Multiple Equilibria in Open Economy Models with Collateral Constraints: Overborrowing Revisited

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Abstract

This paper characterizes equilibria in open economy models with pecuniary externalities due to collateral constraints. It shows analytically that there may exist multiple equilibria. This result holds for models with stock collateral constraints and models with flow collateral constraints. The main result of the paper is to establish the existence of equilibria displaying underborrowing, in the sense that the equilibrium level of external debt in the unregulated economy is smaller than it would be in an economy in which agents internalize the externality. In these equilibria, agents understand that the economy is prone to self-fulfilling financial crises and as a result engage in excessive precautionary savings. The existing related literature has to a large extent emphasize equilibria that display overborrowing. The paper argues that there exist equally plausible calibrations for which the pecuniary externality induces a substantial degree of underborrowing.

JEL classification: E44, F41, G01, H23.
Keywords: Pecuniary externalities, collateral constraints, overborrowing, underborrowing, financial crises, capital controls.

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1 Introduction

In this paper, we study models in which external borrowing is limited by collateral. In some formulations, collateral is assumed to be a stock variable, such as the value of physical capital or the value of real estate. In other formulations, collateral takes the form of a flow, such as the stream of income. The collateral-constraint literature argues that during booms increases in the price of collateral induce agents to borrow and spend excessively, which amplifies the expansionary phase of the cycle. During downturns, the argument continues, the price of collateral falls, forcing agents to deleverage, which aggravates the contraction. This fall in the price of collateral is known as a Fisherian debt deflation, after Fisher (1933), and the excessive borrowing during the boom phase of the cycle is known as overborrowing. Modern formulations of this argument in open economy macroeconomics have been developed by Auernheimer and García-Saltos (2000), Benigno et al. (2013, 2014), Bianchi (2011), Jeanne and Korinek (2010), Korinek (2011), Lorenzoni (2008), Mendoza (2010), and Uribe (2006, 2007), among others.

The theoretical models belonging to this literature predict the existence of a pecuniary externality in the market for external funds. The externality arises because the collateral constraint faced by individual agents depends upon variables that are exogenous to them but endogenous to the economy as a whole. These variables can take various forms. In models in which the collateral is a stock, the variable that causes the externality is the market price of the stock. For instance, if the asset that serves as collateral is physical capital, then the price that causes the externality is Tobin's Q, and if the asset used as collateral is real estate (as in mortgage contracts) then the externality originates in house prices. In models in which the collateral takes the form of a flow, the variable that causes the externality is the price of the different components of the flow in terms of units of debt. For example, if borrowing is limited by a fraction of total income, and income has a nontraded component, the variable that causes the externality is the relative price of nontradables in terms of tradables (or the real exchange rate). The nature of the externality has to do with the fact that during booms the price of collateral is pushed up by the collective decisions of individual agents. Each agent understands that the increase in prices is inefficient because it leads to overborrowing, but can do nothing to prevent it because his consumption and savings are too small to affect market prices.

Amplification of the business cycle is not the only type of fragility caused by the pecuniary externality. A second type of instability, which has been given less attention in the literature, is the emergence of multiplicity of equilibria, that is, equilibria in which in some states the economy is financially unconstrained coexist with other equilibria in which in the same states the economy experiences self-fulfilling Fisherian deflations and a limited ability of agents to allocate resources efficiently over time. This particular source of fragility has been given much less attention in the open economy literature. An exception is Jeanne and Korinek (2010) who present a heuristic analysis of the possibility of multiple equilibria (see also the discussion in Benigno et al., 2014). Even less explored is the issue of optimal macroprudential policy in collateral-constraint models with multiple equilibria. This paper aims at filling this gap. It characterizes analytically the existence of multiple equilibria in models with stock and flow collateral constraints of the form described above. The main result of the paper is that the presence of multiplicity gives rise to equilibria displaying underborrowing, in the sense that the equilibrium level of external debt in the unregulated economy is smaller than it would be in an economy in which agents internalize the externality. In these equilibria, agents understand that the economy is prone to self-fulfilling financial crises and as a result engage in excessive precautionary savings.

The possibility of underborrowing in models with self-fulfilling crises due to collateral constraints is not just a theoretical curiosity. We find that underborrowing arises under plausible calibrations. In an economy calibrated with parameters typically used in the emerging-market business-cycle analysis literature and fed with shocks estimated quarterly Argentine data, we find equilibria in which the unregulated economy underborrows by 6 percentage points of GDP. We find that models are more prone to underborrowing when calibrated at a quarterly frequency as opposed to an annual frequency. We demonstrate this observation by presenting an example in which the same economy described above displays overborrowing when the time period is taken to be a year and all shocks and frequency-sensitive parameters are transformed appropriately (SECTION ON ANNUAL CALIBRATION TO BE ADDED) This result may explain the prevalence of annual calibrations in the applied collateral-constraint literature, in spite of the fact that business-cycle analysis is typically cast at quarterly frequency.

The remainder of the paper is organized as follows. Section 2 presents an open economy with a stock collateral constraint. Section 3 characterizes multiplicity of equilibrium analytically in this economy. Sections 4 and 5 study multiplicity of equilibrium in an economy with a flow collateral constraint, in which external borrowing is limited by the value of tradable and nontradable output. Sections 6 and 7 analytically and quantitatively assess the plausibility of underborrowing in a calibrated stochastic economy.

2 Stock Collateral Constraints

Consider a perfect-foresight small open economy populated by a large number of households with preferences given by the utility function

$$\sum_{t=0}^{\infty} \beta^t \ln c_t,$$

where c_t denotes consumption and $\beta \in (0,1)$ denotes the subjective discount factor. The sequential budget constraint of the household is assumed to be of the form

$$c_t + d_t + q_t(k_{t+1} - k_t) = y_t + \frac{d_{t+1}}{1+r}, \tag{1}$$

where d_t denotes debt acquired in period $t-1$ and due in period t, k_t denotes the stock of physical capital in period t, q_t denotes the price of one unit of capital in terms of consumption in period t, y_t denotes output in period t, and $r > 0$ denotes a constant interest rate on debt. For simplicity, we assume a zero depreciation rate of physical capital. Output is produced with the technology

$$y_t = A_t k_t^{\alpha}, \tag{2}$$

where A_t is an exogenous and deterministic productivity factor, and $\alpha \in (0,1)$ is a parameter. For simplicity we will assume that A_t is constant over time.

Assume that borrowing is limited above by a constant fraction $\kappa > 0$ of the value of physical capital. Formally,

$$d_{t+1} \leq \kappa q_t k_{t+1}. \tag{3}$$

The parameter κ could be interpreted as the fraction of assets that lenders could cease from the borrower in the event of a default. Under this interpretation, the above borrowing constraint is an incentive compatibility restriction, which ensures that the borrower never walks away from his external debt obligations.

The above collateral constraint pertains to the class of stock collateral constraints. The source of externality is the price of capital, q_t. If q_t increases, all other things equal, households can borrow more. Similarly, a fall in q_t can cause households to deleverage, depressing aggregate demand. Individual households fail to internalize this mechanism, because, due to their atomistic nature they take q_t as exogenously given. This externality and its implications for prudential policy was first stressed in the context of an open economy model by Auernheimer and García-Saltos (2000).

The household chooses sequences $c_t > 0$, d_{t+1}, and $k_{t+1} \geq 0$ to maximize its lifetime utility subject to the sequential budget constraint (1), the production technology (2), and the collateral constraint (3), taking as given the sequence of prices q_t and the initial conditions d_0 and k_0. The Lagrangian associated with this optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t + \lambda_t \left[A_t k_t^{\alpha} + \frac{d_{t+1}}{1+r} - c_t - d_t - q_t(k_{t+1} - k_t) \right] + \lambda_t \mu_t \left[\kappa q_t k_{t+1} - d_{t+1} \right] \right\},$$

where $\beta^t \lambda_t$ and $\beta^t \lambda_t \mu_t$ are the Lagrange multipliers associated with the sequential budget constraint and the collateral constraint, respectively. The associated first-order conditions with respect to c_t, d_{t+1}, and k_{t+1} are, respectively,

$$\frac{1}{c_t} = \lambda_t, \tag{4}$$

$$\lambda_t \left[\frac{1}{1+r} - \mu_t \right] = \beta \lambda_{t+1}, \tag{5}$$

and

$$\lambda_t q_t \left[1 - \kappa \mu_t \right] = \beta \lambda_{t+1} \left[q_{t+1} + \alpha A_{t+1} k_{t+1}^{\alpha-1} \right]. \tag{6}$$

In addition, the optimality conditions include the Kuhn-Tucker non-negativity and slackness conditions

$$\mu_t \geq 0, \tag{7}$$

and

$$\mu_t \left(\kappa q_t k_{t+1} - d_{t+1} \right) = 0. \tag{8}$$

Because preferences display no satiation, the optimality conditions also include the terminal condition

$$\lim_{t\to\infty} \frac{d_{t+1}}{(1+r)^t} = \kappa \lim_{t\to\infty} \frac{q_t k_{t+1}}{(1+r)^t}. \tag{9}$$

This is so because if a set of sequences $\{c_t, d_{t+1}, k_{t+1}\}$ satisfies all optimality conditions but (9), then there would exist a welfare dominating set of feasible sequences, that is, sequences satisfying (1)-(3) that generate higher utility.

To facilitate the characterization of equilibrium, assume that the aggregate supply of capital is fixed and equal to k. Therefore, in equilibrium we have

$$k_t = k, \tag{10}$$

for all t. The price of capital must be nonnegative, that is, $q_t \geq 0$. In addition, we restrict attention to equilibria in which the price of capital does not display a bubble, that is, equilibria in which q_t grows at a rate strictly less than r. Formally, we impose

$$\lim_{t\to\infty}(1+r)^{-t}q_t = 0. \tag{11}$$

Conditions (9)-(11) imply that the present discounted value of debt must converge to zero, that is,

$$\lim_{t\to\infty}(1+r)^{-t}d_{t+1} = 0. \tag{12}$$

In turn, this condition together with the sequential budget constraint (1) and the market clearing condition (10) implies $d_0 = \sum_{t=0}^{\infty}\frac{y_t-c_t}{(1+r)^t}$, which states that the present discounted value of the stream of future expected trade balances must be large enough to cover the country's initial net external debt position. Finally, we assume that the subjective and market discount factors are equal,

$$\beta(1+r) = 1.$$

A (bubble-free) competitive equilibrium is then a set of sequences $c_t > 0$, d_{t+1}, $\mu_t \geq 0$, and $q_t \geq 0$ satisfying

$$d_0 = \sum_{t=0}^{\infty}\frac{y-c_t}{(1+r)^t}, \tag{13}$$

$$c_t + d_t = y + \frac{d_{t+1}}{1+r} \tag{14}$$

$$\frac{1}{c_t}[1-\mu_t(1+r)] = \frac{1}{c_{t+1}} \tag{15}$$

$$\frac{q_t}{c_t}[1-\kappa\mu_t] = \frac{\beta}{c_{t+1}}\left[q_{t+1} + \alpha\frac{y}{k}\right] \tag{16}$$

$$\mu_t(\kappa q_t k - d_{t+1}) = 0 \tag{17}$$

$$d_{t+1} \leq \kappa q_t k \tag{18}$$

$$\lim_{t\to\infty}(1+r)^{-t}q_t = 0, \tag{19}$$

given d_0 and $y \equiv Ak^\alpha$. Equation (15) together with the requirement that $c_t > 0$ implies that $\mu_t < 1/(1+r) < 1$.

2.1 The Steady State

A steady-state equilibrium is a set of constant sequences $c_t = c^* > 0$, $d_{t+1} = d^*$, $\mu_t = \mu^* \geq 0$, and $q_t = q^* \geq 0$ that satisfy equilibrium conditions (13)-(19) given d_0. Because consumption is constant over time, equation (15) implies that $\mu^* = 0$. This means that in the steady state, the economy is unconstrained. Then, equation (16) becomes $q_t = \beta q_{t+1} + \beta\alpha y/k$. Since $\beta \in (0,1)$, the unique stationary solution to this expression is

$$q^* = \frac{\alpha y/k}{r} > 0,$$

which intuitively says that the steady-state value of capital equals the present discounted value of current and future marginal products of capital.

Evaluating the sequential budget constraint (14) in any period $t > 0$ implies that the steady-state level of consumption is given by

$$c^* = y - \frac{r}{1+r}d^*.$$

This is a familiar characteristic of open economy models in the steady state. It says that households consume their permanent income, given by the sum of nonfinancial income, y, and interest income, $r/(1+r)d^*$. Using the above expression to eliminate c_0 from the sequential budget constraint in period 0 yields

$$d^* = d_0.$$

Thus, the steady-state level of debt is equal to the initial debt, d_0. Because the net debt position is constant in the steady state, we have that the steady-state current account, denoted ca^*, is nil,

$$ca^* = 0.$$

The steady-state trade balance, $tb^* \equiv y - c^*$, equals the interest obligations on external debt,

$$tb^* = \frac{r}{1+r}d^*.$$

Finally, it is natural to ask what levels of debt are sustainable in the steady state. The above expression for steady-state consumption and the requirement that consumption be positive imposes the following upper bound on external debt

$$d_0 < \frac{1+r}{r}y, \tag{20}$$

which is a natural debt limit, above which servicing the debt would cause households to starve. The collateral constraint introduces a second upper bound on debt, given by

$$d_0 \leq \kappa q^* k \equiv \kappa\frac{\alpha y}{r}. \tag{21}$$

Comparing the debt bounds (20) and (21) we have that as long as $\alpha\kappa < 1 + r$, the latter will be the more restrictive one. A sufficient condition for this to be the case is $\kappa < 1$. Throughout this section, we assume, as in much of the related literature, that

$$\kappa < 1.$$

It then follows that the maximum value of debt sustainable in the steady state is given by condition (21). Any level of debt satisfying this condition can be supported as a steady-state equilibrium.

2.2 An Upper Bound on the Equilibrium Price of Capital, q_t

In this subsection, we establish that in any equilibrium q_t is bounded above by its steady state value q^*. To this end, rewrite the capital Euler equation (16) as

$$q_{t+1} = \tilde{\beta}_t^{-1} q_t - r q^*, \tag{22}$$

where $\tilde{\beta}_t$ is given by

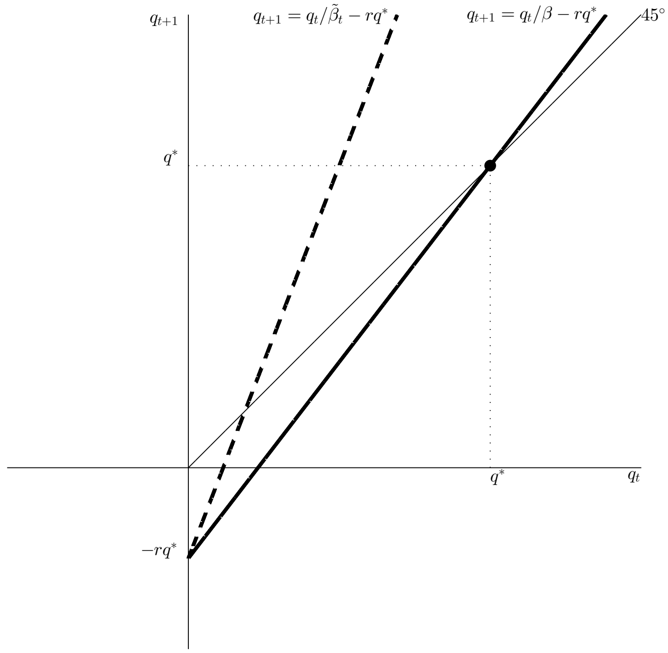
$$\tilde{\beta}_t \equiv \beta \frac{1 - (1+r)\mu_t}{1 - \kappa \mu_t}. \tag{23}$$

Note that $0 < \tilde{\beta}_t \leq \beta$, that $\tilde{\beta}_t = \beta$ when $\mu_t = 0$, and that $\tilde{\beta}_t < \beta$ when $\mu_t > 0$ (recall that we are assuming that $\kappa < 1$ and that $r > 0$). According to this expression, in determining their demand for assets (in this case physical capital), households behave as if they became more impatient in periods in which the collateral constraint binds. Figure 1 displays the phase diagram of the price of capital in the space (q_t, q_{t+1}). The solid line corresponds to the case $\tilde{\beta}_t = \beta$, and the broken line to the case $\tilde{\beta}_t < \beta$. When $\tilde{\beta}_t = \beta$, the stationary state of q_t is given by q^*. It is clear from the phase diagram that, regardless of the value of $\tilde{\beta}_t$, a value of q_t larger than q^* would trigger an explosive path. In principle, a growing path of q_t could be consistent with equilibrium if it does not violate the no-bubble constraint (19). It turns out, however, that this constraint is violated for any initial condition $q_0 > q^*$. To see this, note that since $1/\tilde{\beta}_t > 1 + r$, it suffices to show that any path of q_t with initial condition $q_0 > q^*$ violates the no-bubble constraint for $\tilde{\beta}_t = \beta$. Now evaluate (22) at $\tilde{\beta}_t = \beta$ and divide the left- and right-hand sides by $(1+r)^{t+1}$ and sum for $T-1$ periods to get

$$\sum_{t=0}^{T-1} (\tilde{q}_{t+1} - \tilde{q}_t) = -\frac{rq^*}{1+r} \sum_{t=0}^{T-1} \frac{1}{(1+r)^t},$$

where $\tilde{q}_t \equiv q_t/(1+r)^t$ is the present discounted value of the price of capital. This object must converge to zero in order for the no-bubble constraint to be satisfied. We can write the

Figure 1: Phase Diagram of the Price of Capital



above expression as

$$\tilde{q}_T - q_0 = -\frac{rq^*}{1+r}\sum_{t=0}^{T-1}\frac{1}{(1+r)^t}.$$

Letting $T \to \infty$ in the above expression, we obtain

$$\lim_{T\to\infty}\frac{q_T}{(1+r)^T} = q_0 - q^*.$$

It follows that the no-bubble constraint is violated for any initial condition $q_0 > q^*$. The arguments presented above apply not only in period 0 but in any period $t \geq 0$. Thus we have established that q_t is bounded above by q^* for all $t \geq 0$. As we will see shortly, however, $q_t < q^*$ does not necessarily lead to a violation of the no-bubble constraint, because changes in $\tilde{\beta}_t$ can prevent q_t from imploding.

3 Stock Collateral Constraints and Self-Fulfilling Financial Crises

The equilibrium in economies with collateral constraints is inherently fragile. Essentially, the problem that arises is that if an unconstrained equilibrium exists, often a second equilibrium exists in which the collateral constraint is binding. In the latter equilibrium negative beliefs bring the price of capital down, causing a tightening of the collateral constraint. In turn, the decline in the value of collateral forces agents to deleverage leading to a fire sale of capital. Since the stock of capital is fixed the fire sale depresses asset prices, validating the negative beliefs. To illustrate this point, we show that in the present model, generically there exist self-fulfilling financial crises.

Suppose that

$$d_0 < \kappa q^* k.$$

This restriction guarantees that the initial level of debt does not violate the collateral constraint when $q_0 = q^*$. Then, the analysis presented in section 2.1, implies that there exists an equilibrium in which the economy is at a steady state starting in period 0. In this equilibrium, $c_t = c^* \equiv y - d_0 r/(1+r) > 0$, $d_t = d_0$, $q_t = q^*$, and $\mu_t = 0$ for all $t \geq 0$. Along this equilibrium path, the collateral constraint never binds. We therefore refer to this equilibrium as the unconstrained equilibrium. We wish to show that in general there exists a second equilibrium in which the collateral constraint binds in period 0. In this second equilibrium, the economy suffers a Fisherian deflation and debt deleveraging in the initial period. In addition, the real allocation is welfare inferior to the one associated with the unconstrained equilibrium. We refer to this second equilibrium as the constrained equilibrium.

In the constrained equilibrium we consider here, the economy reaches a steady state in period 1. To see that a steady state equilibrium starting in period 1 exists, note that $d_1 = \kappa q_0 k \leq \kappa q^* k$. The equality follows from our assumption that the collateral constraint is binding in period 0, and the weak inequality follows from the result, derived earlier, that in any equilibrium $q_t \leq q^*$. It follows that in the constrained equilibrium d_1 satisfies condition (21) shifted one period forward. Recall from section 2.1, that this is the only requirement for the existence of a steady-state equilibrium starting in period 1. This steady state is different from the steady state associated with the unconstrained equilibrium, because d_1 in the constrained equilibrium is different from d_0.

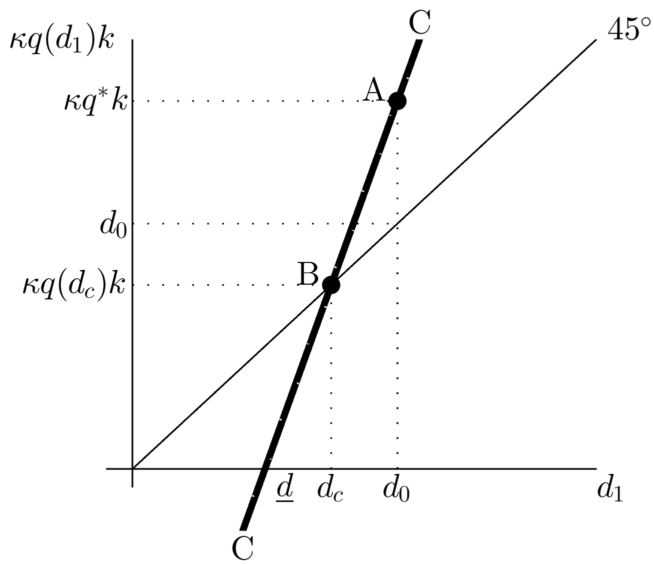
Taking into account that the economy reaches a steady state in period 1, the complete set of equilibrium conditions, equations (13) to (19), collapses to the following system of five equations in the five unknowns, $c_0 > 0$, $c_1 > 0$, d_1, $q_0 \geq 0$, and $\mu_0 \geq 0$

$$d_0 = \frac{1+r}{r} y - \frac{c_1}{r} - c_0 \tag{24}$$

$$c_1 = y - \frac{r}{1+r} d_1 \tag{25}$$

$$\frac{1}{c_0}[1 - (1+r)\mu_0] = \frac{1}{c_1} \tag{26}$$

Figure 2: Collateral Constraints and Multiple Equilibria



$$\frac{q_0}{c_0}(1-\kappa\mu_0) = \frac{\beta}{c_1}(q^* + \alpha y/k) \tag{27}$$

$$\mu_0(\kappa q_0 k - d_1) = 0 \tag{28}$$

$$d_1 \leq \kappa q_0 k. \tag{29}$$

Now solve (24)-(27) for q_0 as a function of d_1 to obtain

$$\kappa q_0 k = \kappa q^* k \left[\frac{(1+r)c^* + d_1 - d_0}{(1+r)c^* + (\kappa - r)(d_1 - d_0)}\right]. \tag{30}$$

Figure 2 displays with a thick solid line the graph of $\kappa q_0 k$ as a function of d_1 implied by this equation. The locus $\overline{CC}$ is the collection of pairs $(d_1, \kappa q_0 k)$ that guarantee that equilibrium conditions (24)-(27) are satisfied. Recalling that $1 + r > 1 > \kappa$, it can readily be shown that $\overline{CC}$ is upward sloping. Also, $\overline{CC}$ crosses the point $(d_0, \kappa q^* k)$, which is labeled A in the figure. Note that point A lies above the 45° line, reflecting the assumption that $d_0 \leq \kappa q^* k$. We have already shown that $d_1 = d_0$ represents a steady state equilibrium. To see that there may exist a second equilibrium, begin by noting that all points of $\overline{CC}$ that lie on or above the 45° line satisfy the collateral constraint (29). Consider now the value of d_1 at which the locus $\overline{CC}$ crosses the horizontal axes. This value of d_1 is denoted $\underline{d}$ in the figure. Suppose that, as shown in the figure, $\underline{d}$ is positive. (We will discuss shortly conditions for this to be the case.) Then, $\overline{CC}$ must necessarily cross the 45° line at some level of debt in the open interval $(\underline{d}, d_0)$. This value of d_1 is denoted d_c and the intersection point is marked with the letter B in the figure. Because B is on $\overline{CC}$ and on the 45° line, it satisfies equilibrium conditions (24) to (27) and the collateral constraint (29). Moreover, at B the collateral constraint holds with equality, which means that the slackness condition (28) is satisfied. To establish that point B represents an equilibrium, it remains to show that $d_1 = d_c$ implies $c_0 > 0$, $c_1 > 0$ and $\mu_0 \geq 0$. To this end, note that the numerator of the expression within brackets in equation (30) is $(1+r)c_0$. At $d_1 = \underline{d}$, the numerator is nil, so $(1+r)c_0 = 0$. At $d_1 = d_0$, $(1+r)c_0 = (1+r)c^*$. Since by (24) and (25), c_0 is increasing in d_1, it follows that at $d_1 = d_c$, $(1+r)c_0$ must be strictly positive and less than $(1+r)c^*$. It follows that $d_1 = d_c$ implies $0 < c_0 < c^*$. Also, the fact that $d_c < d_0$ implies, by the sequential resource constraint (25), that $c_1 > c^*$. So we have that $d_1 = d_c$ implies $0 < c_0 < c^* < c_1$. The debt Euler equation (26) then implies that μ_0 is positive.

This establishes the existence of a second equilibrium in which $q_0 < q^*$ and $d_1 < d_0$, that is, an equilibrium with a Fisherian deflation and debt deleveraging that coexists with the unconstrained equilibrium. We have shown that a sufficient condition for the constrained equilibrium to coexist with the unconstrained equilibrium is that $\underline{d}$ be positive. Since $\underline{d} = (1+r)(d_0 - y)$, this condition is satisfied provided that $d_0/y > 1$. This is not an unrealistic requirement. Suppose that the time unit is one quarter. Then the sufficient condition for the existence of a self-fulfilling financial crises is satisfied as long as net foreign debt is greater than 25 percent of annual output. This result shows that, in the present model, higher external debt makes economies more vulnerable to financial crises driven by nonfundamental revisions in expectations. Finally, the fact that the path of consumption in the self-fulfilling crisis is not flat implies that it is welfare inferior to the flat path associated with the unconstrained equilibrium.

4 Flow Collateral Constraints

A large number of studies of open economies with collateral constraints assume that the object that serves as collateral is a flow rather than a stock. We will focus on the case in which tradable and nontradable output have collateral value, which is the type of flow collateral constraint most frequently studied in the related literature. Under this formulation, the source of pecuniary externalities is the relative price of nontradable goods in terms of tradables, or the real exchange rate. A positive shock that expands aggregate demand pushes the price of nontradables up raising the value of collateral and easing access to credit, which in turn amplifies the expansion in aggregate demand. At the same time, a negative shock that reduces aggregate demand leads to a decline in the relative price of nontradables making the value of nontradable output in terms of tradable goods fall and the collateral constraint tighten, which deepens the contraction.

Like stock collateral constraints, flow collateral constraints create an externality because individual households fail to internalize the effect of their borrowing decision on the relative price of nontradables and hence the value of their own collateral. As a result, the equilibrium features inefficient credit booms and contractions. This type of flow collateral constraint was introduced in open economy models by Mendoza (2002). The externality that emerges when debt is denominated in tradables goods but leveraged on nontradable income and the consequent room for macroprudential policy was emphasized by Korinek (2007) in the context of a three-period model. Bianchi (2011) extends the Korinek model to an infinite-horizon framework and derives quantitative predictions for optimal prudential policy.

4.1 An Open Economy With Output As Collateral

Consider a small open endowment economy in which households have preferences of the form

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \tag{31}$$

where c_t denotes consumption in period t, $U(\cdot)$ denotes an increasing and concave period utility function, $\beta \in (0,1)$ denotes a subjective discount factor, and $\mathbb{E}_t$ denotes the expectations operator conditional on information available in period t. The period utility function takes the form $U(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ with $\sigma > 0$. We assume that consumption is a composite of tradable and nontradable goods of the form

$$c_t = A(c_t^T, c_t^N) \equiv \left[a {c_t^T}^{1-1/\xi} + (1-a) {c_t^N}^{1-1/\xi} \right]^{1/(1-1/\xi)}, \tag{32}$$

where c_t^T denotes consumption of tradables in period t and c_t^N denotes consumption of nontradables in period t. Households are assumed to have access to a single, one-period, risk-free, internationally-traded bond denominated in terms of tradable goods that pays the interest rate r_t when held from periods t to $t+1$. The household's sequential budget constraint is given by

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{1+r_t}, \tag{33}$$

where d_t denotes the amount of debt due in period t and d_{t+1} denotes the amount of debt assumed in period t and maturing in $t+1$. The variable p_t denotes the relative price of nontradables in terms of tradables, and y_t^T and y_t^N denote the endowments of tradables and nontradables, respectively. Both endowments are assumed to be exogenously given. The collateral constraint takes the form

$$d_{t+1} \leq \kappa^T y_t^T + \kappa^N p_t y_t^N, \tag{34}$$

where $\kappa^T, \kappa^N > 0$ are parameters. Households internalize this borrowing limit. However, just as in the case in which the value of capital is used as collateral, this borrowing constraint introduces an externality, because each individual household takes the real exchange rate, p_t, as exogenously determined, even though their collective absorptions of nontradable goods is a key determinant of this relative price.

Households choose a set of processes $\{c_t^T, c_t^N, c_t, d_{t+1}\}$ to maximize (31) subject to (32)-(34), given the processes $\{r_t, p_t, y_t^T, y_t^N\}$ and the initial debt position d_0. The first-order conditions of this problem are (32)-(34) and

$$U'(A(c_t^T, c_t^N))A_1(c_t^T, c_t^N) = \lambda_t, \tag{35}$$

$$p_t = \frac{1-a}{a}\left(\frac{c_t^T}{c_t^N}\right)^{1/\xi}, \tag{36}$$

$$\left(\frac{1}{1+r_t} - \mu_t\right)\lambda_t = \beta \mathbb{E}_t \lambda_{t+1}, \tag{37}$$

$$\mu_t \geq 0, \tag{38}$$

and

$$\mu_t(d_{t+1} - \kappa^T y_t^T - \kappa^N p_t y_t^N) = 0, \tag{39}$$

where $\beta^t \lambda_t$ and $\beta^t \lambda_t \mu_t$ denote the Lagrange multipliers on the sequential budget constraint (33) and the collateral constraint (34), respectively. As usual, the Euler equation (37) equates the marginal benefit of assuming more debt with its marginal cost. During tranquil times, when the collateral constraint does not bind, one unit of debt payable in $t+1$ increases tradable consumption by $1/(1+r_t)$ units in period t, which increases utility by $\lambda_t/(1+r_t)$. The marginal cost of an extra unit of debt assumed in period t and payable in $t+1$ is the marginal utility of consumption in period $t+1$ discounted at the subjective discount factor, $\beta \mathbb{E}_t \lambda_{t+1}$. During financial crises, when the collateral constraint binds, the marginal utility of increasing debt falls to $[1/(1+r_t) - \mu_t]\lambda_t$, reflecting a shadow penalty for trying to increase debt when the collateral constraint is binding.

In equilibrium, the market for nontradables must clear. That is,

$$c_t^N = y_t^N.$$

Then, a competitive equilibrium is a set of processes $\{c_t^T, d_{t+1}, \mu_t\}$ satisfying

$$\left(\frac{1}{1+r_t} - \mu_t\right) U'(A(c_t^T, y_t^N))A_1(c_t^T, y_t^N) = \beta \mathbb{E}_t U'(A(c_{t+1}^T, y_{t+1}^N))A_1(c_{t+1}^T, y_{t+1}^N) \tag{40}$$

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1+r_t} \tag{41}$$

$$d_{t+1} \leq \kappa^T y_t^T + \kappa^N \left(\frac{1-a}{a}\right) c_t^{T\,1/\xi} y_t^{N\,1-1/\xi} \tag{42}$$

$$\mu_t \left[\kappa^T y_t^T + \kappa^N \left(\frac{1-a}{a}\right) c_t^{T\,1/\xi} y_t^{N\,1-1/\xi} - d_{t+1}\right] = 0, \tag{43}$$

$$\mu_t \geq 0, \tag{44}$$

given processes $\{r_t, y_t^T, y_t^N\}$ and the initial condition d_0.

The fact that c_t^T appears on the right-hand side of the equilibrium version of the collateral constraint (42) means that during contractions in which the absorption of tradables falls, the collateral constraint endogenously tightens. Individual agents do not take this effect into account in choosing their consumption plans. This is the nature of the pecuniary externality in this model.

From the perspective of the individual household, equations (33) and (34) define a convex set of feasible debt choices, d_{t+1}. That is, if two debt levels d^1 and d^2 satisfy (33) and (34), then any weighted average $\alpha d^1 + (1-\alpha)d^2$ for $\alpha \in [0,1]$ also satisfies these two conditions. From an equilibrium perspective, however, this ceases to be true in general. The reason is that the relative price of nontradables, p_t, which appears on the right-hand side of the collateral constraint (34) is increasing in consumption of tradables by equation (36), which, in turn, is increasing in d_{t+1} by the resource constraint (41). To see this, use equilibrium condition (41) to eliminate c_t^T from equilibrium condition (42) to obtain

$$d_{t+1} \leq \kappa^T y_t^T + \kappa^N \left(\frac{1-a}{a}\right) \left(y_t^T + \frac{d_{t+1}}{1+r_t} - d_t\right)^{1/\xi} y^{N\,1-1/\xi}.$$

It is clear from this expression that the right-hand side is increasing in the equilibrium level of external debt, d_{t+1}. Moreover, depending on the values assumed by the parameters κ^N, a, and ξ, the equilibrium value of collateral may increase more than one for one with d_{t+1}. In other words, an increase in debt, instead of tightening the collateral constraint may relax it. In this case, the more indebted the economy becomes, the less leveraged it is. As we will see shortly, this possibility can give rise to multiple equilibria and self-fulfilling drops in the value of collateral. Furthermore, if the intratemporal elasticity of substitution ξ is less than unity, which is the case of greatest empirical relevance for many countries (Akinci, 2011), the equilibrium value of collateral is convex in the level of debt. This property may cause the emergence of two distinct values of d_{t+1} for which the collateral constraint binds and two disjoint intervals of debt levels for which the collateral constraint is slack.

5 Flow Collateral Constraints and Self-Fulfilling Financial Crises

The focus of this section is to characterize self-fulfilling financial crises under flow collateral constraints. For analytical convenience, assume that the CRRA period utility function and the CES aggregator function introduced above satisfy $\sigma = 1/\xi = 2$, which is an empirically plausible case. We simplify the economy by assuming that the tradable and nontradable endowments and the interest rate are constant and equal to $y_t^T = y^T$, $y_t^N = 1$, and $r_t = r$, for all t. Further, assume that $a = 0.5$, $\kappa^T = \kappa^N \equiv \kappa$, and

$$\beta(1+r) = 1.$$

Given these assumptions, the equilibrium conditions (40)-(43) can be written as

$$c_{t+1}^T \sqrt{1-(1+r)\mu_t} = c_t^T \tag{45}$$

$$c_t^T + d_t = y^T + \frac{d_{t+1}}{1+r} \tag{46}$$

$$d_{t+1} \leq \kappa[y^T + (y^T + d_{t+1}/(1+r) - d_t)^2] \tag{47}$$

$$\mu_t \left\{\kappa[y^T + (y^T + d_{t+1}/(1+r) - d_t)^2] - d_{t+1}\right\} = 0, \tag{48}$$

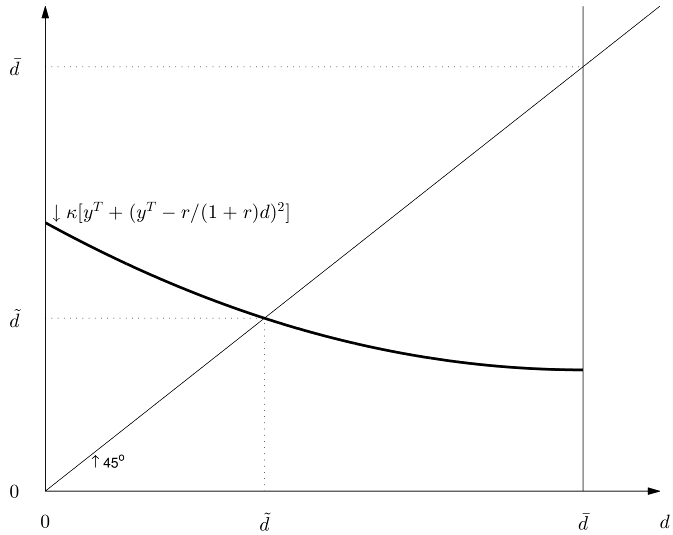
with $c_t^T > 0$ and $\mu_t \geq 0$ and d_0 given.

Let's first characterize conditions under which an equilibrium exists in which traded consumption and debt are constant for all $t \geq 0$, that is, an equilibrium in which $c_t^T = c_0^T$ and $d_t = d_0$ for all $t \geq 0$, where d_0 is given. We refer to this equilibrium as a steady-state equilibrium. By (45), in a steady-state equilibrium $\mu_t = 0$ for all t. This means that in a steady-state equilibrium the slackness condition (48) is also satisfied for all t.

When $d_{t+1} = d_t = d$, the collateral constraint (47) becomes

$$d \leq \kappa\left[y^T + \left(y^T - \frac{rd}{1+r}\right)^2\right]. \tag{49}$$

Figure 3: Feasible Debt Levels in the Steady State Under A Flow Collateral Constraint

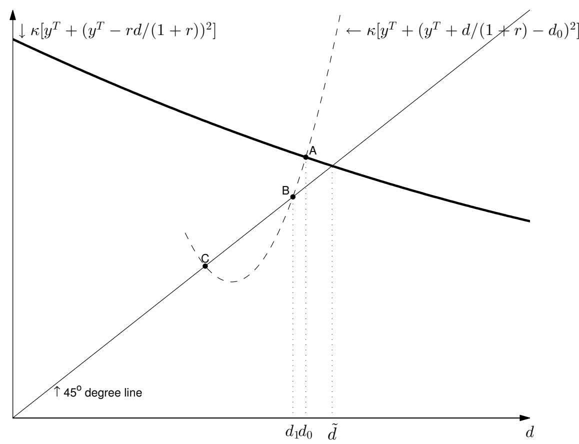


We refer to this expression as the steady-state collateral constraint. Figure 3 displays the left- and right-hand sides of the steady-state collateral constraint (49) as a function of d. The left-hand side is the 45° line. The right-hand side, shown with the thick solid line, is a quadratic expression with a minimum at the natural debt limit $\bar{d} \equiv y^T(1+r)/r$. At the natural debt limit, consumption of tradables is zero. This means that a steady-state equilibrium can exist only for initial values of debt less than $\bar{d}$. At $\bar{d}$, the right-hand side of the collateral constraint equals κy^T and the left-hand side equals $y^T(1+r)/r$. We assume that $\kappa < (1+r)/r$, so that at $\bar{d}$ the left-hand side is larger than the right-hand side, and the steady-state collateral constraint is violated. Let $\tilde{d} < \bar{d}$ be the value of d at which the steady-state collateral constraint (49) holds with equality, that is, the value of d at which the right-hand side of the steady-state collateral constraint crosses the 45° line as indicated in the figure. Any value of initial debt, d_0, less than or equal to $\tilde{d}$ satisfies the steady-state collateral constraint (49). Since we have already shown that a constant value of debt also satisfies all other equilibrium conditions, we have demonstrated that any initial value of debt less than or equal to $\tilde{d}$ can be supported as a steady state equilibrium.

Do there exist other equilibria? The answer is yes. Consider an economy with an initial debt level $d_0 < \tilde{d}$ as shown in figure 4.

The figure reproduces from figure 3 the right-hand side of the steady-state collateral constraint (49) shown with a thick solid line. Because in the graph the initial level of debt, d_0, satisfies $d_0 < \tilde{d}$, we have from the previous analysis that $d_t = d_0$ for all t can be supported

Figure 4: Multiple Equilibria With Flow Collateral Constraints



as an equilibrium. Now consider the collateral constraint in period 0, given by

$$d \leq \kappa \left[y^T + \left(y^T + \frac{d}{1+r} - d_0 \right)^2 \right], \tag{50}$$

expressed as a function of the level of debt in period 1, denoted by d. We refer to expression (50) as the short-run collateral constraint. The figure plots the right-hand side of the short-run collateral constraint with a broken line. The right-hand sides of the short-run and steady-state collateral constraints intersect when $d = d_0$, point A in the figure. At point A, the right-hand side of the short-run collateral constraint (the broken line) is upward sloping, with a slope equal to $2\kappa/(1+r)(y^T - rd_0/(1+r)) > 2\kappa/(1+r)(y^T - r\bar{d}/(1+r)) = 0$.

The right-hand side of the short-run collateral constraint can cross the 45° line to the left of d_0 either zero or two times. Suppose that it crosses the 45° line twice, as shown in figure 4. This is possible for some parameter configurations.[1] At the crossing with the higher debt level, indicated by point B in the figure, the slope of the right-hand side of the short-run collateral constraint must be positive. This means that at point B, c_0^T is positive (recall that the slope of the short-run collateral constraint is $2\kappa/(1+r)c_0^T$. We wish to show that point B can be supported as an equilibrium. In this equilibrium $d_t = d_1 < d_0$ for all $t > 0$. To establish this result, we must show that equilibrium conditions (45)-(48) are satisfied for all $t \geq 0$, with $c_t^T > 0$ and $\mu_t \geq 0$. We have already shown that $c_0^T > 0$ at point B. Now note that at point B the collateral constraint is binding in period 0, since the right-hand side of

[1]A sufficient condition for the existence of two crossings of this type for some range of $d_0 < \tilde{d}$ is that the slope of the right-hand side of the short-run collateral constraint be larger than unity at $d_0 = d_1 = \tilde{d}$. This condition is satisfied as long as $\kappa r/(1+r)(1 - \kappa r/(1+r))y^T > a/(1-a)((1+r)^2 - 1)/4$.

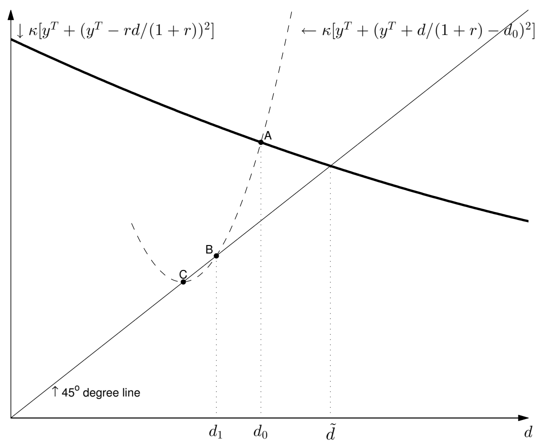
the short-run collateral constraint crosses the 45° line, which is the left-hand side of the short-run collateral constraint. Thus, equilibrium conditions (47) and (48) are satisfied in period 0. Also, the facts that $d_1 < d_0$ and $d_1 = d_2$ imply that $c_0^T < c_1^T$, which can be verified by comparing the resource constraint (46) evaluated at $t = 0$ and $t = 1$. In turn, $c_0^T < c_1^T$ implies, by the Euler equation (45), that a strictly positive value of the Lagrange multiplier μ_0 makes the Euler equation hold with equality in period 0. This establishes that the debt level associated with point B satisfies all equilibrium conditions in period 0. Since $d_1 < \tilde{d}$, we have, from the preceding analysis of steady-state equilibria, that $d_t = d_1$ for all $t \geq 1$ can be supported as an equilibrium. This completes the proof of the existence of multiple equilibria, one with $d_t = d_0$ for all $t \geq 0$ and the collateral constraint never binding, and another one with $d_1 < d_0$ and the collateral constraint binding in period 0 and never binding thereafter. The latter equilibrium takes place at a level of period-1 debt at which, from an aggregate point of view, the collateral constraint behaves perversely in the sense that more borrowing would loosen rather than tighten the borrowing restriction.

What is the intuition behind this second equilibrium? Imagine the economy being originally in a steady state with debt constant and equal to d_0. Unexpectedly, the public becomes pessimistic and aggregate demand contracts. The contraction in aggregate demand means that households want to consume less of both types of good, tradable and nontradable. Because nontradables are in fixed supply, their relative price, p_0, must fall to bring about market clearing. As a result, the value of collateral, given by $\kappa(y^T + p_0 y^N)$, also falls. This reduction in collateral is so large that it forces households to deleverage. This generalized decline in the value of collateral represents the quintessential element of a financial crisis. To reduce their net debt positions, households must cut spending, validating the initial pessimistic sentiments, and making the financial crisis self-fulfilling. The contraction in aggregate demand and the fall in the relative price of nontradables imply that the self-fulfilling financial crisis occurs in the context of a current account surplus and a depreciation of the real exchange rate.

Figure 4 displays a short-run collateral constraint that crosses the 45° line once with a positive slope (point B) and once with a negative slope (point C). In this case point C cannot be an equilibrium because it is associated with negative period-0 consumption, $c_0^T < 0$. To see this, recall that the slope of the short-run collateral constraint in period 0 is given by $2\kappa/(1+r)c_0^T$. So if the slope is negative, so is c_0^T.

However, if the short-run collateral constraint crosses the 45° line twice with a positive slope and before $\tilde{d}$, as shown in figure 5, then a third equilibrium emerges (point C). The proof of this claim is identical to the one establishing that point B is an equilibrium. A third equilibrium of this type entails a larger drop in the value of collateral and more

Figure 5: Third Equilibrium Under Flow Collateral Constraints



deleveraging than in the equilibrium associated with point B. This suggests that in the current environment self-fulfilling financial crisis can come in different sizes.

6 Debt Dynamics in a Stochastic Economy With A Flow Collateral Constraint

We now characterize numerically the debt dynamics in a stochastic version of the flow collateral-constrained economy presented in section 4 above. To this end, we assume a joint stochastic process for the tradable endowment and the country interest rate and calibrate the structural parameters of the model to match certain features of a typical emerging economy.

An equilibrium in the stochastic economy with a flow collateral constraint is a set of stationary processes c_t, c_t^T, d_{t+1}, λ_t, μ_t, p_t satisfying

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1+r_t}, \tag{51}$$

$$c_t = \left[a {c_t^T}^{1-1/\xi} + (1-a) {y^N}^{1-1/\xi} \right]^{\frac{1}{1-1/\xi}},$$

$$a c_t^{-\sigma} \left(\frac{c_t^T}{c_t} \right)^{-1/\xi} = \lambda_t,$$

$$\lambda_t \left[\frac{1}{1+r_t} - \mu_t \right] = \beta \mathbb{E}_t \lambda_{t+1},$$

$$d_{t+1} \leq \kappa^T y_t^T + \kappa^N p_t y^N,$$

$$p_t = \frac{1-a}{a}\left(\frac{c_t^T}{y^N}\right)^{1/\xi},$$

$$\mu_t\left(\kappa^T y_t^T + \kappa^N p_t y^N - d_{t+1}\right) = 0,$$

$$\mu_t \geq 0,$$

given the joint process for (y_t^T, r_t) and the initial condition d_0. In these equilibrium conditions we are invoking market clearing in the nontradable sector, $c_t^N = y^N$, to replace consumption of nontradables with output of nontradables.

6.1 The Unconstrained Economy

For comparison, we also characterize numerically the equilibrium dynamics of an economy without a collateral constraint. We refer to this economy as the unconstrained economy. The unconstrained economy has no externalities. As a result, the competitive equilibrium coincides with the solution to a social planner problem of choosing processes c_t^T and d_{t+1} to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N))$$

subject to the resource constraint (51) and some borrowing limit, such as the natural debt limit, that rules out Ponzi schemes.[2] Because the lifetime utility function is concave and the resource constraint is a convex set in tradable consumption and debt, the solution to the planner's problem is unique, unlike the situation encountered in the economy with the collateral constraint. Furthermore, the numerical approximation of the equilibrium is facilitated by the fact that the above planner's problem can be cast as a Bellman equation. Formally, the competitive equilibrium in the unconstrained economy solves

$$v(y^T, r, d) = \max_{c^T, d'} \left\{ U(A(c^T, y^N)) + \beta \mathbb{E}\left[v(y^{T\prime}, r', d') \,\middle|\, y^T, r \right] \right\}$$

subject to

$$c^T + d = y^T + \frac{d'}{1+r},$$

where a prime superscript denotes next-period values.

[2] The natural debt limit is defined as the level of debt that could be supported with zero consumption of tradables if y_t^T takes forever its lowest possible value, denoted $\underline{y}^T$, and r_t takes forever its highest possible value, denoted $\overline{r}$, that is, the natural debt limit is $\underline{y}^T(1+\overline{r})/\overline{r}$.

6.2 Calibration

We calibrate the model at a quarterly frequency. We assume that $\kappa^T = \kappa^N \equiv \kappa$ and set κ so that the upper limit of net external debt is 30 percent of annual output. This value is in line with those used in the quantitative literature on output-based collateral constraints (e.g., Bianchi, 2011). Because the time unit in the model is a quarter, this calibration restriction implies a value of κ of 1.2 ($= 0.3 \times 4$). The calibration of the other parameters is as follows: $r = 0.0316$, $\beta = 0.9635$, $\sigma = 1/\xi = 2$, $a = 0.26$, and $y^N = 1$.

We assume that y_t^T and r_t follow the bivariate AR(1) process estimated on Argentine quarterly data over the period 1983:Q1 to 2001:Q4. We estimated values are

$$\begin{bmatrix} \ln y_t^T \\ \ln \frac{1+r_t}{1+r} \end{bmatrix} = A \begin{bmatrix} \ln y_{t-1}^T \\ \ln \frac{1+r_{t-1}}{1+r} \end{bmatrix} + \epsilon_t, \tag{52}$$

where $\epsilon_t \sim N(\emptyset, \Sigma_\epsilon)$, with

$$A = \begin{bmatrix} 0.79 & -1.36 \\ -0.01 & 0.86 \end{bmatrix}; \qquad \Sigma_\epsilon = \begin{bmatrix} 0.00123 & -0.00008 \\ -0.00008 & 0.00004 \end{bmatrix}; \qquad r = 0.0316.$$

Table 1 summarizes the calibration.

To approximate the equilibrium, we develop an Euler equation iteration procedure over a discretized state space. The appendix describes the numerical algorithm.

The economy possesses two exogenous states, y_t^T and r_t, and one endogenous state, d_t. We discretize $\ln y_t^T$ using 21 evenly spaced points centered at 0 and discretize $\ln(1+r_t)/(1+r)$ using 11 evenly spaced points centered at 0. Thus, both grids are symmetric. To facilitate convergence of the numerical algorithm, the upper bound of the grids of $\ln y_t^T$ and $\ln((1+r_t)/(1+r))$ are taken to be the maximum realizations of these variables observed in the data used to estimate the driving process given in equation (52), reproduced above. The resulting intervals are $[\ln \underline{y}^T, \ln \overline{y}^T] = [-0.1593, 0.1593]$ and $\left[\ln\left(\frac{1+\underline{r}}{1+r}\right), \ln\left(\frac{1+\overline{r}}{1+r}\right)\right] = [-0.0668, 0.0668]$. We compute the transition probability matrix using the simulation approach of Schmitt-Grohé and Uribe (2009). We use 501 equally spaced points for d_t in the interval $[\underline{d}, \overline{d}]$. In the constrained economy, we set $\underline{d} = 0.5$ and $\overline{d} = 2.3$ and in the unconstrained economy we set $\underline{d} = 2$ and $\overline{d} = 9$. The upper bound in the unconstrained economy is close to the natural debt limit, which under the present calibration equals 9.1427.

The numerical solution must take a stance on how to handle the possibility of indeterminacy of the rational expectations equilibrium of the type identified in section 5. Failing to address this issue may result in nonconvergence of numerical algorithms. Specifically, in searching for an equilibrium we give priority to equilibria with a binding constraint. If for

Table 1: Calibration of the Constrained and Unconstrained Economies

Parameter	Value	Description
κ	1.2	Parameter of collateral constraint
σ	2	Inverse of intertemporal elasticity of consumption
β	0.9635	Quarterly subjective discount factor
r	0.0316	Steady state quarterly country interest rate
ξ	0.5	Elasticity of substitution between tradables and nontradables
a	0.26	Share of tradables in CES aggregator
y^N	1	Nontradable output
y^T	1	Steady-state tradable output
	Discretization of State Space	
n_{y^T}	21	Number of grid points for $\ln y_t^T$, equally spaced
n_r	11	Number of grid points for $\ln[(1+r_t)/(1+r)]$, equally spaced
n_d	501	Number of grid points for d_t, equally spaced
$\left[\ln \underline{y}^T, \ln \overline{y}^T\right]$	[-0.1593,0.1593]	Range for tradable output
$\left[\ln\left(\frac{1+\underline{r}}{1+r}\right), \ln\left(\frac{1+\overline{r}}{1+r}\right)\right]$	[-0.0668,0.0668]	Range for interest rate
$[\underline{d}, \overline{d}]$	[0.5,2.3]	Debt range constrained economy
$[\underline{d}, \overline{d}]$	[2,9]	Debt range unconstrained economy
$[\underline{d}, \overline{d}]$	[1.5,3.5]	Debt range Ramsey economy

Note. The time unit is one quarter.

a given current state (y_t^T, r_t, d_t) there are one or two values of d_{t+1} for which all equilibrium conditions are satisfied and the collateral constraint is binding our algorithm picks the smaller debt value. Thus, the algorithm favors equilibria like point B in figure 4 and point C in figure 5. One could in principle design algorithms to identify other possible equilibria. Our approach is guided by the objective of finding equilibria displaying underborrowing, an issue that we will take up in section 7.2.

6.3 Equilibrium Debt Distributions

Figure 6: External Debt Densities With And Without Collateral Constraints

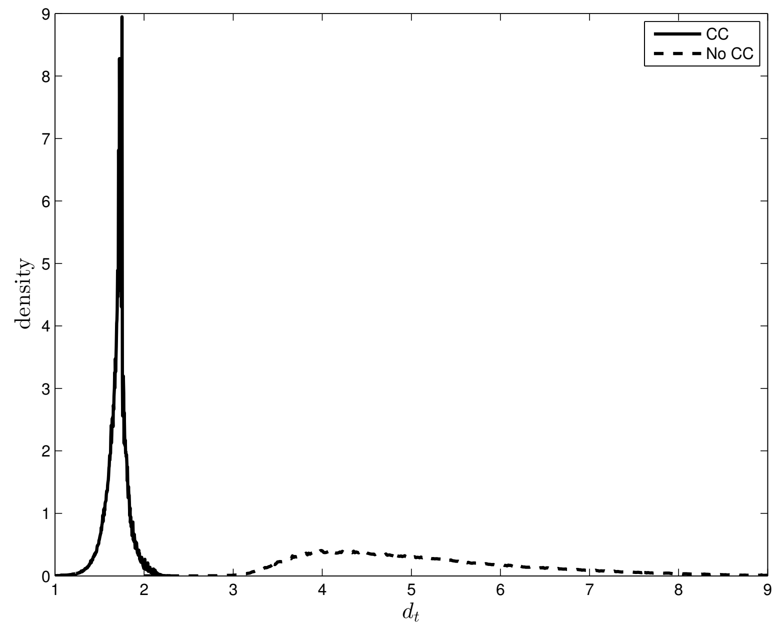
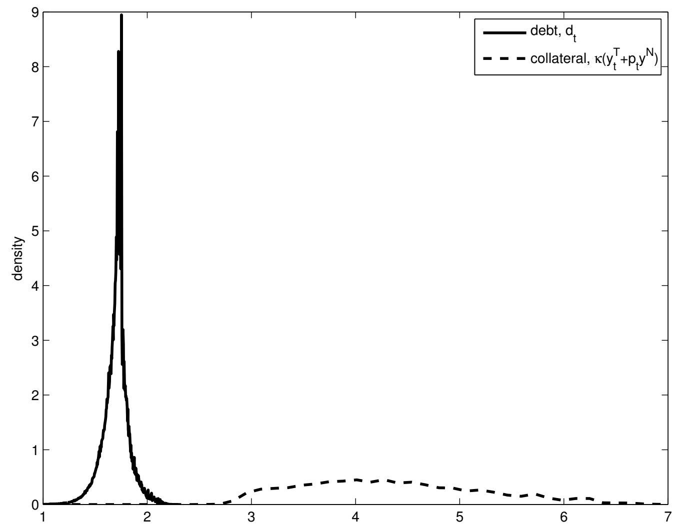


Figure 6 displays with a solid line the unconditional distribution of external debt, d_t, in the collateral-constrained economy and with a broken line the debt distribution associated with the unconstrained economy. As expected, the presence of the collateral constraint shifts the debt distribution to the left. The mean debt level in the constrained economy is about one third as high as in the unconstrained one. The same is roughly true if one measures debt as a fraction of output, 12.2 percent versus 45.8 percent. It follows that the presence of the collateral constraint significantly limits the ability of households to borrow.

The collateral constraint compresses the debt distribution around its mean. The unconditional standard deviation of debt is reduced by a factor of 10 when the collateral constraint is introduced. This does not mean, however, that the collateral constraint is hit frequently. It actually turns out that the contrary is the case. The collateral constraint almost never binds.

Figure 7: The Distributions of Debt and Collateral



On average, the constraint binds only 27 times in one million quarters.[3] This is illustrated in figure 7, which displays the equilibrium distribution of debt, d_t, along with the equilibrium distribution of collateral, $\kappa(y_t^T + p_t y^N)$. The two distributions are disjoint. Households choose to stay clear of the endogenous debt limit virtually all of the time. They manage to avoid being caught with a binding constraint by engaging in precautionary savings. They save because being up against the constraint leaves no room for smoothing income shocks and forces them to deleverage. This collective deleveraging causes the price of collateral to collapse, which reinforces the need to deleverage.

7 Optimal Capital Control Policy

The pecuniary externality created by the presence of the relative price of nontradables in the collateral constraint induces an allocation that is in general suboptimal, not only when compared to the allocation that would result in the absence of a collateral constraint, but also relative to the best allocation possible among all of the ones that satisfy the collateral constraint. As a result, the collateral constraint opens the door for welfare improving policy intervention. We begin by studying capital controls, because they essentially represent a tax on external borrowing, which is the variable most directly affected by the pecuniary externality. In fact, we will show that the optimal capital control policy fully internalizes the pecuniary externality, in the sense that it induces the representative household to behave

[3]Financial crises can be made to occur more frequently by assuming that households are more impatient by lowering the subjective discount factor β.

as if it understood that its own borrowing choices influence the relative price of nontradables and therefore the value of collateral.

We assume that the government is benevolent in the sense that it seeks to maximize the well being of the representative household. Further, we assume that the government has the ability to commit to policy promises. In short, we characterize the Ramsey optimal capital control policy in the context of an open economy with a flow collateral constraint.

Let τ_t be a proportional tax on debt acquired in period t. If τ_t is positive, it represents a proper capital control tax, whereas if it is negative it has the interpretation of a borrowing subsidy. The revenue from capital control taxes is given by $\tau_t d_{t+1}/(1+r_t)$. We assume that the government consumes no goods and that it rebates all revenues from capital controls to the public in the form of lump-sum transfers (lump-sum taxes if $\tau_t < 0$), denoted ℓ_t. The budget constraint of the government is then given by

$$\tau_t \frac{d_{t+1}}{1+r_t} = \ell_t. \tag{53}$$

The household's sequential budget constraint now becomes

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y^N + (1-\tau_t)\frac{d_{t+1}}{1+r_t} + \ell_t.$$

This expression makes it clear that the capital control tax distorts the borrowing decision of the household. In particular, the gross interest rate on foreign borrowing perceived by the private household is no longer $1+r_t$, but $(1+r_t)/(1-\tau_t)$. All other things equal, the higher is τ_t, the higher is the interest rate perceived by households. Thus, by changing τ_t the government can encourage or discourage borrowing. All optimality conditions associated with the household's optimization problem (equations 35-39) are unchanged, except for the debt Euler equation (37), which now takes the form

$$\left(\frac{1-\tau_t}{1+r_t} - \mu_t\right)\lambda_t = \beta \mathbb{E}_t \lambda_{t+1}.$$

A competitive equilibrium in the economy with capital control taxes is then a set of processes c_t^T, d_{t+1}, λ_t, μ_t, and p_t satisfying

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1+r_t} \tag{54}$$

$$d_{t+1} \le \kappa \left[y_t^T + p_t y^N\right] \tag{55}$$

$$\lambda_t = U'(A(c_t^T, y^N))A_1(c_t^T, y^N) \tag{56}$$

$$\left(\frac{1-\tau_t}{1+r_t} - \mu_t\right)\lambda_t = \beta \mathbb{E}_t \lambda_{t+1} \tag{57}$$

$$p_t = \frac{A_2(c_t^T, y^N)}{A_1(c_t^T, y^N)} \tag{58}$$

$$\mu_t[\kappa(y_t^T + p_t y^N) - d_{t+1}] = 0 \tag{59}$$

$$\mu_t \geq 0, \tag{60}$$

given a policy process τ_t, exogenous driving forces y_t^T and r_t, and the initial condition d_0.

The benevolent government sets capital control taxes to maximize the household's lifetime utility subject to the restriction that the optimal allocation be supportable as a competitive equilibrium. It follows that all of the above competitive equilibrium conditions are constraints of the Ramsey government's optimization problem. Formally, the Ramsey-optimal competitive equilibrium is a set of processes τ_t, c_t^T, d_{t+1}, λ_t, μ_t, and p_t that solve the problem of maximizing

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N)) \tag{61}$$

subject to (54)-(60), given processes y_t^T and r_t and the initial condition d_0. In the welfare function above, we have replaced consumption of nontradables, c_t^N, with the endowment of nontradables, y^N, because the Ramsey planner takes into account that in a competitive equilibrium the market for nontradables clears at all times.

The above equilibrium conditions look like a formidable set of constraints. Fortunately, it is possible to reduce the set of constraints considerably. In particular, it turns out that any processes c_t^T and d_{t+1} satisfy equilibrium conditions (54)-(60) if and only if they satisfy (54) and

$$d_{t+1} \leq \kappa \left[y_t^T + \frac{1-a}{a} \left(\frac{c_t^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right]. \tag{62}$$

To see this, suppose c_t^T and d_{t+1} satisfy (54) and (62). We must establish that (54)-(60) are also satisfied. Obviously, the resource constraint (54) holds. Now pick p_t to satisfy (58). This is possible, because the process c_t^T is given. Now use this expression to eliminate p_t from (55). The resulting expression is (62), establishing that (55) holds. Next, pick λ_t to satisfy (56). Now, set $\mu_t = 0$ for all t. It follows immediately that the slackness condition (59) and the non-negativity condition (60) are satisfied. Finally, pick τ_t to ensure that (57) holds, that is,

$$\tau_t = 1 - \beta(1+r_t)\mathbb{E}_t \frac{U'(A(c_{t+1}^T, y^N))A_1(c_{t+1}^T, y^N)}{U'(A(c_t^T, y^N))A_1(c_t^T, y^N)}. \tag{63}$$

Next, we need to show the reverse statement, that is, that processes c_t^T and d_{t+1} that satisfy (54)-(60) also satisfy (54) and (62). Obviously, (54) is satisfied, and combining (55) with (58) yields (62). This completes the proof of the equivalence of the constraint set (54)-(60) and the constraint set (54) and (62).

We can then state the Ramsey problem as

$$\max_{\{c_t^T, d_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t^T, y^N)) \tag{61}$$

subject to

$$c_t^T + d_t = y_t^T + \frac{d_{t+1}}{1+r_t}, \tag{54}$$

$$d_{t+1} \leq \kappa \left[y_t^T + \frac{1-a}{a} \left(\frac{c_t^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right]. \tag{62}$$

Comparing the levels of debt in the Ramsey equilibrium and in the unregulated equilibrium (i.e., the equilibrium without government intervention), we can determine whether the lack of optimal government intervention results in too much or too little debt. The case in which the economy borrows too much is known as overborrowing and the case in which the economy borrows too little as underborrowing. We turn to this issue next.

7.1 Overborrowing or Underborrowing? An Analytical Example

Consider the Ramsey optimal allocation in the perfect-foresight economy analyzed in section 5. Suppose that the initial value of debt, d_0, satisfies $d_0 < \tilde{d}$, as shown in figure 4. Since one possible competitive equilibrium is $d_t = d_0$ and $c_t^T = y^T - rd_0/(1+r)$ for all $t \geq 0$, and since this equilibrium is the first best equilibrium (i.e., the equilibrium that would result in the absence of the collateral constraint), it also has to be the Ramsey optimal equilibrium. What is the capital control tax associated with the Ramsey optimal equilibrium? Take a look at equation (63). Because consumption of tradables is constant over time and because in this analytical example $\beta(1+r) = 1$, it follows that $\tau_t = 0$ for all $t \geq 0$. Compare now the level of debt in the Ramsey optimal allocation with the one associated with the unregulated competitive equilibrium. Here, we must take into account that there are two unregulated competitive equilibria, points A and B. Suppose the unregulated competitive equilibrium happens to be the one in which the collateral constraint binds in period 0, point B in figure 4. In this case the unregulated economy underborrows at all times, since the level of debt at point B is less than the Ramsey optimal level of debt, d_0. If, on the other hand, the unregulated competitive equilibrium happens to be the unconstrained equilibrium (point A in the figure), then there is neither underborrowing nor overborrowing, since its associated level of debt coincides with the Ramsey optimal level, d_0. Thus, in this economy, there is either underborrowing or optimal borrowing, depending on whether the competitive equilibrium happens to be the constrained or the unconstrained one.

Similarly, in the economy depicted in figure 5, the Ramsey optimal equilibrium is at point A, with constant consumption and capital control taxes equal to zero at all times. The economy now possesses three competitive equilibria, points A, B, and C. If the economy coordinates on equilibria B or C, it underborrows, and if it coordinates on equilibrium A, it neither underborrows nor overborrows.

We note further that in general the Ramsey optimal policy is mute with regard to equilibrium implementation. In the context of the present economy, this means that the policy $\tau_t = 0$ does not guarantee that the competitive equilibrium will be the Ramsey optimal one (e.g., point A in figure 4). In particular, it is possible that a policy consisting in setting $\tau_t = 0$ for all t results in the constrained competitive equilibrium given by point B in the figure, which is not Ramsey optimal. Thus a policy of setting $\tau_t = 0$ at all times may fail to implement the Ramsey optimal allocation.

7.2 Overborrowing or Underborrowing? A Quantitative Analysis

We have just shown analytically that under optimal capital controls the perfect-foresight economy of section 5 either underborrows or borrows just the optimal quantity, but does not overborrow. Importantly, that analysis makes clear that the presence or absence of sub-optimal borrowing critically depends on which equilibrium, of possibly multiple ones, materializes. The analysis further suggests that if the equilibrium selection criterion favors picking allocations in which the collateral constraint binds, then the unregulated equilibrium is likely to display underborrowing. We now show that this result is robust to introducing uncertainty.

Consider the Ramsey optimal allocation in the calibrated stochastic economy of section 6. The Ramsey optimal allocation is relatively easy to compute because the Ramsey problem can be cast in the form of a Bellman equation problem. Specifically, the recursive version of the Ramsey problem of maximizing (61) subject to (54) and (62) is given by

$$v(y^T, r, d) = \max_{c^T, d'} \left\{ U(A(c^T, y^N)) + \beta \mathbb{E} \left[v(y^{T'}, r', d') \,\middle|\, y^T, r \right] \right\}$$

subject to

$$c^T + d = y^T + \frac{d'}{1+r}$$

$$d' \leq \kappa \left[y^T + \frac{1-a}{a} \left(\frac{c^T}{y^N} \right)^{\frac{1}{\xi}} y^N \right],$$

where, a prime superscript denotes next-period values. Although the constraints of this control problem may not represent a convex set in tradable consumption and debt, the fact that the Ramsey allocation is the result of a utility maximization problem, implies that its solution is generically unique. The calibration of the economy and the discretization of the state space are the same as the ones used for the unregulated economy, summarized in table 1.

Figure 8: Equilibrium Underborrowing

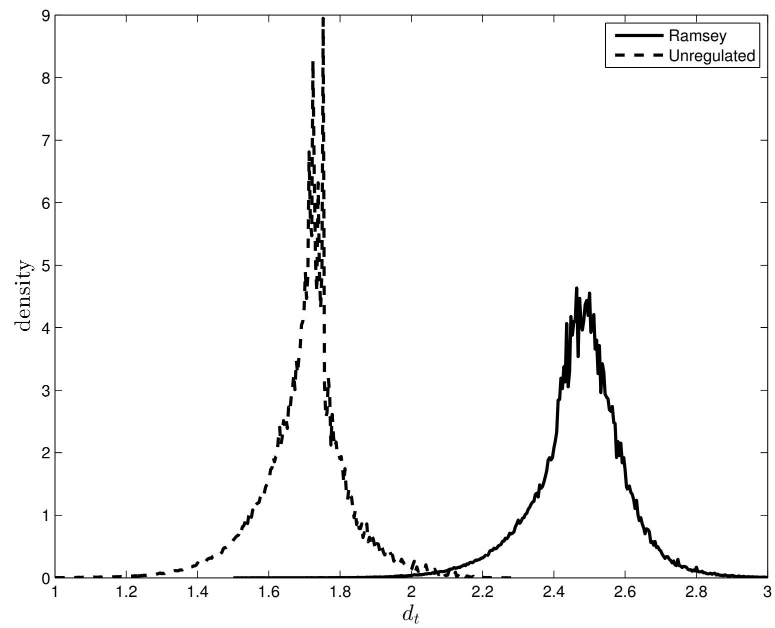


Figure 8 displays with a solid line the unconditional distribution of net external debt, d_t, under Ramsey optimal capital control policy. For comparison, it reproduces from figure 6 the unconditional distributions of debt in the unregulated economy. The figure shows that the analytical result of the previous subsection carries over to the stochastic economy. Namely, the unregulated economy may display underborrowing, in the sense that its debt distribution may lie to the left of the one associated with the Ramsey optimal capital control policy. This occurs when private agents coordinate on equilibria that gravitate toward a binding collateral constraint. The average annual debt-to-output ratio is 12.1 percent in the unregulated equilibrium and 18.4 percent in the Ramsey optimal equilibrium, implying underborrowing of more than 6 percent of GDP. The reason why there is underborrowing is that under the maintained (pessimistic) expectations-coordination environment, households engage in a suboptimally high level of precautionary savings. In turn, precautionary savings is motivated by the fact that the economy is more fragile as it is more prone to financial crises caused by a binding collateral constraint. This does not imply that ex-post the collateral constraint binds more often in the unregulated economy, because through precautionary savings, an economy that is ex-ante more prone to financial crises may ex-post display fewer crises than the Ramsey-optimal economy. This turns out to be the case in the present environment. While the unregulated economy displays only 27 financial crises per one million quarters, the Ramsey-optimal economy experiences 907.

[To be continued.]

Appendix: Numerical Solution Algorithm

This appendix describes the numerical algorithm used to approximate the equilibrium of the stochastic economy with a flow collateral constraint studied in section 6. The algorithm is a modified Euler-equation iteration procedure. The proposed algorithm takes explicitly into account the possibility of multiple equilibria, by allowing for the adoption of the equilibrium selection criteria (a), (b), or (c) described in section **??**. In addition, the algorithm takes into account the possibility that for some states there may exist no choice of next-period debt for which consumption of tradables is positive and the collateral constraint is satisfied. In these states, the procedure modifies the value of collateral to allow the collateral constraint to be satisfied at a minimum subsistence level of tradable consumption.

1. Iteration $n = 1, 2, \ldots$ starts with a guess for the policy function of net external debt $d' = D_n(y^T, r, d)$, where y^T, r, and d denote the state of the economy in the current period and d' denotes debt next period, which is in the information set of the current period.

2. Use the resource constraint (51) to compute consumption of tradables, $C^T(y^T, r, d, d') \equiv y^T + d'/(1+r) - d$. Consumption is then given by

$$C(y^T, r, d, d') \equiv \left[aC^T(y^T, r, d, d')^{1-1/\xi} + (1-a){y^N}^{1-1/\xi} \right]^{1/(1-1/\xi)}.$$

   Compute the marginal utility of tradable consumption as

$$\Lambda(y^T, r, d, d') \equiv C(y^T, r, d, d')^{-\sigma} \left(\frac{C^T(y^T, r, d, d')}{C(y^T, r, d, d'))} \right)^{-1/\xi}.$$

   Compute the relative price of nontradables as

$$P(y^T, r, d, d') \equiv \frac{1-a}{a} \left(\frac{C^T(y^T, r, d, d')}{y^N} \right)^{1/\xi}.$$

3. Compute the value of collateral as

$$M(y^T, r, d, d') \equiv \kappa[y^T + P(y^T, r, d, d')y^N].$$

   There may exist current states (y^T, r, d) for which no level of next-period debt d' exists such that consumption of tradables is positive and the collateral constraint is satisfied, that is, there may exist states (y^T, r, d) for which $C^T(y^T, r, d, d') < 0$ or

$d' > M(y^T, r, d, d')$ or both for all $d' \in \{\underline{d}, \ldots, \overline{d}\}$. For these states, define

$$\hat{d} = \underset{d' \in \{\underline{d}, \ldots, \overline{d}\}}{\arg\min} \{C^T(y^T, r, d, d')\} \text{ subject to } C^T(y^T, r, d, d') > 0$$

and redefine

$$M(y^T, r, d, \hat{d}) = \hat{d}.$$

4. Compute the expected value of the marginal utility of tradable consumption conditional on information available in the current period,

$$\Lambda_n^e(y^T, r, d, d') \equiv \mathbb{E}\Lambda(y^{T'}, r', d', D_n(y^{T'}, r', d')|y^T, r, d, d').$$

Define

$$\mu_n(y^T, r, d, d') = \frac{\Lambda(y^T, r, d, d')}{1+r} - \beta\Lambda_n^e(y^T, r, d, d').$$

5. Adopt one of the following equilibrium selection criteria, corresponding to criteria (a), (b), or (c), discussed in section **??**, respectively:

   (a) Set

   $$D_{n+1}(y^T, r, d) = \underset{d' \in \{\underline{d}, \ldots, \overline{d}\}}{\arg\min} |\mu_n(y^T, r, d, d')|$$

   subject to

   $$C^T(y^T, r, d, d') > 0,$$
   $$\mu_n(y^T, r, d, d') \geq 0,$$
   $$M(y^T, r, d, d') \geq 0,$$
   $$\mu_n(y^T, r, d, d')M(y^T, r, d, d') = 0.$$

   (b) For a given current state (y^T, r, d), define the largest possible choice of next-period debt at which the collateral constraint is binding as

   $$d^b = \max_{d_j \in \{d_2, \ldots, \overline{d}\}} d_j$$

   subject to

   $$[M(y^T, r, d, d_j) - d_j][M(y^T, r, d, d_{j-1}) - d_{j-1}] \leq 0 \text{ and } C^T(y^T, r, d, d_j) > 0,$$

   where d_j is the jth element of the debt grid. If d^b exists for a given current state

(y^T, r, d), check whether $\mu_n(y^T, r, d, d^b) \geq 0$. If so, set $D_{n+1}(y^T, r, d) = d^b$. If d^b does not exist for a given current state (y^T, r, d) or if $\mu_n(y^T, r, d, d^b) < 0$, then pick $D_{n+1}(y^T, r, d)$ following item (a) above.

(c) For a given current state (y^T, r, d), define the smallest possible choice of next-period debt at which the collateral constraint is binding as

$$d^c = \min_{d_j \in \{d_2, \ldots, \overline{d}\}} d_j$$

subject to

$$[M(y^T, r, d, d_j) - d_j][M(y^T, r, d, d_{j-1}) - d_{j-1}] \leq 0 \text{ and } C^T(y^T, r, d, d_j) > 0,$$

where d_j is the jth element of the debt grid.

If d^c exists for a given current state (y^T, r, d), check whether $\mu_n(y^T, r, d, d^c) \geq 0$. If so, set $D_{n+1}(y^T, r, d) = d^c$. If d^c does not exist for a given current state (y^T, r, d) or if $\mu_n(y^T, r, d, d^c) < 0$, then pick $D_{n+1}(y^T, r, d)$ following item (a) above.

6. If

$$\max_{\{y^T, r, d\}} |D_{n+1}(y^T, r, d) - D_n(y^T, r, d)| < \text{Tol}$$

for some tolerance level $\text{Tol} > 0$, the procedure is completed. Else, go to item 1 for a new iteration.

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