Using Elasticities to Derive Optimal Bankruptcy Exemptions

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Abstract

This paper characterizes the optimal bankruptcy exemption for risk averse borrowers who use unsecured contracts but have the possibility of defaulting. It provides a novel general formula for the optimal exemption as a function of a few observable sufficient statistics. Knowledge of borrowers' leverage, interest rate schedule sensitivity to the exemption level, probability of bankruptcy, and change in consumption by bankrupt borrowers is sufficient to determine the optimal exemption. When calibrated to US data, the optimal bankruptcy exemption implied by the model ($100,000) is larger than the average exemption in the US ($70,000), but of the same order of magnitude.

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1 Introduction

Should bankruptcy procedures be harsher or more lenient than they are at the present with borrowers who decide not to repay their debts? What is the socially optimal level of bankruptcy exemptions? These important normative questions are perennial topics of debate that gain relevance after economic downturns, like the recent one. Although the commonly held presumption is that the optimal bankruptcy exemption is neither zero nor infinity, there are no simple and clear guidelines to determine whether the exemption level in a given economy is the appropriate one. In particular, looking at comparable modern economies, it is surprising to observe very large discrepancies in the level of bankruptcy exemptions across similar countries or regions, for instance, across US states.[1] This paper provides a novel analytical characterization that directly addresses these questions. More precisely, this paper shows, using a canonical equilibrium model of unsecured credit, that a few observable variables are sufficient to determine whether the exemption level should be increased or decreased, as well as the optimal exemption level, under a broad array of circumstances.

I initially derive the main results of the paper in a two period model with risk averse borrowers who only have access to a debt contract. I then show that the insights apply very generally. Throughout the paper, the bankruptcy exemption m is defined as the dollar amount that a borrower who declares bankruptcy is allowed to keep.

The main contribution of this paper is to characterize a) the welfare change induced by a marginal change in m, which provides a sharp test for whether it is optimal to increase or decrease the bankruptcy exemption given its current level, and b) the optimal bankruptcy exemption m^*, as a function of a few observable sufficient statistics. This characterization provides a clear interpretation of the forces that determine the optimal exemption for a broad range of primitives and directly links the theoretical tradeoffs to a small set of observable variables.

The best way to introduce the results is by describing the determinants of the optimal exemption m^* — measured in dollars — for logarithmic utility borrowers who claim the full exemption in bankruptcy. The optimal exemption m^* must satisfy the following equation:

$$m^* = \frac{\beta\pi}{\Lambda\varepsilon_{\tilde{r},m}}, \tag{1}$$

where β is the borrowers' rate of time preference, π is the probability of default in equilibrium, Λ is a measure of borrowers' leverage and $\varepsilon_{\tilde{r},m}$ is the sensitivity of the interest rate schedule faced by borrowers with respect to the level of the bankruptcy exemption. Importantly, all four variables in the formula for m^* have direct empirical counterparts.

Equation (1) captures the key tradeoff regarding the optimal determination of the bankruptcy exemption. On the one hand, if borrowing rates rise quickly with the level of the bankruptcy exemption (high $\varepsilon_{\tilde{r},m}$), it is optimal to set a low exemption level, especially when borrowers' leverage is high (high Λ): a low exemption facilitates the access to credit ex-ante in that case. On the other hand, if default is very frequent in equilibrium (high π), especially when borrowers value consumption relatively more in

[1] For brevity, I refer the reader to White (2011) and Claessens and Klapper (2005) for recent detailed discussions of actual institutional features of bankruptcy procedures in the US and around the world. See Skeel (2001) for a historical account of the evolution of the US bankruptcy system.

the terminal period (high β), it is optimal to set a high exemption level, allowing borrowers to consume more when bankrupt. Equation (1) trades off these forces optimally. Although the optimal exemption formula holds at the optimum, I also provide a test for whether it is optimal to increase or decrease the level of the bankruptcy exemption, starting from its current level: in practice, this directional test is the most robust result of this paper.[2]

The decisions of how much to borrow and when to default do not affect the assessment of marginal interventions nor the formula for m^* directly: the fact that borrowers borrow and default optimally guarantees that the effect on these variables induced by a change in the bankruptcy exemption vanishes from the optimal exemption formula. This fact greatly simplifies the characterization of the optimal exemption.

Following the general equilibrium tradition, this paper imposes no restrictions on the shape of contracts used, but takes the number and shape of the contracts available as part of the environment. When the first-best allocation, which implies perfect insurance for the risk averse borrowers, can be achieved, there is no rationale for bankruptcy. However, when the first-best allocation cannot be reached with a given set of contracts, an optimally chosen bankruptcy exemption can improve welfare: a judiciously chosen exemption alleviates the incomplete market friction. Importantly, even if borrowers endogenously choose the shape of contracts used subject to some frictions, I show that the key sufficient condition for my results to be valid is that the shape of the contracts used does not change with the exemption level.

I show the applicability of the theoretical results by calibrating the formula for the optimal exemption to US household bankruptcy data and assessing the magnitude of the welfare gains generated by adjusting the bankruptcy exemption. Using a Kaldor/Hicks welfare criterion to account for household heterogeneity, the preferred calibration to chapter 7 US data implies that the optimal bankruptcy exemption should be in the range of $100,000$ dollars, an amount larger than the average exemption in the US (of approximately $70,000$ dollars). For the same calibration, implementing the optimal exemption achieves welfare gains on the order of approximately 0.03% of ex-ante consumption. This calibration exercise is not meant to settle the debate on the exemption level. Only further work on the measurement of the key required variables will help us refine the optimal prescription for the exemption level. This paper points out that better measurement of consumption and interest rate schedules will be needed.

After introducing the main results of the paper in the simplest possible model, I extensively study the most natural extensions to show the robustness of the main insights of the paper. First, allowing for additional margins of adjustment, like labor supply or unobservable effort choices, does not modify the optimal exemption formula as long as these choices enter separably into borrowers' utility function. Second, allowing for more general utility specifications for consumption — like Epstein-Zin preferences or state dependent utility — only affects the optimal exemption formula through changes in the borrowers' stochastic discount factor. Third, allowing for multiple traded contracts with arbitrary

[2] The right hand side of equation (1) contains endogenous variables. It uses a similar logic to classic characterizations of optimal taxes. For instance, the demand elasticities that appear in the optimal Ramsey commodity tax formulas — see Atkinson and Stiglitz (1980) — are endogenous to the level of taxes. The logic behind equation (1) is also similar to the one behind the CAPM, in which the beta of an asset (an endogenous object) becomes a sufficient statistic to determine expected returns, or consumption based asset pricing models, in which the consumption process, independently of how it is generated, is a sufficient statistic for pricing assets.

payoffs is straightforward: a leverage-weighted average of interest rate sensitivities captures then the marginal cost of increased leniency. Fourth, when borrowers are observably ex-ante heterogeneous but exemptions cannot be individual specific, the optimal exemption becomes a function of a weighted average across active borrowers of the marginal effects present in the baseline case. The same occurs when lenders pool borrowers who are ex-ante unobservably heterogeneous. Finally, I show in a dynamic context that the optimal exemption is given by an average across periods and states of benefits and costs of the bankruptcy exemption, weighted according to the borrowers' own stochastic discount factor. The specifics of borrowers' income processes, involving permanent and transitory shocks, health or family shocks, as well as life cycle considerations, which have been shown to be important to determine borrowers' behavior, are all captured by the identified sufficient statistics.[3]

Two robust insights that emerge from the extensions deserve to be emphasized. First, the precise determination of the region(s) in which borrowers decide to default does not modify the optimal exemption formula, because borrowers default optimally. Hence, evaluating the welfare implications of changing exemptions does not require to explicitly model the multiple factors that may influence default decisions.[4] Second, only measures of consumption of bankrupt borrowers are required to assess the welfare benefits of varying exemptions. Preference parameters, like risk aversion, determine how to translate measures of consumption into welfare. Importantly, the responses of labor supply and other endogenous choice variables are irrelevant once borrowers' consumption is known, as long as their utility of consumption is separable.

Related Literature

This paper directly contributes to the literature on general equilibrium with incomplete markets, which has studied the possibility of default in very general environments. Zame (1993) and Dubey, Geanakoplos and Shubik (2005) are the first to theoretically analyze the core tradeoff present in this paper. They show that allowing for default may be welfare improving in a model with incomplete markets, since it creates insurance opportunities by introducing new contingencies into contracts. These papers take default penalties as exogenous and do not characterize optimal penalties, which is the approach I adopt in this paper.[5] It is well known that default is only beneficial when markets are initially incomplete. Allowing for default when agents can write fully state contingent contracts, as in Kehoe and Levine (1993), Alvarez and Jermann (2000) or Chien and Lustig (2010), only restricts the contracting space, reducing welfare unequivocally. By adopting a classic general equilibrium approach, I circumvent the frictions studied in the literature on optimal contracting that studies borrower-lender relationships.[6]

[3] In the online appendix, I also allow for a) price-taking borrowers, b) exemptions contingent on aggregate shocks, and c) non-zero output gaps and aggregate demand effects.

[4] For instance, default is jointly determined with other endogenous choice variables. Similarly, forward looking borrowers internalize the option value of waiting for uncertainty to be realized before defaulting, which crucially depends on whether shocks are temporary or permanent. Both these considerations, which prevent a simple characterization of default regions, only affect the optimal exemption through the sufficient statistics.

[5] It is interesting that, despite acknowledging in section 7 of their paper that the optimal penalty associated with default is neither zero nor infinite, Dubey, Geanakoplos and Shubik (2005) do not characterize it formally.

[6] Limited commitment or enforcement, moral hazard, imperfect monitoring and secret cash-flow manipulation are examples of these frictions. Townsend (1979), Diamond (1984), Gale and Hellwig (1985), Bolton and Scharfstein (1990), Hart and Moore

This paper complements the well-developed quantitative literature on bankruptcy with unsecured credit. On the one hand, the papers by S. Chatterjee, D. Corbae, M. Nakajima and J.V. Ríos-Rull (2007) or Livshits, MacGee and Tertilt (2007), among several others, provide a careful quantitative structural analysis of unsecured credit and default from a macroeconomic perspective. Due to their rich general equilibrium features, these papers have relied on numerical methods to evaluate the welfare implications of varying exemption levels. This paper takes a different but complementary approach, which allows us to gain analytical insights into this question and to shift our attention to a small set of variables of interest. I refer the reader to Livshits (2014) for a recent survey of this literature, which has not reached a consensus on whether higher or lower leniency in bankruptcy improves welfare. On the other hand, the work by Gropp, Scholz and White (1997), Gross and Souleles (2002), Fay, Hurst and White (2002), Mahoney (2012), Iverson (2013), Dobbie and Song (2013), Severino, Brown and Coates (2013), and Albanesi and Nosal (2015), among others, uses microeconometric methodology to understand the implications of actual bankruptcy policies. See White (2007, 2011) for recent surveys of this body of work.

This paper presents a novel application of the sufficient statistic approach to the problem of bankruptcy and security design. In the spirit of Chetty (2009), this paper derives formulas for the welfare consequences of policies that are functions of high-level observables rather than deep primitives. The optimal policies characterized under this approach are robust to a broad range of environments. Some recent successful applications of this approach are Diamond (1998) and Saez (2001) — hence the title analogy — on income taxation, Shimer and Werning (2007) and Chetty (2008) on unemployment insurance and Arkolakis, Costinot and Rodríguez-Clare (2012) on the welfare gains from trade liberalizations. Within this methodological approach, Alvarez and Jermann (2004) is perhaps the closest paper, given the shared emphasis on pricing specific consumption claims to carry out welfare assessments.

Finally, a version of the environment used in this paper, has become the workhorse model to understand sovereign default. If the ex-post penalties imposed on sovereigns that default were enforceable, the results of this paper would also apply to the international context with minor modifications. Formally closer to this paper, Bolton and Jeanne (2007, 2009) use a two-period model of defaultable credit to study the optimal number of creditors and the ability to refinance. See Aguiar and Amador (2013) for a recent survey.

Outline Section 2 lays out the baseline model and characterizes the equilibrium. Section 3 develops the main results of this paper in the baseline model. Section 4 analyzes multiple extensions and section 5 calibrates the theoretical formulas to US data. Section 6 concludes. All proofs and derivations are in the appendix.

(1994, 1998); Hart (1995), Albuquerque and Hopenhayn (2004), Jappelli, Pagano and Bianco (2005), Rampini (2005), Bisin and Rampini (2006), Hopenhayn and Werning (2008), Grochulski (2010), Gale and Gottardi (2011) are examples of this vast literature. Closer in spirit to this paper, Bizer and DeMarzo (1999), Krasa and Villamil (2000), Bond and Krishnamurthy (2004), Kovrijnykh (2013), Koeppl, Monnet and Quintin (2014), and Drozd and Serrano-Padial (2014) treat the commitment to enforce contracts as choice variables in alternative environments.

2 Baseline model

For clarity, I first introduce the main results of the paper in the simplest possible framework: a two-period version of Eaton and Gersovitz (1981). Then, in section 4, I extend the results in many dimensions and show that the insights of the baseline model are robust and apply generally.

2.1 Environment

Time is discrete, there are two dates, denoted by $t = 0, 1$, and there is a unit measure of borrowers and a unit measure of lenders. Because borrowers are risk averse, the results of this paper are more applicable to household borrowing, rather than to corporate borrowing.[7]

Borrowers Borrowers are risk averse and maximize expected utility of consumption with a rate of time preference $\beta > 0$. Their flow utility $U(C)$ satisfies standard regularity conditions: $U'(\cdot) > 0$, $U''(\cdot) < 0$ and $\lim_{C\to 0} U'(C) = \infty$.

There is a single consumption good (dollar) in this economy, which serves as numeraire. Every borrower is endowed with y_0 units of the consumption good at $t = 0$. At $t = 1$, every borrower receives a stochastic endowment of y_1 units of the consumption good, whose distribution follows a cdf $F(\cdot)$ with support in $\left[\underline{y_1}, \overline{y_1}\right]$, where $\underline{y_1} \geq 0$ and $\overline{y_1}$ could be infinite. y_1 corresponds to net resources available to borrowers after accounting for any form of public social insurance. The realizations of y_1 are iid among borrowers. Therefore, under a law of large numbers, there is no aggregate risk in this economy. The environment and the realization of y_1 are common knowledge. In the baseline model, y_1 should be interpreted as the level of assets, not income. This distinction becomes clear in section 4.4.

Hence, borrowers maximize:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}\left[V(C_1)\right],$$

where $V(C_1) = \max_{\xi \in \{0,1\}} \left\{\xi U\left(C_1^{\mathcal{D}}\right) + (1-\xi) U\left(C_1^{\mathcal{N}}\right)\right\}$ and ξ is an indicator for default for every realization y_1. If a borrower decides to repay, $\xi = 0$, while if he decides to default, $\xi = 1$. $C_1^{\mathcal{D}}$ denotes consumption for a borrower who defaults and $C_1^{\mathcal{N}}$ denotes consumption for a borrower who repays.

Borrowers use a single noncontingent contract, i.e., a debt contract. That is, borrowers issue debt with face value B_0, due at $t = 1$, and receive from lenders $q_0(B_0, m) B_0$ units of the consumption good at $t = 0$. Hence, the gross interest rate faced by borrowers can be defined as $1 + r \equiv \frac{1}{q_0}$. When needed, I denote by $\tilde{r}$ the logarithmic interest rate, i.e., $\tilde{r} \equiv \log(1 + r)$. In the baseline model, borrowers take into account that the interest rate at which they are able to borrow depends on the amount of debt they take. Hence, the budget constraint at $t = 0$ for borrowers is:

$$C_0 = y_0 + q_0(B_0, m) B_0,$$

where the unit price of debt taken $q_0(B_0, m)$ is a function of B_0 and m, as described below.

[7]There is scope to adapt the results of this paper to the corporate context, in which firms — often assumed risk neutral — engage in risk management with imperfect instruments and become endogenously risk averse, along the lines of Froot, Scharfstein and Stein (1993) and Rampini and Viswanathan (2010).

At $t = 1$, once y_1 is realized, borrowers can repay the amount owed B_0 or default.[8] If they default, they consume $C_1^{\mathcal{D}} = \min\{y_1, m\}$, that is, they keep the bankruptcy exemption of m units of the consumption good, unless m is larger than y_1, in which case they only keep y_1 units. Any positive remainder $y_1 - m$ is seized from borrowers and transferred to lenders, although lenders only receive a fraction $\delta \in [0, 1)$ of the transferred resources. This loss captures the resource costs associated to the bankruptcy procedure. There is no ex-post renegotiation of the terms of the contract.

The exemption m takes a value in the interval $[\underline{m}, \overline{m}]$. Because the bankruptcy procedure cannot rely on external funds, $\overline{m} \leq \overline{y_1}$. To simplify the exposition, I further restrict m to be greater than the lowest realization of y_1, that is, $\underline{m} > \underline{y_1}$.[9] Therefore, for a given realization of y_1, the budget constraints at $t = 1$ when a borrower chooses to repay and when it chooses to default are, respectively:

$$C_1^{\mathcal{N}} = y_1 - B_0$$
$$C_1^{\mathcal{D}} = \min\{y_1, m\}$$

Borrowers' rate of time preference β, initial endowment y_0 and distribution of future endowments $F(\cdot)$ are such that borrowers borrow in equilibrium, that is, $B_0 > 0$. The appendix provides an exact sufficient condition.

Lenders Given the exemption level and default behavior of borrowers, lenders supply credit by offering a pricing schedule $q_0(B_0, m)$, which depends on the face value of the debt B_0 and the exemption level m. I focus on the case in which lenders are risk neutral, perfectly competitive and require a given rate of return $1 + r^*$, which can differ from the borrowers' rate of time preference β.[10] Under these assumptions, $q_0(B_0, m)$ takes the form:

$$q_0(B_0, m) = \frac{\delta \int_{\mathcal{D}} \frac{\max\{y_1 - m, 0\}}{B_0} dF(y_1) + \int_{\mathcal{N}} dF(y_1)}{1 + r^*},$$

where $\mathcal{D}$ represents the default region and $\mathcal{N}$ the no default region, which are determined in equilibrium. δ denotes the proportional deadweight loss associated with transferring resources in bankruptcy. As long as lenders make zero profit and $q_0(B_0, m)$ is well-behaved, the insights of this paper are independent of the particular assumptions made regarding the behavior of lenders. For instance, the online appendix shows how to introduce risk averse competitive lenders.

Equilibrium definition and regularity conditions An equilibrium, for a given level of exemption m, is defined as a set of consumption allocations C_0, $\{C_1\}_{y_1}$, default decisions $\{\xi\}_{y_1}$, amount of debt issued B_0, and price q_0 such that borrowers default and borrow optimally internalizing that their choices affect the price of the debt and lenders offer a pricing schedule $q_0(B_0, m)$ while making zero profit.

As usual in problems with continuous distributions of shocks, convexity is in general not guaranteed; see the discussion in Ljungqvist and Sargent (2004). I work under the assumption that the borrowers'

[8]Bankruptcy and default are synonyms in this paper. See White (2011) and Herkenhoff (2012) for how borrowers may default on their obligations without entering in bankruptcy. Allowing borrowers not to repay without declaring bankruptcy does not change the optimal exemption formula, as long as they behave optimally. See the discussion in section 4.7.

[9]The results extend naturally to the case in which $m < \underline{y_1}$, but the analysis becomes more tedious, since multiple cases have to be analyzed then. Those results are available under request.

[10]There is scope to model a richer lending side, potentially including financial intermediaries with funding frictions..

problem is well-behaved, so first-order conditions are necessary and sufficient to characterize the optimum. I further assume that borrowers indirect utility $W(m)$ — defined in equation (7) below — is also well-behaved in m. The appendix provides sufficient conditions for convexity and the online appendix shows numerically that the model is in practice well-behaved for a reasonable set of primitives.

I would like to make a final observation before characterizing the equilibrium. Because the first-best outcome in the baseline model involves risk neutral lenders providing a flat consumption profile (full insurance) to risk averse borrowers at $t = 1$, an exemption level that does not depend on y_1 is optimal in the baseline model, even when a nonlinear bankruptcy scheme that depends on y_1 is feasible. A constant exemption level is not necessarily optimal in some of the extensions: I'll point that out whenever that is the case. In those situations, this paper solves a second best problem with imperfect instruments. In those cases, the choice of a constant exemption is justified by the fact that it is the one used in practice. Allowing for additional policy instruments could further improve welfare, but it does not change the characterization of optimal exemptions developed in this paper.

2.2 Equilibrium characterization

First, I characterize the optimal ex-post default decision by borrowers. Then, I characterize the equilibrium pricing schedules offered by competitive lenders and, finally, I solve for the optimal ex-ante choice of B_0.

Borrowers' default decision At $t = 1$, given his ex-ante choice of B_0, a borrower solves the problem:

$$\max_{\xi \in \{0,1\}} \left\{ \xi U\left(C_1^{\mathcal{D}}\right) + (1-\xi)\, U\left(C_1^{\mathcal{N}}\right) \right\}$$

Because flow utility is strictly monotonic, this problem is equivalent to $\max_{\{\xi\}} \left\{ C_1^{\mathcal{D}}, C_1^{\mathcal{N}} \right\}$. The optimal default decision is given by a threshold on the realization of y_1. When y_1 is high, it is optimal not to default, but when y_1 is sufficiently low, it is preferable to default than to repay the loan. Figure 1 shows graphically the default problem at $t = 1$. The upper envelope of the default and repayment options determines the optimal consumption choice given B_0. The 45 degree line is shown for reference.

Formally, the optimal default decision is:

$$\xi = \begin{cases} 1, & \text{if } y_1 < m + B_0 \qquad \text{Default} \\ 0, & \text{if } y_1 \geq m + B_0 \qquad \text{No Default} \end{cases}$$

The default threshold is determined by the indifference condition between the amount to be repaid B_0 and the amount transferred to lenders $y_1 - m$. I assume that indifferent borrowers decide not to default. Given the default decision, the fraction of borrowers that defaults in equilibrium is deterministic and given by $F(m + B_0)$.

This model incorporates forced default, which occurs when a borrower does not have enough resources to fully pay back its debt (it occurs when $B_0 > y_1$), and strategic default, which happens when borrowers have enough resources to fully repay but they decide not to do it (it occurs when $m + B_0 > y_1 > B_0$). This distinction, which often plays a prominent role in discussions about bankruptcy exemptions, is not relevant for the results of this paper.

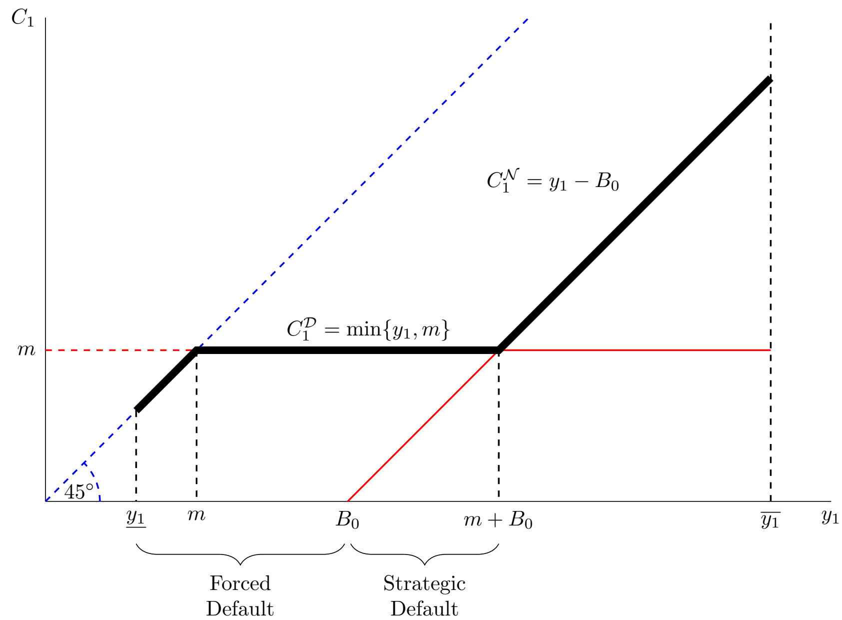

Figure 1: Optimal default decision given B_0

Lenders' pricing/interest rate schedule When lenders are risk neutral and perfectly competitive, given borrowers' default decision, they offer the following pricing schedule for given levels of B_0 and m:

$$q_0\left(B_0, m\right) = \frac{\delta \int_{m}^{m+B_0} \frac{y_1 - m}{B_0} dF\left(y_1\right) + \int_{m+B_0}^{\overline{y_1}} dF\left(y_1\right)}{1 + r^*} \tag{2}$$

Figure 2 graphically shows the repayment to lenders. The upper envelope between $\max\{y_1 - m, 0\}$ and B_0 represents the effective repayment to lenders. The credit spread is positive, that is $r - r^* > 0$, to account for the possibility of default, and approximately equal to the expected unit loss for lenders, because of risk neutrality.

Two properties of the pricing schedule are important for the analysis. First, the pricing schedule decreases (interest rates increase) with the level of debt B_0. Second, the pricing schedule decreases (interest rates increase) with the level of the bankruptcy exemption. Formally (the exact expressions are in the appendix):

$$\frac{\partial q_0\left(B_0, m\right)}{\partial B_0} < 0 \qquad \text{and} \qquad \frac{\partial q_0\left(B_0, m\right)}{\partial m} < 0 \tag{3}$$

For a given level of m, the required interest rate spread increases with the amount of credit issued. This occurs for two reasons. First, the per unit fraction of liabilities recovered by lenders in default states decreases with the total amount of credit. Second, because the default region widens, more resources are lost as bankruptcy costs. Also, for a given level of B_0, the required interest rate spread increases with the level of the bankruptcy exemption. There are again two reasons for this. First, the recovery rate for lenders in default states decreases with the level of the exemption, since borrowers get to keep a higher exemption. Second, because the default region widens, more resources are lost as bankruptcy costs.

Borrowers' optimal choice of B_0 Given the optimal ex-post default decision and taking into account the debt pricing schedule offered by lenders, borrowers optimally choose how much to borrow. The

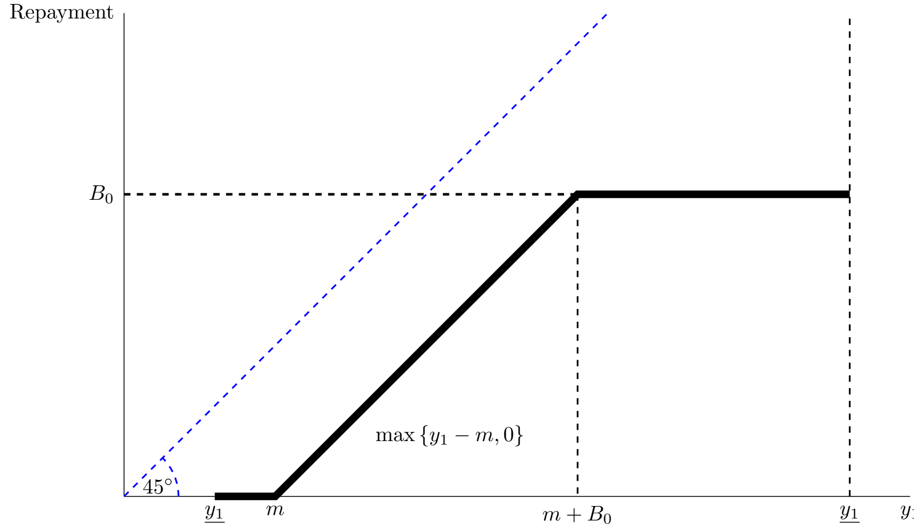

Figure 2: Repayment to lenders given B_0

problem solved by borrowers at $t = 0$, for a given exemption m, is:

$$\max_{B_0} U\left(y_0 + q_0\left(B_0, m\right) B_0\right) + \beta \left[\int_{\underline{y_1}}^{m} U\left(y_1\right) dF\left(y_1\right) + \int_{m}^{m+B_0} U\left(m\right) dF\left(y_1\right) + \int_{m+B_0}^{\overline{y_1}} U\left(y_1 - B_0\right) dF\left(y_1\right)\right] \tag{4}$$

Under the assumed regularity conditions, the following first-order condition fully characterizes the solution to (4):

$$U'\left(C_0\right)\left[q_0\left(B_0, m\right) + \frac{\partial q_0\left(B_0, m\right)}{\partial B_0} B_0\right] = \beta \int_{m+B_0}^{\overline{y_1}} U'\left(y_1 - B_0\right) dF\left(y_1\right) \tag{5}$$

The left hand side of (5) represents the marginal benefit at $t = 0$ of increasing the face value of the debt by a dollar. This marginal benefit is given by q_0, the amount raised at $t = 0$ per dollar promised at $t = 1$, corrected by how the induced interest rate increase affects the total amount borrowed B_0. Borrowers value this change at their marginal utility $U'\left(C_0\right)$. The right hand side of (5) represents the marginal cost of repaying the debt, given by the marginal utility when the payment is due. This cost is only paid in states in which borrowers do not default; debt imposes no effective costs on borrowers in those states in which it does not have to be repaid.

Equation (5) allows us to characterize analytically how the total amount of credit changes in equilibrium with the level of the bankruptcy exemption m. Unsurprisingly, the sign of $\frac{dB_0}{dm}$ is ambiguous. Formally, $\frac{dB_0}{dm}$ has the following sign:

$$\operatorname{sign}\left(\frac{dB_0}{dm}\right) = \operatorname{sign}\left(\underbrace{U''\left(C_0\right)\frac{\partial q_0}{\partial m} B_0\left[q_0 + \frac{\partial q_0}{\partial B_0} B_0\right]}_{\text{Income effect}} + \underbrace{U'\left(C_0\right)\left[\frac{\partial q_0}{\partial m} + \frac{\partial^2 q_0}{\partial B_0 \partial m} B_0\right]}_{\text{Substitution effect}} + \underbrace{\beta U'\left(m\right) f\left(m + B_0\right)}_{\text{Direct effect}}\right)$$

Three distinct effects determine the sign of $\frac{dB_0}{dm}$. First, all else equal, an increase in m reduces q_0, which reduces borrowers' consumption C_0 and increases $t = 0$ marginal utility $U'\left(C_0\right)$; this income effect induces borrowers to increase B_0. Second, all else equal, an increase in m varies the unit amount that can be raised at $t = 0$, given by the direct price effect $\frac{\partial q_0}{\partial m}$ and the change in the derivative of the pricing schedule $\frac{\partial^2 q_0}{\partial B_0 \partial m} B_0$; this substitution effect is in general ambiguous, although the direct price effect induces borrowers to decrease B_0. Third, all else equal, an increase in m reduces the marginal cost of borrowing because the default region widens, which reduces the likelihood of having to pay back the debt. This direct effect induces borrowers to increase B_0. Through this direct effect, borrowers decide to borrow more ex-ante anticipating not having to pay back their debts: this effect is often described as moral hazard.

The empirical literature on this issue finds that high exemptions are often associated with high levels of borrowing. Numerical solutions of the model with standard parametrizations find that B_0 can increase or decrease with m. That said, I show that it is not necessary to take a stance on whether borrowers borrow more or less when exemptions change to understand the effects on welfare of varying bankruptcy exemptions, as long as borrowers choose how much to borrow optimally.

Finally, we can formally express the equilibrium changes in interest rates induced by changing the bankruptcy exemption in the following way:

$$\frac{dq_0(B_0, m)}{dm} = \frac{\partial q_0(B_0, m)}{\partial B_0} \frac{dB_0}{dm} + \frac{\partial q_0(B_0, m)}{\partial m} \tag{6}$$

The last term in equation (6) is negative, but depending on the sign of $\frac{dB_0}{dm}$, $\frac{dq_0(B_0,m)}{dm}$ can take any sign. As long as borrowing increases with m, that is, $\frac{dB_0}{dm} > 0$, observed interest rates increase in equilibrium, that is, $\frac{dq_0(B_0,m)}{dm} < 0$.

3 Welfare and optimal bankruptcy exemptions

After characterizing the equilibrium for a given exemption m, I now study how welfare varies with the level of m and how to determine the welfare maximizing exemption m^*. Because lenders make zero profit in equilibrium, maximizing borrowers' indirect utility is equivalent to maximizing social welfare in this economy.[11]

I denote the indirect utility of borrowers, as a function of m, by $W(m)$:

$$W(m) = \begin{array}{c} U\left(y_0 + q_0\left(B_0(m), m\right) B_0(m)\right) + \\ +\beta\left[\int_{\underline{y_1}}^{m} U(y_1)\, dF(y_1) + \int_{m}^{m+B_0(m)} U(m)\, dF(y_1) + \int_{m+B_0(m)}^{\overline{y_1}} U(y_1 - B_0(m))\, dF(y_1)\right], \end{array} \tag{7}$$

where $B_0(m)$ is given by the solution to equation (5) and $q_0(B_0(m), m)$ is given by:

$$q_0(B_0(m), m) = \frac{\delta \int_{m}^{m+B_0(m)} \frac{y_1 - m}{B_0(m)} dF(y_1) + \int_{m+B_0(m)}^{\overline{y_1}} dF(y_1)}{1 + r^*}$$

Propositions 1 and 2 present the main results of this paper.

Proposition 1. (Marginal effect of varying m on welfare: a test for whether to increase or decrease m)

a) The change in welfare induced by a marginal change in the bankruptcy exemption m is given by:

$$\frac{dW}{dm} = U'(C_0) \frac{\partial q_0(B_0(m), m)}{\partial m} B_0 + \int_{m}^{m+B_0} \beta U'\left(C_1^{\mathcal{D}}\right) dF(y_1) \tag{8}$$

[11] This logic is analogous to assuming that the production sector faces constant returns to scale in optimal taxation problems.

b) The change in welfare induced by a marginal change in the bankruptcy exemption m, expressed as a fraction of $t = 0$ consumption, is given by:

$$\frac{\frac{dW}{dm}}{U'(C_0)\,C_0} = -\Lambda \varepsilon_{\tilde{r},m} + \frac{1}{m}\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}, \tag{9}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$ is a measure of borrowers' leverage, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = -\frac{\frac{\partial q_0(B_0(m),m)}{q_0}}{\partial m}$ denotes the semi-elasticity of the interest rate schedule offered by lenders with respect to the level of the exemption, and $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \int_m^{m+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'(C_1^{\mathcal{D}})}{U'(C_0)} dF(y_1)$ is the price-consumption ratio from the borrowers' perspective of a claim that pays the marginal value of increasing the bankruptcy exemption by one unit.

Proposition 1 characterizes the effect on social welfare of a marginal change in the bankruptcy exemption. The derivation of equation (8) crucially exploits the fact that borrowers borrow and decide when to default optimally. On the one hand, a marginal increase in the exemption m makes borrowing more expensive through a reduction in the price on the debt issued $\frac{\partial q_0}{\partial m}$, which affects the total amount of debt outstanding B_0. This change is valued by borrowers according to their $t = 0$ marginal utility $U'(C_0)$. This increase in borrowing costs is the marginal cost of a more lenient bankruptcy procedure. On the other hand, a marginal increase in the exemption m increases the resources that borrowers can keep when they default while claiming the full exemption. Averaging over the pertinent realizations of y_1 and weighting this gain by the marginal utility $\beta U'\left(C_1^{\mathcal{D}}\right)$ in those states, the marginal welfare gain of a more lenient bankruptcy procedure becomes $\beta \int_m^{m+B_0} U'\left(C_1^{\mathcal{D}}\right) dF(y_1)$.

Equation (9) expresses the change in welfare as a money-metric — dividing by $U'(C_0)$ — before normalizing by initial consumption C_0. It provides a simple test for whether it is optimal to increase or decrease the bankruptcy exemption, starting from its current level.

The term Λ measures borrowers' leverage.[12] The term $\varepsilon_{\tilde{r},m}$ denotes the partial derivative of the interest rate schedule with respect to the bankruptcy exemption. Equation (3) guarantees that $\varepsilon_{\tilde{r},m}$ is strictly positive. $\Pi_m\left\{C_1^{\mathcal{D}}\right\}$ is the price from the borrower's perspective at $t = 0$ of a claim that pays borrowers' consumption only in default states in which borrowers claim the full exemption — those in which $y_1 > m$. $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$ express this price in relative terms to current consumption C_0: this is a measure of the marginal benefit for borrowers of increased leniency. The ratio $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$, which I refer to as the "price-consumption" ratio, is determined by the product of two terms. First, it can be high when the ratio of marginal utilities $\frac{U'(C_1^{\mathcal{D}})}{U'(C_0)}$ is high. Second, it can be high when consumption growth $\frac{C_1^{\mathcal{D}}}{C_0}$ in those states is also high. Both terms are in general related, and tightly linked when utility is CRRA or Epstein-Zin cases, as discussed below.

The optimal bankruptcy exemption m^* can be found as:

$$m^* = \arg\max_m W(m)$$

Under the assumed regularity conditions, the optimal bankruptcy exemption must be a solution to the equation $\frac{dW}{dm} = 0$.

[12] Debt-to-equity ratios, that is, $L \equiv \frac{q_0 B_0}{y_0}$ are the most frequently used measures of leverage. The variable Λ is a monotonic transformation of L: $\Lambda = \frac{1}{\frac{1}{L}+1}$.

Proposition 2. (Optimal bankruptcy exemption) *The optimal exemption m^* — expressed in units of the consumption good, i.e., dollars — is characterized by:*

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}}, \tag{10}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$ is a measure of borrowers' leverage, $\varepsilon_{\tilde{r},m} \equiv \frac{\partial \log(1+r)}{\partial m} = \frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$ denotes the semi-elasticity of the interest rate schedule offered by lenders with respect to the level of the exemption, and $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \int_m^{m+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF\left(y_1\right)$ is the price-consumption ratio from the borrowers' perspective of a claim that pays the marginal value of increasing the exemption by a unit.

The expression for m^* optimally trades off the marginal benefit of increasing consumption in default states (numerator) against the marginal cost of restricting access to credit (denominator). When m^* is high, borrowers face higher interest rates, which makes borrowing less profitable, at the cost of improved insurance when declaring bankruptcy. A low m^* makes ex-ante borrowing less costly while making bankruptcy more painful.

A low value for m^* is optimal when Λ and $\varepsilon_{\tilde{r},m}$ are large. Intuitively, if interest rates schedules are very sensitive to increasing the bankruptcy exemption, making default more attractive by increasing m^* is very costly in terms of curtailed access to credit; this effect is amplified when the amount borrowed Λ is high. A high value for m^* is optimal when $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$, the normalized welfare gain of a marginally higher exemption is large. Although equation (10) must hold at the optimum, it does not provide a characterization of m^* as a function of primitives, because all right hand side variables are endogenous.

CRRA utility To build further intuition, assume that borrowers have constant relative risk aversion utility (CRRA) preferences, that is, $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$, where $\gamma \equiv -C\frac{U''(C)}{U'(C)}$. Assuming a particular utility specification only affects directly the price-consumption ratio $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$. The marginal cost of increased leniency $\Lambda\varepsilon_{\tilde{r},m}$ does not depend directly on the utility function, only through the effects on B_0 and q_0.

We can thus write:

$$\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF\left(y_1\right)$$

Therefore, an increase in leniency m increases $C_1^{\mathcal{D}}$ and has both discount rate and cash flow effects. When the CRRA coefficient γ is greater than one, the discount rate effect $\left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{-\gamma}$ dominates — this is the standard parametrization, see Campbell (2003) — but, if γ is less than one the consumption growth term $\frac{C_1^{\mathcal{D}}}{C_0}$ dominates. With logarithmic utility ($\gamma = 1$) both effects exactly cancel out. We expect $t = 0$ consumption to be higher than the exemption level, that is, $\frac{C_1^{\mathcal{D}}}{C_0} < 1$, so the marginal loss generated by a higher exemption is increasing in the risk aversion parameter γ — see the appendix for a formal argument.

The forces that determine the price-consumption ratio are the same that determine the price-dividend (consumption-wealth) ratio in standard consumption based asset pricing models — see Campbell (2003) for a review. In the classic Lucas (1978) model, price-dividend ratios are constant and equal to the rate of time preference β for investors with logarithmic utility. That same result applies here with one modification: instead of β, the relevant price-consumption ratio becomes $\beta\pi_m$, because we are interested in the price-dividend ratio of a security that only pays in default states with positive recovery by lenders.

Logarithmic utility is an often used benchmark specification for preferences, the price-consumption ratio can be written as:

$$\frac{\Pi_m \left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta\pi_m,$$

where $\pi_m \equiv \int_m^{m+B_0} dF(y_1)$ is the unconditional probability that a borrower consumes the bankruptcy exemption. It can alternatively be written as: $\pi_m = \pi_D \cdot \pi_{m|D}$, the unconditional probability of default π_D times the conditional probability that a borrower claims the full exemption $\pi_{m|D}$. Hence, when borrowers have log utility and lenders always claim the full exemption, only the probability of default is needed to assess the marginal benefit of increasing the bankruptcy exemption. When borrowers' risk aversion is larger than unity and $m < C_0$, the logarithmic utility formula provides a lower bound for the optimal exemption that does not require to measure consumption.

Hence, when borrowers have logarithmic utility, the optimal bankruptcy exemption m^* can be written as:

$$m^* = \frac{\beta\pi_m}{\Lambda\varepsilon_{\tilde{r},m}}$$

In this baseline model with risk neutral lenders, we can further rewrite the equation that characterizes the optimal exemption as:

$$m^* = \left(\frac{\beta(1+r^*)}{\Omega + \delta(1-\Omega)}\right)^{\frac{1}{\gamma}} C_0, \tag{11}$$

where $\Omega = \frac{f(m^*+B_0)B_0}{F(m^*+B_0)-F(m^*)}$ is a measure of curvature of the distribution $F(\cdot)$ that can take values in $[0,\infty]$. When $F(\cdot)$ is a uniform, $\Omega = 1$. Intuitively, the optimal exemption seeks to equalize consumption at $t = 1$ in default states with consumption at $t = 0$, although with a correction that depends on the term $\frac{\beta(1+r^*)}{\delta+(1-\delta)\Omega}$. All else constant, when $\beta(1+r^*) > 1$, it is optimal to consume more at period 1, which calls for lower exemptions. Similarly, when $\delta + (1-\delta)\Omega$, which measures $\frac{\partial \log(1+r)}{\partial m}$, is low, high exemptions are optimal. All else constant, there is no clear comparative static on the bankruptcy cost parameter δ. Two effects compete. On the one hand, a high δ (low bankruptcy costs) amplifies the sensitivity of interest rate schedules to exemption changes because more resources flow from borrowers to lenders in bankruptcy. On the other hand, a high δ dampens the sensitivity of interest rate schedules to exemption changes, because there is no loss associated to defaulting in more states. All these effects are modulated by borrowers' risk aversion: high risk aversion γ pushes towards $\frac{m^*}{C_0} \to 1$. It should not be surprising that a complete closed-form solution cannot be found, even in this simple (but nonlinear) model. For instance, the classic optimal Ramsey commodity taxation result depends on demand elasticities, which in general are endogenous to the level of taxes.

Although equation (11) can be easily solved numerically and it is helpful to provide intuition behind the sufficient statistics, it is only valid in this stylized model. Instead, I focus throughout this paper on equations like (9) and (10), which apply broadly, as shown in section 4.

I conclude this section with four remarks.

Remark. (Sufficient statistics) Three observable variables: leverage, the sensitivity of interest rate schedules with respect to the exemption level and the price-consumption ratio, suffice to determine the optimal exemption, independently of the rest of the structure of the model. For standard preferences, the price-consumption ratio depends on the probability of default and the consumption of borrowers. For instance, the distribution of endowment shocks or the level of interest rates only affect m^* through these sufficient statistics. The logic behind these sufficient statistics is similar to the one behind the CAPM, in which the beta of an asset becomes sufficient to determine expected returns. It is also similar to the logic behind consumption based asset pricing models, in which the consumption process, independently of how it is generated, is sufficient to determine asset prices and expected returns.

Remark. (Applicability of $\frac{dW}{dm}$ versus m^*) Although I emphasize the characterization of m^* in the introduction, because it is a more salient result, the most robust practical insight in this paper comes from the characterization of $\frac{dW}{dm}$, in particular, equation (9). Equation (9) provides a sharp test based on observables for whether it is optimal to increase or to decrease the exemption level, given its current value. This test only uses local information and does not involve any fixed point. By repeatedly applying such test, under appropriate regularity conditions, a policymaker would eventually find the optimal level of the bankruptcy exemption. Following this logic, I carefully qualify the validity of the calibration exercise in my final remarks in section 5.

Remark. (Endogenous contract choice) This paper solves the problem that maximizes the ex-ante welfare of borrowers under commitment, given a set of contracts, which are taken as observable. One interpretation of the results is that agents do not choose the shape of contracts, perhaps because of hysteresis in contract choice. Alternatively, Allen and Gale (1994) show that, if there are fixed costs associated with trading contracts, only a finite number of contracts will be used in equilibrium. Moreover, I show in detail in the appendix and further discuss in section 4.4 that the main characterization of this paper remains valid even if contracts are endogenously chosen, as long as the shape the contracts chosen does not vary with the level of m. In practice, the fact that we observe the same set of contracts (e.g., debt contracts) being traded in economies with high and low exemption levels suggests that that sufficient condition holds in modern economies.

Remark. (Security design interpretation) The key tradeoff in this paper can be interpreted as a security design problem. Assume that borrowers can choose between two fairly priced securities. They can issue either a noncontingent bond, or a bundle of a noncontingent bond with a put option (whose strike is determined by the optimal default decision). Given their endowment process, borrowers will prefer one of these two securities; the choice of m adjusts parametrically between both contracts and m^* selects the optimal traded security. This approach is related to the work in security design motivated by risk sharing, as in Allen and Gale (1994) and Duffie and Rahi (1995). This literature starts by allowing for some form of market incompleteness and then asks the question of which securities should be introduced in the market to improve risk sharing and welfare. The optimal bankruptcy exemption in this paper solves a specific optimal security design problem.

4 Extensions

This section shows that the results derived in the baseline model are robust to many generalizations. I exhaustively address the most natural extensions and relegate several others to the online appendix. To ease the exposition and for brevity, I analyze every extension separately and omit equilibrium definitions and regularity conditions.

A reader more interested in the practical applicability of the results can jump directly to section 5, keeping in mind that many claims made there follow formally from the extensions studied here.

4.1 Endogenous income: elastic labor supply and effort choice in frictionless markets

In the baseline model, borrowers' income is exogenously determined. However, it is often argued that creditor friendly bankruptcy procedures reduce borrowers' welfare by distorting labor supply decisions: seizing borrowers' labor income can have a negative effect on labor supply. Similarly, large exemptions may distort ex-ante effort choices by borrowers. However, the optimal exemption formula does not change when effort and labor supply are affected by changes in exemptions, as long as those decisions are made optimally.

To capture these concerns, I modify the baseline model in two dimensions. First, I allow borrowers to choose labor supply in both periods. Second, I assume that the distribution of income $F(Y_1; a)$ is a differentiable function of a noncontractible effort choice a, made by borrowers at $t = 0$. This effort choice creates another form of moral hazard. Borrowers can work at given wages w_0 and w_1 — in the background, perfectly competitive/constant returns to scale firms provide labor demand curves. Borrowers' flow utility is now given by a well-behaved function $U(C, N; a)$, where N denotes hours worked and a is the quantity of effort exerted. For now, I make no assumptions on the separability between consumption, leisure and effort. For simplicity, I abstract from limits on wage garnishments and assume that all labor income is transferred from borrowers to lenders in bankruptcy.

Borrowers now solve:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}, N_0, \{N_1\}_{y_1}, a} U(C_0, N_0; a) + \beta \mathbb{E}_a \left[V(C_1, N_1) \right]$$

$$\text{s.t.} \qquad C_0 = y_0 + w_0 N_0 + q_0 B_0; \quad C_1^{\mathcal{N}} = y_1 + w_1 N_1^{\mathcal{N}} - B_0; \quad C_1^{\mathcal{D}} = \min\{y_1, m\},$$

where $V(C_1, N_1) = \max_{\xi \in \{0,1\}} \left\{ \xi \max_{C_1^{\mathcal{D}}, N_1^{\mathcal{D}}} U\left(C_1^{\mathcal{D}}, N_1^{\mathcal{D}}\right) + (1 - \xi) \max_{C_1^{\mathcal{N}}, N_1^{\mathcal{N}}} U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) \right\}$. As in the baseline model, there are three regions depending on the realization of y_1. First, the no default region, denoted by $\mathcal{N}$. Second, the default region in which borrowers keep the full exemption, denoted by $\mathcal{D}_m$. Third, the default region in which borrowers do not exhaust the full exemption, denoted by $\mathcal{D}_y$. Borrowers decide not to work at all when bankrupt, because all their labor income can be garnished — a cap to wage garnishments would only reduce their labor supply partially, but the conclusions would remain unchanged.

The logic used to characterize the default region is identical to the baseline model. First, I define the $t = 1$ indirect utility of borrowers after choosing optimally consumption and labor supply as $\tilde{V}(y_1; B_0, w_1)$, that is:

$$\tilde{V}(y_1; B_0, w_1) \equiv \max U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) \text{ s.t. } \quad C_1^{\mathcal{N}} = y_1 + w_1 N_1^{\mathcal{N}} - B_0$$

Given y_1, a static consumption-leisure choice characterizes borrowers' labor supply, that is:

$$w_1 \frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) = -\frac{\partial U}{\partial N}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)$$

Second, the default region is characterized by a threshold $\tilde{y}_1$ such that:

$$U(m) = \tilde{V}(\tilde{y}_1; B_0, w_1)$$

When $y_1 \geq \tilde{y}_1$, it is optimal for a borrower to repay, but when $y_1 < \tilde{y}_1$, it is optimal to default — see figure A.1 in the appendix for a graphical representation. In this case, lenders pricing schedules also depend directly on borrowers effect choice a, that is, we have $q_0(B_0, m, a)$. Given the optimal default decision, borrowers' behavior is characterized by three additional optimality conditions. First, an Euler equation for borrowing:

$$\frac{\partial U}{\partial C}(C_0, N_0; a)\left[q_0 + \frac{\partial q_0(B_0, m, a)}{\partial B_0}B_0\right] = \beta \int_{\mathcal{N}} \frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) dF(y_1; a)$$

Second, an optimal effort choice:

$$\frac{\partial U}{\partial a}(C_0, N_0; a) + \frac{\partial U}{\partial C}(C_0, N_0; a)\frac{\partial q_0(B_0, m, a)}{\partial a}B_0 + \beta \int V(C_1, N_1)\frac{\partial f(y_1; a)}{\partial a} dy_1 = 0$$

Third, an optimal consumption-leisure choice at $t = 0$:

$$w_0 \frac{\partial U}{\partial C}(C_0, N_0; a) = -\frac{\partial U}{\partial N}(C_0, N_0; a)$$

From the optimality conditions, it is easy to show that changes in the exemption m modify both borrower's consumption, labor supply, and effort choices. In particular, when m increases, borrowers' labor supply is lower because the default region, in which no labor is supplied, grows.

Proposition 3. (Endogenous income: frictionless markets)

a) *The marginal welfare change from varying the optimal exemption m when labor supply is endogenous and borrowers have an effort choice is given by:*

$$\frac{dW}{dm} = \frac{\partial U}{\partial C_0}(C_0, N_0; a)\frac{\partial q_0(B_0, m, a)}{\partial m}B_0 + \beta \int_{D_m} \frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right) dF(y_1; a) \tag{12}$$

b) The optimal exemption m^* when labor supply is endogenous and borrowers have an effort choice is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{13}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv -\frac{\frac{\partial q_0(B_0,m,a)}{q_0}}{\partial m}$, *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right)}{\frac{\partial U}{\partial C_0}(C_0, N_0; a)} dF(y_1; a)\, dF(y_1)$.

The intuition behind proposition (13) is simple and powerful: as long as borrowers optimally choose their labor supply and effort, changes in these variables do not modify the optimal exemption formula — the same argument applies to any other static endogenous variable. If the utility of consumption is separable from the disutility of working and providing effort, equations (12) and (13) are identical to their counterparts in the baseline model. In general, the marginal benefit of the bankruptcy exemption depends on the values of N and a through its effect on the marginal utility of consumption. Although there are is a rich literature studying the separability properties of the utility function, e.g., Attanasio and Weber (1989) or Aguiar and Hurst (2007), separable utility of consumption is often seen as a reasonable benchmark.

This is an important takeaway of this paper: we only need to measure consumption to determine the welfare consequences of bankruptcy policies. Labor supply responses to changes in exemptions are irrelevant once the consumption response has been accounted for.[13]

4.2 Non-pecuniary utility loss

In the baseline model, borrowers ruthlessly default whenever the pecuniary benefits of doing so are greater than the pecuniary cost. Perhaps because of stigma, social pressure or to avoid being hounded by lenders, it is often argued that borrowers may experience a non-pecuniary loss if they do not pay back their liabilities. In particular, White (1998) and Fay, Hurst and White (2002), among others, calculate that the fraction of households that would benefit from bankruptcy, based on pecuniary considerations, far exceeds the fraction of households who actually file for bankruptcy. I now capture that possibility by allowing borrowers to have state dependent utility. In particular, I assume that the utility of a borrower who consumes C units of the consumption good in bankruptcy is given by $U(\phi C)$, where $\phi \in [0,1)$. As in the baseline mode, renegotiation remains unfeasible.

Borrowers now solve:

$$\max_{C_0, \{C_1\}_{y_1}, B_0, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}\left[V(C_1)\right],$$

where $V(C_1) = \max_{\xi \in \{0,1\}} \left\{ \xi U\left(\phi C_1^{\mathcal{D}}\right) + (1-\xi) U\left(C_1^{\mathcal{N}}\right) \right\}$. The logic used to characterize the default region is identical to the baseline model. The optimal default decision is given by:

$$\begin{cases} \xi = 1, & \text{if } y_1 < \phi m + B_0 \qquad \text{Default} \\ \xi = 0, & \text{if } y_1 \geq \phi m + B_0 \qquad \text{No Default} \end{cases}$$

With the exception of the change in the default region, the expression that characterizes B_0 is analogous to the one in the baseline model, that is:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0\right] = \beta \int_{\phi m + B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1)$$

Proposition 4. (Non-pecuniary utility loss)

a) *The marginal welfare change from varying the optimal exemption m when bankrupt borrowers experience a non-pecuniary utility loss is given by:*

$$\frac{dW}{dm} = U'(C_0) \frac{\partial q_0(B_0(m), m)}{\partial m} B_0 + \beta \int_{m}^{\phi m + B_0} \phi U'\left(\phi C_1^{\mathcal{D}}\right) dF(y_1)$$

b) The optimal exemption m^* when bankrupt borrowers experience a non-pecuniary utility loss is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}},$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv -\frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$, *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_m^{\phi m + B_0} \frac{\phi C_1^{\mathcal{D}}}{C_0} \frac{U'\left(\phi C_1^{\mathcal{D}}\right)}{U'(C_0)} dF(y_1)$.

[13] Similar insights are widely used in the consumption based asset pricing literature. Marginal utility of consumption is the only information needed to price any asset when utility is separable.

As expected, for the standard case of $\gamma > 1$, low values of ϕ push towards higher optimal exemptions because, all else constant, borrowers' marginal utility is higher when bankrupt. The presence of non-pecuniary costs of default modifies the default region and the expression for the price-consumption ratio, but the formula for the optimal bankruptcy exemption remains unchanged. This conclusion extends to any form of state dependent utility. State dependent utility only modifies m^* through the sufficient statistics identified in this paper.

4.3 Epstein-Zin utility

The results of the baseline model remain valid when borrowers do not have expected utility. I analyze here the Epstein-Zin case to disentangle the effects of risk aversion versus intertemporal substitution. While risk aversion plays an important role in pricing the marginal benefit of a bankruptcy exemption increase, intertemporal substitution plays an important role shaping the sensitivity of credit demand to interest rates. All results can be easily extended to more general Kreps-Porteus preferences or other types of well-behaved nonexpected utility preferences.

Borrowers' utility is now given by:

$$V_0 = \left[\left(1 - \hat{\beta}\right) C_0^{1-\frac{1}{\psi}} + \hat{\beta} \left(\mathbb{E}\left[C_1^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}},$$

where the parameter γ is the coefficient of relative risk aversion and ψ represents the elasticity of intertemporal substitution for a given nonstochastic consumption path. Given B_0, borrowers default decision is identical to the baseline model. Taking that into account, borrowers now solve:

$$\max_{B_0} \left[\begin{array}{c} \left(1 - \hat{\beta}\right) \left(y_0 + q_0 B_0\right)^{1-\frac{1}{\psi}} + \\ \hat{\beta} \left(\int_{\underline{y_1}}^{m} \left(y_1\right)^{1-\gamma} dF\left(y_1\right) + \int_{m}^{m+B_0} \left(m\right)^{1-\gamma} dF\left(y_1\right) + \int_{m+B_0}^{\overline{y_1}} \left(y_1 - B_0\right)^{1-\gamma} dF\left(y_1\right) \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \end{array} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

Borrowers choose how much to borrow as in the baseline model. Their choice of B_0 is given by:

$$\left(C_0\right)^{-\frac{1}{\psi}} \left[q_0 + \frac{\partial q_0 \left(B_0, m\right)}{\partial B_0} B_0 \right] = \beta Q^{\gamma - \frac{1}{\psi}} \int_{m+B_0}^{\overline{y_1}} \left(y_1 - B_0\right)^{-\gamma} dF\left(y_1\right), \tag{14}$$

where Q denotes the certainty equivalent of consumption at $t = 1$ (the appendix contains the exact expression) and $\beta \equiv \frac{\hat{\beta}}{1-\hat{\beta}}$. It is easy to show that the sensitivity of credit demand to interest rates crucially depends on ψ (see the derivation in the appendix).

Proposition 5. (Epstein-Zin utility)

a) *The marginal welfare change from varying the optimal exemption m when borrowers have Epstein-Zin preferences is given by:*

$$\frac{dW}{dm} = V_0^{\frac{1}{\psi}} \left[\left(1 - \hat{\beta}\right) \left(C_0\right)^{-\frac{1}{\psi}} \frac{\partial q_0 \left(B_0\left(m\right), m\right)}{\partial m} B_0 + \hat{\beta} Q^{\frac{\gamma - \frac{1}{\psi}}{1-\gamma}} \int_{m}^{m+B_0} \left(C_1^{\mathcal{D}}\right)^{-\gamma} dF\left(y_1\right) \right]$$

b) The optimal exemption m^* when borrowers have Epstein-Zin preferences is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{15}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0+q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv -\frac{\frac{\partial q_0(B_0,m)}{q_0}}{\partial m}$, *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}} \beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF\left(y_1\right)$.

Using a more general form of preferences only modifies the expression for the price-consumption ratio $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}$, which is now given by the product of two terms. The first term $\beta \int_m^{m+B_0} \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma} dF\left(y_1\right)$ is the same as in the CRRA case studied above. The second term $\left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}}$ can be interpreted as a correction to the rate of time preference that captures a preference for early versus late resolution of uncertainty. Although Q is a complex object, equation (15) provides intuition on the optimal exemption. For instance, when $Q < C_0$, a situation in which borrowers face large consumption risks, high values of $\gamma - \frac{1}{\psi}$, which represent a preference for early resolution of uncertainty, make consumption at $t = 0$ more valuable, which calls for lower exemptions all else constant. Intuitively, borrowers enjoy more early consumption, so they value more cheaper access to credit ex-ante. As expected, when $\gamma = \frac{1}{\psi}$, the second term of the price-consumption ratio cancels out, recovering the CRRA formulas.

Intuitively, it is fair to say that the elasticity of intertemporal substitution is important to determine the demand for credit — because ψ controls directly the sensitivity of the demand for credit to interest rates — but risk aversion, through the classic CRRA term, and the preference for early versus later resolution of uncertainty, play an important role when assessing the welfare effects of varying bankruptcy exemptions.[14] More generally, changes in preferences only modify m^* through the sufficient statistics identified in this paper.

4.4 Multiple contracts

In the baseline model, borrowers only have access to a single noncontingent contract. Now borrowers have access to an arbitrary number of contracts with arbitrarily general payoffs. Every contract $j = 1, \ldots, J$ has an arbitrary payoff scheme $z_j\left(y_1\right)$, which can take positive or negative values depending on the realization of y_1, which now exclusively represents borrowers' income. In this more general model, it becomes explicit that the bankruptcy exemption applies to the level of assets held by borrowers. Borrowers optimally choose both positive and negative values of B_{0j}. Given these new assumptions, the distinction between borrowers and lenders is now blurred, although I continue to use the same nomenclature. As shown in the appendix, as long as the shapes of the contracts used, independently of whether such shapes are optimally chosen or not, do not vary with the exemption level, there is no loss of generality on taking the set of contracts as a primitive for the purpose of understanding the welfare implications of varying exemption levels. This logic motivates the choice of the debt contract for the baseline model, since debt contracts are pervasively used in economies with high and low levels of exemptions.

I make two simplifying assumptions about the behavior of lenders. First, if needed, lenders can fully commit to pay borrowers at $t = 1$. Second, lenders are able to observe borrowers' portfolio choices at $t = 0$, allowing the pricing schedule offered for contract j to depend on the whole portfolio of a borrower, that is:

$$q_{0j}\left(B_{01}, \ldots, B_{0J}, m\right)$$

[14] These results relate to the findings of Tallarini (2000), who shows that the EIS determines quantity dynamics but risk aversion is the relevant parameter for asset valuations.

Borrowers' budget constraints now read as:

$$C_0 = y_0 + \sum_{j=1}^{J} q_{0j} \left(B_{01,\ldots} B_{0J}, m\right) B_{0j}$$
$$C_1^{\mathcal{N}} = y_1 + \sum_{j=1}^{J} \max \left\{-z_j \left(y_1\right) B_{0j}, 0\right\} - \sum_{j=1}^{J} \max \left\{z_j \left(y_1\right) B_{0j}, 0\right\}$$
$$C_1^{\mathcal{D}} = \min \left\{ y_1 + \sum_{j=1}^{J} \max \left\{-z_j \left(y_1\right) B_{0j}, 0\right\}, m \right\} \tag{16}$$

Some features of this general environment deserve to be emphasized. First, the baseline model is a special case of this one when $J = 1$ and $z_1 \left(y_1\right) = 1$. Second, if the set of securities spans all realizations of y_1, this formulation also nests the complete markets benchmark.[15] Third, because bankruptcy exemptions are often linked to house ownership (homestead exemptions), we can interpret one of these assets as positive equity in a house. To interpret m exactly as a homestead exemption, equation (16) must be written as $C_1^{\mathcal{D}} = \min \left\{\max \left\{-z_h \left(y_1\right) B_{0h}, 0\right\}, m\right\}$, where the asset h represents home equity. Finally, it is easy to see how allowing for fully or partially secured contracts in this environment, as well as having assets with different liquidity properties, as in for instance Kaplan and Violante (2014), leaves the insights of the paper unchanged.

The logic used to characterize the default region is identical to the baseline model. Borrowers default for those realizations of y_1 in which $C_1^{\mathcal{D}} > C_1^{\mathcal{N}}$. A set of equalities given by $\min \left\{ y_1 + \sum_{j=1}^{J} \max \left\{-z_j \left(y_1\right) B_{0j}, 0\right\}, m \right\} = y_1 + \sum_{j=1}^{J} \max \left\{-z_j \left(y_1\right) B_{0j}, 0\right\} - \sum_{j=1}^{J} \max \left\{z_j \left(y_1\right) B_{0j}, 0\right\}$ define thresholds for three regions that depend on the realization of y_1. First, the no default region, denoted by $\mathcal{N}$. Second, the default region in which borrowers keep the full exemption, denoted by $\mathcal{D}_m$. Third, the default region in which borrowers do not exhaust the full exemption, denoted by $\mathcal{D}_y$. Figure 3, which is the counterpart of figure 1 in the baseline model, shows a possible scenario. This figure illustrates how the optimal exemption formula is invariant to whether borrowers default for high or low realizations of y_1. Both forced and strategic default also occur in this more general case too.

Assuming that all claimants split borrowers' payments proportionally in bankruptcy, the pricing schedules offered by risk neutral lenders are:

$$q_{0j} \left(B_{01}, \ldots, B_{0J}, m\right) = \frac{\eta_j \int_{\mathcal{D}_m} \left(\frac{y_1 - m}{B_{0j}}\right) dF \left(y_1\right) + \int_{\mathcal{N}} z \left(y_1\right) dF \left(y_1\right)}{1 + r^*}, \quad \forall j,$$

where $\eta_j \equiv \frac{z_j(y_1) B_{0j}}{\sum_{j=1}^{J} z_j(y_1) B_{0j}}$ is the recovery rate in bankruptcy. Different assumptions about η_j do not affect the main insights.

Borrowers' behavior can be characterized by a set of J optimality conditions:

$$U' \left(C_0\right) \left[q_{0j} + \sum_{j=1}^{J} \frac{\partial q_{0j} \left(B_{01,\ldots} B_{0J}, m\right)}{\partial B_{0j}} B_{0j} \right] = \beta \left[\begin{array}{c} \int_{\mathcal{N}} z_j \left(y_1\right) U' \left(C_1^{\mathcal{N}}\right) dF \left(y_1\right) + \\ \int_{\mathcal{D}_y} z_j \left(y_1\right) U' \left(C_1^{\mathcal{D}}\right) dF \left(y_1\right) \mathbb{I} \left[z_j \left(y_1\right) B_{0j} < 0\right] \end{array} \right], \quad \forall j$$

A marginal change in B_{0j} affects the interest rate charged to all other contracts.

[15] The complete markets extension is more natural when $F(\cdot)$ has finite support. Starting from the complete markets benchmark, when all Arrow-Debreu contracts are available, it is easy to show that allowing for bankruptcy is welfare reducing. Intuitively, the contingent contracts for states in which borrowers default cease to be traded, because lenders would require an arbitrarily large interest rate, so $\frac{\partial \log(1+r_j)}{\partial m} \approx \infty$ and $m^* \to 0$.

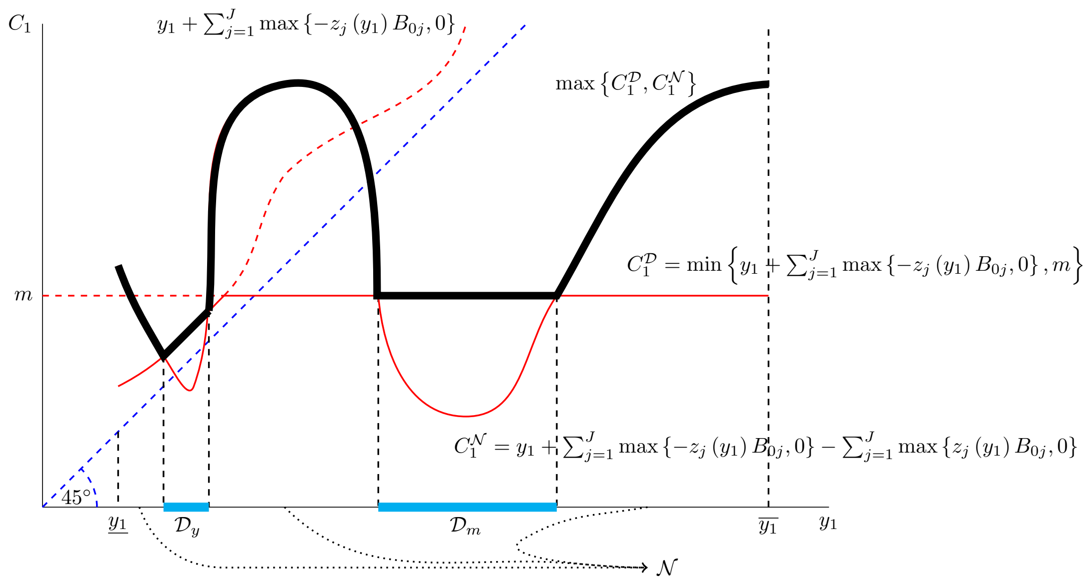

Figure 3: Optimal default decision with multiple contracts with arbitrary payoffs

Proposition 6. (Multiple contracts with arbitrary payoffs)

a) *The marginal welfare change from varying the optimal exemption m when borrowers can use J contracts with arbitrary payoffs is given by:*

$$\frac{dW}{dm} = U'(C_0) \sum_{j=1}^{J} \frac{\partial q_{0j}(B_{01,\ldots} B_{0J}, m)}{\partial m} B_{0j} + \beta \int_{D_m} U'\left(C_1^{\mathcal{P}}\right) dF(y_1) \tag{17}$$

b) The optimal exemption m^* when borrowers can use J contracts with arbitrary payoffs is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{P}}\right\}}{C_0}}{\sum_{j=1}^{J} \Lambda_j \varepsilon_{\tilde{r}_j,m}} \tag{18}$$

where $\Lambda_j \equiv \frac{q_{0j} B_{0j}}{y_0 + \sum_{j=1}^{J} q_{0j} B_{0j}}$, $\varepsilon_{\tilde{r}_j,m} \equiv -\frac{\frac{\partial q_0\left(B_{01,\ldots} B_{0J}, m\right)}{q_0}}{\partial m}$, *and* $\frac{\Pi_m\left\{C_1^{\mathcal{P}}\right\}}{C_0} \equiv \beta \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{P}}}{C_0} \frac{U'\left(C_1^{\mathcal{P}}\right)}{U'(C_0)} dF(y_1)$.

Proposition 6 shows that the cost of increasing the exemption level is in general given by a weighted average of the interest rate sensitivities with respect to the bankruptcy exemption for all contracts traded. The weight given to a contract j is increasing in how much borrowers rely on that contract to raise funds. Intuitively, the welfare costs associated with higher rates are larger for contracts which account for a larger fraction of borrowers portfolios. Allowing for multiple contracts with arbitrary payoffs only changes directly the benefits of varying exemptions through changes in the default region.

4.5 Heterogeneous borrowers: observable heterogeneity with an extensive margin choice

In the baseline model, borrowers are ex-ante symmetric. However, borrowers could be ex-ante heterogeneous on observable or unobservable characteristics. I first analyze the case of observable heterogeneity to show that my results are robust to extensive margin choices on whether to borrow or not to borrow at all. I study unobservable heterogeneity next.

Borrowers are now ex-ante heterogeneous regarding preferences, endowments, distribution of shocks, etc. I index the different types of ex-ante borrowers by i and assume that they are distributed in the population according to a well-behaved distribution with cdf $G(i)$. For now, borrowers' heterogeneity is observable, so lenders are able to price each (group of) borrower(s) i separately. In the baseline model, for clarity, I imposed a regularity condition guaranteeing that borrowers would always borrow a strictly positive amount for any level of m. I now relax that assumption, so it is sometimes optimal for borrowers to choose $B_0 = 0$.

Importantly, I restrict the analysis to a single bankruptcy exemption. Conditioning the exemption level on observable characteristics or allowing for nonlinear exemptions is optimal in this environment, although these policies are not used in practice. I calculate welfare using a social welfare function that maximizes a weighted sum of individual utilities with arbitrary welfare weights $\lambda(i)$. Hence, social welfare $W(m)$ is now given by:

$$W(m) = \int \lambda(i) W(i) \, dG(i)$$

I denote by $q_{0i}(B_{0i}, m)$ the pricing schedule offered by lenders to a borrower of type i. When $B_{0i} > 0$, $q_{0i}(B_{0i}, m)$ is identical to the schedule in equation (2), although the particular distribution for a borrower of type i, $F_i(y_{1i})$. In general, the pricing schedule offered to borrowers when they can save is discontinuous at $B_{0i} = 0$. Formally:

$$q_{0i}(B_{0i}, m) = \begin{cases} \frac{1}{1+r^*}, & B_{0i} \leq 0 \\ \frac{\delta \int_m^{m+B_{0i}} \frac{y_{1i}-m}{B_{0i}} dF_i(y_{1i}) + \int_{m+B_{0i}}^{\overline{y_{1i}}} dF_i(y_{1i})}{1+r^*}, & B_{0i} > 0 \end{cases}$$

Figure 4 represents the new (discontinuous) pricing schedule as well as the objective function of a given borrower, which is continuous but non-differentiable at $B_{0i} = 0$. Depending on their characteristics and the level of m, it is optimal for some types of borrowers to choose $B_{0i} = 0$. Formally, I denote the set of active borrowers for a given level of m by $I_A(m)$ and the set of excluded borrowers by $I_N(m)$, that is:

$$\begin{aligned} I_A(m) &= \{i | B_{0i}(m) > 0\}, \quad \text{Active borrowers} \\ I_N(m) &= \{i | B_{0i}(m) = 0\}, \quad \text{Inactive borrowers} \end{aligned}$$

It is easy to show — see the appendix — that the measure of excluded borrowers, under the assumed regularity conditions, increases with the exemption level. Therefore, varying m causes borrowers to adjust their decisions both on the intensive and the extensive margin.

Proposition 7. (Heterogeneous borrowers: observable heterogeneity)

a) *The marginal welfare change from varying the optimal exemption m when borrowers are ex-ante observably heterogeneous is given by:*

$$\frac{dW}{dm} = \int_{I_A(m)} \lambda(i) \left[U_i'(C_{0i}) \frac{\partial q_{0i}(B_{0i}, m)}{\partial m} B_{0i} + \beta_i \int_m^{m+B_{0i}} U_i'\left(C_{1i}^{\mathcal{D}}\right) dF_i(y_{1i}) \right] dG(i)$$

b) The optimal exemption m^* when borrowers are ex-ante observably heterogeneous is given by:

$$m^* = \frac{\int_{I_A(m^*)} h_i \frac{\Pi_{m,i}\left\{C_{1i}^{\mathcal{D}}\right\}}{C_{0i}} dG(i)}{\int_{I_A(m^*)} h_i \Lambda_i \varepsilon_{\tilde{r}_i, m} dG(i)}$$

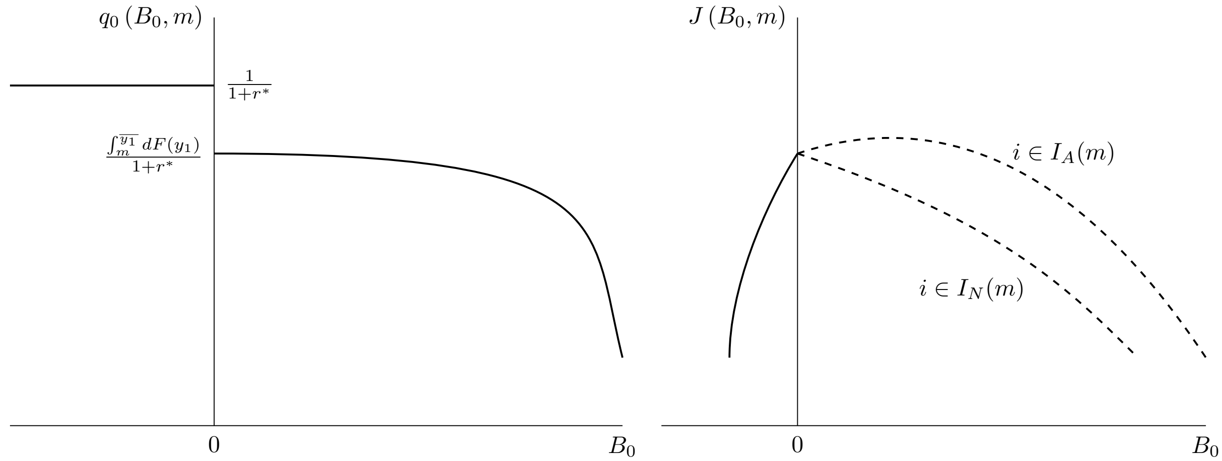

Figure 4: Pricing schedule $q_{0i}(B_{0i}, m)$ and objective function $J_i(B_{0i}, m)$ for a borrower i

where $I_A(m)$ denotes the set of active borrowers, $h_i \equiv \lambda(i) U_i'(C_{0i})$, $\Lambda_i \equiv \frac{q_{0i}B_{0i}}{y_{0i}+q_{0i}B_{0i}}$, $\varepsilon_{\tilde{r}_i,m} \equiv -\frac{\frac{\partial q_{0i}(B_{0i},m)}{q_{0i}}}{\partial m}$ and $\frac{\Pi_{m,i}\{C_{1i}^{\mathcal{D}}\}}{C_0} \equiv \int_m^{m+B_{0i}} \frac{C_{1i}^{\mathcal{D}}}{C_{0i}} \frac{\beta_i U_i'(C_{1i}^{\mathcal{D}})}{U_i'(C_{0i})} dF_i(y_{1i})$.

The optimal exemption now contains a weighted average of marginal costs and benefits in the cross-section of borrowers. The weights h_i are a combination of the social welfare weight and the marginal utility of consumption. When $h_i = 1$ for all borrowers, social welfare is simply the sum of certainty equivalents. $h_i = 1$ emerges endogenously when ex-transfers across borrowers are available and it corresponds to adopting a Kaldor/Hicks welfare criterion. This is an often used benchmark in welfare analysis. In that case, the optimal m^* can written as:

$$m^* = \frac{\mathbb{E}_{G,A}\left[\frac{\Pi_{m,i}\{C_{1i}^{\mathcal{D}}\}}{C_{0i}}\right]}{\mathbb{E}_{G,A}\left[\Lambda_i \varepsilon_{\tilde{r}_i,m}\right]}, \tag{19}$$

where $\mathbb{E}_{G,A}[\cdot]$ denotes cross-sectional averages for active borrowers. Importantly, because changing m does not affect excluded borrowers' welfare and because it is optimal for borrowers who move at the margin from being active to excluded to do so, all averages are taken using the set of active borrowers. This is an important takeaway of this paper: only information about the set of active borrowers is needed to assess welfare changes induced by varying the level of the bankruptcy exemption.[16]

4.6 Heterogeneous borrowers: unobservable heterogeneity

I now study the same environment as in the previous extension, but make borrowers' heterogeneity unobservable: lenders do not know the characteristics of individual borrowers, but they do know the distribution of the characteristics of the population of borrowers. Once asymmetric information is introduced, many phenomena can arise. Analyzing all of them is outside of the scope of this paper. I focus on an environment in which lenders compete using a single pricing schedule and commit to it

[16]When $\mathbb{C}ov_{G,A}[\Lambda_i \varepsilon_{\tilde{r}_i,m}] \approx 0$, equation (19) justifies using the product of cross-sectional averages $\mathbb{E}_{G,A}[\Lambda_i]\,\mathbb{E}_{G,A}[\varepsilon_{\tilde{r}_i,m}]$ as denominator.

— once the price schedule is posted, lenders must lent the amount demanded at the posted rate. This is a natural choice to understand how changes in the level of the exemptions affect the welfare of pooled and excluded borrowers.

As above, we expect some borrowers to borrow in equilibrium, those in $I_A(m)$, while others decide not to, those in $I_N(m)$. Hence, it is optimal for lenders to condition the pricing schedule they offer on the set of active borrowers in the economy. Therefore, the pricing schedule faced by any borrower is given by:

$$q_{0i}(B_0, m) = \begin{cases} \frac{1}{1+r^*}, & B_{0i} \leq 0 \\ \int_{I_A(m)} \frac{\tilde{q}_{0i}(B_0,m)}{\int_{I_A(m)} dG(i)} dG(i), & B_{0i} > 0, \end{cases}$$

where $\tilde{q}_{0i}(B_0, m) = \frac{\delta \int_m^{m+B_0} \frac{y_{1i}-m}{B_0} dF_i(y_{1i}) + \int_{m+B_0}^{\overline{y_{1i}}} dF_i(y_{1i})}{1+r^*}$ and $I_A(m)$ denotes the set of active borrowers, defined as in the previous extension. In this equilibrium, lenders offer a pricing schedule that corresponds to the weighted average of borrowers, taking into account that the level of m and how their choice of schedules changes the composition of borrowers in the economy. Given this price schedule, borrowers choose whether to borrow or not. Under regularity conditions, an equilibrium exists in which the pricing schedules offered by lenders are consistent with set of active borrowers in the economy — I solve a specific example with heterogeneity in discount factors in the appendix. Given the price schedule that they face, borrowers decide how much to borrow, as in the baseline model, and whether to borrow at all, as in the previous extension. Because the composition of borrowers in the economy $G(\cdot)$ determines which borrowers have access to credit in this economy, we can argue that the equilibrium features a form of credit rationing/redlining — see Jaffee and Stiglitz (1990) for a classic discussion of these effects.

Proposition 8. (Heterogeneous borrowers: unobservable heterogeneity)

a) *The marginal welfare change from varying the optimal exemption m when borrowers are ex-ante unobservably heterogeneous is given by:*

$$\frac{dW}{dm} = \int_{I_A(m)} \lambda(i) \left[U_i'(C_{0i}) \frac{\partial q_{0i}(B_{0i}, m)}{\partial m} B_{0i} + \beta_i \int_m^{m+B_{0i}} U_i'\left(C_{1i}^{\mathcal{D}}\right) dF_i(y_{1i}) \right] dG(i)$$

b) The optimal exemption m^* when borrowers are ex-ante unobservably heterogeneous is given by:

$$m^* = \frac{\int_{I_A(m^*)} h_i \frac{\Pi_{m,i}\left\{C_{1i}^{\mathcal{D}}\right\}}{C_{0i}} dG(i)}{\int_{I_A(m^*)} h_i \Lambda_i \varepsilon_{\tilde{r}_i,m} dG(i)},$$

where $I_A(m)$ denotes the set of active borrowers, $h_i \equiv \lambda(i) U_i'(C_{0i})$, $\Lambda_i \equiv \frac{q_{0i}B_{0i}}{y_{0i}+q_{0i}B_{0i}}$, $\varepsilon_{\tilde{r}_i,m} \equiv -\frac{\frac{\partial q_{0i}(B_{0i},m)}{q_{0i}}}{\partial m}$, and $\frac{\Pi_{m,i}\left\{C_{1i}^{\mathcal{D}}\right\}}{C_0} \equiv \int_m^{m+B_{0i}} \frac{C_{1i}^{\mathcal{D}}}{C_{0i}} \frac{\beta_i U_i'\left(C_{1i}^{\mathcal{D}}\right)}{U_i'(C_{0i})} dF_i(y_{1i})$.

There is no difference between observable and unobservable heterogeneity for the purpose of understanding the welfare implications of changing the level of the bankruptcy exemption. With unobservable heterogeneity, varying m changes the composition of borrowers not only directly but also through endogenous changes in the pricing schedules faced by borrowers in equilibrium, which creates feedback and modifies borrowers choices on the extensive and intensive margins. However, all the information needed to assess whether a change in m increases or decreases welfare is embedded in the sufficient statistics identified in this paper, in particular, the sensitivity of the interest rate schedule. An important takeaway of this extension is that unobserved heterogeneity complicates the equilibrium characterization of the economy, but does not change the sufficient statistics for the welfare implications of changing exemptions. As in the case with observed heterogeneity, excluded borrowers are inframarginal from a welfare perspective.

4.7 Dynamics

Finally, I extend the results to a dynamic environment. I assume that time is discrete and there is a finite horizon: $t = 0, \ldots, T$.[17] At every point in time, borrowers trade a single noncontingent one period contract. The income process for y_t has a Markov structure. Employment, health or family shocks, which previous literature has shown to be key drivers of bankruptcy, as well as social insurance transfers, are all captured in the stochastic process for y_t. In case of default, borrowers' debt is fully discharged but they can't borrow in the defaulting period and recover access to credit markets stochastically, with probability α. S. Chatterjee, D. Corbae, M. Nakajima and J.V. Ríos-Rull (2007) argue that these are reasonable assumptions in the context of the US unsecured credit system. I again restrict my attention to a noncontingent exemption which is not time varying. More sophisticated instruments, as choosing the fraction of debt discharged or the time of borrowers' exclusion from financial markets, can be welfare improving in this environment. Independently of whether these features are determined optimally or not, they do not affect the optimal characterization of exemptions.[18]

Hence, borrowers maximize:

$$\max \mathbb{E}\left[\sum_{t=0}^{T} \beta^t U\left(C_t\right)\right]$$

Borrowers' consumption when they do not default is given by $C_t^{\mathcal{N}} = y_t + q_t B_t - B_{t-1}$. In default states, borrowers consume $C_t^{\mathcal{D}} = \min\left\{y_t, m\right\}$ and, when excluded from credit markets, they simply consume their income y_t.

From a $t = 0$ perspective, we can write borrowers' problem recursively as:

$$V_{\mathcal{N},0}\left(B_{-1}, y_0; m\right) = \max_{B_0} U\left(y_0 + q_0 B_0\right) + \beta \mathbb{E}\left[\max\left\{V_{\mathcal{N},1}\left(B_0, y_1; m\right), V_{\mathcal{D},1}\left(y_1; m\right)\right\}\right],$$

where $V_{\mathcal{N},1}\left(B_0, y_1; m\right)$ and $V_{\mathcal{D},1}\left(B_0, y_1; m\right)$, which denote indirect utility at $t = 1$ for borrowers' who repay and default, respectively, are explicitly characterized in the appendix. Indirect utility in default $V_{\mathcal{D},1}$ is independent of B_0 because of the assumption that all debt is fully discharged.

I characterize default at $t = 1$ and borrowing at $t = 0$. The results for other periods follow directly. The logic used to characterize the default region is identical to the baseline model. Borrowers default decision is determined by an indifference condition, given by:

$$V_{\mathcal{N},1}\left(B_0, \hat{y}_1; m\right) = V_{\mathcal{D},1}\left(\hat{y}_1; m\right), \tag{20}$$

where $\hat{y}_1$ denote indifference thresholds. Once again, I denote the repayment region by $\mathcal{N}_t$, the default region in which borrowers claim the full exemption by $\mathcal{D}_{m,t}$ and the default region in which borrowers

[17] Under appropriate regularity conditions, the results extend easily to infinite horizon economies.

[18] There is scope to extend the results of this section allowing for longer maturity contracts and explicit renegotiation.

do not claim the full exemption by $\mathcal{D}_{y,t}$. All three regions are formed in general by a union of intervals. The value functions in equation (20) embed the forward looking nature of the default decision.

Borrowers choose B_0 according to:

$$U'\left(y_0+q_0B_0\right)\left[q_0+\frac{\partial q_0\left(B_0,m\right)}{\partial B_0}B_0\right]=\beta\int_{\mathcal{N},1}U'\left(y_1+q_1B_1-B_0\right)dF\left(y_1\right)$$

The characterization of default and borrowing decisions is identical in other periods.

Proposition 9. (Dynamics)

a) *The marginal welfare change from varying the optimal exemption m in the dynamic model is given by:*

$$\frac{dW}{dm}=\sum_{t=0}^{T-1}\mathbb{E}_{\mathcal{N},t}\left[\beta^tU'\left(C_t^{\mathcal{N}}\right)\frac{\partial q_t}{\partial m}B_t\right]+\sum_{t=1}^{T}\mathbb{E}_{\mathcal{D}_m,t}\left[\beta^tU'\left(C_t^{\mathcal{P}}\right)\right]$$

b) The optimal exemption m^* in the dynamic model is given by:

$$m^*=\frac{\sum_{t=1}^{T}\frac{\Pi_{m,t}\left\{C_t^{\mathcal{P}}\right\}}{C_0}}{\sum_{t=0}^{T-1}\Pi_{\mathcal{N},t}\left\{g_t\Lambda_t\varepsilon_{\tilde{r}_t,m}\right\}},$$

where $\mathbb{E}_{\mathcal{N}_t}\left[\cdot\right]$ denotes the $t=0$ expectation of being in a no default state in which borrowers have access to credit, $\mathbb{E}_{\mathcal{D}_m,t}\left[\cdot\right]$ denotes the $t=0$ expectation of defaulting in a given state, $\Lambda_t\equiv\frac{q_tB_t}{y_t+q_tB_t-B_{t-1}}$, $g_t\equiv\frac{C_t}{C_0}$, $\varepsilon_{\tilde{r}_t,m}\equiv-\frac{\frac{\partial q_t}{q_t}}{\partial m}$, $\frac{\Pi_{m,t}\left\{C_t^{\mathcal{P}}\right\}}{C_0}\equiv\mathbb{E}_{\mathcal{D}_m,t}\left[\frac{C_t^{\mathcal{P}}}{C_0}\frac{\beta^tU'\left(C_t^{\mathcal{P}}\right)}{U'\left(C_0\right)}\right]$, and $\Pi_{\mathcal{N},t}\left\{x\right\}\equiv\mathbb{E}_{\mathcal{D},t}\left[\frac{\beta^tU'\left(C_t^{\mathcal{P}}\right)}{U'\left(C_0\right)}x\right]$.

The optimal exemption becomes a weighted average across periods/states of the marginal benefits/losses. The numerator of m^*, which captures the marginal benefit of increased leniency, is the price of a claim to an asset that pays consumption in the default states in which borrowers claim the full exemption as a ratio of initial consumption. The denominator, which captures marginal losses, is the $t=0$ price of a weighted average of interest rate schedule sensitivities with respect to the bankruptcy exemption — with weights given by the product of consumption growth g_t and leverage ratios Λ_t. Intuitively, the optimal formula in the dynamic model trades off marginal welfare gains and losses using borrowers' stochastic discount factor. Therefore, the results of the static model can be interpreted as the steady state of a dynamic model when $g_t\approx1$.

I would like to make four final observations. First, previous literature has shown that the magnitude of income shocks, family shocks or health shocks faced by households are important drivers of borrowing and bankruptcy decisions. For instance, Livshits, MacGee and Tertilt (2007) show that the nature of income shocks — whether they are transitory or permanent — as well as life-cycle borrowing motives play an important role. Similarly, taxation and social insurance programs affect borrowing and bankruptcy decisions. All these considerations are captured in the framework of this paper and only affect the exemption level through the characterized sufficient statistics.

Second, the precise conditions that determine the decision of declaring bankruptcy, for instance, whether borrowers' go bankrupt after high or low income realizations, does not change the formula for the optimal exemption. The option value to wait before defaulting complicates the characterization of the default decision, but it only affects the optimal exemption through the characterized sufficient statistics.

Third, if borrowers are allowed to save part of their exemption when bankrupt, the price-consumption ratio must be corrected by a term that depends on the propensity to consume of bankrupt borrowers — see the appendix for an exact expression. Hence, the baseline model implicitly assumes a propensity to consume close to unity for bankrupt borrowers. This entails little loss of generality, since we expect bankrupt borrowers to have done poorly and have high propensities to consume.

Finally, from the derivation of proposition 9, it is easy to see that allowing borrowers to stop repayments (default) without declaring bankruptcy does not affect the formula for the optimal exemption. That possibility will be priced by lenders and borrowers will exercise it optimally, so it will only affect exemptions through the characterized sufficient statistics, in particular the sensitivity of the interest rate schedule.

5 Calibration exercise and implications for measurement

Although the main contribution of this paper is theoretical, the upshot of the approach followed is that the key theoretical tradeoffs can be quantified by measuring a few observables. To show the applicability of my results in practice, I first calibrate the optimal exemption to chapter 7 US data and then study quantitatively and qualitatively the welfare effects induced by varying the exemption level. Before concluding, I discuss additional practical implications.

Optimal exemption

I calibrate the optimal exemption assuming that borrowers have CRRA utility and using a Kaldor/Hicks welfare criterion to account for heterogeneity, as described in section 4.5: this allows us to use cross-sectional averages provided the cross-sectional covariances of terms in numerator and denominator are not large, which I'll assume.[19] As shown in proposition 2, the optimal exemption for CRRA utility borrowers is given by:[20]

$$m^* = \frac{\beta \pi_m \left(\frac{C_1^{\mathcal{D}}}{C_0}\right)^{1-\gamma}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{21}$$

As just shown, we should interpret this formula as the optimal exemption in the steady state of a dynamic economy. I use additional insights from the extensions to determine the correct variables to target. Therefore, four observable variables and two preference parameters, the rate of time preference β and the coefficient of relative risk aversion γ, need to be calibrated. I adopt a yearly calibration and choose appropriate values for the US economy as a whole.

Calibrating β, γ and Λ is straightforward. I use $\beta = 0.96$ as the annual rate of time preference: this is a standard choice. I adopt $\gamma = 10$ for the baseline calibration for risk aversion, but report the results for other levels of risk aversion. To calibrate Λ, I target the same average ratio of unsecured debt to

[19] If borrowers happened to trade assets that directly reveal their willingness to pay for a dollar in bankrupt states, we could calculate their price-consumption ratio without the need to make assumptions about their preferences. Alvarez and Jermann (2004) use that approach to measure the cost of business cycles. Unfortunately, that approach is not feasible in this case.

[20] Allowing for Epstein-Zin preferences and assuming that the certainty equivalent of $t = 1$ consumption equals C_0 yields identical results. The optimal exemption with $\gamma = 10$ and $\psi = 1.5$ would be approximately 10% larger if expected consumption growth in certainty equivalent terms is 1%.

personal disposable income as Livshits, MacGee and Tertilt (2007), which is 8.4%. This value implies that $\Lambda = 0.0775$.[21]

Finding appropriate values for π_m , $C_1^{\mathcal{P}}/C_0$, and $\varepsilon_{\tilde{r},m}$ requires further work. First, π_m can be written as the product of the unconditional probability of default π_D with the probability of claiming the full bankruptcy exemption conditional on defaulting $\pi_{m|D}$. To determine π_D, I use the average probability of filling for chapter 7 bankruptcy, also from Livshits, MacGee and Tertilt (2007), which is 0.8%. This choice should raise no concerns. I determine $\pi_{m|D}$ by using the fact that roughly 90% of bankruptcies are filed as "no-asset" bankruptcies — see Lupica (2012). This implies that 10% of bankrupt borrowers fully claim their exemption.

Second, I use $C_1^{\mathcal{P}}/C_0 = 0.9$ for the change in consumption by bankrupt borrowers. A 10% reduction in consumption may be considered as a large change, but it is within the range of variation documented in Filer and Fisher (2005) using PSID data on changes in food consumption for bankrupt individuals. I purposefully pick a relatively large number for $C_1^{\mathcal{P}}/C_0$ to make m^* sensitive to varying risk aversion; otherwise, the results when $C_1^{\mathcal{P}}/C_0 \approx 1$ are nearly identical to the log utility results as long as γ is within a reasonable range.

Finally, calibrating the sensitivity of the interest rate schedule with respect to the exemption $\varepsilon_{\tilde{r},m}$ is not easy. Ideally, we would have direct access to lenders' pricing schedules or we would identify the shape of the pricing schedules offered to the same borrower using multiple changes in the exemption level. Unfortunately, because the required pricing data is sensitive for lenders and because changes in exemptions are rare, no paper has been able to follow these approaches. Instead, I use the average estimate from Gropp, Scholz and White (1997), who estimate the equilibrium effect on interest rates of changes in bankruptcy exemptions over time and across states. A value of $\varepsilon_{\tilde{r},m} = 2.5 \cdot 10^{-7}$ is within the upper end of reasonable estimates for the average response of interest rates in the population of active borrowers — many of the estimates have significance issues. As shown in section 4, we do not have to directly account for extensive margin choices. This choice of $\varepsilon_{\tilde{r},m}$ implies that increasing the bankruptcy exemption by a \$100,000 increases the equilibrium interest rate by 250 basis points. Conceptually, this value corresponds to $-\frac{\frac{dq_0}{q_0}}{dm}$, as defined in equation (6), and not to $-\frac{\frac{\partial q_0}{q_0}}{\partial m}$, since it does not control for the change in interest rate caused by changes in the level of borrowing $\frac{dB_0}{dm}$, which is found to be positive in the data and may bias $\varepsilon_{\tilde{r},m}$ upwards. Alternatively, we can think that we are calibrating the model in which borrowers are price takers. As shown in the online appendix, the equilibrium semielasticity is the correct one in that case.[22]

Table 1 summarizes the choices of parameters and variables used in the baseline calibration.

[21] In actual economies, a substantial fraction of borrowing is collateralized. I work under the assumption that collateralized borrowing is fully secured, which implies that the interest rate charged is not sensitive to m. Therefore, equation (18) implies that the denominator of the optimal exemption only has to account for the fraction of unsecured credit. Since the bulk of exemptions in the US are homestead exemptions, and the papers that identify $\varepsilon_{\tilde{r},m}$ use changes in those, I'm implicitly extrapolating from borrowers' with positive home equity.

[22] It is hard to make a case for whether borrowers take interest rates as given or not for unsecured borrowing. A previous version of this paper used price taking as the baseline case.

Parameter/Variable		Value	Parameter/Variable		Value
β	Rate of time preference	0.96	π_D	Probability of default	0.008
$C_1^{\mathcal{D}}/C_0$	Consumption change	0.9	$\pi_{m\|D}$	Probability of no-asset bankruptcy	0.1
Λ	Leverage	0.0775	$\varepsilon_{\tilde{r},m}$	Interest rate sensitivity	$2.5 \cdot 10^{-7}$

Table 1: Calibrated variables

Using equation (21), table 2 presents the optimal exemption m^* for different values of the risk aversion coefficient γ.

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$
39,638	60,416	**102,314**	293,435

Table 2: Optimal exemption m^* (measured in dollars)

Taking at face value the choices for the sufficient statistics, this analysis concludes that the actual average bankruptcy exemption across US states, of roughly 70,000 dollars, is of the same order of magnitude as the one predicted by the model. The preferred calibration suggests that the optimal exemption should approximately be 100,000 dollars, although the level of risk aversion has an important effect on m^*.

On the size of welfare gains The results regarding m^* show that the optimal exemption should be larger than the average exemption across US states. Given that result, a natural question is how large are the welfare gains from changing m at the current exemption level for an average US state. To answer this question, I focus again in the CRRA case and define $\sigma(m) \equiv \frac{\frac{dW}{dm}}{U'(C_0)C_0}$ as the welfare gain, expressed as a fraction of initial consumption, of increasing the bankruptcy exemption starting from a level m. The value of $\sigma(m)$, as shown in section 3, is given by:

$$\sigma(m) = -\Lambda \varepsilon_{\tilde{r},m} + \frac{1}{m} \beta \pi_m \left(\frac{C_1^{\mathcal{D}}}{C_0} \right)^{1-\gamma}$$

For the purposes of this exercise, I freely use $\sigma(m)$ to assess changes of any magnitude, subject to the caveats discussed in my remarks. Using the same set of parameters described in table 2, and starting from the average exemption of $m = 70,000$ dollars, I show in table 3 the welfare gains associated to a 10,000 dollars increase in m.

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$
-0.0084%	-0.0027%	**0.0089**%	0.062%

Table 3: Welfare gain/loss (10,000 dollars change) measured as a fraction of initial consumption

Therefore, table 3 provides the welfare gains, measured as a percent of initial consumption C_0, from increasing the bankruptcy exemption m by 10,000 dollars when $m = 70,000$, for the same parametrization used to derive m^*. As expected, the magnitude of the welfare gain grows with the value of m^*, when $m^* > 70,000$ and viceversa. For instance, for the benchmark calibration, a 10,000 dollars increase in m increases welfare by 0.0089%. Increasing m by 30,000 dollars, which moves the exemption level from the average 70,000 to the optimum of 100,000, yields a welfare gain of roughly 0.03%. This measures the welfare loss incurred by not having the optimal exemption.

Remarks on applicability and measurement

I would like to conclude with two remarks on the practical applicability of the results.

Remark. (Validity of the calibration exercise) Although the theoretical characterizations of $\frac{dW}{dm}$ and m^* are everywhere exact, the calibration exercise incurs in an approximation error when it imposes that the values assigned to the sufficient statistics for the set of active borrowers do not change with the exemption level. Ideally, if we could obtain empirical counterparts of the different sufficient statistics for all values of m, we could calculate the exact welfare gain or loss induced by a nonmarginal change in m by integrating $\frac{dW}{dm}$ over the relevant range of exemptions. Formally:

$$W(m_1) - W(m_0) = \int_{m_0}^{m_1} \frac{dW}{dm}(\tilde{m})\, d\tilde{m}$$

Similarly, we could find the fixed point that determines m^*. Although better measurement can make this possible in the future, it is not feasible as of today. Therefore, the quantitative results are more precise for small interventions and should be interpreted otherwise with caution.

However, the sign of $\sigma(m)$, which uses information measured at the current exemption level, provides a directional answer to whether exemptions should be increased or decreased without any approximation error. This is the most conservative way of reading the calibration results, but one that provides exact answers if all variables are correctly measured. Hence, if $\sigma(m)$ is positive (negative), this paper guarantees that locally increasing the bankruptcy exemption is welfare improving and viceversa. Repeatedly applying this logic for different levels allows us to identify m^*.

Remark. (Cross sectional implications/importance of pricing and consumption data) The results of this paper are normative, not positive. However, if actual exemptions happen to be chosen optimally according to the model developed in the paper, sharp cross sectional implications arise.[23] In particular, a cross-sectional regression of (log) exemption levels on (log) values of the right hand side variables in equation (10) using different economies as units of observation, as described here:

$$\log m_i^* = \alpha_1 \log(\beta_i) + \alpha_2 \log(\pi_{m,i}) + \alpha_3 \log\left({}^{C_{1i}^{\mathcal{D}}}/_{C_{0i}}\right) + \alpha_4 \log(\Lambda_i) + \alpha_5 \log(\varepsilon_{\tilde{r}_i,m}) + u_i,$$

should find that $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 1 - \gamma_i$, as well as $\alpha_4 = \alpha_5 = -1$. Unfortunately, detailed information on consumption and interest rate elasticities for different regions/countries is not easily available, but there is no reason why it couldn't be in the future. This paper implies that gathering detailed information on consumption for bankrupt borrowers and understanding the shape of interest rate schedules offered by lenders, especially how these schedules vary with possible changes in exemption levels, are necessary measurement efforts to determine optimal bankruptcy exemptions in the future.

[23] There is no presumption that actual exemption levels, which in practice are the outcome of political processes as described by Skeel (2001), are optimally chosen following the prescriptions of this paper.

6 Conclusion

This paper provides a novel general theoretical characterization of optimal bankruptcy exemptions. Importantly, this characterization uses measurable sufficient statistics. Knowledge of borrowers' leverage, the sensitivity of the interest rate schedule faced by borrowers with respect to the exemption level, the probability of default and the change in the consumption of bankrupt borrowers is sufficient to assess the adequacy of changes in the level of the bankruptcy exemption and to characterize the optimal exemption in a wide variety of circumstances. Other features of the environment need not be specified once these variables are known. In an application of the theoretical results, for the preferred calibration to US data, this paper suggests that the optimal bankruptcy exemption should be approximately $100,000$ dollars. This value is larger than the average exemption level in the US, but it is of same order of magnitude.

The conclusions of this paper will help to foster further research on both structural modeling and microeconometric work on bankruptcy. Regarding structural macroeconomic modeling, computing the sufficient statistics found in this paper for different models can shed light on through which channels different assumptions on primitives affect the level of optimal exemptions. Empirical work that identifies the sensitivity of interest rate schedules with respect to changes in exemptions or that provides precise measures of consumption for bankrupt borrowers will be essential to assess which exemption level is appropriate for every economy. Understanding how varying bankruptcy exemptions affects the balance sheets of banks and other intermediaries using a richer model of the lending side of the economy is another promising avenue for further research.

Appendix

Proofs: Section 2

Differentiating the pricing schedule in equation (2) yields:

$$\frac{\partial q_0\left(B_0,m\right)}{\partial B_0}=\frac{-\delta\int_m^{m+B_0}\frac{y_1-m}{B_0^2}dF\left(y_1\right)-\left(1-\delta\right)f\left(m+B_0\right)}{1+r^*}<0$$
$$\frac{\partial q_0\left(B_0,m\right)}{\partial m}=\frac{-\delta\int_m^{m+B_0}\frac{1}{B_0}dF\left(y_1\right)-\left(1-\delta\right)f\left(m+B_0\right)}{1+r^*}<0$$

Given their optimal ex-post default decision, borrowers solve $\max_{B_0} J\left(B_0;m\right)$, where:

$$J\left(B_0,m\right)=U\left(y_0+q_0\left(B_0,m\right)B_0\right)+\beta\left[\begin{array}{c}\int_{\underline{y_1}}^{m}U\left(y_1\right)dF\left(y_1\right)+\int_m^{m+B_0}U\left(m\right)dF\left(y_1\right)\\+\int_{m+B_0}^{\overline{y_1}}U\left(y_1-B_0\right)dF\left(y_1\right)\end{array}\right]\tag{22}$$

A sufficient condition for borrowers to borrow in equilibrium, when combined with the SOC in equation (24), is the following:

$$\lim_{B_0\to 0^+}\frac{dJ}{dB_0}=\frac{U'\left(y_0\right)}{1+r^*}\int_m^{\overline{y_1}}dF\left(y_1\right)-\beta\int_{\underline{y_1}}^{\overline{y_1}}U'\left(y_1\right)dF\left(y_1\right)>0,\ \forall m\in\left[\underline{m},\overline{m}\right]$$

This is a condition on primitives. It holds, for instance, when the initial endowment y_0 is sufficiently low, when borrowers are very impatient, $\beta\to 0$, or when the level of expected future income is sufficiently large and not too stochastic (to prevent the precautionary savings effect from dominating). Figure 4 in the text illustrates the result.

The first-order condition to the borrowers' problem is the following:[24]

$$\frac{dJ}{dB_0}=U'\left(C_0\right)\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right]-\beta\int_{m+B_0}^{\overline{y_1}}U'\left(y_1-B_0\right)dF\left(y_1\right)=0\tag{23}$$

Exploiting lenders' risk neutrality, this expression can be written as:

$$U'\left(C_0\right)\left[\frac{\left(\delta-1\right)f\left(m+B_0\right)B_0+\int_{m+B_0}^{\overline{y_1}}dF\left(y_1\right)}{1+r^*}\right]=\beta\int_{m+B_0}^{\overline{y_1}}U'\left(y_1-B_0\right)dF\left(y_1\right)$$

Note that, when $m+B_0\to\overline{y_1}$, $\frac{dJ}{dB_0}<0$, which guarantees that there is an interior solution to the problem. Note also that, when bankruptcy is costless ($\delta\to 1$), by choosing m^* equal to the full insurance benchmark, we can replicate the perfect insurance outcome between risk neutral lenders and risk averse borrowers.

The second order condition, which establishes convexity and I assume that is satisfied everywhere, is the following:

$$\frac{dJ}{dB_0^2}=\underbrace{U''\left(C_0\right)\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right]^2}_{<0}+\underbrace{U'\left(C_0\right)\frac{\partial\left(q_0+\frac{\partial q_0}{\partial B_0}B_0\right)}{\partial B_0}}_{\gtreqless 0}+\underbrace{\beta U'\left(m\right)f\left(m+B_0\right)}_{>0}\tag{24}$$
$$+\underbrace{\beta\int_{m+B_0}^{\overline{y_1}}U''\left(y_1-B_0\right)dF\left(y_1\right)}_{<0}\leq 0$$

[24] I write $\frac{\partial q_0}{\partial B_0}$ throughout instead of $\frac{\partial q_0\left(B_0,m\right)}{\partial B_0}$ to simplify the notation.

Note that $\frac{\partial\left(q_0+\frac{\partial q_0}{\partial B_0}B_0\right)}{\partial B_0} = 2\frac{\partial q_0}{\partial B_0} + \frac{\partial q_0^2}{\partial B_0^2}B_0$, which in the case of risk neutral lenders equals $\frac{f(m+B_0)\left[-(2-\delta)-(1-\delta)B_0\frac{f'(m+B_0)}{f(m+B_0)}\right]}{1+r^*}$. Hence, a sufficient condition for the second term to be negative is that $B_0\frac{f'(m+B_0)}{f(m+B_0)} \geq -\frac{2-\delta}{1-\delta}$. When numerically solving the model, for usual distributions, the borrowers' problem is convex, despite the positive third term in equation (24). The curvature of the utility function induced by borrowers' risk aversion prevents to find simple sufficient conditions for convexity, as the one used by Bernanke, Gertler and Gilchrist (1999) in a related environment.

By differentiating equation (23), we can find $\frac{dB_0}{dm}$:

$$\frac{dB_0}{dm} = \frac{\overbrace{U''\left(C_0\right)\frac{\partial q_0}{\partial m}B_0\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right]}^{>0} + \overbrace{U'\left(C_0\right)\left[\frac{\partial q_0}{\partial m}+\frac{\partial^2 q_0}{\partial B_0 \partial m}B_0\right]}^{\gtrless 0} + \overbrace{\beta U'\left(m\right)f\left(m+B_0\right)}^{>0}}{-\frac{dJ}{dB_0^2}}$$

When numerically solving the model, for usual distributions, the level of borrowing B_0 increases with m. The amount raised by borrowers at $t=0$ can also increase or decrease with m. Formally, $\frac{d(q_0B_0)}{dm} = \left[\frac{\partial q_0}{\partial m}+\frac{\partial q_0}{\partial B_0}\frac{dB_0}{dm}\right]B_0 + q_0\frac{dB_0}{dm}$, which can take any sign.

To determine whether $\frac{C_1^{\mathcal{D}}}{C_0} > 1$, we can rewrite the optimality condition for debt as $U'\left(C_0\right) = \beta\left(1+\hat{r}\right)\pi_{\mathcal{N}}\mathbb{E}\left[U'\left(C_1^{\mathcal{N}}\right)|\mathcal{N}\right]$, where $1+\hat{r} = \left(q_0+\frac{\partial q_0(B_0,m)}{\partial B_0}B_0\right)^{-1}$. We can also express $\mathbb{E}\left[U'\left(C_1^{\mathcal{N}}\right)|\mathcal{N}\right] \approx U'\left(\mathbb{E}\left[C_1^{\mathcal{N}}|\mathcal{N}\right]\right)\left(1+\kappa\right)$, where $\kappa > 0$ corrects for Jensen's inequality. Combining both expressions, we can conclude that $U'\left(C_0\right) = \alpha U'\left(\mathbb{E}\left[C_1^{\mathcal{N}}|\mathcal{N}\right]\right) < \alpha U'\left(C_1^{\mathcal{D}}\right)$, where $\alpha \equiv \beta\left(1+\hat{r}\right)\pi_{\mathcal{N}}\left(1+\kappa\right)$. So a sufficient (not necessary) condition for $\frac{C_1^{\mathcal{D}}}{C_0} < 1$ is that $\alpha \leq 1$. Borrowers impatience (low β) and small precautionary savings motives (low κ), which are the conditions needed for borrowers to borrow, guarantee that $\alpha \leq 1$.

Proofs: Section 3

Proposition 1. (Marginal effect of varying m on welfare)

a) Starting from equation (7), we can write, applying repeatedly Leibniz rule and using the optimality condition for B_0 and the default decision:

$$\begin{aligned}\frac{dW}{dm} = & \left[U'\left(C_0\right)\frac{\partial q_0}{\partial m}B_0 + \beta\int_m^{m+B_0}U'\left(C_1^{\mathcal{D}}\right)dF\left(y_1\right)\right] \\ & + \underbrace{\left[U'\left(C_0\right)\left[q_0+\frac{\partial q_0}{\partial B_0}B_0\right] - \beta\int_{m+B_0}^{\overline{y_1}}U'\left(y_1-B_0\right)dF\left(y_1\right)\right]}_{=0}\frac{dB_0}{dm} \\ & + \beta\underbrace{\left[U\left(m\right)f\left(m+B_0\right)\left(1+\frac{dB_0}{dm}\right) - U\left(m\right)f\left(m+B_0\right)\left(1+\frac{dB_0}{dm}\right)\right]}_{=0},\end{aligned} \tag{25}$$

where we need to use equation (3) in the paper. As in most normative exercises, it is hard to guarantee the convexity of the problem in general, although numerical solutions are well-behaved when $m > \underline{y_1}$. Formally, we can write:

$$\frac{d^2J}{dm^2} = d\frac{\left(U'\left(C_0\right)\frac{\partial q_0}{\partial m}B_0\right)}{dm} + \beta\int_m^{m+B_0}U''\left(C_1^{\mathcal{D}}\right)dF\left(y_1\right) + \beta U'\left(C_1^{\mathcal{D}}\right)\left(f\left(m+B_0\right)\left(1+\frac{dB_0}{dm}\right) - f\left(m\right)\right),$$

where:

$$d\frac{\left(U'\left(C_0\right)\frac{\partial q_0}{\partial m}B_0\right)}{dm}=U''\left(C_0\right)\frac{d\left(q_0B_0\right)}{dm}\frac{\partial q_0}{\partial m}B_0+U'\left(C_0\right)\left[\frac{\partial q_0}{\partial m}\frac{dB_0}{dm}+\left[\frac{\partial^2 q_0}{\partial m^2}+\frac{\partial^2 q_0}{\partial m\partial B_0}\frac{dB_0}{dm}\right]B_0\right]$$

b) By rewriting equation (8), we can find:

$$\frac{\frac{dW}{dm}}{U'\left(C_0\right)C_0}=\underbrace{\frac{\frac{\partial q_0}{q_0}}{\partial m}}_{\equiv-\varepsilon_{\tilde{r},m}}\underbrace{\frac{q_0B_0}{C_0}}_{\equiv\Lambda}+\frac{1}{m}\underbrace{\beta\int_m^{m+B_0}\frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'\left(C_0\right)}dF\left(y_1\right)}_{\equiv\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}},$$

where $\varepsilon_{\tilde{r},m}\equiv\frac{\partial\log(1+r)}{\partial m}=-\frac{\frac{\partial q_0\left(B_0(m),m\right)}{q_0}}{\partial m}$.

Proposition 2. (Optimal bankruptcy exemption)

Setting $\frac{dW}{dm}=0$ characterizes the optimal exemption. If the problem is well-behaved, m^* determines the unique optimal solution. Formally, solving for m^* in equation (9) immediately yields (10).

In the logarithmic utility case:

$$\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}=\beta\int_m^{m+B_0}\frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'\left(C_0\right)}dF\left(y_1\right)=\beta\underbrace{\int_m^{m+B_0}dF\left(y_1\right)}_{\equiv\pi_m},$$

since $U'\left(C\right)=\frac{1}{C}$.

Note that we can write: $-\frac{\partial q_0(\cdot,\cdot)}{\partial m}B_0=\frac{\pi_m}{1+r^*}\left[\varepsilon+\left(1-\varepsilon\right)\Omega\right]$, where $\Omega=\frac{f(m+B_0)B_0}{\pi_m}$. Substituting into equation (8), we can find:

$$\frac{\left(1+r^*\right)\frac{dW}{dm}}{U'\left(C_0\right)\pi_m}=-\left[\varepsilon+\left(1-\varepsilon\right)\Omega\right]+\beta\left(1+r^*\right)\left(\frac{m}{C_0}\right)^{-\gamma}=0$$

Solving for m^* yields equation (11) in the paper. When the distribution $F\left(\cdot\right)$ is a uniform, $F\left(m+B_0\right)-F\left(m\right)=f\left(m+B_0\right)B_0$, so $\Omega=1$.

Endogenous contract response

Following the general equilibrium tradition, this paper takes the shape and number of contracts as part of the environment. I now show that the crucial underlying assumption for the results to be valid is that the shape of contracts used remains invariant to the level of m. Rather than specifying some ad-hoc assumption on contracting frictions, I adopt a more abstract approach. I relegate this discussion to the appendix because I will argue that the main sufficient condition that justifies the results of the paper is satisfied in modern economies.

Borrowers now solve, accounting already for the possibility of defaulting:

$$\max_{B_0}U\left(C_0\right)+\beta\int_{\mathcal{D}}U\left(C_1^{\mathcal{D}}\right)dF\left(y_1\right)+\beta\int_{\mathcal{N}}U\left(C_1^{\mathcal{N}}\right)dF\left(y_1\right),$$

subject to

$$\begin{aligned}C_0&=y_0+q_0\left(B_0,m,\left\{\rho\left(y_1,m\right)\right\}\right)B_0\\C_1^{\mathcal{N}}&=y_1-\rho\left(y_1,m\right)B_0\\C_1^{\mathcal{D}}&=\min\left\{y_1,m\right\}\end{aligned}$$

Borrowers repayment at date 1 is given by $\rho(y_1, m) B_0$, where the function $\rho(y_1, m)$, that allows for a full nonlinear contract shape, can be potentially negative, meaning that borrowers could receive insurance from lenders at date 1. I derive the results using a general $\rho(y_1, m)$ and discuss below how choosing $\rho(y_1, m)$ optimally may affect the results.

Note that now the pricing schedule offered by lenders $q_0(\cdot)$ depends on the shape of the contract $\rho(y_1, m)$, potentially in a complicated nonlinear way. To continue using differential techniques, I assume that the shape of repayments $\rho(y_1, m)$, now contingent on the $t = 1$ realization y_1 and potentially on the exemption choice, is well-behaved. The pricing schedule offered by lenders is given by

$$q_0(B_0, m, \{\rho(y_1, m)\}) B_0 = \frac{\delta \int_{\mathcal{D}_m} (y_1 - m) dF(y_1) + \int_{\mathcal{N}} \rho(y_1, m) B_0 dF(y_1)}{1 + r^*}$$

The indifference condition between the default and no-default regions is $m = y_1 - \rho(y_1, m) B_0$. I denote by y_1^* the solution to this equation. Given the shape of the contract $\rho(y_1, m)$ and the exemption level, borrowers choose B_0 optimally according to

$$U'(C_0)\left[q_0(B_0, m) + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0\right] = \beta \int_{\mathcal{N}} \rho(y_1, m) U'(y_1 - B_0) dF(y_1)$$

Proposition 10. (Endogenous contract response)

a) *The marginal welfare change from varying the optimal exemption m when when the shape of contracts can vary with m is given by:*

$$\frac{dW}{dm} = U'(C_0)\left|\frac{dq_0}{dm}\right|_{B_0} B_0 + \beta \int_{\mathcal{D}_m} U'\left(C_1^{\mathcal{D}}\right) dF(y_1) - \beta \int_{\mathcal{N}} U'\left(C_1^{\mathcal{N}}\right) \frac{\partial \rho(y_1, m)}{\partial m} B_0 dF(y_1), \tag{26}$$

where $\left|\frac{dq_0}{dm}\right|_{B_0}$ *corresponds to the chance in price schedule induced by a change in m, holding B_0 constant.*

b) The optimal exemption m^* when borrowers are price takers is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} - \frac{\Pi_{\mathcal{N}}\left\{\partial \rho(y_1, m)/\partial m B_0\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r}, m}},$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r}, m} \equiv \left.\frac{\partial \log(1+r)}{\partial m}\right|_{B_0} = -\left.\frac{\frac{dq_0}{q_0}}{dm}\right|_{B_0}$, $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{D}}}{C_0} \frac{U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF(y_1)$, *and* $\frac{\Pi_{\mathcal{N}}\left\{\frac{\partial \rho(y_1, m)}{\partial m} B_0\right\}}{C_0} \equiv \beta \int_{\mathcal{N}} \frac{\frac{\partial \rho(y_1, m)}{\partial m} B_0}{C_0} \frac{U'\left(C_1^{\mathcal{N}}\right)}{U'(C_0)} dF(y_1)$.

Now a change in the exemption level endogenously changes the shape of the contract used. This creates first-order effects on welfare either by changing the interest rate schedule faced by borrowers and by varying the borrowers' consumption in no default states. In the default regions, the shape of the contract is irrelevant for welfare. It is clear from equation (26) that we recover the results of the baseline model whenever $\frac{\partial \rho(y_1, m)}{\partial m} = 0, \forall y_1, m$. In that case, $\left|\frac{dq_0}{dm}\right|_{B_0} = \frac{\partial q_0}{\partial m}$.

Note that this result is independent of how $\rho(y_1, m)$ is determined. If $\rho(y_1, m)$ can be chosen optimally by borrowers without constraints, they would be able to implement the full insurance first-best outcome. This is not necessarily the case if they face constraints. For instance, we could assume that there is a well-behaved cost function with the form $\mathcal{O}(\{\rho(y_1, m)\})$ that borrowers face when choosing contracts. After $\rho(y_1, m)$ is determined, m can be then chosen costlessly, as assumed throughout the paper. Solving different constrained contracting problems is outside the scope of this paper, but it is easy to see that $\frac{\partial \rho(y_1, m)}{\partial m} = 0$ is sufficient for the results of the text to hold.

Corollary. *Independently of how the shape of the traded contract $\rho(y_1, m)$ is determined, a sufficient condition for the results of proposition 1 (and all others in the paper) to be valid is that the shape of the contract does not change with the level of the exemption, formally:*

$$\frac{\partial \rho(y_1, m)}{\partial m} = 0, \forall y_1, m. \tag{27}$$

To provide more intuition, using further the structure of the two period case, we can rewrite equation (26) as

$$\frac{dW}{dm} = U'(C_0)\frac{\partial q_0}{\partial m}B_0 + \beta\int_{\mathcal{D}_m} U'\left(C_1^{\mathcal{D}}\right)dF(y_1)$$
$$+ \frac{U'(C_0)}{1+r^*}\left[(\delta-1)f(y_1^*)\rho(y_1^*, m)\frac{\frac{\partial\rho}{\partial m}(y_1^*, m)B_0}{1-\frac{\partial\rho}{\partial y_1}(y_1^*, m)B_0} + \int_{\mathcal{N}}\left(1 - \frac{(1+r^*)\beta U'\left(C_1^{\mathcal{N}}\right)}{U'(C_0)}\right)\frac{\partial\rho(y_1, m)}{\partial m}dF(y_1)\right] \tag{28}$$

For the the second line in equation (28) to be nonzero, it is necessary that there are costs associated with bankruptcy, $\delta < 1$, and that the starting point does not entail full insurance, $1 \neq \frac{(1+r^*)\beta U'\left(C_1^{\mathcal{N}}\right)}{U'(C_0)}$, in addition to have $\frac{\partial\rho}{\partial m} \neq 0$. If the changes on the shape of the contract used induced by changes in the exemption level affect the probability of default or improve the insurance opportunities of borrowers, this will have a first order effect on welfare.

How likely is equation (27) to hold? Theoretically, the literature in financial innovation and optimal security design, as in Allen and Gale (1994), studies which contracts are endogenously traded in general equilibrium. They show that, if there are fixed costs associated with trading contracts, only a finite number of contracts will be traded in equilibrium. In that framework, we expect equation (27) to hold in general. It is also the case that many optimal contracting papers find their solution at boundaries, which also supports equation (27). In practice, the fact that we observe the same set of contracts (e.g., debt contracts) being traded in economies with high and low exemption levels suggests that equation (27) holds in modern economies.

Summing up, the framework developed in this paper can accommodate reactions of the shapes of contracts used in equilibrium to changes in exemption levels. There are two important takeaways of this generalization. First, a sufficient condition for all the results of this paper to be valid is that $\frac{\partial\rho(y_1,m)}{\partial m} = 0$, that is, the shape of the contracts used does not change with the level of exemptions, independently of how the shape of the contract is chosen. Second, if this condition does not hold, having additional information about changes in the shape of the contract in no default states and incorporating how the change in the shape of the contract affects the pricing schedule sensitivity, as in equation (26), is sufficient to determine the optimal exemption level.

Proofs: Section 4

Proposition 3. (Endogenous income: frictionless markets)

a) Applying repeatedly Leibniz' rule and using the optimality condition B_0, the default decision and first-order condition on labor supply $N_1^{\mathcal{N}}$ and N_0 and the optimal effort choice a, we can write:

$$\frac{dW}{dm} = \frac{\partial U}{\partial C_0}(C_0, N_0; a)\frac{\partial q_0(B_0, m, a)}{\partial m}B_0 + \beta\int_{D_m}\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right)dF(y_1; a),$$

where $\frac{\partial U}{\partial C_0}(C_0, N_0; a)$ denotes the partial derivative or flow utility with respect to consumption — equivalently with $\frac{\partial U}{\partial C_1}\left(C_1^{\mathcal{D}}, 0\right)$. Note that the derivation of the optimality conditions in the text use the fact that $\frac{d\tilde{V}}{dB_0} = -\frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)$.

Figure A.1 illustrates graphically how to characterize the default region when borrowers' labor supply decision is nontrivial. Although it is impossible to solve explicitly for the default region, its characterization is conceptually identical to the baseline model.

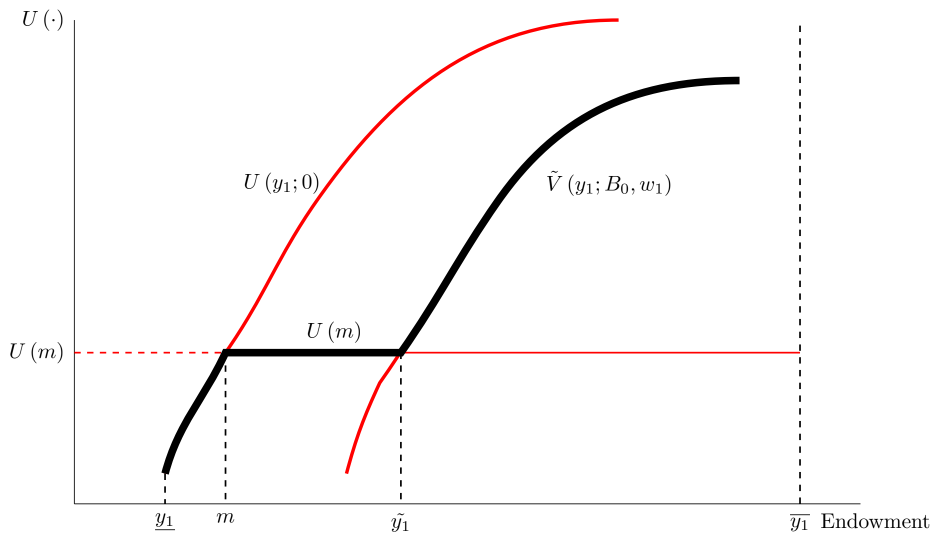

Figure A.1: Optimal default decision given B_0

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 4. (Non-pecuniary penalties/state contingent utility)

a) The bankruptcy region is characterized a new indifference condition $\phi m = y_1 - B_0$. The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model.

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 5. (Epstein-Zin utility)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. I define Q, which denotes the certainty equivalent of consumption at $t=1$, as:

$$Q \equiv \left(\int_{\underline{y_1}}^{m} (y_1)^{1-\gamma} dF(y_1) + \int_{m}^{m+B_0} (m)^{1-\gamma} dF(y_1) + \int_{m+B_0}^{\overline{y_1}} (y_1 - B_0)^{1-\gamma} dF(y_1) \right)^{\frac{1}{1-\gamma}}$$

Taking derivatives in equation (14), assuming that $\frac{\partial q_0}{\partial B_0} \approx 0$, yields:

$$\left.\frac{dB_0}{dq_0}\right|_{\frac{\partial q_0}{\partial B_0}\approx 0} = \frac{(C_0)^{-\frac{1}{\psi}}\left[1-\frac{\Lambda}{\psi}\right]}{-SOC},$$

where

$$SOC = \beta\left[(q_0)^2 \frac{1}{\psi}(y_0 + q_0 B_0)^{-\frac{1}{\psi}-1} + \int_{m+B_0}^{\overline{y_1}} \gamma (y_1 - B_0)^{-\gamma-1} dF(y_1) + (y_1 - B_0)^{-\gamma} f(m+B_0)\right] < 0.$$

This shows that a change in interest rates (keeping interest rate schedule elasticities fixed) has an income and a substitution effect. The substitution effect, which determines the sensitivity of borrowing to interest rate changes, dominates when $\psi > \Lambda$. Hence, $\psi > 1$, the standard parametrization in asset pricing models, implies that low interest rates increase borrowing and viceversa.

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

Proposition 6. (Multiple contracts with arbitrary payoffs)

a) As stated in the paper, there are three regions depending on the realization of y_1. The indifference condition between the regions $\mathcal{D}_m$ and $\mathcal{N}$ in the range $y_1 + \sum_{j=1}^{J} \max\left\{-z_j(y_1) B_{0j}, 0\right\} > m$ is given by $m = y_1 + \sum_{j=1}^{J} \max\left\{-z_j(y_1) B_{0j}, 0\right\} - \sum_{j=1}^{J} \max\left\{z_j(y_1) B_{0j}, 0\right\}$. The indifference condition between the regions $\mathcal{D}_y$ and $\mathcal{N}$ in the range $y_1 + \sum_{j=1}^{J} \max\left\{-z_j(y_1) B_{0j}, 0\right\} \leq m$ is given by $0 = \sum_{j=1}^{J} \max\left\{z_j(y_1) B_{0j}, 0\right\}$. When $\sum_{j=1}^{J} \max\left\{z_j(y_1) B_{0j}, 0\right\} > 0$, at $y_1 = m$ there is a new boundary separating the $\mathcal{D}_m$ and $\mathcal{D}_y$ regions — this doesn't occur for instance in figure 3 in the text. This characterization decomposes the possible set of realizations in $\left[\underline{y_1}, \overline{y_1}\right]$ into multiple nonoverlapping intervals.

Indirect utility as a function of the exemption is written as:

$$W(m) \equiv U\left(y_0 + \sum_j q_{0j}(B_{01}, \ldots, B_{0J}, m) B_{0j}\right)$$
$$+\beta \left[\begin{array}{c} \int_{\mathcal{D}_y} U\left(y_1 + \sum_{j=1}^{J} \max\left\{-z_j(y_1) B_{0j}, 0\right\}\right) dF(y_1) + \int_{\mathcal{D}_m} U(m)\, dF(y_1) \\ + \int_{\mathcal{N}} U\left(y_1 + \sum_{j=1}^{J} \max\left\{-z_j(y_1) B_{0j}, 0\right\} - \sum_{j=1}^{J} \max\left\{z_j(y_1) B_{0j}, 0\right\}\right) dF(y_1) \end{array}\right]$$

The derivation of equation (17) follows the same steps as the baseline model. It requires the use of J optimality conditions for the J different contracts traded.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 7. (Heterogeneous borrowers: observable heterogeneity)

a) Only for simplicity, I still assume a) that $\lim_{B_0 \to 0^-} \frac{dJ}{dB_0} > 0$, which implies that it is never optimal to save, and b) that the problem is well-behaved in the region $B_0 > 0$. Formally, the price schedule becomes discontinuous at $B_0 = 0$, because:

$$\lim_{B_0 \to 0^+} q_{0i}(B_0, m) = \frac{\int_m^{\overline{y_1}} dF(y_1)}{1 + r^*} \qquad \text{and} \qquad \lim_{B_0 \to 0^-} q_{0i}(B_0, m) = \frac{1}{1 + r^*}$$

It is easy to show that $J(B_0, m)$ is continuous everywhere, but it is not differentiable at $B_0 = 0$. Given these assumptions, we can define:

$$I_A(m) = \left\{ i \,\middle|\, \frac{U_i'(y_{0i})}{1 + r^*} \int_m^{\overline{y_{1i}}} dF_i(y_{1i}) - \beta_i \int_{\underline{y_{1i}}}^{\overline{y_{1i}}} U'(y_{1i})\, dF_i(y_{1i}) > 0 \right\}$$
$$I_N(m) = \left\{ i \,\middle|\, \frac{U_i'(y_{0i})}{1 + r^*} \int_m^{\overline{y_{1i}}} dF_i(y_{1i}) - \beta_i \int_{\underline{y_{1i}}}^{\overline{y_{1i}}} U'(y_{1i})\, dF_i(y_{1i}) \leq 0 \right\}$$

Therefore, for every level of m we can define two regions of borrowers. Some of those will decide to borrow, but others will decide not to. Because $\int_m^{\overline{y_{1i}}} dF_i(y_{1i}) = 1 - F(m)$ is decreasing in m, the measure of active borrowers decreases with the exemption level.

The derivation of $\frac{dW}{dm}$ in this case uses the fact that, for an indifferent borrower, $\frac{U_i'(y_{0i})}{1+r^*} \frac{\partial q_{0i}(B_0,m)}{\partial m} = \beta_i \int_{\underline{y_{1i}}}^{\overline{y_{1i}}} U'(y_{1i})\, dF_i(y_{1i})$.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 8. (Heterogeneous borrowers: unobservable heterogeneity)

See online appendix.

Proposition 9. (Dynamics)

See online appendix.

References

Aguiar, Mark, and Erik Hurst. 2007. "Measuring Trends in Leisure: The Allocation of Time Over Five Decades." *The Quarterly Journal of Economics*, 122(3): 969–1006.

Aguiar, Mark, and Manuel Amador. 2013. "Sovereign debt." *Handbook of International Economics*, 4.

Albanesi, Stefania, and Jaromir Nosal. 2015. "Insolvency after the 2005 Bankruptcy Reform." *Working Paper*.

Albuquerque, R., and H.A. Hopenhayn. 2004. "Optimal lending contracts and firm dynamics." *Review of Economic Studies*, 71(2): 285–315.

Allen, F., and D. Gale. 1994. *Financial innovation and risk sharing.* MIT press.

Alvarez, Fernando, and Urban J. Jermann. 2000. "Efficiency, equilibrium, and asset pricing with risk of default." *Econometrica*, 68(4): 775–797.

Alvarez, Fernando, and Urban J Jermann. 2004. "Using Asset Prices to Measure the Cost of Business Cycles." *Journal of Political Economy*, 112(6): 1223–1256.

Arkolakis, Costas, Arnaud Costinot, and Andrés Rodríguez-Clare. 2012. "New Trade Models, Same Old Gains?" *American Economic Review*, 102(1): 94–130.

Atkinson, Anthony Barnes, and Joseph E Stiglitz. 1980. "Lectures on public economics."

Attanasio, O.P., and G. Weber. 1989. "Intertemporal substitution, risk aversion and the Euler equation for consumption." *The Economic Journal*, 99(395): 59–73.

Bernanke, B.S., M. Gertler, and S. Gilchrist. 1999. "The financial accelerator in a quantitative business cycle framework." *Handbook of macroeconomics*, 1: 1341–1393.

Bisin, Alberto, and Adriano Rampini. 2006. "Exclusive contracts and the institution of bankruptcy." *Economic Theory*, 27(2): 277–304.

Bizer, D.S., and P.M. DeMarzo. 1999. "Optimal incentive contracts when agents can save, borrow, and default." *Journal of Financial Intermediation*, 8(4): 241–269.

Bolton, P., and D.S. Scharfstein. 1990. "A theory of predation based on agency problems in financial contracting." *The American Economic Review*, 93–106.

Bolton, Patrick, and Olivier Jeanne. 2007. "Structuring and restructuring sovereign debt: the role of a bankruptcy regime." *Journal of Political Economy*, 115(6): 901–924.

Bolton, Patrick, and Olivier Jeanne. 2009. "Structuring and restructuring sovereign debt: The role of seniority." *The Review of Economic Studies*, 76(3): 879–902.

Bond, Philip, and Arvind Krishnamurthy. 2004. "Regulating exclusion from financial markets." *The Review of Economic Studies*, 71(3): 681–707.

Campbell, J.Y. 2003. "Consumption-based asset pricing." *Handbook of the Economics of Finance*, 1: 803–887.

Chari, VV, P.J. Kehoe, and E.R. McGrattan. 2007. "Business Cycle Accounting." *Econometrica*, 781–836.

Chatterjee, S., D. Corbae, M. Nakajima, and J.V. Ríos-Rull. 2007. "A quantitative theory of unsecured consumer credit with risk of default." *Econometrica*, 75(6): 1525–1589.

Chetty, Raj. 2008. "Moral Hazard versus Liquidity and Optimal Unemployment Insurance." *Journal of Political Economy*, 116(2): 173–234.

Chetty, Raj. 2009. "Sufficient Statistics for Welfare Analysis: A Bridge Between Structural and Reduced-Form Methods." *Annual Review of Economics*, 1(1): 451–488.

Chien, Y.L., and H. Lustig. 2010. "The market price of aggregate risk and the wealth distribution." *Review of Financial Studies*, 23(4): 1596–1650.

Claessens, Stijn, and Leora F Klapper. 2005. "Bankruptcy around the world: Explanations of its relative use." *American Law and Economics Review*, 7(1): 253–283.

Diamond, D.W. 1984. "Financial intermediation and delegated monitoring." *The Review of Economic Studies*, 51(3): 393–414.

Diamond, P.A. 1998. "Optimal income taxation: an example with a U-shaped pattern of optimal marginal tax rates." *American Economic Review*, 83–95.

Dobbie, Will, and Jae Song. 2013. "Debt Relief and Debtor Outcomes: Measuring the Effects of Consumer Bankruptcy Protection." *Harvard University Working Paper*.

Dobbie, Will, and Paul Goldsmith-Pinkham. 2014. "Debtor Protections and the Great Recession."

Drozd, Lukasz A., and Ricardo Serrano-Padial. 2014. "Financial Contracting with Enforcement Externalities." *Working Paper*.

Dubey, P., J. Geanakoplos, and M. Shubik. 2005. "Default and Punishment In General Equilibrium." *Econometrica*, 73(1): 1–37.

Duffie, D., and R. Rahi. 1995. "Financial market innovation and security design: An introduction." *Journal of Economic Theory*, 65(1): 1–42.

Eaton, Jonathan, and Mark Gersovitz. 1981. "Debt with potential repudiation: Theoretical and empirical analysis." *The Review of Economic Studies*, 48(2): 289–309.

Farhi, Emmanuel, and Iván Werning. 2014. "A theory of macroprudential policies in the presence of nominal rigidities." NBER Working Paper.

Fay, S., E. Hurst, and M.J. White. 2002. "The household bankruptcy decision." *American Economic Review*, 706–718.

Filer, Larry H, and Jonathan D Fisher. 2005. "The consumption effects associated with filing for personal bankruptcy." *Southern Economic Journal*, 837–854.

Froot, Kenneth A., David S. Scharfstein, and Jeremy C. Stein. 1993. "Risk management: coordinating corporate investment and financing decisions." *Journal of Finance*, 48(5): 1629–1658.

Gale, D., and M. Hellwig. 1985. "Incentive-compatible debt contracts: The one-period problem." *The Review of Economic Studies*, 52(4): 647–663.

Gale, D., and P. Gottardi. 2011. "Bankruptcy, Finance Constraints, and the Value of the Firm." *American Economic Journal: Microeconomics*, 3(2): 1–37.

Gali, J., M. Gertler, and J.D. Lopez-Salido. 2007. "Markups, gaps, and the welfare costs of business fluctuations." *the Review of Economics and Statistics*, 89(1): 44–59.

Grochulski, B. 2010. "Optimal personal bankruptcy design under moral hazard." *Review of Economic Dynamics*, 13(2): 350–378.

Gropp, R., J.K. Scholz, and M.J. White. 1997. "Personal bankruptcy and credit supply and demand." *The Quarterly Journal of Economics*, 112(1): 217–251.

Gross, D.B., and N.S. Souleles. 2002. "Do Liquidity Constraints and Interest Rates Matter for Consumer Behavior? Evidence from Credit Card Data*." *The Quarterly journal of economics*, 117(1): 149–185.

Hart, O. 1995. *Firms, contracts, and financial structure.* Oxford University Press, USA.

Hart, O., and J. Moore. 1994. "A Theory of Debt Based on the Inalienability of Human Capital." *The Quarterly Journal of Economics*, 841–879.

Hart, O., and J. Moore. 1998. "Default and renegotiation: A dynamic model of debt." *The Quarterly Journal of Economics*, 113(1): 1–41.

Herkenhoff, Kyle. 2012. "The Role of Default (not Bankruptcy) as Unemployment Insurance: New Facts and Theory." *Working Paper*.

Hopenhayn, H., and I. Werning. 2008. "Equilibrium default." *Manuscript, MIT*.

Iverson, Benjamin. 2013. "Get in Line: Chapter 11 Restructuring in Crowded Bankruptcy Courts." *Harvard University Working Paper*.

Jaffee, Dwight, and Joseph Stiglitz. 1990. "Credit rationing." *Handbook of Monetary Economics*, 2: 837–888.

Jappelli, T., M. Pagano, and M. Bianco. 2005. "Courts and Banks: Effects of Judicial Enforcement on Credit Markets." *Journal of Money, Credit and Banking*, 223–244.

Kaplan, Greg, and Giovanni L Violante. 2014. "A model of the consumption response to fiscal stimulus payments." *Econometrica*, 82(4): 1199–1239.

Kehoe, Timothy J., and David K. Levine. 1993. "Debt-constrained asset markets." *The Review of Economic Studies*, 60(4): 865.

Koeppl, Thorsten, Cyril Monnet, and Erwan Quintin. 2014. "Efficient contract enforcement." *Economic Theory*, 55(1): 161–183.

Kovrijnykh, Natalia. 2013. "Debt contracts with partial commitment." *The American Economic Review*, 103(7): 2848–2874.

Krasa, Stefan, and Anne P Villamil. 2000. "Optimal contracts when enforcement is a decision variable." *Econometrica*, 68(1): 119–134.

Livshits, I. 2014. "Recent Developments in Consumer Credit and Default Literature." *Working Paper*.

Livshits, I., J. MacGee, and M. Tertilt. 2007. "Consumer bankruptcy: A fresh start." *The American Economic Review*, 97(1): 402–418.

Ljungqvist, L., and T.J. Sargent. 2004. *Recursive macroeconomic theory.* The MIT press.

Lucas, R.E. 1978. "Asset prices in an exchange economy." *Econometrica: Journal of the Econometric Society*, 1429–1445.

Lupica, Lois R. 2012. "The Consumer Bankruptcy Fee Study: Final Report." *Am. Bankr. Inst. L. Rev.*, 20: 17–759.

Mahoney, Neale. 2012. "Bankruptcy as implicit health insurance."

Rampini, A., and S. Viswanathan. 2010. "Collateral, risk management, and the distribution of debt capacity." *The Journal of Finance*, 65(6): 2293–2322.

Rampini, Adriano. 2005. "Default and aggregate income." *Journal of Economic Theory*, 122(2): 225–253.

Saez, Emmanuel. 2001. "Using elasticities to derive optimal income tax rates." *The Review of Economic Studies*, 68(1): 205–229.

Severino, Felipe, Meta Brown, and Brandi Coates. 2013. "Personal Bankruptcy Protection and Household Debt." *Working Paper*.

Shimer, Robert. 2009. "Convergence in macroeconomics: The labor wedge." *American Economic Journal: Macroeconomics*, 1(1): 280–297.

Shimer, Robert, and Ivan Werning. 2007. "Reservation wages and unemployment insurance." *The Quarterly Journal of Economics*, 122(3): 1145–1185.

Skeel, David A. 2001. *Debt's dominion: A history of bankruptcy law in America.* Princeton University Press.

Tallarini, Thomas D. 2000. "Risk-sensitive real business cycles." *Journal of Monetary Economics*, 45(3): 507–532.

Townsend, R. 1979. "Optimal contracts and competitive markets with costly state verification." *Journal of Economic Theory*, 21(2): 265–93.

White, Michelle J. 1998. "Why it pays to file for bankruptcy: A critical look at the incentives under US personal bankruptcy law and a proposal for change." *The University of Chicago Law Review*, 685–732.

White, Michelle J. 2007. "Bankruptcy Reform and Credit Cards." *The Journal of Economic Perspectives*, 21(4): 175–200.

White, Michelle J. 2011. "Corporate and Personal Bankruptcy Law." *Annu. Rev. Law Soc. Sci*, 7: 139–64.

Zame, W.R. 1993. "Efficiency and the role of default when security markets are incomplete." *The American Economic Review*, 1142–1164.

Online appendix (not for publication)

A Numerical example

I illustrate the equilibrium of the Epstein-Zin version of the model with a numerical example, using the parameters given in table 4. The distribution $F(\cdot)$ is a log-normal with parameters μ and σ. This is not a calibration exercise and merely an illustration of the analytical results derived in the paper.

Preferences	$\gamma = 10$	$\psi = 1.5$	$\beta = 0.96$	$r^* = 4\%$
Endowments	$y_0 = 55$	$\mu = 4.9$	$\sigma = 0.095$	
Bankruptcy	$\delta = 0.1$			

Table 4: Parameters numerical example

Figure A.1 shows how the interest rate schedule $q_0(B_0, m)$ changes with B_0 and m respectively, keeping one of the two variables constant. For the first figure, I assume that $m = 80$ and, for the second one, I assume that $B_0 = 23$. It corresponds with the maximand in equation (2) in the text. As shown analytically is equation (3), both functions are downward sloping.

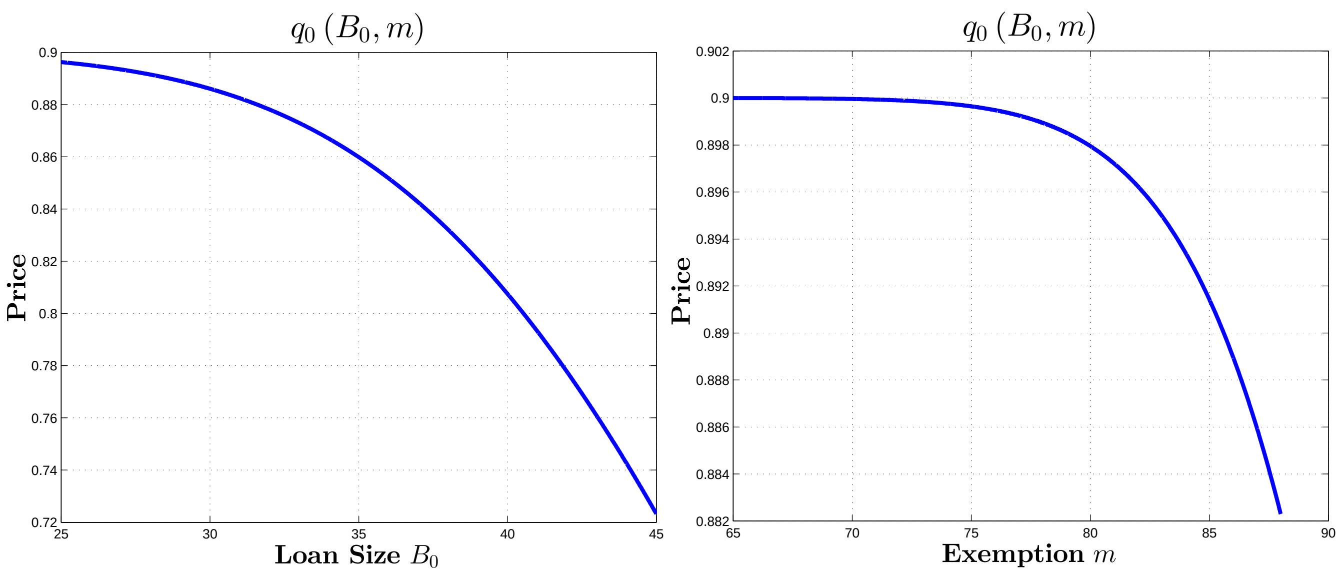

Figure A.1: Interest rate schedule $q_0(B_0, m)$

Figure A.2 shows the function maximized by borrowers at $t = 0$ when they have to choose B_0 for a given level of m. I assume $m = 80$. It corresponds with equation (4) in the text and (22) in the appendix.

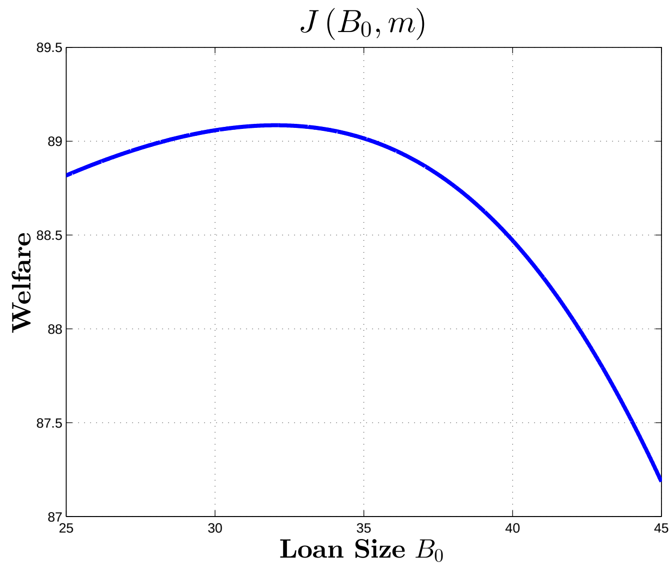

Figure A.2: Borrowers objective function $J(B_0, m = 80)$

The left plot in figure A.3 shows the optimal choice of B_0 for different levels of m. It corresponds with equation (5) in the text. As described in the text, $\frac{dB_0}{dm}$ can take any sign depending on the strength of the effects mentioned. I chose this particular parametrization to illustrate how $B_0(m)$ can increase or decrease on m — we observe a form of "Laffer curve". The right plot in figure A.3 shows borrowers welfare for a different levels of m. It corresponds with equation (7) in the text. The optimal exemption m^* is the value that maximizes that expression.

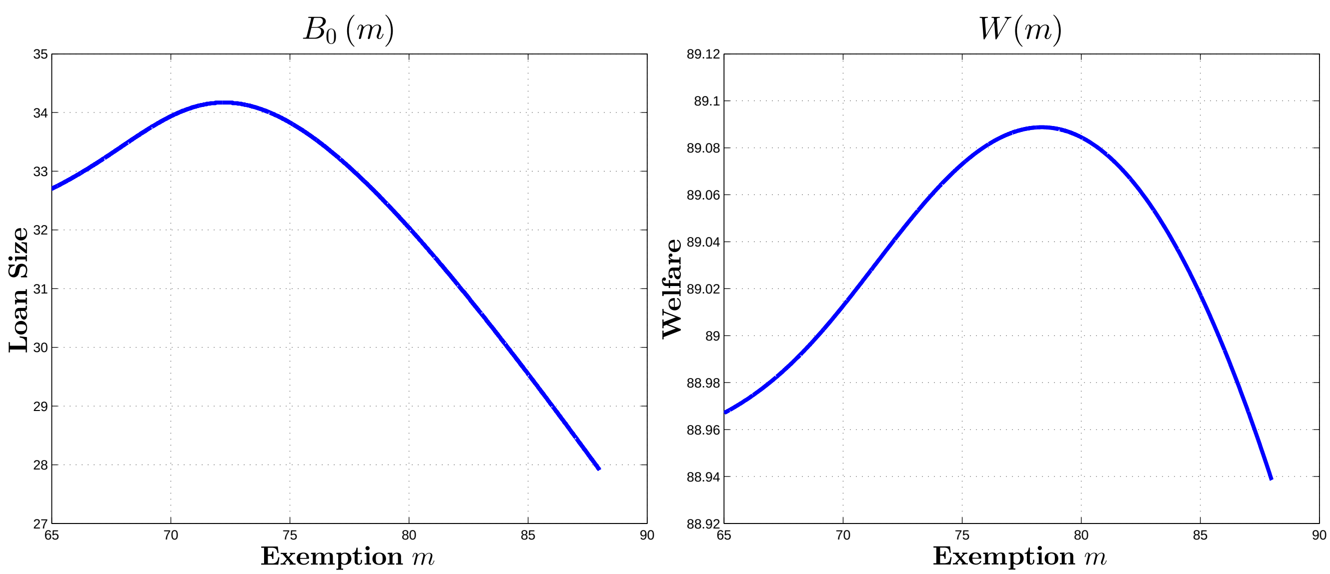

Figure A.3: Optimal borrowing choice $B_0(m)$ and indirect utility $W(m)$

In this example, $m^* = 78.35$, $B_0(m^*) = 32.74$, $\Lambda = 0.34$ and $\pi_D = 2.29\%$.

B Additional extensions

B.1 Price taking borrowers

In the baseline model, borrowers take into account the pricing schedule offered by lenders when they choose how much to borrow. Alternatively, we can assume that borrowers perceive that they can borrow at a constant rate, at least within a given range. I now explore that possibility. I adopt an equilibrium notion in which borrowers provide a loan demand, given interest rates, and lenders provide a loan supply schedule, which determines a standard competitive equilibrium.[25]

The ex-post default decision is identical to the baseline model. The ex-ante behavior of borrowers is now captured by the following Euler equation:

$$U'(C_0)\, q_0 = \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1) \tag{29}$$

This condition provides a demand for credit, which combined the supply schedule $q_0(B_0, m)$ characterizes the equilibrium. In this case, when borrowers' EIS is greater than unity ($\psi > 1$), it can be shown that the demand for credit increases with low interest rates and viceversa. It can also be shown that the equilibrium interest rates always increase with the exemption level, that is, $\frac{dq_0}{dm} < 0$. Unsurprisingly, the sign of $\frac{dB_0}{dm}$ remains indeterminate.

Proposition 11. (Price taking borrowers)
a) *The marginal welfare change from varying the optimal exemption m when when borrowers are price takers is given by:*

$$\frac{dW}{dm} = U'(C_0)\frac{dq_0}{dm}B_0 + \int_m^{m+B_0} \beta U'\left(C_1^{\mathcal{D}}\right) dF(y_1)$$

b) The optimal exemption m^* when borrowers are price takers is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}},$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm} = -\frac{\frac{dq_0}{q_0}}{dm}$ *and* $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \beta \int_m^{m+B_0} \frac{C_1^{\mathcal{D}}}{C_0}\frac{U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF(y_1)$.

There is an important difference between the results of the baseline model, in which borrowers internalize the effect of borrowing on interest rates, and the results in which borrowers are price takers: the relevant variable to measure the sensitivity of interest rates to changes in exemptions is now the full equilibrium response of interest rates, which also includes the change in interest rates induced by equilibrium changes in total borrowing. Equation (6) relates both terms in equilibrium. Intuitively, because price taking borrowers fail to internalize that higher borrowing increases interest rates, this effect must be accounted for when measuring welfare. This extension shows that the assumptions on the behavior of agents are important to derive the results in this paper.

B.2 Bankruptcy exemptions contingent on aggregate risk

I now allow for aggregate shocks and ask how bankruptcy exemptions should vary depending on the state of economy. As long as agents use contracts that do not condition on aggregate shocks, it is optimal

[25] The problem of price taking borrowers has two local optima. One is characterized by an interior first-order condition. The second one entails borrowing as much as possible. When the feasible set of B_0 is unbounded above, borrowers credit demand is infinite at any given rate, preventing the existence of an equilibrium. I assume instead that borrowers judiciously set an upper bound on the total amount of credit so that the interior optimum is also global. Eaton and Gersovitz (1981) use a similar approach to guarantee an interior solution in a related environment. In practice, we observe that credit card lenders offer fixed rates and credit limits, which matches this behavior. A previous version of this paper used the price taking case as the benchmark: detailed results for that case are available under request.

to set bankruptcy exemptions which vary with the realization of the aggregate state. Assume that borrowers income now takes the form $y_1 = A(\omega) + \tilde{y}_1(\omega)$. The realization of the aggregate shock is denoted by ω, which can take a finite number of values $\omega \in \Omega$ with probability $p(\omega)$. $A(\omega)$ and $\tilde{y}_1(\omega)$ denote aggregate and idiosyncratic endowment shocks. The distribution of idiosyncratic shocks $F_\omega(\cdot)$ can vary with ω, freeing the correlation patterns between aggregate and idiosyncratic shocks. I also allow for exemptions $m(\omega)$ contingent on the value of the aggregate shock.

Given ω, the logic used to characterize the default region is identical to the baseline model. Hence, borrowers solve:

$$\max_{B_0} U(y_0 + q_0 B_0) + \beta \sum_{\omega} p(\omega) \begin{bmatrix} \int_{\underline{y_1(\omega)}}^{m(\omega)} U(y_1(\omega))\, dF_\omega(y_1(\omega)) \\ + \int_{m(\omega)}^{m(\omega)+B_0} U(m(\omega))\, dF_\omega(y_1(\omega)) \\ + \int_{m(\omega)+B_0}^{\overline{y_1(\omega)}} U(y_1 - B_0)\, dF_\omega(y_1(\omega)) \end{bmatrix}$$

So they borrow according to:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, \{m(\omega)\})}{\partial B_0} B_0\right] = \beta \sum_{\omega} p(\omega) \left[\int_{m(\omega)+B_0}^{\overline{y_1(\omega)}} U'(y_1 - B_0)\, dF_\omega(y_1(\omega))\right]$$

Under natural assumptions, for instance, under lenders' risk neutral, the pricing schedule offered can be written as:

$$q_0(B_0, \{m(\omega)\}) = \sum_{\omega} p(\omega)\, q_0^{\omega}(B_0, m(\omega)),$$

where $q_0^{\omega}(B_0, m(\omega))$ is the pricing schedule of a security that only pays in state ω. This allows us to write $\frac{\partial q_0(B_0,\{m(\omega)\})}{\partial m(\omega)} = p(\omega) \frac{\partial q_0^{\omega}(B_0,m(\omega))}{\partial m(\omega)}$

Proposition 12. (Bankruptcy exemption contingent on aggregate risk)

a) *The marginal welfare change from varying the optimal state contingent exemption $m(\omega)$ is given by:*

$$\frac{\partial W(\{m(\omega)\})}{\partial m(\omega)} = p(\omega)\left[U'(C_0)\frac{\partial q_0^{\omega}(B_0, m(\omega))}{\partial m(\omega)} B_0 + \beta \int_{m(\omega)}^{m(\omega)+B_0} U'\left(C_1^{\mathcal{D}}\right) dF_\omega(y_1(\omega))\right], \qquad \forall \omega \quad (30)$$

b) The optimal state contingent exemptions $m^*(\omega)$ are given by:

$$m^*(\omega) = \frac{\frac{\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m(\omega)}}, \quad \forall \omega$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m(\omega)} \equiv -\frac{\frac{\partial q_0^{\omega}(B_0,m(\omega))}{q_0^{\omega}}}{\partial m}$ *and* $\frac{\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}}{C_0} \equiv \int_{m(\omega)}^{m(\omega)+B_0} \frac{C_1^{\mathcal{D}}}{C_0} \frac{\beta U'\left(C_1^{\mathcal{D}}\right)}{U'(C_0)} dF_\omega(y_1(\omega))$.

The set of optimal state contingent exemptions is jointly determined by (30). The marginal gain/loss of varying the exemption in state ω is now proportional to the probability $p(\omega)$ of that state occurring. Conditional on the state occurring, the tradeoff is identical to the baseline model. The pro- or countercyclicality of exemptions depends on how price-consumption ratios and interest rate sensitivities vary across states ω. If, all else constant, bankruptcies increase in recessions, as we expect to occur, $\Pi_{m(\omega)}\left\{C_1^{\mathcal{D}}\right\}$ is high in aggregate downturns. This calls for countercyclical penalties under standard utility specifications, as long as interest rate sensitivities $\varepsilon_{\tilde{r},m(\omega)}$ do not vary significantly across aggregate states, the natural assumption for risk neutral lenders.[26]

[26]There is scope to explore how optimal exemptions must be determined in a coinsurance problem between which risk averse borrowers and lenders who face credit constraints (which makes them effectively risk averse).

B.3 Endogenous income: labor wedges and aggregate demand

The derivation of the optimal exemption formula in section 4.1 assumed no frictions on the production side of the economy. I now show how the existence of a welfare relevant labor wedge modifies the optimal exemption formula. I keep the same structure for aggregate shocks as in the previous extension, but return to the constant exemption benchmark. For simplicity, output is only produced at $t = 1$ with a constant returns to scale production function $Y = AN$, where A is a constant productivity parameter. At the first-best, we expect the condition $w_1(\omega) = A$ to hold. I assume instead that $w_1(\omega) = (1+\tau(\omega))A$, where $\tau(\omega)$ is the labor wedge in this economy, which can emerge under nominal rigidities and imperfect monetary policy.[27] There is one-to-one relation between the labor wedge and the output gap. When the labor wedge is positive, the economy is infra utilizing its resources, and viceversa when negative. Firms profits are returned back to households. As before, all labor income is garnished in bankruptcy, so borrowers opt for not working then.[28]

Given ω, the logic used to characterize the default region is identical to the baseline model. Hence, borrowers solve:

$$\max_{B_0} U(C_0) + \beta \sum_{\omega} p(\omega) \mathbb{E}\left[V(C_1(\omega), N_1(\omega))\right| \omega],$$

$$\text{s.t.} \qquad C_0 = y_0 + q_0 B_0; \quad C_1^{\mathcal{N}}(\omega) = y_1 + w_1(\omega) N_1^{\mathcal{N}}(\omega) - B_0 + \pi_1(\omega); \quad C_1^{\mathcal{D}} = \min\{y_1, m\},$$

$V(C_1, N_1) = \max_{\xi\in\{0,1\}} \left\{\xi \max_{C_1^{\mathcal{D}}, N_1^{\mathcal{D}}} U\left(C_1^{\mathcal{D}}, N_1^{\mathcal{D}}\right) + (1-\xi)\max_{C_1^{\mathcal{N}}, N_1^{\mathcal{N}}} U\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)\right\}$. As before, conditional on the aggregate shock, three regions for default arise. Borrowers' labor supply and borrowing choices are characterized by:

$$U'(C_0)\left[q_0 + \frac{\partial q_0(B_0, m)}{\partial B_0} B_0\right] = \beta \sum_{\omega} p(\omega)\left[\int_{\mathcal{N}(\omega)} U\left(C_1^{\mathcal{N}}(\omega), N_1^{\mathcal{N}}(\omega)\right) dF_\omega(y_1(\omega))\right]$$

$$w_1(\omega)\frac{\partial U}{\partial C}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right) = -\frac{\partial U}{\partial N}\left(C_1^{\mathcal{N}}, N_1^{\mathcal{N}}\right)$$

This last condition, combined with the (effective) labor demand function $w_1(\omega) = (1+\tau(\omega))A$, pins down the equilibrium on the production side economy.

Proposition 13. (Endogenous income: labor wedges and aggregate demand)

a) *The marginal welfare change from varying the optimal exemption m when there is a nonzero labor wedge is given by:*

$$\frac{dW}{dm} = U'(C_0)\frac{\partial q_0}{\partial m} B_0 + \beta \sum_{\omega} p(\omega)\left[\begin{array}{c} \int_{\mathcal{D}_m(\omega)} U'\left(C_1^{\mathcal{D}}(\omega), 0\right) dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}(\omega)} \frac{\partial U(C_1(\omega), N_1(\omega))}{\partial C_1(\omega)} dF_\omega(y_1(\omega))\, \tau(\omega)\frac{dY(\omega)}{dm}\end{array}\right],$$

b) The optimal exemption m^* when there is a nonzero labor wedge is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} + \frac{\Pi_{\mathcal{N}}\left\{\tau(\omega)\frac{dY(\omega)}{\frac{dm}{m}}\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{31}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv -\frac{\frac{\partial q_0}{q_0}}{\partial m}$, $\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0} = \beta\sum_\omega p(\omega)\left[\int_{\mathcal{D}_m(\omega)} \frac{C_1^{\mathcal{D}}(\omega)}{C_0}\frac{\frac{\partial U\left(C_1^{\mathcal{D}}(\omega), 0\right)}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega(y_1(\omega))\right]$ *and* $\frac{\Pi_{\mathcal{N}}\left\{\tau(\omega)\frac{dY(\omega)}{\frac{dm}{m}}\right\}}{C_0} = \beta\sum_\omega p(\omega)\left[\int_{\mathcal{N}} \frac{\frac{\partial U(C_1(\omega), N_1(\omega))}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega(y_1(\omega))\frac{\tau(\omega)}{C_0}\frac{dY(\omega)}{\frac{dm}{m}}\right]$.

[27] A full-fledged New-Keynesian microfoundation is straightforward to develop, so I use the New-Keynesian terminology when describing the results. See, for instance, Gali, Gertler and Lopez-Salido (2007), Chari, Kehoe and McGrattan (2007) or Shimer (2009) to understand how labor wedges may arise. For the analysis of this section, I am assuming that labor wedges are caused by welfare reducing frictions and not by (efficient) preference shocks.

[28] There is scope to further analyze the interaction of bankruptcy exemptions and aggregate demand effects in a quantitative dynamic model. See Dobbie and Goldsmith-Pinkham (2014) for microeconometric work along those lines.

A new term $\frac{\Pi_{\mathcal{N}}\left\{\tau(\omega)\frac{dY(\omega)}{\frac{dm}{m}}\right\}}{C_0}$ that was not present in the baseline model emerges in the numerator of the optimal exemption formula. In general, high exemptions are welfare improving when $\mathbb{C}ov\left(\tau\left(\omega\right), \frac{dY(\omega)}{dm}\right) > 0$. For instance, when there is a demand shortfall, that is, $\tau\left(\omega\right) > 0$, and high bankruptcy exemptions boost aggregate demand, that is, $\frac{dY(\omega)}{dm} > 0$, there is an additional rational to increase bankruptcy exemptions, since they alleviate macroeconomic fluctuations. Using the language of Farhi and Werning (2014), the second term in the numerator of equation (31) can be interpreted as a macroprudential correction to the optimal exemption.[29]

B.4 Externalities

Assume that social welfare now includes an additional cost in terms of utility that is only paid when there is default. Hence the optimal exemption solves:

$$\max_{m} W^S \equiv W\left(m\right) - \Theta F\left(B_0 + m\right)$$

Where $W\left(m\right)$ is the same as in equation (7), $\Theta > 0$ is a scalar capturing the internalized social cost and $F\left(B_0 + m\right)$ is the probability of default. The change in welfare induced by a change in m is given by:

$$\frac{dW}{dm} = U'\left(C_0\right)\frac{\partial q_0}{\partial m}B_0 + \beta\int_{\underline{y_1}}^{m+B_0} y_1 U'\left(C_1^{\mathcal{D}}\right)dF\left(y_1\right) - \Theta f\left(m+B_0\right)$$

The optimal exemption is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda\varepsilon_{q_0} + \frac{\Theta f(m+B_0)}{U'(C_0)C_0}}$$

Because, $\Theta f\left(m+B_0\right) > 0$, when there are externalities associated with bankruptcy, the optimal m^* is lower, to reduce bankruptcy in equilibrium.

B.5 Risk averse lenders

Allowing for competitive risk averse lenders is straightforward. We can think of many competitive lenders owned by the same set of well diversified investors, who pin down the stochastic discount factor effectively used by lenders. The interest rate schedule becomes:

$$q_0\left(B_0, m\right) = \frac{\delta\int_{\mathcal{D}}\frac{\max\{y_1-m,0\}}{B_0}dH\left(y_1\right) + \int_{\mathcal{N}}dH\left(y_1\right)}{1+r^*},$$

where $H\left(\cdot\right)$ is a well-behaved cdf (an absolutely continuous change of measure with respect to $F(\cdot)$). A similar approach can be followed in the case with aggregate shocks. Allowing for risk averse competitive lenders directly affects the equilibrium through changes in interest rates schedules.

[29] Aggregate demand externalities are only one type of externality. Any other externality associated with bankruptcy introduces a wedge in the optimal exemption formula, with negative externalities pushing for lower exemptions — see the appendix for a full derivation. Only spillovers which are not priced by lenders modify the optimal exemption formula. For instance, if default creates a social loss of Θ units (expressed in borrowers' ex-ante utility), social welfare is given by $W\left(m\right) - \Theta F\left(m+B_0\right)$ and the optimal exemption becomes:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^{\mathcal{D}}\right\}}{C_0}}{\Lambda\varepsilon_{q_0} + \frac{\Theta f(m+B_0)}{U'(C_0)C_0}}$$

Similarly, a planner that acknowledges that borrowers make mistakes when borrowing — for instance, because they are too optimistic or pessimistic about their income realizations — would also tilt the optimal exemption. These internalities would also introduce wedges into the optimal exemption formula.

Proofs

Proposition 8. (Heterogeneous borrowers: unobservable heterogeneity) a) To be concrete, I focus on the case in which β_i is the only source of unobservable heterogeneity. I choose this variable because it makes the solution easy, not because it is particularly relevant. The proofs for the general case are available under request. Let's assume that $\beta_i \sim G(i)$, with support in $\left[\underline{\beta}, \overline{\beta}\right]$, and $\underline{\beta} > 0$, $\overline{\beta} < 1$. We can conjecture and verify that there exists a threshold $\hat{\beta}$ such that borrowers with $\beta_i < \hat{\beta}$ decide to borrow and those with $\beta_i > \hat{\beta}$ do not. See figure A.4 for illustration. In that case, the pricing schedule is given by:

$$q_{0i}(B_0, m) = \begin{cases} \frac{1}{1+r^*}, & B_{0i} \leq 0 \\ \int_{\underline{\beta}}^{\hat{\beta}(m)} \frac{\tilde{q}_{0i}(B_0,m)}{\int_{\underline{\beta}}^{\hat{\beta}(m)} dG(i)} dG(i), & B_{0i} > 0, \end{cases}$$

where $\tilde{q}_{0i}(B_0, m) = \frac{\delta \int_m^{m+B_0} \frac{y_{1i}-m}{B_0} dF_i(y_{1i}) + \int_{m+B_0}^{\overline{y_{1i}}} dF_i(y_{1i})}{1+r^*}$. We can show that $q_0^+\left(0, m, \hat{\beta}\right) \equiv \lim_{B_0 \to 0^+} q_{0i}(B_0, m) = \int_{\underline{\beta}}^{\hat{\beta}} \frac{\frac{\int_m^{\overline{y_{1i}}} dF(y_{1i})}{1+r^*}}{\int_{\underline{\beta}}^{\hat{\beta}} dG(i)} dG$. Hence, the condition that makes a borrower indifferent between borrowing and not borrowing is given by:

$$\lim_{B_0 \to 0^+} \frac{dJ}{dB_0} = \frac{U'(y_0)}{1+r^*} q_0^+\left(0, m, \hat{\beta}\right) - \beta \int_{\underline{y_1}}^{\overline{y_1}} U'(y_1)\, dF(y_1) = 0 \tag{32}$$

A sufficient condition so that the solution to equation (32) defines a unique thresholds is that $\frac{\partial q_0^+(0,m,\beta)}{\partial \beta} < 0$. In that case, we can solve for $\hat{\beta}(m)$ from:

$$\hat{\beta}(m) = \frac{U'(y_0)}{\int_{\underline{y_1}}^{\overline{y_1}} U'(y_1)\, dF(y_1)} \frac{q_0^+\left(0, m, \hat{\beta}(m)\right)}{1+r^*}$$

Once $\hat{\beta}(m)$ is determined, the pricing schedule and the optimal choice for borrowers are determined as in the baseline model. The derivation of $\frac{dW}{dm}$ uses again the fact that borrowers are indifferent in their extensive margin choice. In the general case, regularity conditions are required to have a well-behaved equilibrium.

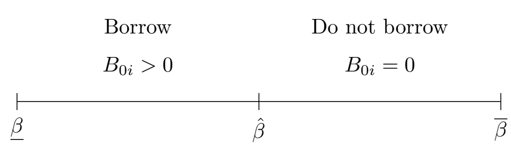

Figure A.4: Threshold $\hat{\beta}$

For reference, we can decompose the partial derivative of lenders' offered price schedule $\frac{\partial q_{0i}(B_0,m)}{\partial m}$ into different components:

$$\frac{\partial q_{0i}(B_0, m)}{\partial m} = \underbrace{\int_{I_A(m)} \frac{\frac{\partial \tilde{q}_{0i}(B_0,m)}{\partial m}}{\int_{I_A(m)} dG(i)} dG(i)}_{\text{Marginal Price Effect}} + \underbrace{\frac{\tilde{q}_{0i}(B_0, m)}{\int_{I_A(m)} dG(i)} g(I_A(m)) \frac{\partial I_A(m)}{\partial m}}_{\text{Extensive Margin}}$$
$$\underbrace{- \frac{\partial I_A(m)}{\partial m} g(I_A(m)) \int_{I_A(m)} \frac{\tilde{q}_{0i}(B_0, m)}{\left(\int_{I_A(m)} dG(i)\right)^2} dG(i)}_{\text{Reweighting Effect}}$$

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 9. (Dynamics) a) The problem solved by borrowers at $t = 0$ can be written as:

$$V_{\mathcal{N},0}(0, y_0; m) = \max_{B_0} U(y_0 + q_0 B_0) + \beta \mathbb{E}[\max\{V_{\mathcal{N},1}(B_0, y_1; m), V_{\mathcal{D},1}(y_1; m)\}],$$

where $V_{\mathcal{N},1}(B_0, y_1; m)$ denotes indirect utility at $t = 1$ when a borrower repays and $V_{D,1}(y_1; m)$ denotes the indirect utility at $t = 1$ when a borrower just defaulted, given by:

$$V_{\mathcal{N},1}(B_0, y_1; m) = \max_{B_1} U(y_1 + q_1 B_1 - B_0) + \beta \mathbb{E}[\max\{V_{\mathcal{N},2}(B_1, y_2; m), V_{\mathcal{D},2}(B_1, y_2; m)\}]$$

$$V_{\mathcal{D},1}(y_1; m) = U(\min\{y_1, m\}) + \beta[\alpha \mathbb{E}[V_{\mathcal{D}_\alpha,2}(y_2; m)|\, y_1] + (1-\alpha)\mathbb{E}[V_{\mathcal{N},2}(0, y_2; m)|\, y_1]],$$

where $V_{\mathcal{D}_\alpha,2}(y_2; m)$ denotes the indirect utility at $t = 2$ for a borrower who has no access to credit markets.

$$V_{\mathcal{D}_\alpha,2}(y_2; m) = U(y_2) + \beta[\alpha \mathbb{E}[V_{\mathcal{D}_\alpha,3}(y_3; m)|\, y_2] + (1-\alpha)\mathbb{E}[V_{\mathcal{N},3}(0, y_3; m)|\, y_2]]$$

The definition of these value functions for future periods is straightforward.

Hence, the derivative $\frac{dW}{dm}$ can be written, after using extensively optimality conditions, as:

$$\frac{dW}{dm} = U'(C_0)\frac{dq_0}{dm}B_0 + \beta\left[\int_{\mathcal{N},1}\frac{dV_{\mathcal{N},1}(B_0, y_1; m)}{dm}dF(y_1) + \int_{\mathcal{D},1}\frac{dV_{\mathcal{D},1}(y_1; m)}{dm}dF(y_1)\right], \tag{33}$$

where

$$\frac{dV_{\mathcal{N},1}(B_0, y_1; m)}{dm} = U'\left(C_1^{\mathcal{D}}\right)\frac{dq_0}{dm}B_1 + \beta\begin{bmatrix} \int_{\mathcal{N},2}\frac{dV_{\mathcal{N},2}(B_1, y_2; m)}{dm}dF(y_2|y_1) \\ + \int_{\mathcal{D},2}\frac{dV_{\mathcal{D},2}(y_2; m)}{dm}dF(y_2|y_1)\end{bmatrix}$$

$$\frac{dV_{\mathcal{D},1}(y_1; m)}{dm} = U'(m)\,\mathbb{I}(y_1 > m) + \beta\begin{bmatrix} \alpha\mathbb{E}\left[\left.\frac{dV_{\mathcal{D}_\alpha,2}(y_2; m)}{dm}\right| y_1\right] \\ + (1-\alpha)\mathbb{E}\left[\left.\frac{dV_{\mathcal{N},2}(0, y_2; m)}{dm}\right| y_1\right]\end{bmatrix}$$

$$\frac{dV_{\mathcal{D}_\alpha,2}(y_2; m)}{dm} = \beta\left[\alpha\mathbb{E}\left[\left.\frac{dV_{\mathcal{D}_\alpha,3}(y_3; m)}{dm}\right| y_2\right] + (1-\alpha)\mathbb{E}\left[\left.\frac{dV_{\mathcal{N},3}(0, y_3; m)}{dm}\right| y_2\right]\right]$$

Substituting the last three expressions into equation (33) and iterating forward, we can write:

$$\frac{dW}{dm} = \sum_{t=0}^{T-1}\mathbb{E}_{\mathcal{N},t}\left[\beta^t U'\left(C_t^{\mathcal{N}}\right)\frac{dq_t}{dm}B_t\right] + \sum_{t=1}^{T}\mathbb{E}_{D_m,t}\left[\beta^t U'\left(C_t^{\mathcal{D}}\right)\right],$$

where $\mathbb{E}_{\mathcal{N},t}[\cdot]$ denotes the $t = 0$ expectation of being in a no default situation at a given state/period and $\mathbb{E}_{D_m,t}[\cdot]$ denotes the $t = 0$ expectation of defaulting while consuming the bankruptcy exemption in a given state.

The previous equation can be rewritten as:

$$\frac{\frac{dW}{dm}}{U'(C_0)C_0} = -\sum_{t=0}^{T-1}\Pi_{\mathcal{N},t}\{g_t\Lambda_t\varepsilon_{\tilde{r}_t,m}\} + \frac{1}{m}\sum_{t=1}^{T}\frac{\Pi_{m,t}\{C_t^{\mathcal{D}}\}}{C_0}, \tag{34}$$

where all new variables are defined in proposition 9. If part of the exemption can be saved, that is, $C_t^{\mathcal{D}} + S_t = \min\{y_t, m\}$, the second term in equation (34) becomes $\frac{1}{m}\sum_{t=1}^{T}\frac{\Pi_{m,t}\{\mu_t C_t^{\mathcal{D}}\}}{C_0}$, where $\mu_t = \frac{1}{1-\frac{S_t}{m}}$. When borrowers' propensity to consume is close to one, $\mu_t \approx 1$, recovering the baseline results.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 11. (Price taking borrowers)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. Note that now the term corresponding to $\frac{\partial q_0}{\partial B_0} B_0 \frac{dB_0}{dm} U'(C_0)$ does not cancel out in equation (25). By differentiating equation (29), and denoting by $B_0^b(q_0)$ the solution to that loan demand equation, we can write:

$$\frac{\partial q_0}{\partial B_0^b} = \frac{-\beta\left[(q_0)^2 U''(C_0) + \int_{m+B_0}^{\overline{y_1}} U''\left(y_1 - B_0^b\right) dF(y_1) + U'(m) f\left(m + B_0^b\right)\right]}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Delta}{\psi}\right]},$$

where the numerator is the second order condition to the borrowers' problem. And:

$$\frac{\partial q_0}{\partial m} = \frac{-\beta U'(m) f\left(m + B_0^b\right)}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Delta}{\psi}\right]} < 0$$

Combining these two equations that characterize loan demand with equation (3), which characterizes loan supply, it is easy to show that $\frac{dq_0}{dm} < 0$ and $\frac{dB_0}{dm}$ is indeterminate.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 12. (Bankruptcy exemption contingent on aggregate risk)

a) The problem that characterizes the optimal set of exemptions is given by: $\max_{\{m_\omega\}} W(\{m_\omega\})$. The derivation of $\frac{\partial W}{\partial m_\omega}$, $\forall \omega$ follows the same steps as the baseline model. It uses borrowers' optimality of default decisions given the realization of ω and optimality of the choice of B_0.

b) Setting $\frac{dW}{dm_\omega} = 0$ and solving for m_ω yields directly m_ω^* for every ω.

Proposition 13. (Endogenous income: labor wedges and aggregate demand)

a) We can write ex-ante welfare as:

$$W(m) = \max U(y_0 + q_0 B_0) + \beta \sum_\omega p(\omega) \left[\begin{array}{c} \int_{\mathcal{D}_y} U(y_1, 0)\, dF_\omega(y_1(\omega)) + \int_{\mathcal{D}_m} U(m, 0)\, dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}} U\left(y_1 + f(N(\omega)) - B_0, N_1^{\mathcal{N}}(\omega)\right) dF_\omega(y_1(\omega)) \end{array}\right]$$

Using optimality conditions we can write:

$$\frac{dW}{dm} = U'(C_0)\frac{\partial q_0}{\partial m} B_0 + \beta \sum_\omega p(\omega) \left[\begin{array}{c} \int_{\mathcal{D}_m} U'\left(C_1^{\mathcal{D}}(\omega), 0\right) dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}} \left[\frac{\partial U\left(C_1(\omega), N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)} A + \frac{\partial U\left(C_1(\omega), N_1^{\mathcal{N}}(\omega)\right)}{\partial N_1(\omega)}\right] \frac{d\overline{N}(\omega)}{dm} dF_\omega(y_1(\omega)) \end{array}\right]$$

$$\frac{dW}{dm} = U'(C_0)\frac{\partial q_0}{\partial m} B_0 + \beta \sum_\omega p(\omega) \left[\begin{array}{c} \int_{\mathcal{D}_m} U'\left(C_1^{\mathcal{D}}(\omega), 0\right) dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}} \frac{\partial U\left(C_1(\omega), N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)} dF_\omega(y_1(\omega))\, \tau(\omega) \frac{dY(\omega)}{dm} \end{array}\right]$$

Where we can use the following facts: $\frac{dY}{dm} = A\frac{dN}{dm}$, and because of constant returns to scale $\frac{dY}{dN_1^{\mathcal{N}}} = \frac{dY}{d\overline{N}} = A$, where $\overline{N}$ is aggregate labor supply.

b) We can thus write:

$$\frac{\frac{dW}{dm}}{C_0 U'(C_0)} = \frac{\frac{\partial q_0}{q_0}}{\partial m}\frac{q_0 B_0}{C_0} + \frac{1}{m}\beta \sum_\omega p(\omega) \left[\begin{array}{c} \int_{\mathcal{D}_m} \frac{C_1^{\mathcal{D}}(\omega)}{C_0} \frac{\frac{\partial U\left(C_1^{\mathcal{D}}(\omega), 0\right)}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega(y_1(\omega)) \\ + \int_{\mathcal{N}} \frac{\frac{\partial U\left(C_1(\omega), N_1^{\mathcal{N}}(\omega)\right)}{\partial C_1(\omega)}}{U'(C_0)} dF_\omega(y_1(\omega)) \frac{\tau(\omega)}{C_0} \frac{dY(\omega)}{\frac{dm}{m}} \end{array}\right]$$

And solving for m yields directly m^* in equation (31).