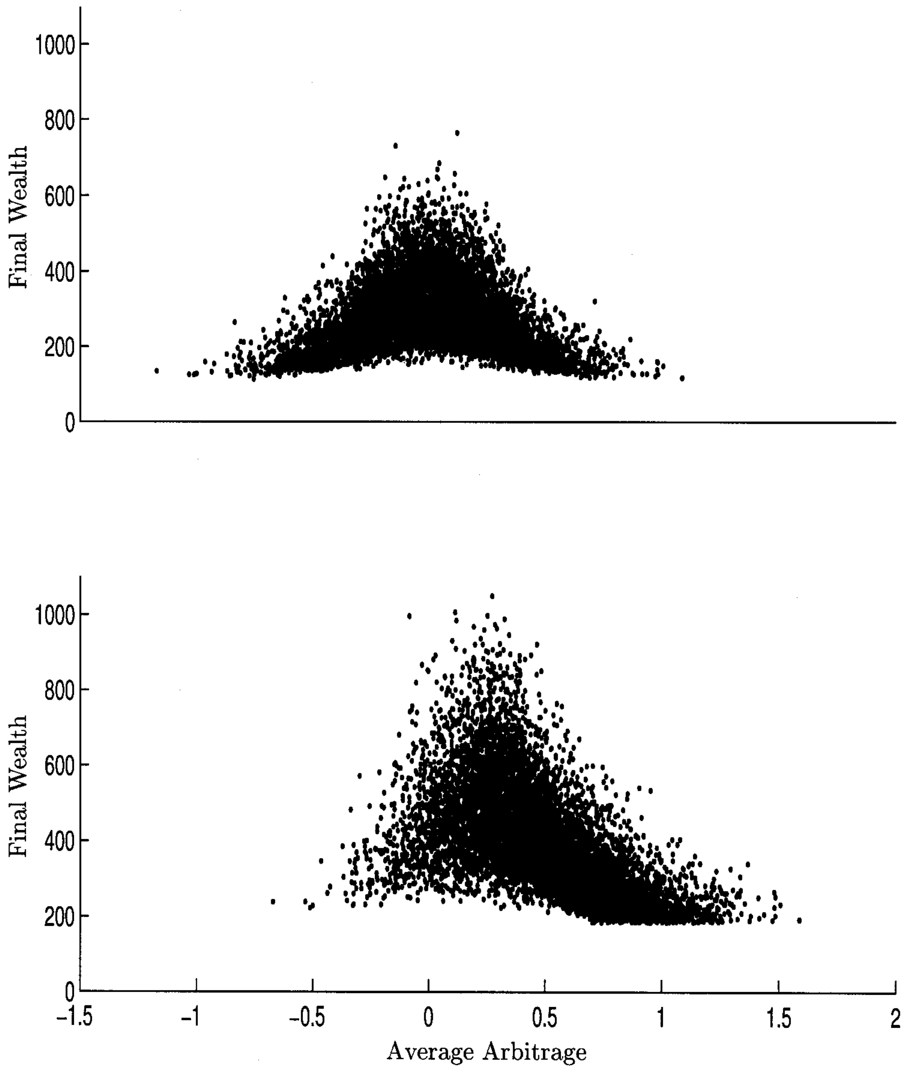

Figure 6. Graph of Wealth and the Average of the Arbitrage. The graph shows the relation between the final value of the optimal portfolio and the average of the arbitrage. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0 and 1 respectively in the the top and bottom graph. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

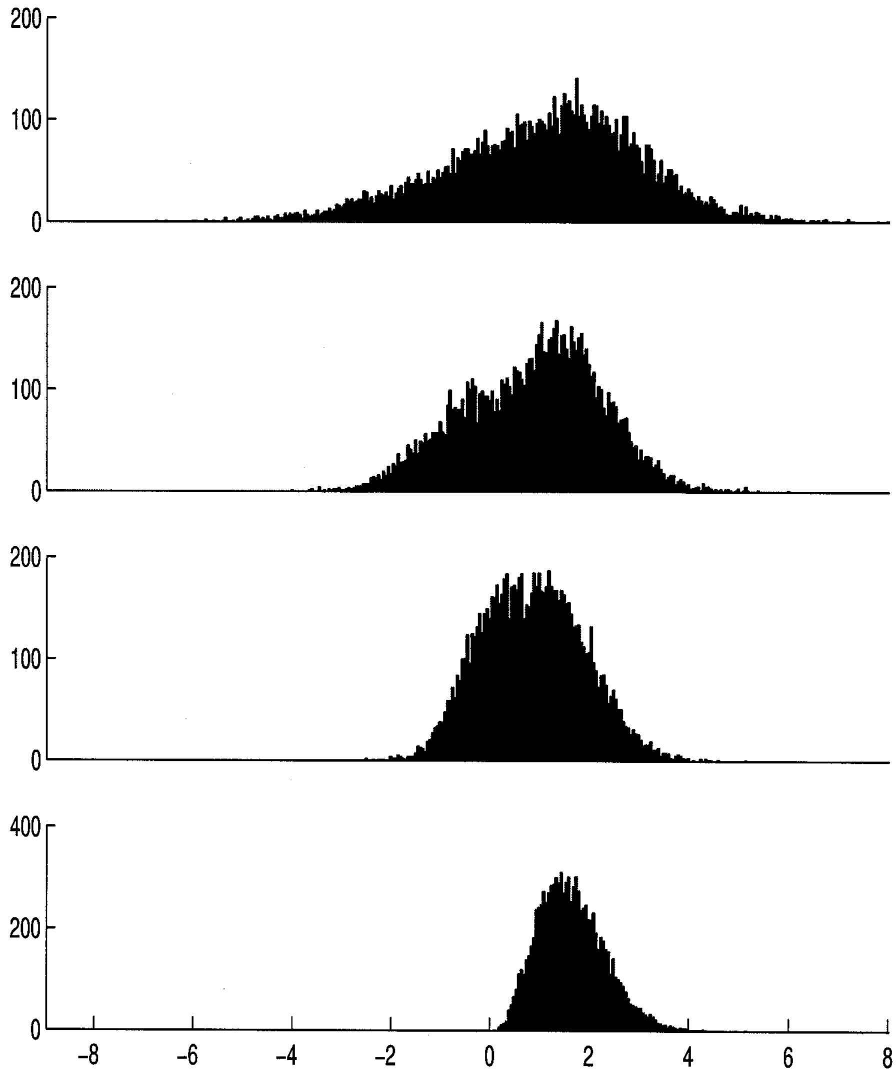

Figure 7. Distribution of the Sharpe Ratio. From top down, the graph shows the distribution of the annualized Sharpe ratio at time t=0.25, 0.50, 0.75, and 1.00. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 0. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.

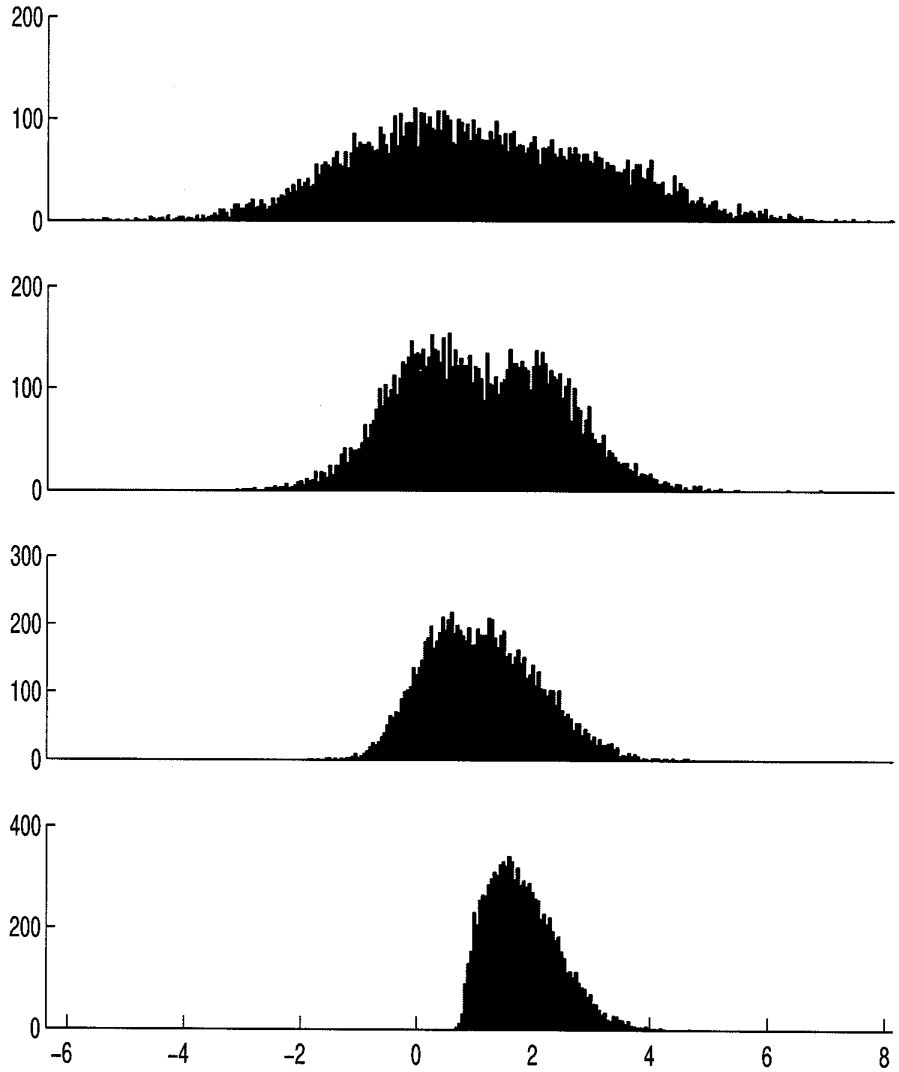

Figure 8. Distribution of the Sharpe Ratio. From top down, the graph shows the distribution of the annualized Sharpe ratio at time t=0.25, 0.50, 0.75, and 1.00. The initial value of the portfolio is 100. The initial value of the arbitrage A_0 is 1. The parameter values are $\alpha = 1$, $\lambda = 1$, and $\sigma = 1$.