Uncertainty Traps[*]

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Abstract

We develop a theory of endogenous uncertainty and business cycles in which short-lived shocks can generate long-lasting recessions. In the model, higher uncertainty about fundamentals discourages investment. Since agents learn from the actions of others, information flows slowly in times of low activity and uncertainty remains high, further discouraging investment. The unique equilibrium of this economy displays uncertainty traps: self-reinforcing episodes of high uncertainty and low activity. While the economy recovers quickly after small shocks, large temporary shocks may have nearly permanent effects on the level of activity. The economy is subject to an information externality but uncertainty traps remain even in the efficient allocation. We extend our framework to include additional features of standard business cycle models and show, in that context, that uncertainty traps can substantially worsen recessions and increase their duration, even under optimal policy interventions.

JEL Classification: E32, D80

1 Introduction

We develop a theory of endogenous uncertainty and business cycles. The theory combines two forces: higher uncertainty about economic fundamentals deters investment, and uncertainty evolves endogenously because agents learn from the actions of others. The unique rational expectation equilibrium of the economy features uncertainty traps: self reinforcing episodes of high uncertainty and low activity. Because of uncertainty traps, short-lived shocks can generate long-lasting recessions, and low activity may persist even under good fundamentals. Thus, the theory rationalizes salient features of U.S. macroeconomic activity that are not easily explained by standard business cycle models, such as the slow recovery of output after recessions despite typically faster improvements in measured productivity.

We first build a model that includes the essential features that give rise to uncertainty traps, and then embed it into a standard real business cycle model. In the model, firms decide whether to undertake an irreversible investment whose return depends on an imperfectly observed fundamental that evolves randomly according to a persistent process. Firms are heterogeneous in the cost of undertaking this investment and hold common beliefs about the fundamental. Beliefs are regularly updated with new information, and in particular firms learn by observing the return on the investment of other producers. We use the variance of these beliefs as our notion of uncertainty.

This environment naturally produces an interaction between beliefs and economic activity. Firms are more likely to invest if their beliefs about the fundamental have higher mean, but also if they have smaller variance (lower uncertainty). At the same time, the laws of motion for the mean and variance of beliefs depend on the investment rate. When few firms invest, little information is released, so uncertainty rises.

The key feature of the model is that this interaction between information and investment leads to uncertainty traps, formally defined as the coexistence of multiple stationary points in the dynamics of uncertainty and economic activity. The economy will converge to either a good regime (with low uncertainty and high economic activity) if the current level of uncertainty is sufficiently low, or to a bad regime (with high uncertainty and low activity) if the current level of uncertainty is sufficiently high. As a result of this multiplicity, the economy exhibits strong nonlinearities in its response to shocks. The economy quickly recovers after small temporary shocks, but it may shift into a low-activity regime after a large temporary shock. Once it has fallen in the low regime, only a large enough positive shock can put the economy back in the high-activity regime.

An important feature of the model is that, despite the presence of uncertainty traps, there is a unique recursive competitive equilibrium. That is, multiplicity of stationary points does not mean multiplicity of equilibria. Therefore, unlike in other macro models with complementarities, there is no room in our model for multiple equilibria or sunspots.[1]

The model features an inefficiently low level of investment because agents do not internalize the effect of their actions on common information. This inefficiency naturally creates room for

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[1]For recent examples of business cycle models with multiple equilibria see Benhabib et al. (2012), Farmer (2013) or Kaplan and Menzio (2013).

welfare-enhancing policy interventions. We study the problem of a constrained planner that is subject to the same informational constraints as private agents. The socially constrained-efficient allocation can be implemented with state-dependent subsidies. For example, it could be desirable to subsidize investment in times of high uncertainty and low activity. But, surprisingly, the optimal policy does not necessarily eliminate the uncertainty traps. Therefore, while policy interventions are desirable, they do not eliminate the nonlinearities generated by the complementarity between uncertainty and economic activity.

After characterizing the baseline model, we embed the mechanism into a standard model of business cycles. We show that uncertainty traps survive in that context, and explore numerically their ability to generate deep and persistent recessions for a range of parameter values. For that, we compare the behavior of a real business cycle model where the level of uncertainty is fixed with another that allows uncertainty to adjust as in our baseline model. We find that uncertainty traps make recessions substantially deeper and longer relative to a framework with fixed uncertainty.

The theory is motivated by the strong countercyclicality of several measures of uncertainty and evidence of high levels of uncertainty in the aftermath of the 2007-2009 recession.[2] Also consistent with our mechanism is the increase in firm inactivity during recessions. Because firms face irreversible investment choices, their incentives to invest are low when uncertainty is high. Gourio and Kashyap (2007) show that in the US and Chile the share of exact or near-zeros in firm-level investment is strongly counter-cyclical with a correlation with aggregate investment of -0.94 in the US and -0.56 in Chile.

This paper is closely related to the literature on "uncertainty shocks" as in Bloom (2009) and more recent contributions such as Arellano et al. (2012), Bachmann and Bayer (2009), Bloom et al. (2012), and Schaal (2012), on how changes in the volatility of productivity shocks affects the economy. Two features set us apart from this literature. First, these papers define uncertainty as the volatility of either aggregate or idiosyncratic productivity shocks. In contrast, we define uncertainty as the variance of firms' beliefs about the returns to their investment. This enables us to dissociate subjective uncertainty from volatility, so that, in our setup there can be periods of high uncertainty with low volatility. Indeed, several measures of volatility have receded after the 2007-2009 recession, while subjective uncertainty remains high.[3] Second, in our model the movements in uncertainty are endogenous and depend on agents' decisions. The main implication of endogenous uncertainty is a higher persistence in macroeconomic series in response to fundamental shocks. In contrast, models that focus on exogenous volatility shocks, such as Bloom et al. (2012), are hard

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[2]The National Federation of Independent Business (2012) reports that 40% of firms rank "economic uncertainty" as the most critical problem that they faced in 2012. Using the Confederation of British Industry's Industrial Trends Survey and the Michigan Survey of Consumers, Leduc and Liu (2012) show that firms' perceived uncertainty in the U.K. and the fraction of consumers who report "uncertain future" as the main reason to postpone purchases of durable goods in the United States peaks during recessions. Other measures of subjective uncertainty with strong counter-cyclical patterns are the variance of ex-post forecast errors about economic conditions and the dispersion of beliefs constructed by Bachmann et al. (2013). Baker et al. (2012) report that measures have of economic and policy uncertainty have remained high after the 2007-2009 recession. See also Fernández-Villaverde et al. (2011).

[3]Measures of aggregate and idiosyncratic volatility such as the VIX volatility index have substantially declined since 2009.

to reconcile with the persistence of recessions because high volatility events are short-lived.[4]

Our analysis also relates to a theoretical macroeconomic literature that studies environments characterized by learning from market outcomes such as Amador and Weill (2010), Caplin and Leahy (1993), Ordoñez (2009), Rob (1991), Veldkamp (2005) and Zeira (1994). Closely related to our paper is the analysis of Van Nieuwerburgh and Veldkamp (2006). They focus on explaining business-cycle asymmetries in an RBC model with incomplete information in which agents receive signals with procyclical precision about the economy's fundamental. During recessions, agents discount new information more heavily and the mean of their beliefs is slow to recover. Since the fundamental follows a two-state Markov process, beliefs are fully described by a single sufficient statistic, so that uncertainty recovers whenever the mean does. As a result, uncertainty does not provide an independent propagation mechanism and uncertainty traps do not arise. In contrast, our approach builds on a standard model of irreversible investment under uncertainty as in Dixit and Pindyck (1994) and Stokey (2008), and is able to disentangle the effects of mean vs. variance. The interaction between the option value of waiting due to irreversibilities and endogenous countercylical uncertainty is unique to our model, and essential to generate uncertainty traps.

This paper is also related to the literature on fads and herding in the tradition of Banerjee (1992), Bikhchandani et al. (1992), and Chamley and Gale (1994). Articles in that tradition consider economies with an unknown fixed fundamental and study a one-shot evolution towards a stable state, whereas we study the full cyclical dynamics of an economy that fluctuates between regimes.

The dynamics generated by the model, with endogenous fluctuations between regimes, is reminiscent of the literature on dynamic coordination games such as Angeletos et al. (2007), Chamley (1999) and Frankel and Pauzner (2000). These papers study repeated games in which complementarity in payoffs leads to multiple equilibria under complete information. The introduction of strategic uncertainty through noisy observation of the fundamental leads to a departure from common knowledge that eliminates the multiplicity. In contrast, the complete-information version of our model does not feature multiplicity, and complementarity only arises under incomplete information through social learning. Uniqueness does not obtain through strategic uncertainty but through slow adjustment of beliefs.

The paper is structured as follows. Section 2 presents the baseline model and the definition of the recursive equilibrium. Section 3 characterizes the partial-equilibrium investment decision of an individual firm and demonstrates the uniqueness of the equilibrium. Section 4 shows the existence of uncertainty traps, examines the non-linearities that they generate, and discusses the planner's problem. Section 5 describes the extended model and shows how uncertainty traps influence the response of the economy to various shocks. Section 6 concludes. Proofs can be found in the appendix.

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[4]Some recent papers discuss alternative channels that give rise to endogenous volatility over the business cycle. See Bachmann and Moscarini (2011) and D'Erasmo and Boedo (2011).

2 Baseline Model

We begin by presenting a stylized model that only features the necessary ingredients to generate uncertainty traps. The intuitions from this simple model as well as the laws of motion governing the dynamics of uncertainty carry through to the extended model that we use for numerical analysis.

2.1 Population and Technology

Time is discrete. There is a large, fixed number of firms $\overline{N}$ indexed by $j \in \{1, \ldots, \overline{N}\}$. Each firm holds a single investment opportunity that produces output x_j which is the sum of two components: a persistent common component θ, which denotes the economy's fundamental, as well as an idiosyncratic transitory component ε_j^x,

$$x_j = \theta + \varepsilon_j^x.$$

The common component follows a random walk, so that the next period's fundamental is

$$\theta' = \theta + \varepsilon^\theta. \tag{1}$$

The innovations $\left(\varepsilon^\theta, \varepsilon_j^x \right)$ are independent and normally distributed over time and across firms,[5]

$$\varepsilon^\theta \sim \mathcal{N}\left(0, \gamma_\theta^{-1}\right) \text{ and } \varepsilon_j^x \sim \mathcal{N}\left(0, \gamma_x^{-1}\right).$$

To produce, a firm must pay a fixed cost f, drawn each period from the continuous cumulative distribution F with mean $\bar{f}$ and standard deviation σ^f. Once production has taken place, the firm exits the economy and is immediately replaced by a new firm holding an investment opportunity. This assumption ensures that the mass of firms in the economy remains constant.[6]

Upon investment, the firm receives the payoff x_j. Firms have constant absolute risk-aversion,[7]

$$u\left(x_j\right) = \frac{1}{a}\left(1 - e^{-ax_j}\right),$$

where a is the coefficient of absolute risk aversion.

2.2 Timing and Information

At the beginning of each period, firms decide whether to invest or not without knowing their return on investment x_j. This decision therefore depends on their beliefs about the unobserved

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[5]The assumption of a random walk for the aggregate productivity θ can be relaxed and our results extend to an AR(1) process as long as the persistence is sufficiently high. Similarly, idiosyncratic productivity ε^x is assumed to be iid for simplicity, but the theory could also accommodate persistence in this component.

[6]This assumption is made for tractability and is relaxed for the numerical exercise of Section 5.

[7]Agents can be thought of as entrepreneurs. However, risk aversion is not crucial for the results and is only assumed for technical reasons in the proofs. In the numerical exercises we show that the mechanism carries through with risk neutrality

fundamental θ. As time unfolds, they learn about θ in various ways. First, they learn from a public signal Y observed at the end of each period,

$$Y = \theta + \varepsilon^y \tag{2}$$

where $\varepsilon^y \sim$ iid $\mathcal{N}(0, \gamma_y^{-1})$. This signal captures the information released by statistical agencies or the media. Second, they learn by observing production in the economy. Social learning takes place through this channel: when firm j invests, its output x_j is observed by all the other firms. Since θ cannot be distinguished from the idiosyncratic term ε_j^x, production x_j acts as a noisy signal about the fundamental. Because of the normality assumption, a sufficient statistic for the information provided by each firm's individual output is the public signal
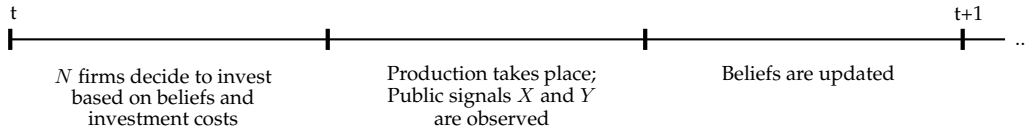
$$X \equiv \frac{1}{N} \sum_{j \in I} x_j = \theta + \varepsilon_N^X, \tag{3}$$

where $N \in \{1, \ldots, \overline{N}\}$ is the endogenous number of firms that invest, I is the set of such firms, and

$$\varepsilon_N^X \equiv \frac{1}{N} \sum_{j \in I} \varepsilon_j^x \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right).$$

Importantly, the precision $N\gamma_x$ of this signal increases with the number of investing firms N.

The timing of events is summarized in Figure 1.

Figure 1: Timing of events

2.3 Beliefs

Under the assumption of a common initial prior, and because all signals become public at the end of a period, beliefs are common across firms. The normality assumptions about the signals and the fundamental imply that beliefs are also normally distributed

$$\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right),$$

where $\mathcal{I}$ is the information set at the beginning of the period. The mean of the distribution μ captures the optimism of agents about the state of the economy, while γ represents the precision of their beliefs about the fundamental. Precision γ is inversely related to the amount of uncertainty: as γ increases, the variance of beliefs decreases (uncertainty declines).

Our notion of uncertainty, the dispersion of the subjective beliefs about the fundamental, differs from what is used in most of the uncertainty-driven business cycle literature where uncertainty is

associated to volatility in aggregate or idiosyncratic conditions. A key difference is that, in our approach, there can be uncertainty without volatility being observed in the data.

Firms start the period with beliefs (μ, γ). By the end of the period, they have observed the public signals X and Y. Firms use all the information available to update their beliefs according to Bayes' rule. Therefore, beliefs about next period's fundamental θ' are normally distributed with mean and precision equal to

$$\mu' = \frac{\gamma\mu + \gamma_y Y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x}, \tag{4}$$

$$\gamma' = \left(\frac{1}{\gamma + \gamma_y + N\gamma_x} + \frac{1}{\gamma_\theta}\right)^{-1} \equiv \Gamma(N, \gamma). \tag{5}$$

These standard updating rules have straightforward interpretations: the mean of future beliefs μ' is a precision-weighted average of the present belief μ and the new signals, Y and X, whereas γ' depends on the precision of current beliefs, the precision of the signals and the variance of the shock to θ. Importantly, the precision of future beliefs does not depend on the realization of public signals, but only on N and γ. The higher is N, the more precise is the public signal X, and the lower is next period uncertainty. We use $\Gamma(N, \gamma)$ in (5) to denote the law of motion of the precision of information.

2.4 Firm Problem

We now describe the problem of a firm. In each period, given fixed cost f and beliefs about the fundamental, a firm can either wait or invest. It solves the Bellman equation

$$V(\mu, \gamma, f) = \max\left\{V^W(\mu, \gamma), V^I(\mu, \gamma) - f\right\}, \tag{6}$$

where $V^W(\mu, \gamma)$ is the value of waiting and $V^I(\mu, \gamma)$ is the value of investing after incurring the investment cost f. We assume that the number of firms $\overline{N}$ is large enough so that firms behave competitively. Specifically, they do not internalize the impact of their decisions on aggregate information.

If a firm waits, it starts the next period with updated beliefs (μ', γ') about the fundamental and a new draw of the fixed cost f'. Therefore, the value of waiting is

$$V^W(\mu, \gamma) = \beta \mathbb{E}_{\mu', \gamma'}\left[\int V(\mu', \gamma', f')\, dF(f') \mid \mu, \gamma\right]. \tag{7}$$

In turn, when a firm invests it receives output x and exits. Therefore,

$$V^I(\mu, \gamma) = \mathbb{E}[u(x) \mid \mu, \gamma] = \mathbb{E}\left[\frac{1}{a}\left(1 - e^{-a \cdot x}\right) \mid \mu, \gamma\right]. \tag{8}$$

The firm's optimal investment decision takes the form of a cutoff rule $f_c(\mu, \gamma)$ such that a firm

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invests if and only if $f \leq f_c(\mu, \gamma)$. The cutoff is defined by the following indifference condition

$$f_c(\mu, \gamma) = V^I(\mu, \gamma) - V^W(\mu, \gamma). \tag{9}$$

2.5 Law of Motion for the Number of Investing Firms N

We now aggregate the individual decisions of the firms. From (9), the process for the number of investing firms N satisfies

$$N\left(\mu, \gamma, \{f_j\}_{1 \leq j \leq \bar{N}}\right) = \sum_{j=1}^{\overline{N}} \mathbb{I}\left(f_j \leq f_c(\mu, \gamma)\right). \tag{10}$$

Since investment depends on a random fixed cost, the number of investing firms is a random variable that depends on the realization of the shocks $\{f_j\}_{1 \leq j \leq \bar{N}}$. As these costs are i.i.d., the ex-ante probability of investment is identical across firms. Therefore, the ex-ante distribution of N, as perceived by firms, is binomial,

$$N \mid \mu, \gamma \sim \mathrm{Bin}\left(\overline{N}, p(\mu, \gamma)\right), \tag{11}$$

where $p(\mu, \gamma)$ captures the perceived probability of investment for other firms. In equilibrium, firms' expectations must be consistent with the actual probability of investing:

$$p(\mu, \gamma) = F\left(f_c(\mu, \gamma)\right). \tag{12}$$

Note that N is only a function of the beliefs (μ, γ) and the shocks $\{f_j\}_{1 \leq j \leq \overline{N}}$. Since these shocks are independent from the fundamental θ and investment decisions are made before the observation of $\{x_j\}_{j \in I}$, there is nothing to learn from the non-investment of firms, nor from the realization of N itself.

2.6 Recursive Competitive Equilibrium

We define a recursive rational-expectation equilibrium as follows:

Definition 1. A recursive competitive equilibrium consists of a cutoff rule $f_c(\mu, \gamma)$, value functions $V(\mu, \gamma, f)$, $V^W(\mu, \gamma)$, $V^I(\mu, \gamma)$, a perceived ex-ante investment probability $p(\mu, \gamma)$, laws of motions for aggregate beliefs $\{\mu', \gamma'\}$, and a number of investing firms $N\left(\mu, \gamma, \{f_j\}_{1 \leq j \leq \bar{N}}\right)$, such that

1. The value function $V(\mu, \gamma, f)$ solves (6), with $V^W(\mu, \gamma)$ and $V^I(\mu, \gamma)$ defined according to (7) and (8), yielding the cutoff rule $f_c(\mu, \gamma)$ in (9);

2. The aggregate beliefs (μ, γ) evolve according to (4) and (5) where N is given by (11);

3. The ex-ante investment probability $p(\mu, \gamma)$ and the cutoff rule $f_c(\mu, \gamma)$ satisfy (12) and

4. The number $N\left(\mu, \gamma, \{f_j\}_{1 \leq j \leq \bar{N}}\right)$ of investing firms is given by (10).

3   Equilibrium Characterization

We first characterize the optimal investment decision of a firm. We provide conditions such that, due to the the irreversibility of investment, firms are less likely to invest when uncertainty is high. Then, we prove the existence and uniqueness of the recursive equilibrium and characterize its key properties.

3.1   Investment Rule Given the Evolution of Beliefs

The optimal investment rule $f_c(\mu, \gamma)$ depends on how beliefs evolve. We begin by establishing two simple lemmas about the dynamics of aggregate beliefs.

Evolution of the Mean of Beliefs

Using (4), we can express the stochastic process for the mean of beliefs as follows.

Lemma 1. *For a given N, mean beliefs μ follow a random walk with time-varying volatility s,*

$$\mu' = \mu + s(N, \gamma)\,\varepsilon,$$

where $s(N, \gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}}$ and $\varepsilon \sim \mathcal{N}(0, 1)$.

Proof. The full statements and proofs of propositions are in the Appendix. □

The mean of beliefs captures the optimism of agents about the fundamental and evolves stochastically due to the the arrival of new information. It inherits the random-walk properties of the fundamental, and its volatility $s(N, \gamma)$ is time-varying because the amount of information that firms collect over time is endogenous. The volatility is decreasing with γ and increasing with N. In times of low uncertainty (γ high) agents place more weight on their current information and less on new signals, making the mean of beliefs more stable. In contrast, in times of high activity (N high) more information is released, making beliefs more likely to fluctuate.

Evolution of Uncertainty

The precision of beliefs γ captures the (inverse of) uncertainty about the fundamental and its dynamics play a key role for the existence of uncertainty traps. Its law of motion satisfies the following properties.

Lemma 2. *The precision of next-period beliefs γ' increases with N and γ. For a given number of investing firms N, the law of motion for the precision of beliefs $\gamma' = \Gamma(N, \gamma)$ admits a unique stable stationary point in γ.*

The thin solid curves on Figure 2 depict $\Gamma(N, \gamma)$ for different values of N. An increase in the level of activity raises the next period precision of information γ' for each level of γ in the current
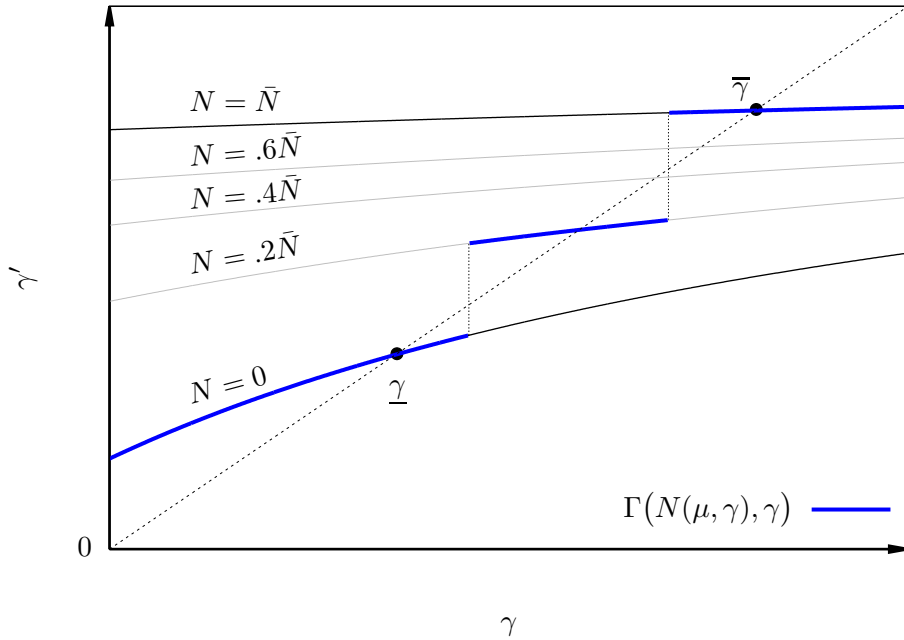
Figure 2: Example of dynamics for belief precision γ

period. Since N is between 0 and $\overline{N}$, the support of the ergodic distribution of γ must lie between the two bounds $\underline{\gamma}$ and $\bar{\gamma}$ defined by $\underline{\gamma} \equiv \Gamma(0, \underline{\gamma})$ and $\bar{\gamma} \equiv \Gamma(\bar{N}, \bar{\gamma})$. In other words, $\underline{\gamma}$ is the stationary level of precision when no firm invests, while $\bar{\gamma}$ is the one reached when all firms invest.

In equilibrium, N varies with μ and γ. Suppose, as an example, that N is a deterministic and increasing step function of γ, and let us keep μ fixed for the moment. Figure 2 illustrates how the feedback from uncertainty to investment opens up the possibility of multiple stationary points in the dynamics of the precision of beliefs, and therefore uncertainty. In this example, the function $\gamma' = \Gamma(N(\mu, \gamma), \gamma)$, depicted by the solid curve, has three fixed points. We formally establish, in part 4, that this type of multiplicity is a generic feature of the equilibrium.

Optimal Timing of Investment

With the laws of motion for aggregate beliefs at hand we can characterize the individual investment decision as a function of beliefs. Naturally, a more optimistic firm (higher μ) is more likely to invest. In turn, uncertainty (lower γ) may reduce the returns to investment for two reasons. First, risk averse firms dislike uncertain payoffs. Second, since investment is costly and irreversible, there is an option value of waiting: in the face of uncertainty, firms prefer to delay investment to gather additional information and avoid downside risks.

The next proposition formally establishes the validity of these intuitions. Specifically, it provides a partial-equilibrium characterization of the optimal investment behavior of a firm who, consistently with (11), perceives N as following a binomial distribution $Bin\left(\overline{N}, p(\mu, \gamma)\right)$ for some sufficiently smooth function $p(\mu, \gamma) \in \mathcal{P}$ as defined in the Appendix.

Proposition 1. *Under mild regularity conditions stated in the Appendix, given a random number of investing firms $N \sim Bin\left(\overline{N}, p\left(\mu, \gamma\right)\right)$ for some $p\left(\mu, \gamma\right) \in \mathcal{P}$, and for γ_x sufficiently low, there exists a unique solution $V\left(\mu, \gamma, f\right)$ to the firm's Bellman equation and the resulting cutoff $f_c\left(\mu, \gamma\right)$ is strictly increasing in μ and γ.*

The properties satisfied by the optimal investment rule are typical of optimal stopping time models of investment, but in our context they are not straightforward. As expected, investment follows a cutoff rule which is strictly increasing in μ (because it inherits the monotonicity of the firm's objective function), but the challenging aspect of the analysis lies in establishing the last property, crucial to our mechanism, that the cutoff strictly increases in γ, i.e., that the probability of investment decreases with uncertainty. On the one hand, uncertainty directly discourages investment through risk aversion and an option value of waiting. On the other hand, these effects may be offset by an opposing general equilibrium effect through social learning. If the number of investing firms N declines with uncertainty, then firms have less incentives to wait because less information is released. Our proposition ensures that this indirect effect is second order when the social learning channel is not too strong, which specifically happens when the precision of the individual signals γ_x is low.

3.2 Existence and Uniqueness

We have described in Lemmas 1 and 2 how beliefs depend on the number of investing firms, and, in Proposition 1, how firms' investment decisions are affected by beliefs. In the latter, firms make their decisions taking the aggregate investment probability p as given. We now close the equilibrium by requiring that the perceived investment behavior of firms, summarized by $N \sim Bin\left(\overline{N}, p\left(\mu, \gamma\right)\right)$, is consistent with their actual investment decisions: $p\left(\mu, \gamma\right) = F\left(f_c\left(\mu, \gamma\right)\right)$. The next proposition shows that such a general equilibrium exists and is unique.

Proposition 2. *Under mild regularity conditions stated in the Appendix and for γ_x small enough, a recursive equilibrium exists and is unique. The equilibrium expected fraction of firms investing $p(\mu, \gamma)$ is increasing in the mean of beliefs μ and the precision γ.*

Showing uniqueness of the fixed point $p\left(\mu, \gamma\right) = F\left(f_c\left(\mu, \gamma\right)\right)$ is challenging due to the ambiguous feedback from uncertainty to investment discussed above. Formally, this leads to a failure in the monotonicity of the mapping from the perceived investment probability $p(\mu, \gamma)$ to the investment probability of each firm, which prevents us from using Blackwell's theorem. Fortunately, we can explicitly show that the main fixed point problem is a contraction when γ_x is low. This assumption ensures that the complementarity between information and economic activity is not strong enough to support multiple equilibria. Uniqueness of equilibrium is an attractive feature as it leads to unambiguous predictions and makes the model amenable to quantitative work. Despite uniqueness of the equilibrium, the model features interesting non-linear dynamics and multiple regimes, as we show in part 4.

Figure 3 depicts the expected fraction of investing firms as a function of beliefs (μ, γ). The partial equilibrium results from Proposition 1 carry through to the general equilibrium: the number of investing firms increases as they become more optimistic about the fundamental (μ high) or less uncertain (γ high).
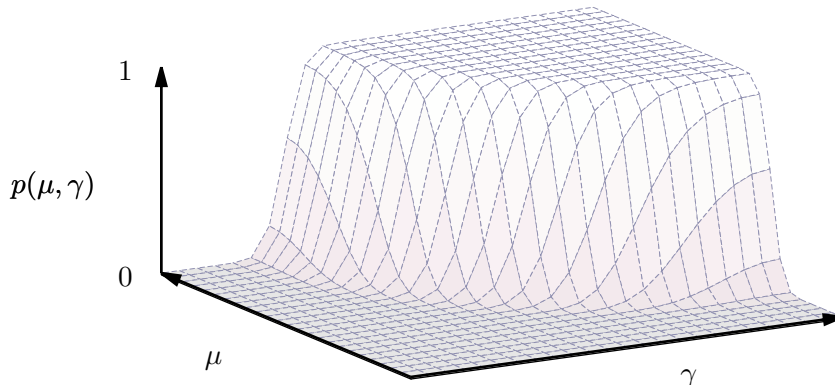

Figure 3: Example of aggregate investment pattern

4   Uncertainty Traps

We now examine the interaction between firms' behavior in the face of uncertainty and social learning. This interaction leads to episodes of self-sustaining uncertainty and low activity, which we call uncertainty traps. We provide sufficient conditions on the parameters that guarantee the existence of such traps and discuss the type of aggregate dynamics that they imply. The response of the economy to shocks is highly non-linear: it quickly recovers after small shocks, but large, short-lived shocks may plunge the economy into long-lasting recessions. We also characterize the constrained planner's problem and discuss policy implications.

4.1   Definition and Existence

We assume at this point that the total number of firms $\overline{N}$ is large enough, so that

$$n\left(\mu, \gamma\right) \equiv \frac{N\left(\mu, \gamma\right)}{\overline{N}} \simeq p(\mu, \gamma). \tag{13}$$

With this assumption, we can treat n as a deterministic function of beliefs, ignoring fluctuations due to the finiteness in the number of firms. We formally describe the limit economy in Appendix

B. The model's equations remain the same except that we must substitute $N(\mu, \gamma)$ with $n(\mu, \gamma)$.[8] We are now ready to define an uncertainty trap:

Definition 2. For a given mean of beliefs μ, there is an *uncertainty trap* if there are at least two locally stable fixed points in the dynamics of beliefs precision $\gamma' = \Gamma(n(\mu, \gamma), \gamma)$.

The definition of an uncertainty trap captures the situation depicted in Figure 2: for a given mean of beliefs, the economy may find itself in distinct fixed points of the dynamics of uncertainty. We refer to these stationary points as *regimes.* Note that multiplicity of regimes does not imply multiple equilibria. This distinction is important because it highlights that the model is not subject to indeterminacy. While multiple values of γ may satisfy the equation $\gamma = \Gamma(n(\mu, \gamma), \gamma)$ for a given μ, the regime that prevails at any given time is unambiguously determined by the history of past aggregate shocks, summarized by the current beliefs (μ, γ). The definition also emphasizes the notion of *stability,* which is required for the type of self-enforcing dynamics that we propose. Notice, however, that we only require local stability along the dimension γ while μ is fixed.

The following proposition formally establishes that uncertainty traps exist for a range of mean of beliefs μ under some condition on the dispersion of investment costs.

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $[\mu_l, \mu_h]$ such that, for all $\mu \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l(\mu) < \gamma_h(\mu)$. Regime γ_l is characterized by high uncertainty and low investment while regime γ_h is characterized by low uncertainty and high investment.*

Figure 4 presents examples for the law of motion of γ when the investment costs f are normally distributed. The solid curves represent the function $\gamma' = \Gamma(n(\mu, \gamma), \gamma)$ evaluated at five different values of μ, with the thick solid curve corresponding to an intermediate value of μ. In all cases, for small precision γ, uncertainty is high and firms do not invest. As a result, they do not learn from observing aggregate activity and the precision of beliefs γ' remains low. As precision increases, uncertainty decreases and firms become sufficiently confident about the fundamental to start investing. As that happens, uncertainty decreases further.

In our example, the thick curve intersects the 45° line three times. The second intersection corresponds to an unstable regime, but the other two are locally stable. We denote these regimes by γ_l and γ_h. In regime γ_l, uncertainty is high and investment is low, while the opposite is true in regime γ_h.

Proposition 3 shows that this situation is a generic feature of the equilibrium when the dispersion of investment costs σ^f is small. This condition ensures that the feedback of investment on information is strong enough to sustain distinct stationary points.

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[8] To prevent uncertainty from vanishing completely as $\overline{N} \to \infty$, we assume that the precision of firms' individual signals decreases with $\overline{N}$: $\gamma_x(\overline{N}) = \gamma_x / \overline{N}$. The details of the limit and the corresponding economy are explained in appendix B.
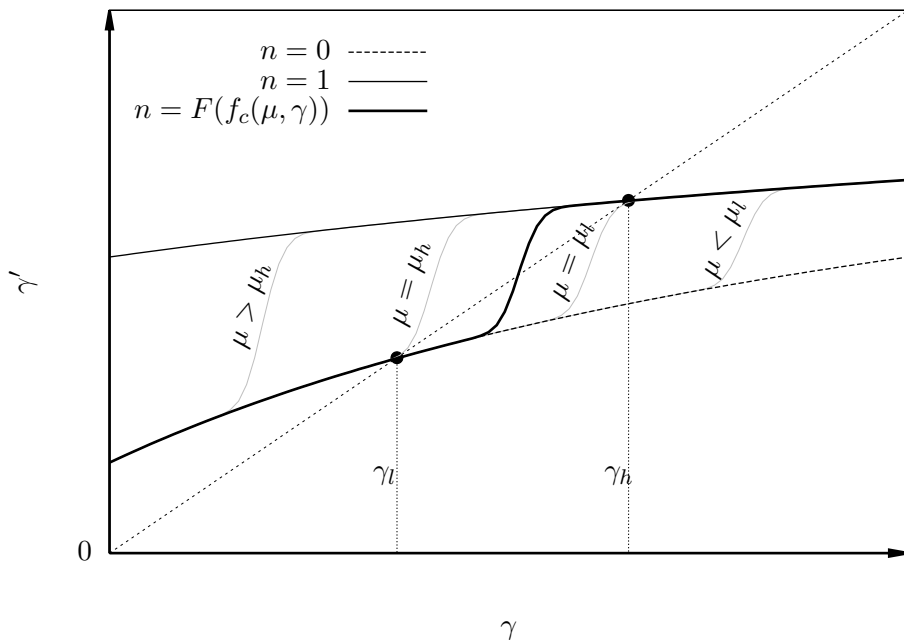
Figure 4: Dynamics of precision $\gamma' = \Gamma(n(\mu, \gamma), \gamma)$ for different values of μ

4.2   Dynamics: Non-linearity and Persistence

We now describe the full dynamics of the economy by taking into account the evolution of μ in response to the arrival of new information. Figure 4 shows that, as long as μ stays between the values μ_l and μ_h, defined in Proposition 3, the two regimes $\gamma_l(\mu)$ and $\gamma_h(\mu)$ preserve their stability. As a result, uncertainty and the fraction of active firms n are relatively unaffected. In contrast, for values of μ above μ_h, a large enough fraction of firms invest, so the dynamics of beliefs only admits the high-activity regime as a stationary point. Similarly, for values below μ_l, the economy only admits the low-activity regime. Therefore, sufficiently large shocks to μ can make one regime unstable and trigger a regime switch.

The economy displays non-linear dynamics: it reacts very differently to large shocks in comparison to small ones. Figure 5 shows various simulations to illustrate this feature using the example from Figure 4. The top panel presents three different series of shocks to the mean of beliefs μ. The three series start from the high-activity/low-uncertainty regime. At $t = 5$, the economy is hit by a negative shock to μ, due to a bad realization of either the public signals or the fundamental. The mean of beliefs then returns to its initial value at $t = 10$. Across the three series, the magnitude of the initial shock is different.

The middle and bottom panels show the response of belief precision γ and the fraction of investing firms n. The economy starts from a regime in which all firms invest. The solid gray line represents a small temporary shock, such that μ remains within $[\mu_l, \mu_h]$. Despite the negative shocks to the mean of beliefs, all firms keep investing and the precision of beliefs is unaffected. When the economy is hit by a temporary shock of medium size (dashed line), some firms stop investing,
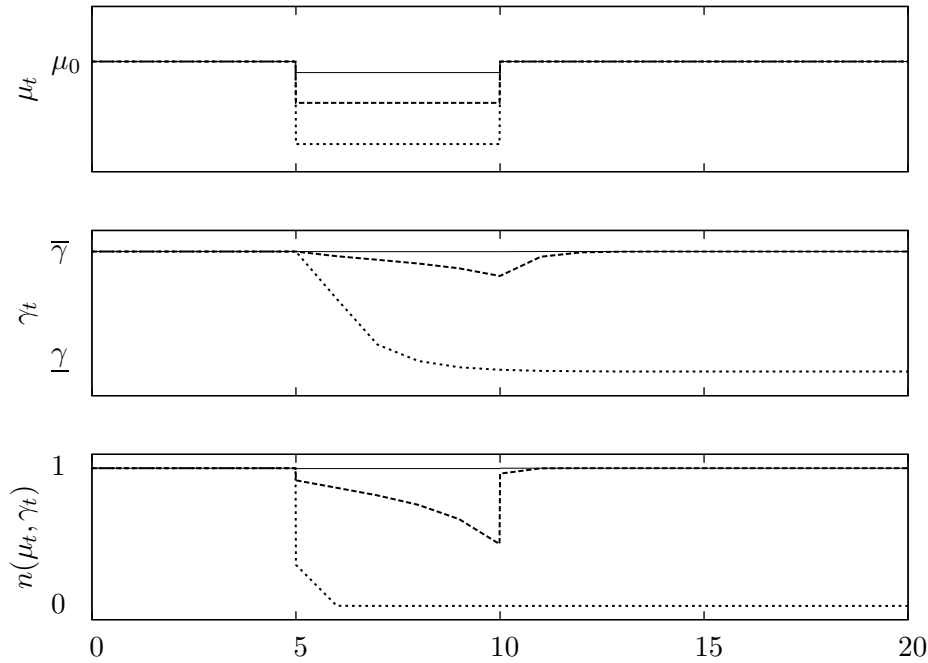
14

Figure 5: Persistent Effects of Temporary Shocks

leading to a gradual increase in uncertainty. As uncertainty rises, investment falls further and the economy starts to drift towards the low regime. However, when the mean of beliefs recovers, the precision of information and the number of active firms quickly return to the high-activity regime. In contrast, when the economy is hit by a large temporary shock (dotted line), the number of firms delaying investment is large enough to produce a self-sustaining increase in uncertainty. The economy quickly shifts to the low-activity regime and remains there even after the mean of beliefs recovers.

We now show how the economy escapes from the trap in which it fell in Figure 5. Figure 6 shows the effect of positive shocks when the economy starts from the low regime. The economy receives positive signals that lead to a temporary increase in mean beliefs between periods 20 and 25, possibly because of a recovery in the fundamental. When the temporary increase in average beliefs is not sufficiently strong, the recovery is interrupted as μ returns to its initial value. However, when the temporary increase is sufficiently large, the economy reverts back to the high-activity regime. Once again, temporary shocks of sufficient magnitude to the fundamental may lead to nearly permanent effects on the economy.

4.3 Additional Remarks

A number of additional lessons can be drawn from these simulations. First, in this framework, uncertainty is a by-product of recessions. This result echoes the empirical findings of Bachmann et al. (2013) who show that uncertainty is caused by recessions and conclude, by that, that it is of secondary importance for the business cycle. We show, however, that uncertainty may still have a
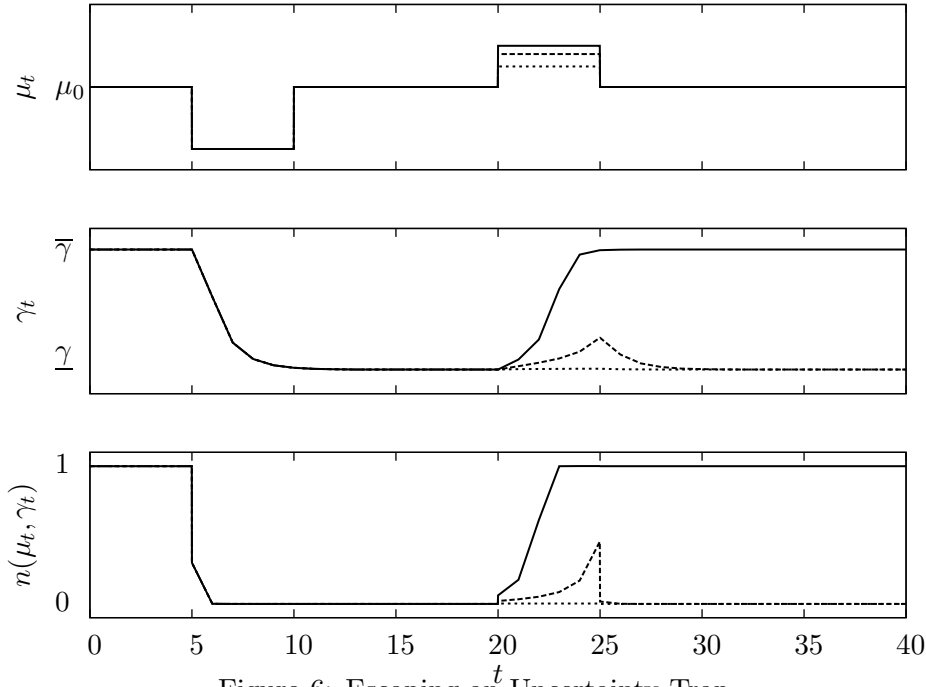
15

Figure 6: Escaping an Uncertainty Trap

large impact on the economy by affecting the persistence and depth of recessions, even if it is not what triggers them.

Second, as in models with learning in the spirit of Van Nieuwerburgh and Veldkamp (2006), this theory provides an explanation for asymmetries in business cycles. In good times, since agents receive a large flow of information, they react faster to shocks than in bad times.

Third, our economy may feature high uncertainty without volatility, consistent with evidence in the aftermath of the 2007-2009 recession. For instance, in the low regime, agents are highly uncertain about the fundamental but the volatility of economic aggregates is low. Therefore, according to our theory, subjective uncertainty may affect economic fluctuations even if no volatility is observed in the data. This distinguishes our approach from the existing uncertainty-driven business cycle literature in the spirit of Bloom (2009). In particular, direct measures of subjective uncertainty rather than measures of volatility may be needed to capture the full amount of uncertainty in the economy.

Finally, a recent literature uses survey data to derive measures of uncertainty based on ex-ante forecast error. Our model highlights a potential difficulty about this approach, as uncertainty about *fundamentals* differs from uncertainty about *endogenous variables*, such as output or investment. For example, when the economy is trapped in the low activity regime, firms know that all firms are uncertain, and therefore that output and investment are likely to be low. As a result, their forecasts about economic aggregates are accurate even though their uncertainty about the fundamental is high. As implied by the model, forecast errors about variables like output may possibly be a bad proxy for uncertainty about fundamentals.

16

4.4 Policy Implications

The economy is subject to an information externality: in the decentralized equilibrium, firms invest less often than they should because they do not internalize the release of information to the rest of the economy caused by their investment. In Appendix D, we solve the problem of a constrained planner subject to the same information technology as agents in the economy. Proposition 4 shows that the decentralized economy is constrained inefficient and that a simple policy instrument such as an investment subsidy that depends on current beliefs (μ, γ) is sufficient to restore constrained efficiency. Despite internalizing information flows, the constrained optimum still features uncertainty traps.

Proposition 4. *The recursive competitive equilibrium is constrained inefficient. The efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$ and a uniform tax. When γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

The subsidy that implements the optimal allocation takes a simple form to align social and private incentives. As shown in the proof of the proposition, it is simply the sum of the social value of releasing an additional signal to the economy and the private value of delaying investment.

The optimal policy being a subsidy, proposition 4 implies that firms are more likely to invest in the efficient allocation than in the laissez-faire economy. However, uncertainty traps can still arise in the efficient allocation. This may be surprising if one believes that the planner should always be able to implement the high regime. As long as the planner does not have more information than individual agents, it is still optimal to wait when uncertainty is high enough. Hence, there still exists a sufficiently strong complementarity between information and the level of activity in the constrained-efficient allocation to generate uncertainty traps. While uncertainty traps remain present in the efficient allocation, they are less persistent than in the laissez-faire economy because the region in which investing is dominant is wider.[9]

5 Extended Model and Numerical Example

We embed the mechanism into a standard model of business cycles. The purpose of this exercise is twofold. First, we show that uncertainty traps survive in a more general context. Second, we explore numerically the ability of the mechanism to generate deep and persistent recessions for a range of parameter values.

We enrich the benchmark theory in several dimensions. First, we allow mean reversion in the stochastic process followed by the fundamental. Second, infinitely-lived firms produce every period using a Cobb-Douglas production function that combines labor and capital as inputs. They also accumulate capital over time by investing through intensive and extensive margins. Third, a representative household consumes a single good and supplies labor inelastically. The ensuing model

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[9]In contrast to models with multiple equilibria, γ is a predetermined state variable that summarizes past information, not a forward-looking variable that the planner can pick to select equilibria.

nests a standard real business cycle model when investment irreversibilities and social learning are shut down.

We solve the planner's problem for tractability reasons and because it provides conservative estimates of the impact of uncertainty traps. Since the planner internalizes the information externality, firms invest more often in the optimal allocation than in the competitive equilibrium. Therefore, our numerical results provide a lower bound to the persistence and amplification produced by uncertainty traps in the competitive allocation.

5.1 Extended model

Preferences and Technology

There is a unit measure of firms indexed by $j \in [0, 1]$ that operate a Cobb-Douglas technology and produce a single consumption good. Firm j employing l_j units of labor and k_j units of capital produces output

$$(A + Y) \, k_j^{\alpha} l_j^{1-\alpha},$$

where

$$
\begin{aligned}
Y &= \theta + \varepsilon^y \\
\theta' &= \rho_\theta \theta + \varepsilon^\theta
\end{aligned}
$$

with $\varepsilon^\theta \sim$ iid $\mathcal{N}\left(0, \left(1 - \rho_\theta^2\right) \gamma_\theta^{-1}\right)$ and $\varepsilon^y \sim$ iid $\mathcal{N}\left(0, \gamma_y^{-1}\right)$, and where $A > 0$ is the unconditional mean of total factor productivity.[10] As in the baseline model, the stochastic process θ is the fundamental of the economy. However, we relax the assumption of a random walk and adopt the more standard assumption of an AR(1) process. Variable Y, which was assumed to be a purely informational signal in the baseline model, is now the aggregate TFP of the economy. This specification is convenient as it guarantees that Y enters the laws of motion of beliefs in the same way as before and, at the same time, prevents the aggregation of individual outputs from revealing the fundamental as would have been the case with idiosyncratic noise.

There is a risk-neutral representative household who supplies one unit of labor inelastically and discounts future utility at rate β. We assume risk neutrality to establish, through numerical examples, that risk aversion is not necessary for uncertainty traps to arise, and that the mechanism may operate solely through the option value of waiting. The introduction of risk aversion would naturally provide an additional incentive for firms to delay investment during recessions.

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[10]Our specification of TFP ensures that the perceived variance of Y has no effect on expected output. While productivity could theoretically be negative, the variance of the distributions we consider in our numerical exercise are such that it never happens in any of the simulations.

Investment

To introduce an option value of waiting we assume that only a subset of firms holds an investment opportunity in each period. A firm without an opportunity receives one with probability $\overline{q}$ at the end of the period and keeps it until exercised. Only one investment opportunity can be held at a time. A firm that uses its investment opportunity must pay a fixed cost $f > 0$ and a variable cost $c(i)$ that depends on its investment rate i, where $c'(i) > 0$ and $c''(i) > 0$. To obtain aggregation, both costs are proportional to the stock of capital owned by the firm. Therefore, firm j must pay a total cost of $(f + c(i))\,k_j$ to increase its capital stock to

$$k_j' = (1 - \delta + i)\,k_j,$$

where δ denotes the depreciation rate.[11]

Timing and information

The information structure and the timing of events are as follows:

1. All firms share the same prior distribution over the fundamental $\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right)$, where μ and γ denote the mean and the precision of beliefs.

2. Firms that hold an investment opportunity decide whether to invest or not. Firms that invest pay f and chose their investment rate i.

3. Firms choose labor and production takes place.

4. Aggregate productivity Y is observed. In addition, each investing firm generates a signal x_j about θ with precision $\gamma_x k_j$.

5. Firms that do not hold an investment opportunity receive one with probability $\overline{q}$.

6. Agents update their beliefs for the next period.

As in the baseline model, firms' investment decision is solely based on the common beliefs (μ, γ). Each firms that invests generates a signal that is then observed by all firms. We assume that the individual signals x_j are observed by everyone and that their precision is proportional to the capital stock of the firm. This assumption allows aggregation of the economy; also, it seems realistic to assume that investment by large corporations reveal more information than investment by a small mom-and-pop stores.

5.2 Social Planner

We now define the social planner's problem. To lighten the exposition, we use the notation $\{z_j\} \equiv \{z_j\}_{j \in [0,1]}$ for any variable z. We let q_j be a dummy for firm j equal to 1 if it has an

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[11]Note that, in contrast to the baseline model, here all firms have the same fixed cost f.

investment opportunity and 0 otherwise. Similarly, we let n_j be the probability that firm j invests and $\chi_j \sim \mathcal{B}(n_j)$ a Bernoulli variable that captures the realization of this decision. Summing up individual outputs across all firms, the problem of the planner is

$$V(\mu, \gamma, \{k_j, q_j\}) = \max_{\{i_j, n_j, l_j\}} \mathbb{E}\left[(A+Y) \int_0^1 k_j^\alpha l_j^{1-\alpha} dj \right.$$

$$\left. - \int_0^1 (f + c(i_j)) k_j q_j \chi_j dj + \beta V\left(\mu', \gamma', \{k'_j, q'_j\}\right) | \mu, \gamma \right]$$

subject to the following conditions:

$$1 = \int_0^1 l_j dj$$

$$k'_j = q_j \chi_j k_j (1 - \delta + i_j) + (1 - q_j \chi_j) k_j (1 - \delta)$$

$$q'_j = q_j (1 - \chi_j) + (1 - q_j + q_j \chi_j) \begin{cases} 0 & \text{with prob } (1 - \bar{q}) \\ 1 & \text{with prob } \bar{q} \end{cases}$$

$$\mu' = \rho \frac{\gamma \mu + \gamma_y Y + \left(\gamma_x \int q_j \chi_j k_j dj\right) X}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj}$$

$$\gamma' = \left(\frac{\rho^2}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj} + (1 - \rho^2)\sigma_\theta^2 \right)^{-1}.$$

Thanks to constant returns to scale in production and the overall linear structure of the model, the economy admits aggregation. In particular, the investment decision of firms on the intensive and extensive margins are independent of their capital stock and identical across firms, i.e., $n_j = n$ for all j. We provide the details of the aggregation in Appendix C. In particular, the aggregate capital stock $K \equiv \int k_j dj$ and the capital-weighted stock of investment opportunities $Q \equiv \int k_j q_j dj$, together with μ and γ, fully characterize the aggregate state of the economy. The laws of motion for information become

$$\mu' = \rho \frac{\gamma \mu + \gamma_y Y + nQ\gamma_x X}{\gamma + \gamma_y + nQ\gamma_x} \tag{14}$$

$$\gamma' = \left(\frac{\rho^2}{\gamma + \gamma_y + nQ\gamma_x} + (1 - \rho^2)\sigma_\theta^2 \right)^{-1} \tag{15}$$

where, as in the baseline model, X denotes the public signal that results from the aggregation of the individual signals released by investing firms. In particular, X is a signal about θ with precision $nQ\gamma_x$. Equations (14) and (15) are analog to the laws of motion governing beliefs in the baseline case, adjusted for mean reversion in the process for θ and the scaling in the precision of X.

The aggregated extended model nests a standard real business cycle model when the probability of receiving an investment opportunity $\bar{q}$ equals 1 and the fundamental is perfectly observed. Also, because of the irreversibility in investment, which arises when $\bar{q} < 1$, the planning problem retains the key features of the baseline model that led to uncertainty traps. In particular, the precision

of the information obtained through social learning, $nQ\gamma_x$, increases linearly with the fraction of investing firms n. In addition, the option value of waiting is captured by the dynamics in the stock of investment opportunities Q. When faced with an increase in uncertainty, the planner is reluctant to deplete its stock of opportunities and prefers to delay investment.

5.3 Simulations

Parameterization

We use a quadratic function $c(i) = i + \phi i^2$ for the variable cost of investment and parameterize the model with the values shown in Table 1. The time period is one month. The discount rate β is chosen to match a yearly value of 0.95. The depreciation rate δ is set to 10% per year and the share α of capital in production is set to 0.4 to match the average capital income share in postwar US. We parameterize the incoming rate of investment opportunities $\overline{q}$, the investment cost parameter ϕ, and the fixed cost of investment f to match the fraction of firms with an investment spike in a quarter, the average investment rate and the frequency of spikes at the firm level observed in Compustat.[12] We choose the parameters governing the dynamics of the fundamental $(\rho_\theta, \gamma_\theta)$ to match a yearly correlation of 0.9 and a standard deviation of 5%. Unfortunately, the literature provides little guidance on the information-related parameters (γ_y, γ_x). We choose a value $\gamma_x = 1500$, which corresponds to a standard deviation for the individual signal of about 2.5%, and a value $\gamma_y = 100$ to obtain a standard deviation of the public signal of 10%.[13] We perform a sensitivity analysis in Appendix A, which shows impulse responses for different parameterizations of the information parameters.

Benchmark Simulations

We first examine the properties of the policy function. Figure 7 presents the fraction of investing firms as a function of the mean of beliefs μ for three levels of uncertainty. As in the benchmark model, firms are more likely to invest when μ is high and uncertainty is low. Notice also that when μ is at its steady-state value of 0, firms invest significantly more when uncertainty is low. This feature is crucial for uncertainty traps to exist. Figure 8 shows the dynamics of the precision of beliefs γ from Equation (15). It suggests that there are indeed two stable stationary points similar to those in the baseline model. These points are however only stable as long as K and Q remain in the neighborhood of their steady-state values. In a recession, after a negative shock, firms reduce their investment and the stock of capital K decreases. As that happens, the returns to capital increase, firms resume their investment, and more information is released, so that the low regime

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[12]Following the lumpy investment literature, we focus on investment spikes. We define a spike as an investment above 5%. The average fraction of firms experiencing a spike in Compustat in a quarter is about $20\% \simeq 3 \times nQ/K$. The average size of investment over the period 1974-2012 conditioning on having a peak in investment is about 10% and the frequency of spikes at the firm level is about a year.

[13]Our benchmark parametrization for γ_x and γ_y implies a standard deviation of beliefs $1/\sqrt{\gamma}$ of 1% at the steady state. In the Survey of Professional Forecasters, over the period 1981-2012, the average standard deviation in ex-ante forecasts for annual real GDP growth is 1.02%, a similar order of magnitude to our parametrization.

Parameter	Value
Time period	month
Total factor productivity	$A = 1$
Discount factor	$\beta = (0.95)^{1/12}$
Depreciation rate	$\delta = 1 - (0.9)^{1/12}$
Share of capital in production	$\alpha = 0.4$
Probability of receiving an investment opportunity	$\overline{q} = 0.2$
Fixed cost of investment	$f = 0.10$
Variable cost of investment	$\phi = 10$
Persistence of fundamental	$\rho_\theta = 0.99$
Precision of ergodic distribution of fundamental	$\gamma_\theta = 400$
Precision of public signal	$\gamma_y = 100$
Precision of aggregated private signals when $n = 1$	$\gamma_x = 1500$

Table 1: Parameters values for the numerical simulations

may disappear. This feature is an important difference between the benchmark and the extended model. Here, because of the joint dynamics of K and γ, the low regime is not globally stable. It is, however, a well-defined attractor of the dynamic system during a transitory period. We can thus still expect this mechanism to increase the duration of recessions. We now investigate the strength of this effect through a series of simulations.
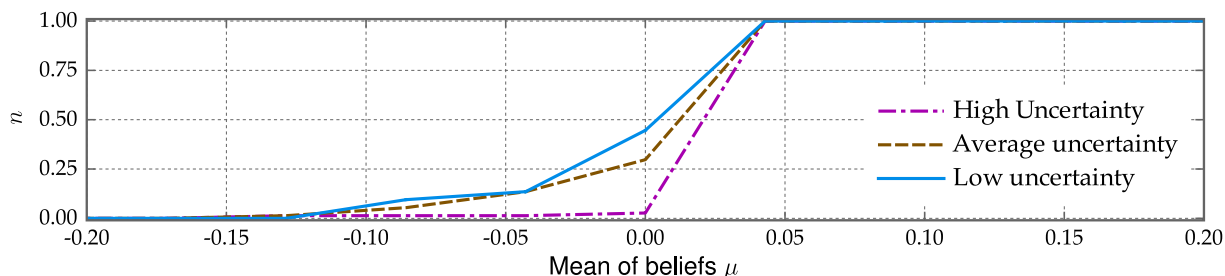
Figure 7: Investment decision $n(\mu, \gamma, K, Q)$ for K and Q constant at their steady-state level.

Since primitive shocks only affect the planner's decision through their effects on beliefs, we examine directly the response of the economy after a belief shock. We consider the evolution of an economy that suffers from a negative 5% shock to the mean of beliefs μ resulting from bad realizations of the signals. The impulse response functions are represented by the solid curves in Figure 9. The dashed curves represent the response of a control economy with fixed information flow and no endogenous uncertainty. More precisely, in the control economy, we keep constant the precision of aggregate signal X at its steady-state value so that the precisions of beliefs γ coincide in the steady states of the two economies. This allows us to isolate the impact of the endogenous uncertainty channel.

Let us consider the full model first. On impact, firms believe that productivity is low. The expected return of adding capital becomes lower than its cost and firms cut back on investment.

Figure 8: Dynamics of the precision γ for μ, K and Q constant at their steady-state value.

As a result, fewer private signals are generated and the precision of beliefs starts falling, as seen in Panel (c). Once the shock is over, agents start receiving signals suggesting that the fundamental is actually better than what they expected. Firms update their beliefs accordingly and, as shown in Panel (b), the mean of their b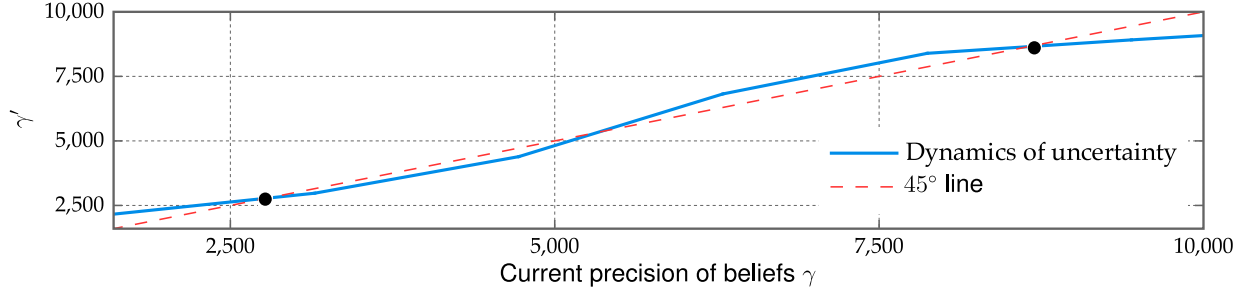eliefs starts to recover. The recovery in output is, however, delayed by the high uncertainty. Once the stock of capital has sufficiently declined and a large enough stock of opportunities is accumulated, firms resume investing. This triggers a large release of information, μ recovers quickly and uncertainty declines sharply, which increases investment further.

In comparison, the recession is less severe in the fixed-information-flow economy. In this case, uncertainty does not rise after the initial shock. Thus, as the mean of beliefs μ recovers, firms resume their investments earlier and the downturn is shorter. We see a drop in output of about 1.2% in the control economy while production shrinks by 4.2% in the full model. The trough of the recession also happens 8 months later in the full model and the economy takes substantially more time to recover to its steady state.

Further applications

To understand the nonlinear nature of the model, it is useful to compare this 5% shock to μ with a shock of only 1%. The impulse response functions of output to both shocks are shown in Figure 10. For the small shock, the uncertainty trap mechanism doubles both the depth of the recession and the time it takes for the economy to reach the trough of the recession. For the large shock, the uncertainty trap mechanism triples these quantities. As in the benchmark theory, endogenous uncertainty introduces important non-linearities in the economy.

Finally, to allow comparison with the uncertainty shock literature, we consider shocks to the precision of beliefs.[14] Figure 11 shows the impulse response functions of output, the extensive margin of investment and the precision of beliefs when the standard deviation of beliefs doubles while keeping the mean constant. Output drops in both the full model and the fixed information economy. In both economies, γ recovers according to equation (15), but the recovery is faster in the control economy since the information flow is fixed. Thus, exogenous uncertainty shocks get

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[14]These shocks are exogenous zero-probability events that raise uncertainty and that the planner does not anticipate.

propagated and amplified in our model as they slow the learning process.

Sensitivity Analysis

Appendix A shows impulse response functions for different parameterizations of the information parameters. Figure 12 displays the response of output after a negative 5% shock to μ for three different values of γ_x: 500, 1500 (benchmark) and 5000. Lower values of γ_x tend to make the downturn more protracted and deeper as beliefs take longer to catch up. The overall impact is, however, moderate. On the other hand, changes in γ_y have a large impact on the economy. Figure 13 presents the response of output for different values of γ_y: 100 (benchmark), 1000 and 5000. Larger values of γ_y limit the effect of endogenous uncertainty by reducing the overall level of uncertainty in the economy and therefore lowering the incentives to wait.

This result highlights the importance that the public signal Y would have in a full quantitative evaluation of the mechanism. In our current setup, its precision is directly related to the volatility in aggregate productivity. Since the standard deviation in the *level* of TFP in the US ranges from only 2 to 3%, the precision parameter γ_y cannot be too low. This may possibly limit the impact of uncertainty traps.[15] However, the importance of uncertainty traps can be restored if agents are uncertain about fundamentals that fluctuate more in the data. Hence, natural extensions for a full quantitative analysis would be to allow for uncertainty on the *growth rate* of TFP or on sector-specific productivity in a multi-sector economy, both of which display high volatility in the data.

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[15]This finding is reminiscent of Bloom et al. (2012), who found that reasonably calibrated exogenous uncertainty shocks to aggregate productivity have only a moderate impact on the economy.
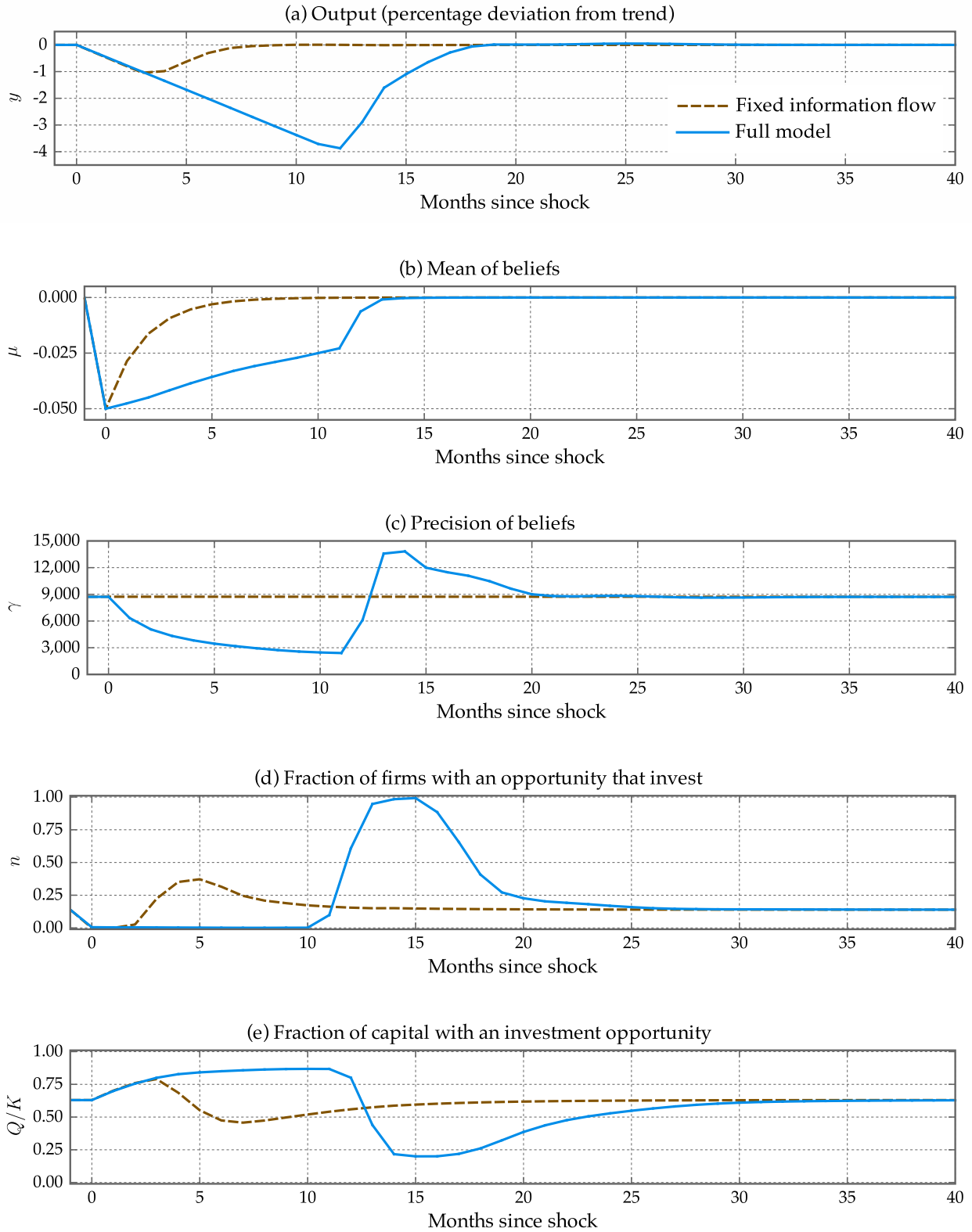
Figure 9: Evolution of the economy after a one-period 5% negative shock to μ. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.
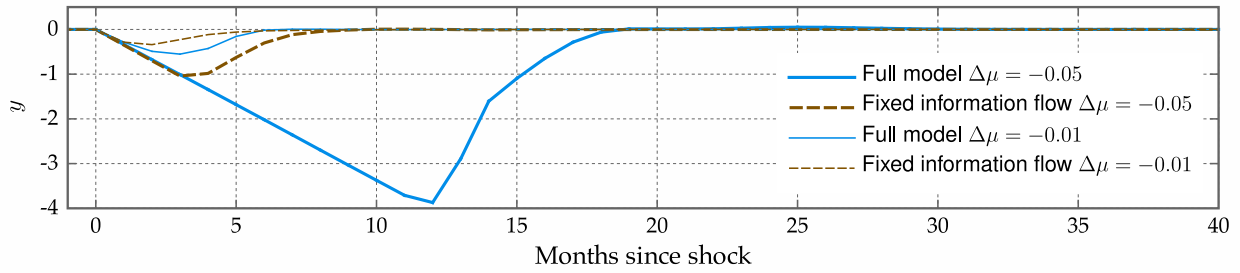
Figure 10: Evolution of output (percentage deviation from trend) after a 5% and 1% negative shock to μ in the full model and the control economy.
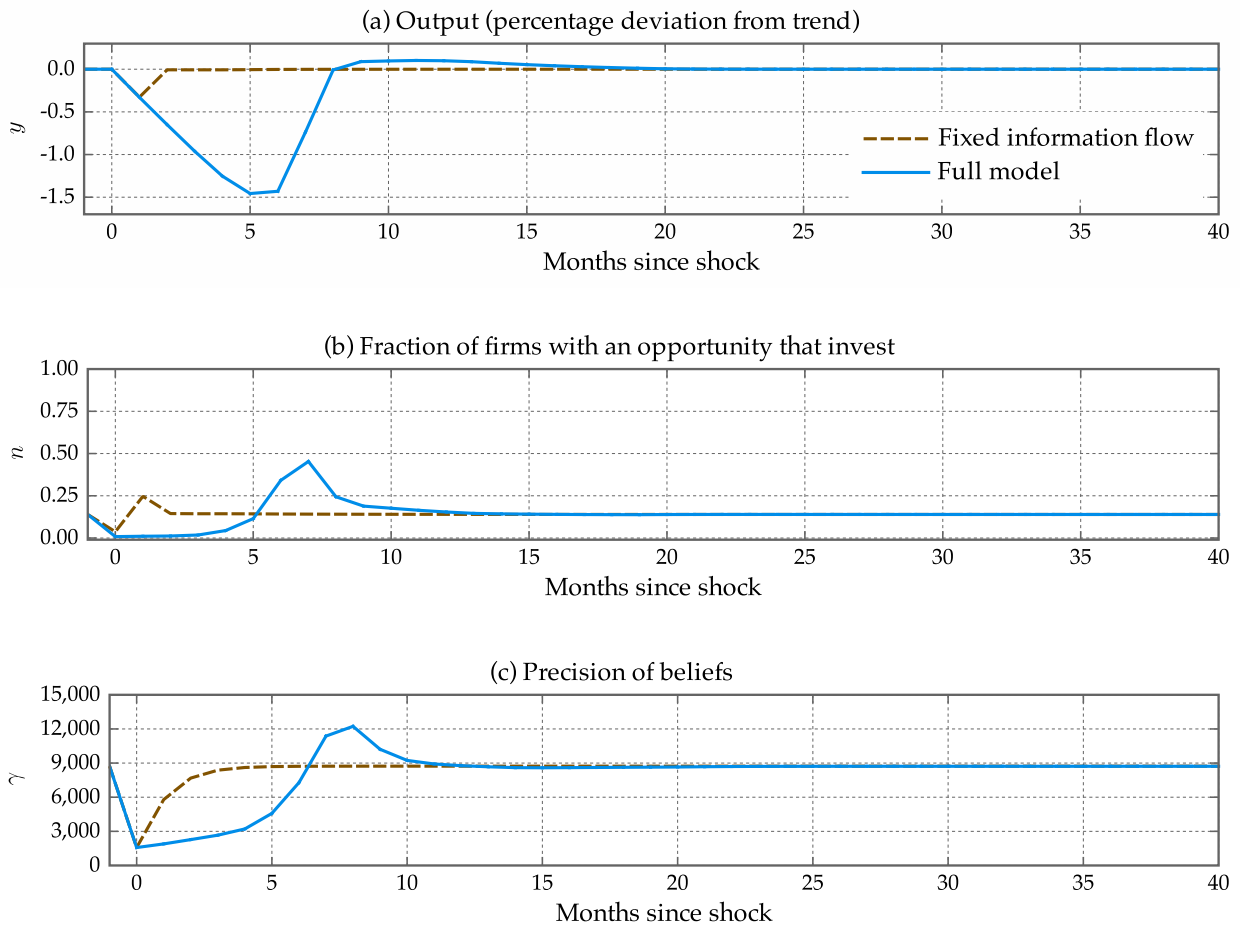
Figure 11: Endogenous vs exogenous uncertainty. Evolution of the economy after a sudden increase in the standard deviation of the prior. The solid curve shows the evolution of the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

6   Conclusion

We develop a theory of endogenous uncertainty and business cycles that combines two forces: higher uncertainty about economic fundamentals deters investment, and uncertainty evolves endogenously because agents learn from the actions of others. The interaction between investment and uncertainty leads to *uncertainty traps*: episodes in which high uncertainty leads firms to delay investment, further raising uncertainty. In the unique equilibrium of the model, the economy fluctuates between a high-activity/low-uncertainty regime and a low-activity/high-uncertainty regime and is subject to strong nonlinear dynamics in which large shocks can have near permanent effects.

To explore the robustness of this mechanism, we embed it into a standard model of the business cycle. Uncertainty traps survive in that context and we find that recessions may become substantially deeper and longer relative to a framework with fixed exogenous uncertainty.

We believe that the novel channel proposed in this paper is important for several reasons. First, the emphasis on subjective uncertainty — and beliefs about fundamentals in particular — implies that not only exogenous volatility shocks, but also other sources of uncertainty, matter for the economy. Thus, we view recent empirical work using survey data on forecasts or consumer and business expectations as an important step towards a more complete understanding of the role of uncertainty in business cycles. Second, we believe that our framework may be useful as a theoretical benchmark for empirical and quantitative studies seeking to estimate the direct and feedback effects of uncertainty on economic activity. Despite the multiplicity of regimes and strong nonlinearities, the model features a single competitive equilibrium, which makes it amenable to applied work. Third, we have shown that allowing uncertainty to fluctuate endogenously may lead to a significant propagation and amplification mechanism. The type of non-linearities and the multiplicity in regimes that we obtain may be of broader interest for business cycle modeling in general and could also shed light on some particularly large historical downturns.

For the sake of clarity, we have exposited the mechanism in a purposely simple framework, but a number of generalizations may be worth investigating. In particular, it would be interesting to understand how uncertainty traps interact with frictions that could magnify their impact, such as financial frictions, demand externalities or belief heterogeneity. A full quantitative evaluation of the model is also needed, potentially using newly available data on business expectations. We leave these questions to future research.

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A   Sensitivity Analysis

This appendix shows how economies with different information parameters respond to the main shock introduced in Section 5. Figure 12 shows the response of the economy to a 5% negative shock to μ when the parameters are the same as those of Table 1 except for γ_x. As we can see on the figure, reasonable variations in γ_x have little impact on the depth and length of the recession. Figure 13, however, shows that an increase in γ_y, which implies that agents learn faster, makes the downturn shorter and less protracted. The rest of this appendix compares various impulse response functions for these economies in the full model and in the fixed information control model.

A.1   High precision of individual signals

Figure 14 shows the impulse response function to a 5% negative shock to μ to an economy with the same parameters as those of Table 1 except for the precision of the individual signals which is at $\gamma_x = 5000$.

These impulse response functions are quite similar to those of the benchmark simulation of Figure 9. Here, output recovers faster and the peak precision of beliefs is also higher. Notice that the steady-state γ is also much higher than in the benchmark simulations.

A.2   Low precision of individual signals

Figure 15 shows the impulse response function to a 5% negative shock to μ to an economy with the same parameters as those of Table 1 except for the precision of the individual signals which is at $\gamma_x = 500$.

Again, these impulse response functions are quite similar to those of the benchmark simulation of Figure 9. However here, since agents get less information, the recovery in output is slower and the peak precision of beliefs is lower.
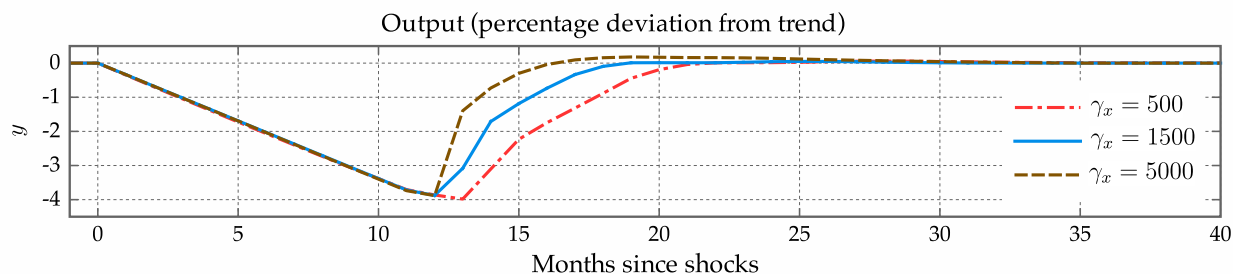
Figure 12: Evolution of the economy after a 5% negative shock to μ with various precisions of the individual signal γ_x
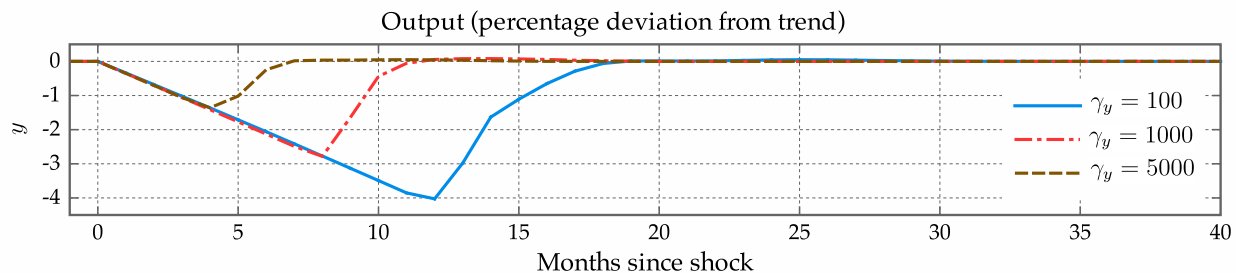
Figure 13: Evolution of the economy after a 5% negative shock to μ with various precisions of the public signal γ_y

A.3 High precision of public signal

Figure 16 shows the impulse response function to a 5% negative shock to μ to an economy with the same parameters as those of Table 1 except for the precision of the public signal which is at $\gamma_y = 1000$.

These impulse response functions are qualitatively similar to those of the benchmark simulation of Figure 9 but here the recession is shallower and shorter. As the public signal is relatively accurate, agents quickly gather information about the fundamental. This prevents the precision of beliefs from dropping substantially.

A.4 Very high precision of public signal

Figure 17 shows the impulse response function to a 5% negative shock to μ to an economy with the same parameters as those of Table 1 except for the precision of the public signal which is at $\gamma_y = 5000$.

These impulse response functions are qualitatively similar to those of the benchmark simulation of Figure 9 but here the recession is must shallower and shorter. As the public signal is very accurate, agents quickly gather information about the fundamental and the endogenous uncertainty channel does not matter much.
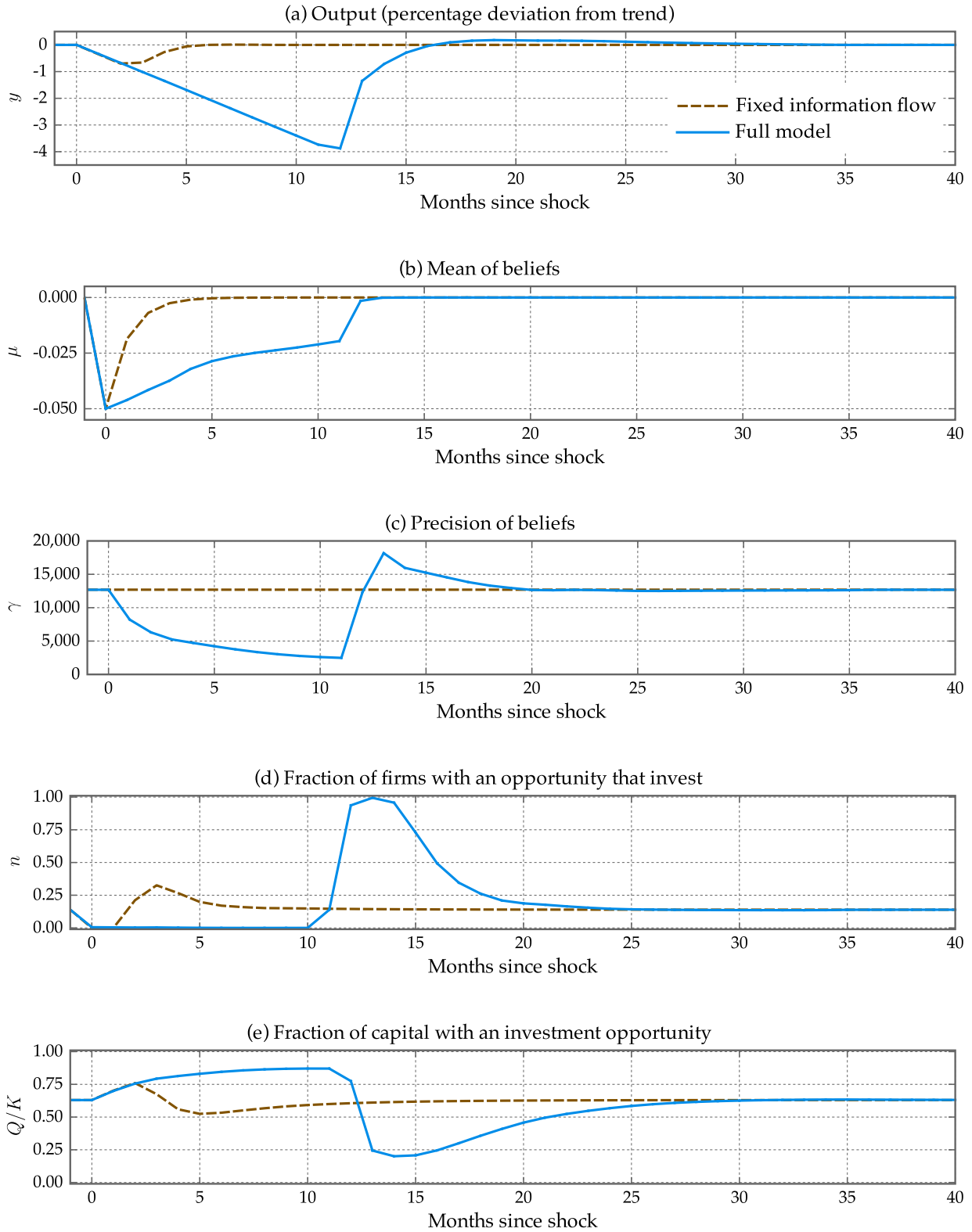
Figure 14: Evolution of the economy after a 5% negative shock to μ with $\gamma_x = 5000$. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.
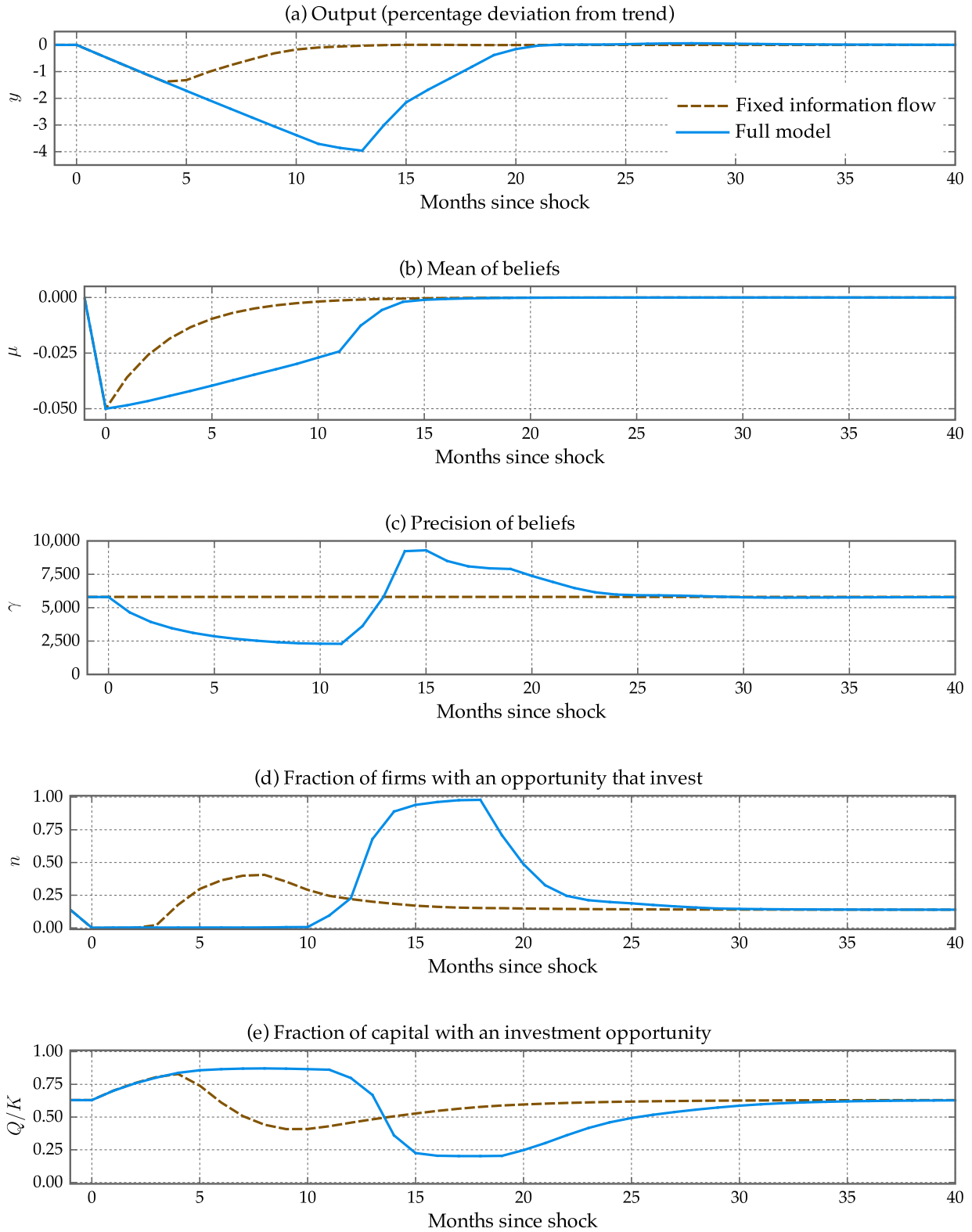
Figure 15: Evolution of the economy after a 5% negative shock to μ with $\gamma_x = 500$. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.
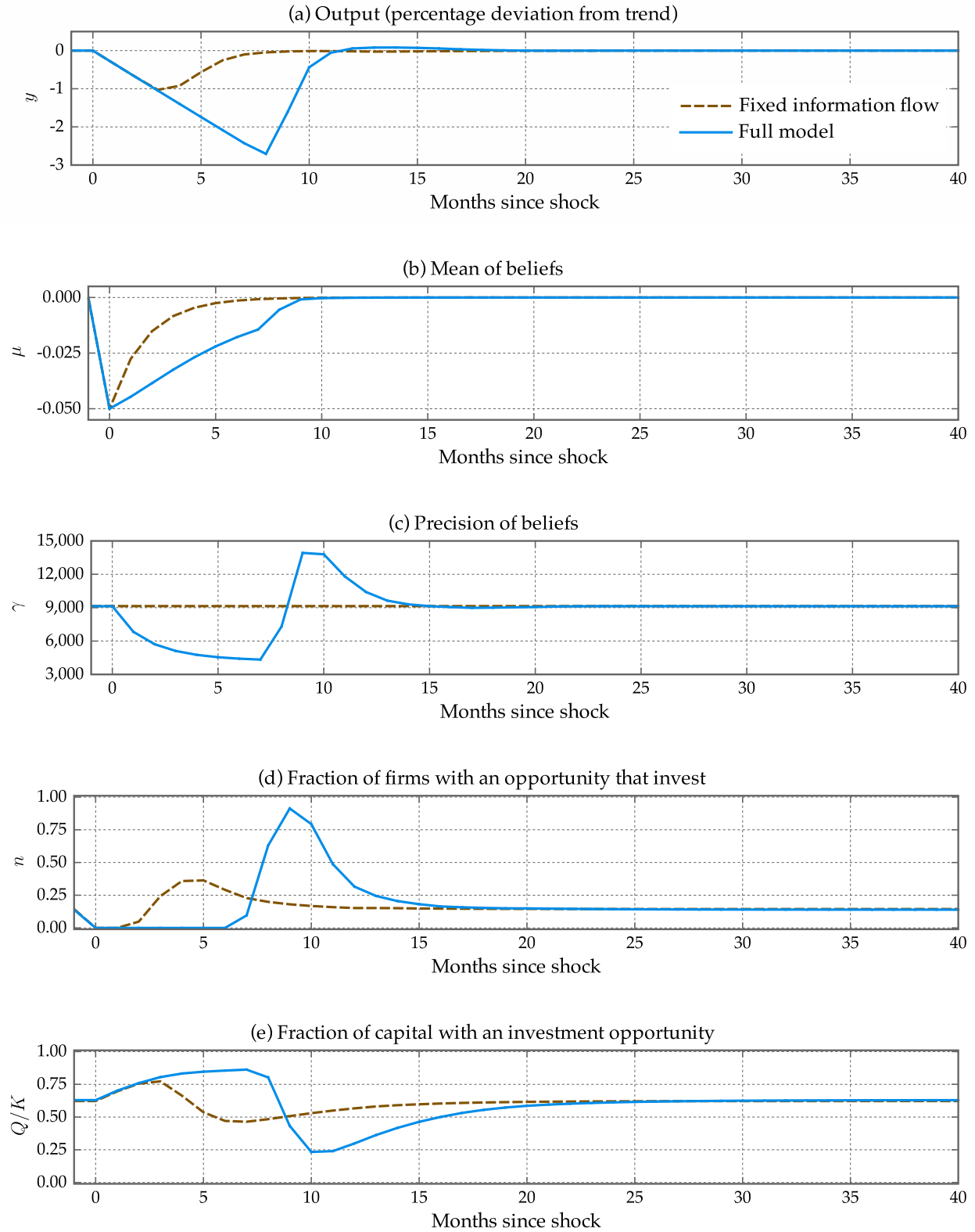
Figure 16: Evolution of the economy after a 5% negative shock to μ with $\gamma_y = 1000$. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.
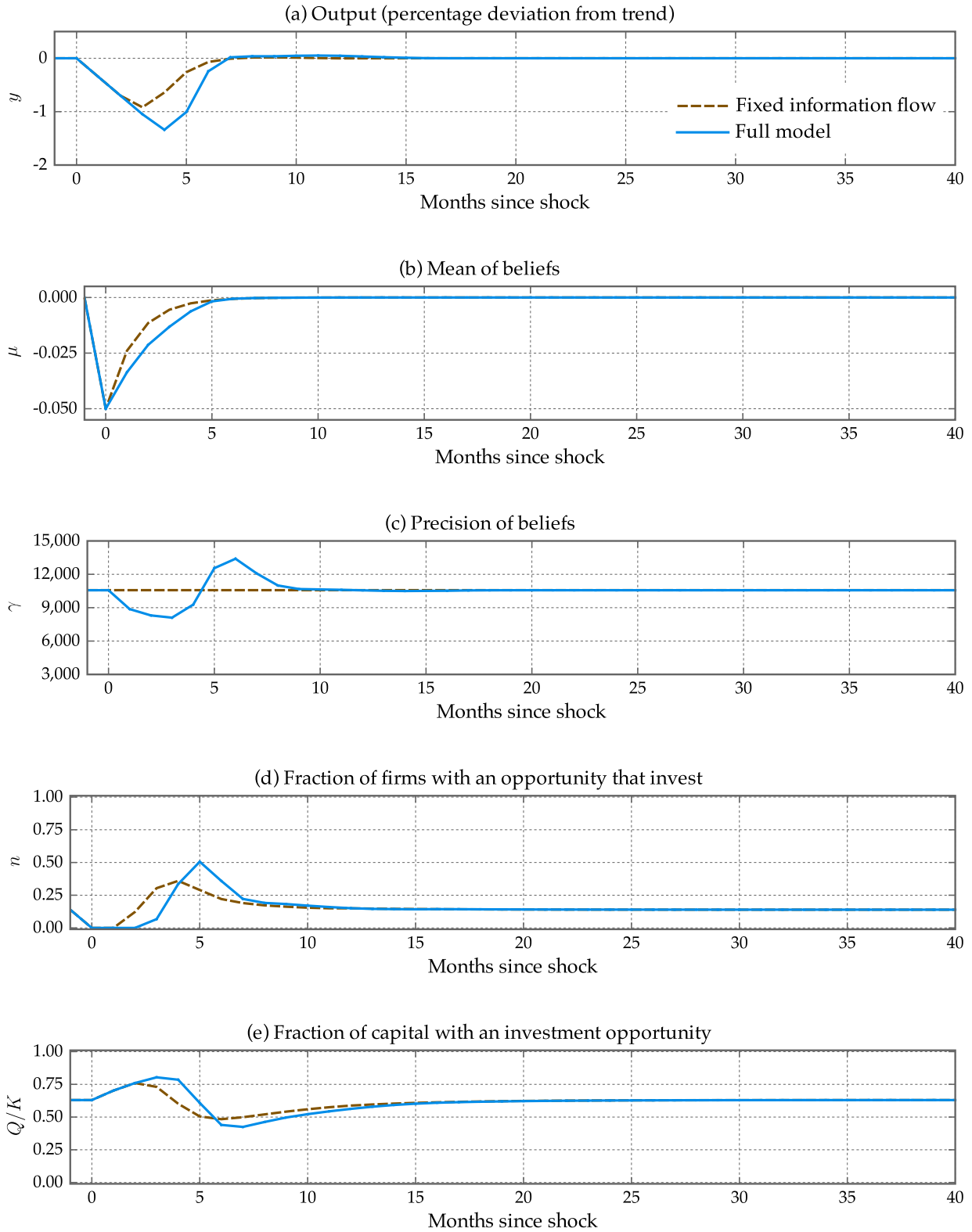
Figure 17: Evolution of the economy after a 5% negative shock to μ with $\gamma_y = 5000$. The solid curve shows the evolution of the economy according to the full model while the dashed curve shows the evolution of a control economy in which the flow of public information is fixed at its steady-state level.

B Limit economy when $\overline{N} \to \infty$

In the baseline model introduced in part 2, firms face the same ex-ante probability of investing. Yet, the aggregate number of investing firms N is random because of the finiteness in the number of firms. This sample risk is of no relevance in respect to the uncertainty trap mechanism that we propose. We thus take the limit $\overline{N} \to \infty$ in part 4, so that the fraction of investing firms becomes a deterministic function of the state variables (μ, γ). This section exposes in detail how this limit is to be taken and how the limit economy is defined.

The only potential difficulty that we face as $\overline{N} \to \infty$ is that the social learning channel could become fully revealing. To be more precise, if we kept the precision of each individual signal γ_x constant, a law of large number would apply and $X_N \equiv \frac{1}{N} \sum_{j=1}^{N} x_j \to \theta$. The fundamental θ would be revealed for sure and no uncertainty would remain in the economy.

Instead, we assume that $\gamma_x \left(\overline{N} \right)$ evolves with $\overline{N}$ in the following manner,

$$\gamma_x \left(\overline{N} \right) = \gamma_x / \overline{N}. \tag{16}$$

This assumption captures the idea that the information gathered by each agent is proportional to its size. Possible microfoundations may include: i) the amount of information is proportional to the market size of the firm, its number of clients, etc; ii) agents use similar sources of information and information is correlated, implying that the precision of information brought by each agent decreases with $\overline{N}$.

Specification (16) displays the great advantage that the Bayesian updating rules for beliefs do not depend on $\overline{N}$. In particular, the learning dynamics follow the same rule as in the finite $\overline{N}$ case, thus ensuring that the intuition behind uncertainty traps remains the same. We define $n(\mu, \gamma) = N(\mu, \gamma) / \overline{N}$ the fraction of investing firm. The law of motion for beliefs satisfy

$$\mu' = \frac{\gamma\mu + \gamma_y Y + N\gamma_x \left(\overline{N} \right) X}{\gamma + \gamma_y + N\gamma_x \left(\overline{N} \right)} = \frac{\gamma\mu + \gamma_y Y + n\gamma_x X}{\gamma + \gamma_y + n\gamma_x}, \tag{17}$$

$$\gamma' = \left(\frac{1}{\gamma + \gamma_y + N\gamma_x \left(\overline{N} \right)} + \frac{1}{\gamma_\theta} \right)^{-1} = \left(\frac{1}{\gamma + \gamma_y + n\gamma_x} + \frac{1}{\gamma_\theta} \right)^{-1}, \tag{18}$$

where $X = \frac{1}{N} \sum_{j=1}^{N} x_n \sim \theta + \mathcal{N} \left(0, (n\gamma_x)^{-1} \right)$. The number of investing firms is now deterministic as

$$n = \frac{N}{\overline{N}} = \frac{1}{\overline{N}} \sum_{j=1}^{\overline{N}} \mathbb{1}\left(f_j \leq f_c(\mu, \gamma) \right) \xrightarrow[\text{a.s}]{} p(\mu, \gamma) = F\left(f_c(\mu, \gamma) \right),$$

by a law of large number since the investment costs $\{f_j\}_{j \geq 1}$ are i.i.d and distributed according to F.

We may now define an equilibrium for the limit economy:

Definition 3. A recursive equilibrium of the limit economy consists of a policy function $f_c(\mu, \gamma)$,

value functions $V(\mu, \gamma, f)$, $V^W(\mu, \gamma)$, $V^I(\mu, \gamma)$, laws of motions for aggregate beliefs $\{\mu', \gamma'\}$, and a fraction of investing firms $n(\mu, \gamma)$, such that

1. The value function $V(\mu, \gamma, f)$ solves (6), with $V^W(\mu, \gamma)$ and $V^I(\mu, \gamma)$ defined according to (7) and (8), with the corresponding cutoff rule $f_c(\mu, \gamma)$;

2. The aggregate beliefs (μ, γ) evolve according to (17) and (18);

3. The number $n(\mu, \gamma)$ of firms that invest is given by $n(\mu, \gamma) = F(f_c(\mu, \gamma))$.

Since the limit economy is in many aspects simpler than the economy considered in part 2, all the lemmas and propositions derived in appendix D extend to this new environment. Given the similarity in the arguments, the proofs are omitted.

C   Aggregation of the Social Planner's Problem

This appendix derives the aggregation of the social planner's problem from part 5. To lighten the exposition, we use the notation $\{z_j\} \equiv \{z_j\}_{j \in [0,1]}$ for some variable z. We denote q_j a dummy for firm j equal to 1 if it has an investment opportunity and 0 otherwise. Similarly, we let n_j be the probability that firm j invests and $\chi_j \sim \mathcal{B}(n_j)$ a Bernoulli variable that captures the realization of this decision.

Step 1. We first derive the Bayesian updating rules when the precision of each individual signal received by an investing firm is proportional to its capital stock. As in the baseline model, we take the limit of the case in which a finite number of firms $\overline{N}$ receives individual signals given by $x_j = \theta + \varepsilon_j^x$, where $\varepsilon_j^x \sim \mathcal{N}\left(0, (\gamma_x(\overline{N}) k_j)^{-1}\right)$ and $\gamma_x(\overline{N}) = \gamma_x/\overline{N}$. The law of motion for beliefs is

$$\mu' = \rho \frac{\gamma \mu + \gamma_y Y + \left(\gamma_x(\overline{N}) \sum q_j \chi_j k_j\right) X_{\overline{N}}}{\gamma + \gamma_y + \gamma_x(\overline{N}) \sum q_j \chi_j k_j}$$

$$\gamma' = \left(\frac{\rho^2}{\gamma + \gamma_y + \gamma_x(\overline{N}) \sum q_j n_j k_j} + (1-\rho^2)\sigma_\theta^2\right)^{-1},$$

where the aggregated information $X_{\overline{N}} = \frac{1}{\sum_{j=1}^{\overline{N}} q_j \chi_j k_j} \sum_{j=1}^{\overline{N}} q_j \chi_j k_j x_j$ follows the normal distribution $\mathcal{N}\left(\theta, \left(\frac{\gamma_x}{\overline{N}} \sum_{j=1}^{\overline{N}} q_j \chi_j k_j\right)^{-1}\right)$ when conditioning on the beliefs (μ, γ) and the realization of $\{\chi_j\}$. Taking the limit $\overline{N} \to \infty$, variable $X_{\overline{N}}$ converges to $X_{\overline{N}} \xrightarrow{\text{d}} X$ where $X \mid \mu, \gamma, \{\chi_j\} \sim \mathcal{N}\left(\theta, (\gamma_x \int q_j \chi_j k_j dj)^{-1}\right)$ and the updating rules become:

$$\mu' = \rho \frac{\gamma \mu + \gamma_y Y + \left(\gamma_x \int q_j \chi_j k_j dj\right) X}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj}$$

$$\gamma' = \left(\frac{\rho^2}{\gamma + \gamma_y + \gamma_x \int q_j \chi_j k_j dj} + (1-\rho^2)\sigma_\theta^2\right)^{-1} \equiv \Gamma\left(\int q_j \chi_j k_j dj, \gamma\right).$$

In particular, it is possible to show, as we did in the baseline case, that $\mu' = \rho\mu + s\left(\int q_j\chi_j k_j dj, \gamma\right)\varepsilon$ with $\varepsilon \sim \mathcal{N}(0,1)$ and $s\left(\int q_j\chi_j k_j dj, \gamma\right) = \rho\left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + \gamma_x \int q_j\chi_j k_j dj}\right)^{\frac{1}{2}}$.

Step 2. We now write the social planner's problem. Summing up individual output across all firms $j \in [0,1]$, the problem is

$$V(\mu, \gamma, \{k_j\}, \{q_j\}) = \max_{\{i_j\},\{n_j\},\{l_j\}} \mathbb{E}\left[(A+Y)\int_0^1 k_j^\alpha l_j^{1-\alpha} dj\right.$$

$$\left. -\int_0^1 (f + c(i_j)) k_j q_j \chi_j dj + \beta V\left(\mu', \gamma', \{k_j'\}, \{q_j'\}\right) |\mu, \gamma\right]$$

subject to the following conditions:

$$n_j \in [0,1], \chi_j \sim \mathcal{B}(n_j)$$

$$1 = \int_0^1 l_j dj$$

$$k_j' = q_j \chi_j k_j (1 - \delta + i_j) + (1 - q_j\chi_j) k_j (1-\delta) \tag{19}$$

$$q_j' = q_j(1-\chi_j) + (1 - q_j + q_j\chi_j)\begin{cases} 0 & \text{with prob } (1-\bar{q}) \\ 1 & \text{with prob } \bar{q} \end{cases} \tag{20}$$

$$\mu' = \rho\mu + s\left(\int q_j\chi_j k_j dj, \gamma\right)\varepsilon$$

$$\gamma' = \Gamma\left(\int q_j\chi_j k_j dj, \gamma\right).$$

To understand why aggregation obtains in this economy, notice that if the planner's decisions had no impact on the information, the structure of the adjustment cost would imply that the investment i_j and the entry decision n_j would be independent of k_j and aggregation would follow. This result also holds when the impact of the decisions on the information is taken into account since the extra informational benefit provided by an entering firm is proportional to that firm's current capital.

More formally, guess that the value function can be written $V(\mu, \gamma, \{k_j\}, \{q_j\}) = V(\mu, \gamma, K, Q)$ where $K = \int k_j dj$ and $Q = \int k_j q_j dj$ is a capital-weighted measure of investment opportunities. Then the first order conditions of the above problem are:

$$[n_j]_{q_j=1} \quad 0 = -(c(i_j) + f) k_j + \beta\mathbb{E}\{V_\mu \varepsilon s_1 k_j + V_\gamma \Gamma_1 k_j$$

$$+V_K i_j k_j + V_Q\left[-(1-\bar{q}) k_j (1-\delta) + \bar{q} i_j k_j\right]\}$$

$$[i_j]_{q_j=1} \quad 0 = -c'(i_j) k_j + \beta\mathbb{E}\left[V_K n_j k_j + V_Q \bar{q} n_j k_j\right]$$

$$[l_j] \quad 0 = (1-\alpha)(A+\mu) k_j^\alpha l_j^{-\alpha} - \lambda$$

where λ is the Lagrange multiplier on the labor constraint. These equations tell us the following: i) the labor decision l_j is proportional to capital k_j, ii) the investment decision on the intensive

margin i_j and extensive margin n_j are independent of the capital stock k_j. Thus, plugging in the solution for l_j, the problem aggregates as follows:

$$V(\mu, \gamma, K, Q) = \max_{i, n \in [0,1]} (A + \mu) K^\alpha - nQ(f + c(i))$$
$$+ \beta \mathbb{E}\left[V(\mu', \gamma', K', Q') | \mu, \gamma\right]$$

subject to the following conditions:

$$K' = (1 - \delta) K + inQ \tag{21}$$
$$Q' = (1 - \delta)(1 - \overline{q})(1 - n) Q + (1 - \delta)\overline{q}K + \overline{q}inQ \tag{22}$$
$$\mu' = \rho\mu + s(nQ, \gamma)\varepsilon$$
$$\gamma' = \Gamma(nQ, \gamma),$$

where the aggregate laws of motion (21) and (22) result from the aggregation of (19) and (20) using the independence of n_j and i_j of k_j. Thus, our guess that V is a function K and Q is satisfied and we obtain aggregation. $\square$

D   Proofs

D.1   Assumptions and Definitions

Assumption 1. *Parameters are such that* $\beta e^{\frac{a^2}{2\gamma\theta}} < 1$.

This assumption guarantees that a number of effects highlighted in the baseline model are unambiguous and in particular that the option value of waiting is strong enough to dominate other forces.

Assumption 2. *F is a continuous, twice differentiable cumulative distribution function with bounded first and second derivatives. F has bounded support* $\left[\underline{f}, \overline{f}\right]$, *mean* ω^f *and standard deviation* σ^f.

These regularity conditions on the cumulative distribution of investment costs guarantee that the equilibrium number of investing firms $N(\mu, \gamma) \sim Bin\left(\overline{N}, F(f_c(\mu, \gamma))\right)$ is well-behaved.

It is useful for the propositions and definitions below to define the following mapping which sums over the distribution of investment costs:

Definition 4. *Let* S *be the mapping such that*

$$[S(G)](\mu, \gamma) = \int G(\mu, \gamma, f) \, dF(f),$$

where $G : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $S(G) : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Definition 5. Define the following bounds and set:

1. Let $\overline{\gamma}$ be the unique strictly positive solution of

$$\overline{\gamma} = \left(\frac{1}{\overline{\gamma} + \gamma_y + \overline{N}\gamma_x} + \frac{1}{\gamma_\theta} \right)^{-1} = \Gamma\left(\overline{N}, \overline{\gamma}\right), \tag{23}$$

and $\underline{\gamma}$ the unique strictly positive solution of

$$\underline{\gamma} = \left(\frac{1}{\underline{\gamma} + \gamma_y} + \frac{1}{\gamma_\theta} \right)^{-1} = \Gamma\left(0, \underline{\gamma}\right), \tag{24}$$

2. Let $\mathcal{S} = [\underline{\mu}, \overline{\mu}] \times [\underline{\gamma}, \overline{\gamma}]$, where $\underline{\mu}$ and $\overline{\mu}$ are some arbitrary but large bounds on μ.

We define the set $\mathcal{P}$ in which the probability $p(\mu, \gamma) = F(f_c(\mu, \gamma))$ that a firm invests will lie:

Definition 6. Let $\mathcal{P}$ be the set of twice-differentiable functions $p : (\mu, \gamma) \in \mathcal{S} \longrightarrow \mathbb{R}$ such that p has bounded first and second derivatives: $\forall (\mu, \gamma) \in \mathcal{S}, |p_\mu(\mu, \gamma)| \leq \overline{p}_\mu$, $|p_\gamma(\mu, \gamma)| \leq \overline{p}_\gamma$, and $|p_{xy}(\mu, \gamma)| \leq \overline{p}_{xy}$ for $(x, y) \in \{\mu, \gamma\}^2$.

We also define the set $\mathcal{G}$ in which the firm's surplus of waiting compared to investing will lie:

Definition 7. Let $\mathcal{G}$ be the set of continuous functions G of $(\mu, \gamma, f) \in \mathcal{S} \times [\underline{f}, \overline{f}] \longrightarrow \mathbb{R}$ such that

1. G is bounded by $\overline{G}$,

2. G is weakly decreasing and convex in μ,

3. G is weakly decreasing in γ,

4. G is Lipschitz continuous of constant 1 in f, and

5. G is such that $[S(G)](\mu, \gamma)$ is twice-differentiable with bounded first and second derivatives: $\forall (\mu, \gamma), \left| \frac{\partial}{\partial x} [S(G)](\mu, \gamma) \right| \leq \overline{G}_x$ and $\left| \frac{\partial}{\partial xy} [S(G)](\mu, \gamma) \right| \leq \overline{G}_{xy}$ for $(x, y) \in \{\mu, \gamma\}^2$.

We define the mapping $\mathcal{T}$ that corresponds to the waiting decision of a firm in partial equilibrium, taking a probability of investment for other firms $p \in \mathcal{P}$ as given:

Definition 8. For a given probability of investment $p \in \mathcal{P}$, define the mapping $\mathcal{T}^p : G \in \mathcal{G} \longrightarrow \mathcal{G}$

$$[\mathcal{T}^p G](\mu, \gamma, f) = \max\left\{ C^p(G(\mu, \gamma, f)), 0 \right\},$$

where $C^p(G)$ is the value in the continuation region, defined by:

$$[C^p(G)](\mu, \gamma, f) = \frac{1}{a}\left[e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) - (1 - \beta) \right] + f - \beta\omega^f$$

$$+ \beta\mathbb{E}_p\left\{ [S(G)](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \right\}.$$

In the recursive equilibrium from 1, the probability that each firm invests satisfies $p(\mu, \gamma) = F(f_c(\mu, \gamma))$. Therefore, we define the following mapping:

Definition 9. Let $\mathcal{M}$ be the mapping from $p : \mathcal{P} \longrightarrow \mathcal{P}$ such that, for all $\mu, \gamma \in \mathcal{S}$,

$$(\mathcal{M}p)(\mu, \gamma) = F(f_c^p(\mu, \gamma))$$

where $f_c^p(\mu, \gamma)$ is defined by

$$f_c^p(\mu, \gamma) = -\frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \frac{1}{a}(1 - \beta) + \beta \omega^f - \beta \mathbb{E}_p \left\{[S(G^p)](\mu', \gamma')\right\},$$

where G^p is the unique fixed point of the mapping

$$G^p = \mathcal{T}^p G^p.$$

D.2   Two Useful Lemmas

Lemma 1. *For a given N, mean beliefs μ follow a random walk with time-varying volatility s,*

$$\mu' = \mu + s(N, \gamma)\varepsilon,$$

where $s(N, \gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}}$ and $\varepsilon \sim \mathcal{N}(0, 1)$.

Proof. We use (4), (3) and (2) to compute the mean and the variance of the next period mean beliefs μ' given current-period information, (μ, γ), and a given realization of N:

$$\mathbb{E}\left[\mu'|\mu, \gamma, N\right] = \mu,$$
$$V\left[\mu'|\mu, \gamma, N\right] = \frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}.$$

Being the sum of normally distributed variables, μ' is also normally distributed and can therefore be expressed by $\mu' = \mu + \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}} \varepsilon$. □

Lemma 2. *The precision of next-period beliefs, γ', increases with N and γ. For a given N, there exists a unique positive fixed point in the law of motion for the precision of beliefs $\gamma' = \Gamma(N, \gamma)$.*

Proof. The fact that γ' increases with N and γ follows by inspection of (5). Given N, uniqueness of a positive fixed point follows from noting that the all fixed points γ must satisfy:

$$0 = \gamma^2 + \gamma(\gamma_y + N\gamma_x) - (\gamma_\theta \gamma_y + N\gamma_\theta \gamma_x).$$

Because the quadratic function of γ on the right-hand side is negative at $\gamma = 0$, it necessarily has a unique positive root. □

D.3 Propositions

We start in Proposition 1 by demonstrating that the individual firm problem is well defined for a given $p(\mu, \gamma)$ and characterize its properties. Then we show that there is a unique $p(\mu, \gamma)$ in Proposition 2.

Proposition 1. *Under Assumption 1, given a random number of investing firms $N \sim Bin\left(\overline{N}, p(\mu, \gamma)\right)$ for some $p \in \mathcal{P}$, and for γ_x sufficiently low, there exists a unique solution to the firm's problem and the resulting cutoff $f_c(\mu, \gamma)$ is strictly increasing in μ and γ.*

Proof. First, we demonstrate that the difference between the value of waiting and investing is uniquely determined. Then, we characterize properties of that difference that guarantee the existence of the cutoff and its properties.

To proceed it is useful to define some notation. Note that the distributions of μ' and γ' defined in (4) and (5) depend on the random variable N with binomial distribution given by (11). In particular, the probability that N firms invest when the total number of firms in the economy is $\overline{N}$ and the individual probability of investing is p is

$$\pi_N^{\overline{N}}(p) = \binom{\overline{N}}{N} p^N (1-p)^{\overline{N}-N}. \tag{25}$$

Since $\pi_N^{\overline{N}}(p)$ is a polynomial in p of degree $\overline{N}$, it is bounded on $[0, 1]$. Denote $\overline{\pi}$ its upper bound, as well as $\overline{\pi}_p$ ($\overline{\pi}_{pp}$) the upper bound of its first (second) derivative.

For a given value function V, we define the surplus of waiting $G(\mu, \gamma, f) \equiv V(\mu, \gamma, f) - \left[V^I(\mu, \gamma) - f\right]$. In particular, using the definition of V from (6), G must satisfy the recursive relation

$$G(\mu, \gamma, f) = \max\left\{\beta\mathbb{E}\left[G(\mu', \gamma', f') + V^I(\mu', \gamma') - f'\right] - \left(V^I(\mu, \gamma) - f\right), 0\right\}.$$

Substituting the stopping value $V^I(\mu, \gamma) = \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\right)$, using (1) and (5) and some manipulations give

$$G(\mu, \gamma, f) = \max\left\{C^p(G(\mu, \gamma, f)), 0\right\}, \tag{26}$$

where C^p is the value in the continuation region, defined by:

$$[C^p(G)](\mu, \gamma, f) = \frac{1}{a}\left[e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) - (1-\beta)\right] + f - \beta\omega^f$$
$$+ \beta\mathbb{E}_{p,f}\left[G(\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma), f')\right]. \tag{27}$$

In other words, G is a fixed point of the mapping $\mathcal{T}^p$ from Definition 8.[16] The expectation in the last term is with respect to the shock ε to average beliefs, the number of investing firms N and the

---

[16]Note that G is defined for μ over the interval $[\underline{\mu}, \overline{\mu}]$ but the expectation for μ' is computed using a normal distribution with unbounded support. Therefore, for the expectation term in the second line of (27), we extend the

fixed cost f'. Notice for future reference that the term $\mathbb{E}_{p,f}\left[G\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right), f'\right)\right]$ is equal to $\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right\}$. It depends on the individual probability of investing, p:

$$\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu + s\varepsilon, \Gamma\right)\right\} = \sum_{N=1}^{\overline{N}} \pi_N^{\overline{N}}\left(p\right) g_N\left(\mu,\gamma\right), \tag{28}$$

where

$$g_N\left(\mu,\gamma\right) \equiv \int \left[S\left(G\right)\right]\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) d\Phi\left(\varepsilon\right) \tag{29}$$

where $\Phi\left(\varepsilon\right)$ is the CDF of a standard normal, and where $\Gamma = \Gamma\left(N,\gamma\right)$.

Note that $\mathcal{T}^p$ trivially satisfies the Blackwell conditions for a contraction so that, if it is a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$, it admits a unique fixed point. To prove uniqueness of $\mathcal{T}^p$ it remains to show that it is indeed a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$, i.e. that if G is an element of the set $\mathcal{G}$ defined in (7) then so is $\mathcal{T}^p G$. We do so next:

1. $\mathcal{T}^p G$ **is bounded and continuous**: continuity follows easily from the definition of the mapping $\mathcal{T}^p$ as it is the maximum of two continuous functions. Boundedness follows from the fact that we can bound $C^p\left(G\right)$ as follows:

$$\left|\left[C^p\left(G\right)\right]\left(\mu,\gamma,f\right)\right| \leq \frac{1}{a}e^{-a\underline{\mu} + \frac{a^2}{2}\left(\frac{1}{2} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \frac{1-\beta}{a} + \overline{f} + \beta\omega^f + \beta\overline{G}.$$

Thus, $\mathcal{T}^p G$ is bounded as long as $\overline{G}$ is chosen large enough that

$$\overline{G} \geq \left(1 - \beta\right)^{-1}\left(\frac{1}{a}e^{-a\underline{\mu} + \frac{a^2}{2}\left(\frac{1}{2} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \frac{1-\beta}{a} + \overline{f} + \beta\omega^f\right).$$

2. $\mathcal{T}^p G$ **is decreasing with** μ: within the continuation region,

$$\frac{\partial}{\partial\mu}\left[C^p\left(G\right)\right]\left(\mu,\gamma,f\right) = -e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta\mathbb{E}_p\left[\frac{\partial}{\partial\mu}S\left(G\right)\right]$$
$$+ \beta\frac{\partial p}{\partial\mu}\frac{\partial}{\partial p}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu + s\varepsilon, \Gamma\right)\right\}. \tag{30}$$

We must prove that this expression is negative. The first term is negative and bounded away from 0 since

$$-e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) \leq -e^{-a\overline{\mu} + \frac{a^2}{2}\frac{1}{\gamma}}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) < 0.$$

The second term, $\frac{\partial}{\partial\mu}S\left(G\right)$, is also negative because $G \in \mathcal{G}$. To conclude, we show that the

definition of G to values of μ not in $\left[\underline{\mu}, \overline{\mu}\right]$ by assuming that the bounds are absorbing, i.e., that $\forall \mu > \overline{\mu}$, $G\left(\mu,\gamma,f\right) = G\left(\overline{\mu},\gamma,f\right)$ and $\forall \mu < \underline{\mu}$, $G\left(\mu,\gamma,f\right) = G\left(\underline{\mu},\gamma,f\right)$. This assumption guarantees the validity of our proofs. In numerical simulations, the bounds can be chosen to be sufficiently large so as to have no impact on the results.

last term is $O(\gamma_x)$ and therefore negligible compared to the first term, so that $\frac{\partial}{\partial\mu}\left[C^p\left(G\right)\right]<0$ when γ_x is small. For that, note first that from Definition 6 the term $\frac{\partial p}{\partial\mu}$ is bounded above by some constant $\overline{p}_\mu$. Therefore, it remains to show that $\frac{\partial}{\partial p}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}=O\left(\gamma_x\right)$. For that, let $\Pi_N^{\overline{N}}\left(p\right)=\sum_{n=1}^{N}\pi_n^{\overline{N}}\left(p\right)$, and sum by parts in (28) to write:

$$\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}=g_{\overline{N}}\left(\mu,\gamma\right)-\sum_{N=1}^{\overline{N}-1}\Pi_N^{\overline{N}}\left(p\right)\cdot\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right),\qquad(31)$$

which implies:

$$\frac{\partial}{\partial p}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}=-\sum_{N=1}^{\overline{N}-1}\left[\frac{\partial}{\partial p}\Pi_N^{\overline{N}}\left(p\right)\right]\cdot\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right).\qquad(32)$$

Note in addition that

$$\left|\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right)\right|=\int\left[S\left(G\right)\left(\mu+s\left(N+1,\gamma\right)\varepsilon,\Gamma\left(N+1,\gamma\right)\right)\right.$$
$$\left.-S\left(G\right)\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right)\right)\right]d\Phi\left(\varepsilon\right)$$
$$\leq\overline{G}_\mu\left|\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right|+\overline{G}_\gamma\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|,$$

where the last line follows from the fact that G has bounded derivatives. From the expressions for s and Γ obtained in lemmas (1) and (2) and using the concavity of s, we note that the terms in absolute value on the second inequality are $O\left(\gamma_x\right)$,

$$\left|s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right|\leq s_N\left(N,\gamma\right)=\frac{1}{2}\left(\frac{1}{\gamma}-\frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{-\frac{1}{2}}\frac{\gamma_x}{\left(\gamma+\gamma_y+N\gamma_x\right)^2}$$
$$\leq\frac{1}{2}\left(\frac{\gamma_y}{\overline{\gamma}\left(\overline{\gamma}+\gamma_y\right)}\right)^{-\frac{1}{2}}\frac{\gamma_x}{\left(\underline{\gamma}+\gamma_y\right)^2}\equiv B_s\gamma_x=O\left(\gamma_x\right),$$
$$\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|=\frac{\gamma_\theta^2\gamma_x}{\left(\gamma_\theta+\gamma+\gamma_y+N\gamma_x\right)\left(\gamma_\theta+\gamma+\gamma_y+\left(N+1\right)\gamma_x\right)}$$
$$\leq\frac{\gamma_\theta^2\gamma_x}{\left(\gamma_\theta+\underline{\gamma}+\gamma_y\right)^2}\equiv B_\Gamma\gamma_x=O\left(\gamma_x\right),$$

implies that

$$\left|\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right)\right|=O\left(\gamma_x\right)\qquad(33)$$

and therefore $\left|\frac{\partial}{\partial p}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}\right|=O\left(\gamma_x\right)$, where we have used the fact that

$$\left|\frac{\partial}{\partial p}\Pi_N^{\overline{N}}\left(p\right)\right|\leq\overline{N}\overline{\pi}_p.$$

3. $\mathcal{T}^pG$ **is decreasing in** γ: This follows from the same argument as the one developed above

to show that $\mathcal{T}^p G$ is decreasing in μ. Following these arguments, we have that

$$\frac{\partial}{\partial\gamma}\left[C^p\left(G\right)\right](\mu,\gamma,f) = \underbrace{-\frac{a}{2\gamma^2}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{\leq -\frac{a}{2\bar{\gamma}}e^{-a\bar{\mu}+\frac{a^2}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)<0}$$

$$+\beta\mathbb{E}_p\underbrace{\left[\frac{\partial}{\partial\gamma}S\left(G\right)\right]}_{\leq 0} + \beta\underbrace{\frac{\partial}{\partial p}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}}_{O(\gamma_x)}\underbrace{\frac{\partial p}{\partial\gamma}}_{\leq\bar{p}_\gamma},$$

so that, for γ_x small enough, the derivative is strictly negative and bounded away from 0.

4. **$\mathcal{T}^p G$ is Lipschitz in f of constant 1:** Choosing $f_1 < f_2$ then from (26) this is trivially satisfied because

$$\left|\mathcal{T}^p G\left(\mu,\gamma,f_2\right) - \mathcal{T}^p G\left(\mu,\gamma,f_1\right)\right| \leq \left|C^p\left(G\left(\mu,\gamma,f_2\right)\right) - C^p\left(G\left(\mu,\gamma,f_1\right)\right)\right|$$
$$= \left|f_2 - f_1\right|.$$

5. **$\mathcal{T}^p G$ is convex in μ:** From (30), the second derivative of the continuation value with respect to μ is:

$$\frac{\partial^2}{\partial\mu^2}C^p\left(G\right) = \underbrace{ae^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{\geq ae^{-a\bar{\mu}+\frac{a^2}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)>0} + \beta\mathbb{E}_p\underbrace{\left[\frac{\partial^2}{\partial\mu^2}S\left(G\right)\right]}_{\geq 0}$$

$$+\beta\underbrace{\frac{\partial p}{\partial\mu}}_{\leq\bar{p}_\mu}\underbrace{\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}}_{O(\gamma_x)} + \beta\underbrace{\frac{\partial^2 p}{\partial\mu^2}}_{\leq\bar{p}_{\mu\mu}}\underbrace{\frac{\partial^2}{\partial p^2}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\}}_{O(\gamma_x)},$$

where $\frac{\partial^2}{\partial p^2}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = O\left(\gamma_x\right)$ follows from (32) and the fact that $\Pi_N''\left(p\right)$ is a polynomial of degree $\overline{N}-2$ in p and is therefore bounded on $[0,1]$. To see that

$$\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] = O\left(\gamma_x\right),$$

note from (31) that

$$\frac{\partial}{\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = \frac{\partial g_{\overline{N}}}{\partial\mu} - \sum_{N=1}^{\overline{N}-1}\Pi_N^{\overline{N}}\left(p\right)\cdot\left(\frac{\partial g_{N+1}}{\partial\mu} - \frac{\partial g_N}{\partial\mu}\right)(\mu,\gamma),$$

which, taking derivative with respect to p and using (29), gives

$$\frac{\partial^2}{\partial p\partial\mu}\mathbb{E}_p\left\{\left[S\left(G\right)\right]\left(\mu+s\varepsilon,\Gamma\right)\right\} = -\sum_{N=1}^{\overline{N}-1}\frac{\partial\Pi_N^{\overline{N}}\left(p\right)}{\partial p}\cdot\left(\frac{\partial g_{N+1}}{\partial\mu} - \frac{\partial g_N}{\partial\mu}\right)(\mu,\gamma)$$

45

Following the same arguments as before, we obtain that

$$\left| \frac{\partial g_{N+1}}{\partial \mu} - \frac{\partial g_N}{\partial \mu} \right| \leq \overline{G}_{\mu\mu} \left| \left(s \left(N+1, \gamma \right) - s \left(N, \gamma \right) \right) \right|$$
$$+ \overline{G}_{\mu\gamma} \left| \Gamma \left(N+1, \gamma \right) - \Gamma \left(N, \gamma \right) \right| = O \left(\gamma_x \right),$$

which completes the proof.

6. $[S \left(\mathcal{T}^p G \right)] \left(\mu, \gamma \right)$ **is twice-differentiable with bounded first and second derivatives:** Notice, first, that since $G \in \mathcal{G}$, then $C^p \left(G \right)$ is twice-differentiable in $\left(\mu, \gamma \right)$ and linear in f:

$$C^p \left(G \right) = \frac{1}{a} \left[e^{-a\mu + \frac{a^2}{2} \left(\frac{1}{\gamma} + \frac{1}{\gamma_x} \right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}} \right) - \left(1 - \beta \right) \right] + f - \beta \omega^f$$
$$+ \beta \mathbb{E}_p \left\{ \left[S \left(G \right) \right] \left(\mu + s\varepsilon, \Gamma \left(N, \gamma \right) \right) \right\}.$$

We show below that investment takes the form of a cutoff rule $f_c^p \left(\mu, \gamma \right)$ that satisfies $f_c^p \left(\mu, \gamma \right) = f - \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right)$. Hence, $f_c^p \left(\mu, \gamma \right)$ is twice-differentiable in $\left(\mu, \gamma \right)$ and independent of f. Therefore,

$$\left[S \left(\mathcal{T}^p G \right) \right] \left(\mu, \gamma \right) \equiv \int \left[\mathcal{T}^p G \right] \left(\mu, \gamma, f \right) dF \left(f \right)$$
$$= \int \max \left\{ \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right), 0 \right\} dF \left(f \right)$$
$$= \int_{f_c^p \left(\mu, \gamma \right)}^{\infty} \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right) dF \left(f \right).$$

Since $C^p \left(G \right)$ and f_c^p are twice-differentiable in $\left(\mu, \gamma \right)$, so is $S \left(\mathcal{T}^p G \right)$. To finish the proof, it only remains to show that the first and second derivatives of $S \left(\mathcal{T}^p G \right)$ are bounded. For $\left(x, y \right) = \left\{ \mu, \gamma \right\}$,

$$\frac{\partial}{\partial x} S \left(\mathcal{T}^p G \right) = \int_{-\infty}^{f_c^p \left(\mu, \gamma \right)} \frac{\partial}{\partial x} \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right) dF \left(f \right).$$

According to previous results and after some manipulations,

$$\left| \frac{\partial}{\partial \mu} \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right) \right| \leq e^{-a\underline{\mu} + \frac{a^2}{2} \left(\frac{1}{\underline{\gamma}} + \frac{1}{\gamma_x} \right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}} \right) + \beta \overline{G}_\mu$$
$$+ \beta \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x \left(B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma \right),$$

$$\left| \frac{\partial}{\partial \gamma} \left[C^p \left(G \right) \right] \left(\mu, \gamma, f \right) \right| \leq \frac{a}{2\underline{\gamma}^2} e^{-a\underline{\mu} + \frac{a^2}{2} \left(\frac{1}{\underline{\gamma}} + \frac{1}{\gamma_x} \right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}} \right) + \beta \overline{G}_\gamma$$
$$+ \beta \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x \left(B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma \right).$$

Boundedness of the derivatives is guaranteed if

$$
X\left(\gamma_x\right) + \beta A\left(\gamma_x\right) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \leq \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \Leftrightarrow X\left(\gamma_x\right) \leq \left(I - \beta A\left(\gamma_x\right)\right) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix}
$$

where

$$
A\left(\gamma_x\right) = \begin{bmatrix} 1 + \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x B_s & \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma \\ \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma & 1 + \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x B_\Gamma \end{bmatrix}
$$

and

$$
X\left(\gamma_x\right) = \begin{bmatrix} 1 \\ \frac{a}{2\underline{\gamma}^2} \end{bmatrix} e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{2}+\frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right).
$$

Note that the matrix $I - \beta A\left(\gamma_x\right)$ satisfies $I - \beta A\left(\gamma_x\right) \xrightarrow[\gamma_x \to 0]{} \left(1 - \beta\right) I$. Thus, with γ_x small enough, the following is satisfied

$$
\left[I - \beta A\left(\gamma_x\right)\right] \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \geq \frac{1}{2}\left(1 - \beta\right) \begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix}.
$$

We can then choose positive bounds $\{\overline{G}_\mu, \overline{G}_\gamma\}$ such that

$$
\begin{bmatrix} \overline{G}_\mu \\ \overline{G}_\gamma \end{bmatrix} \geq 2\left(1 - \beta\right)^{-1} X\left(\gamma_x\right) \geq 0,
$$

which guarantees the boundedness of the first derivatives. Regarding the second derivatives, manipulations of a similar nature as above yield the following inequalities:

$$
\begin{aligned}
\left| \frac{\partial^2}{\partial \mu^2} \left[C^p\left(G\right)\right]\left(\mu, \gamma, f\right) \right| \leq\ & a e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{2}+\frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta \overline{G}_{\mu\mu} \\
& + \beta \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x \left(B_s \overline{G}_{\mu\mu} + B_\Gamma \overline{G}_{\mu\gamma}\right) \\
& + \beta \overline{p}_{\mu\mu} \overline{\pi}_{pp} \overline{N}^2 \gamma_x \left(B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma\right),
\end{aligned}
$$

$$
\begin{aligned}
\left| \frac{\partial^2}{\partial \gamma^2} \left[C^p\left(G\right)\right]\left(\mu, \gamma, f\right) \right| \leq\ & \left(\frac{a}{\underline{\gamma}^3} + \frac{a^3}{4\underline{\gamma}^4}\right) e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{2}+\frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta \overline{G}_{\gamma\gamma} \\
& + \beta \overline{p}_\gamma \overline{\pi}_p \overline{N}^2 \gamma_x \left(B_s \overline{G}_{\mu\gamma} + B_\Gamma \overline{G}_{\gamma\gamma}\right) \\
& + \beta \overline{p}_{\gamma\gamma} \overline{\pi}_{pp} \overline{N}^2 \gamma_x \left(B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma\right),
\end{aligned}
$$

$$
\begin{aligned}
\left| \frac{\partial^2}{\partial \mu \partial \gamma} \left[C^p\left(G\right)\right]\left(\mu, \gamma, f\right) \right| \leq\ & \frac{a^2}{2\underline{\gamma}^2} e^{-a\underline{\mu}+\frac{a^2}{2}\left(\frac{1}{2}+\frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta \overline{G}_{\mu\gamma} \\
& + \beta \overline{p}_\mu \overline{\pi}_p \overline{N}^2 \gamma_x \left(B_s \overline{G}_{\mu\gamma} + B_\Gamma \overline{G}_{\gamma\gamma}\right) \\
& + \beta \overline{p}_{\mu\gamma} \overline{\pi}_{pp} \overline{N}^2 \gamma_x \left(B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma\right).
\end{aligned}
$$

We can then conclude with an argument similar to the previous one that there exists bounds $\{\overline{G}_{\mu\mu}, \overline{G}_{\mu\gamma}, \overline{G}_{\gamma\gamma}\}$ for the second derivatives of $S(G)$ as long as γ_x is small enough.

It remains to show existence and monotonicity of $f_c(\mu, \gamma)$. A firm invests if and only if

$$
\begin{aligned}
[C^p(G)](\mu, \gamma, f) &= \frac{1}{a}\left[e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) - (1 - \beta)\right] \\
&\quad + f - \beta\omega^f + \beta\mathbb{E}_p\{[S(G)](\mu + s\varepsilon, \Gamma(N, \gamma))\} \leq 0,
\end{aligned}
$$

i.e., when its fixed cost satisfies

$$
\begin{aligned}
f &\leq -\frac{1}{a}\left[e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) - (1 - \beta)\right] - \beta\omega^f \\
&\quad + \beta\mathbb{E}_p\{[S(G)](\mu + s\varepsilon, \Gamma(N, \gamma))\} \\
&\equiv f_c^p(\mu, \gamma)
\end{aligned}
\tag{34}
$$

Notice, furthermore, that $f_c^p(\mu, \gamma) = f - C^p(G(\mu, \gamma, f))$. Thus, the threshold inherits a number of properties from the continuation value. In particular, $f_c(\mu, \gamma)$ is strictly increasing in μ and γ, and strictly concave in μ for γ_x small enough.

$\square$

Proposition 2. *Under assumptions 1 and 2 and for γ_x small enough, a recursive equilibrium exists and is unique. The equilibrium $p(\mu, \gamma)$ is increasing in the mean of beliefs μ and the precision γ.*

Proof. Proving uniqueness of the recursive equilibrium is equivalent to showing that there is a unique fixed point $p^*(\mu, \gamma) \in \mathcal{P}$ such that $\mathcal{M}p^* = p^*$ for the mapping $\mathcal{M}$ in Definition 9.

We establish first that $\mathcal{M}$ is a well-defined mapping from $\mathcal{P}$ to $\mathcal{P}$. This follows from the definition of $f_c^p(\mu, \gamma)$, which inherits the properties of C^p. In particular, it is twice-differentiable with bounded first and second derivatives. Under assumption 2, $\mathcal{M}p = F(f_c^p)$ preserves these properties and it is possible to find bounds on the first and second derivatives of p that are preserved by the mapping using a similar argument as the one developed in proposition (1).

Next, we show that $\mathcal{M}$ defines a contraction from $\mathcal{P}$ to $\mathcal{P}$.[17] For $p_1, p_2 \in \mathcal{P}$, by the mean value theorem, the mapping $\mathcal{M}$ satisfies

$$
\begin{aligned}
|(\mathcal{M}p_2 - \mathcal{M}p_1)(\mu, \gamma)| &= |F(f_c^{p_2}(\mu, \gamma)) - F(f_c^{p_1}(\mu, \gamma))| \\
&= \left|F'\left(\tilde{f}\right)(f_c^{p_2}(\mu, \gamma) - f_c^{p_1}(\mu, \gamma))\right|
\end{aligned}
$$

for some $\tilde{f} \in [f_c^{p_1}(\mu, \gamma), f_c^{p_2}(\mu, \gamma)]$. Therefore, if

$$
|f_c^{p_2}(\mu, \gamma) - f_c^{p_1}(\mu, \gamma)| \leq A\gamma_x \| p_2 - p_1 \|
\tag{35}
$$

---

[17] We cannot prove that $\mathcal{M}$ satisfies monotonicity and therefore we cannot apply the Blackwell conditions. Instead, we directly show that $\mathcal{M}$ satisfies the definition of a contraction.

for some constant A, we reach

$$|(\mathcal{M}p_2 - \mathcal{M}p_1)(\mu, \gamma)| \le A\gamma_x \parallel F'\left(\tilde{f}\right) \parallel \cdot \parallel p_2 - p_1 \parallel, \tag{36}$$

implying that the mapping $\mathcal{M}$ is continuous as long as F' is bounded, which is guaranteed by assumption 2. We can then choose γ_x such that $A\gamma_x \parallel F'\left(\tilde{f}\right) \parallel < 1$ and use (36) to guarantee that $\mathcal{M}$ is indeed a contraction. By the contraction mapping theorem, this implies that the equilibrium exists and is unique for γ_x sufficiently small.

Therefore, to prove existence and uniqueness it remains to establish that (35) holds for some constant A. From the definition of $f_c^p(\mu, \gamma)$, the left-hand side of (35) can be expressed as:

$$|f_c^{p_2}(\mu, \gamma) - f_c^{p_1}(\mu, \gamma)|$$
$$= \beta \left| \mathbb{E}_{p_2} \{ [S(G^{p_2})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} - \mathbb{E}_{p_1} \{ [S(G^{p_1})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} \right|$$
$$\le \beta \left| \mathbb{E}_{p_2} \{ [S(G^{p_2})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) - [S(G^{p_1})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} \right| \tag{37}$$
$$+ \beta \left| \mathbb{E}_{p_2} \{ [S(G^{p_1})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} - \mathbb{E}_{p_1} \{ [S(G^{p_1})](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} \right| \tag{38}$$

To prove that (35) holds we will control each term in this expression. We start with the term in (38). For any $G \in \mathcal{G}$, we can use (34) to write

$$\left| \begin{array}{c} \mathbb{E}_{p_2} \{ [S(G)](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} \\ -\mathbb{E}_{p_1} \{ [S(G)](\mu + s(N, \gamma)\varepsilon, \Gamma(N, \gamma)) \} \end{array} \right| = \left| \sum_{N=1}^{\overline{N}-1} \left[\Pi_N^{\overline{N}}(p_2) - \Pi_N^{\overline{N}}(p_1) \right] \cdot (g_{N+1} - g_N)(\mu, \gamma) \right|$$

$$\le \gamma_x \left[B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma \right] \sum_{N=1}^{\overline{N}-1} \left| \Pi_N^{\overline{N}}(p_2) - \Pi_N^{\overline{N}}(p_1) \right|$$

$$\le B\gamma_x \parallel p_2 - p_1 \parallel \tag{39}$$

where B is some constant. The second line follows from the results established in Proposition 1, and the third line follows from noting that $\Pi_N^{\overline{N}}(p)$ is a polynomial in p of degree $\overline{N}$ and therefore continuous on the compact set $[0, 1]$. In particular, we can control the term by $\parallel \Pi_N^{\overline{N}}(p_2) - \Pi_N^{\overline{N}}(p_1) \parallel \le \parallel \frac{\partial}{\partial p} \Pi_N^{\overline{N}} \parallel \parallel p_2 - p_1 \parallel$.

To conclude, we move to the term in (37). For that, we need to evaluate the norm of $\parallel G^{p_2} - G^{p_1} \parallel$. We first consider the term $[\mathcal{T}^{p_2}(G) - \mathcal{T}^{p_1}(G)](\mu, \gamma, f)$, starting from some common function $G \in \mathcal{G}$. Assuming w.l.o.g. that

$$[\mathcal{T}^{p_2}(G)](\mu, \gamma, f) \ge [\mathcal{T}^{p_1}(G)](\mu, \gamma, f),$$

from the definition of $\mathcal{T}^p$ it follows that only the next scenarios are possible:

1. $[\mathcal{T}^{p_2}(G)](\mu, \gamma, f) = [\mathcal{T}^{p_1}(G)](\mu, \gamma, f) = 0$;

2. $[\mathcal{T}^{p_2}(G)](\mu, \gamma, f) = [C^{p_2}(G)](\mu, \gamma, f)$, in which case (39) implies

$$|[\mathcal{T}^{p_2}(G) - \mathcal{T}^{p_1}(G)](\mu, \gamma, f)|$$
$$\le \beta \{ \mathbb{E}_{p_2} \{ [S(G)](\mu', \gamma') \} - \mathbb{E}_{p_1} \{ [S(G)](\mu', \gamma') \} \}$$
$$\le \beta B\gamma_x \parallel p_2 - p_1 \parallel.$$

Following similar arguments, we can recursively show that, for $k > 1$,

$$\| \left(\mathcal{T}^{p_2}\right)^k G - \left(\mathcal{T}^{p_1}\right)^k G \| \leq \beta \frac{1 - \beta^k}{1 - \beta} B\gamma_x \| p_2 - p_1 \|$$

Since, from Proposition 1, the operator $\mathcal{T}^p$ is a contraction, we have in the limit that:

$$\| G^{p_2} - G^{p_1} \| \leq \frac{\beta}{1 - \beta} B\gamma_x \| p_2 - p_1 \|, \tag{40}$$

implying

$$\left| \mathbb{E}_{p_2} \left\{ [S\left(G^{p_2}\right)]\left(\mu + s\left(N, \gamma\right)\varepsilon, \Gamma\left(N, \gamma\right)\right) - [S\left(G^{p_1}\right)]\left(\mu + s\left(N, \gamma\right)\varepsilon, \Gamma\left(N, \gamma\right)\right) \right\} \right|$$
$$\leq \frac{\beta}{1 - \beta} B\gamma_x \| p_2 - p_1 \| \tag{41}$$

Combining (39) with (37) and (38) implies

$$\left| f_c^{p_2}\left(\mu, \gamma\right) - f_c^{p_1}\left(\mu, \gamma\right) \right| \leq \frac{1}{1 - \beta} B\gamma_x \| p_2 - p_1 \|$$

so that (35) indeed holds, implying uniqueness.

Showing that the expected number of investing firms is increasing in the mean of beliefs μ and in precision γ is equivalent to showing the threshold $f_c^{p^*}\left(\mu, \gamma\right)$ is strictly increasing in μ and γ. This immediately follows from the properties of the continuation value C^p demonstrated in the proof of the previous proposition. $\qquad \square$

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $[\mu_l, \mu_h]$ such that, for all $\mu \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l\left(\mu\right) < \gamma_h\left(\mu\right)$. Regime γ_l is characterized by high uncertainty and low investment while regime γ_h is characterized by low uncertainty and high investment.*

Proof. In the limit case where the number of firms is large enough that the approximation $n = N/\overline{N} = F\left(f_c\right)$ is valid, we can define the function

$$\varphi_\mu^n\left(\gamma\right) = \Gamma\left(n\left(\mu, \gamma\right), \gamma\right) - \gamma$$
$$= \left(\frac{1}{\gamma + \gamma_y + n\left(\mu, \gamma\right)\gamma_x} + \frac{1}{\gamma_\theta} \right)^{-1} - \gamma$$

where Γ is the law of motion for γ defined in (5). By continuity of $n = F\left(f_c\left(\mu, \gamma\right)\right)$, the function $\varphi_\mu^n\left(\gamma\right)$ is continuous in γ. From the definition of $\{\overline{\gamma}, \underline{\gamma}\}$ in (23) and (24), we have that $\varphi_\mu^n\left(\underline{\gamma}\right) \geq 0 \geq \varphi_\mu^n\left(\overline{\gamma}\right)$.

Consider a distribution of fixed investment costs F^1 with mean ω^f and standard deviation 1

that satisfies Assumption 2. Let

$$F^{\sigma^f}(f) = F^1\left[\left(\sigma^f\right)^{-1}\left(f - \omega^f\right) + \omega^f\right]$$

be a mean-preserving, rescaled version of that distribution with standard deviation σ^f. We are going to show that when σ_f is low, there exists a range $[\mu_l, \mu_h]$ such that for any $\mu^* \in (\mu_l, \mu_h)$, we can always find two points $\gamma_1 < \gamma_2$ with $\gamma_1, \gamma_2 \in \left(\underline{\gamma}, \overline{\gamma}\right)$ such that $\varphi^n_{\mu^*}(\gamma_1) < 0$ and $\varphi^n_{\mu^*}(\gamma_2) > 0$. This will imply, by the Intermediate Value Theorem, that there exist two values $\gamma^*_l < \gamma^*_h$ with $\underline{\gamma} \leq \gamma^*_l < \gamma_1$ and $\gamma_2 < \gamma^*_h \leq \overline{\gamma}$ such that $\varphi^n_\mu(\gamma^*_l) = \varphi^n_\mu(\gamma^*_h) = 0$, i.e. two distinct stationary points in the dynamics of precision γ.

An important step in this proof is established in lemma 3 from Appendix E, where we prove that as σ^f goes to 0 the cutoff $f_c^{\sigma^f}$ corresponding to the variance σ^f of the fixed-cost distribution converges uniformly towards some limit f_c^0 and that the number of investing firms converges pointwise to the limit $n^0(\mu, \gamma) = \mathbb{I}\left(\omega^f \leq f_c^0(\mu, \gamma)\right)$.

We must first find a range of values for μ in which we are guaranteed to have multiple stationary points for γ. We are going to use the fact that $f_c^{\sigma^f}$ is strictly increasing in μ and γ at a bounded rate. In what follows, we denote G^{σ^f} the general equilibrium surplus of investing for a given dispersion of costs σ^f, i.e., $\mathcal{T}^{p(\sigma^f)}G^{\sigma^f} = G^{\sigma^f}$ where $\mathcal{M}^{\sigma^f}\left[p\left(\sigma^f\right)\right] = p\left(\sigma^f\right)$. Recall the definition:

$$
\begin{aligned}
f_c^{\sigma^f}(\mu, \gamma) &= -\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + (1 - \beta)\frac{1}{a} + \beta\omega^f \\
&\quad - \beta\mathbb{E}\left\{\left[S^{\sigma^f}\left(G^{\sigma^f}\right)\right](\mu', \gamma')\right\}.
\end{aligned}
$$

Since G^{σ^f} has bounded derivatives, we can find upper and lower bounds for the derivatives of $f_c^{\sigma^f}$ in μ and γ that are strictly positive, as we did in proposition 1 for γ_x low enough. Denote these bounds $\overline{f}_\mu$, $\underline{f}_\mu$ and $\overline{f}_\gamma$, $\underline{f}_\gamma$. The derivatives are:

$$
\begin{aligned}
0 < \underline{f}_\mu &\leq \frac{\partial}{\partial\mu}f_c^{\sigma^f}(\mu, \gamma) \\
&= e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial\mu}G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\mu \\
0 < \underline{f}_\gamma &\leq \frac{\partial}{\partial\gamma}f_c^{\sigma^f}(\mu, \gamma) \\
&= \frac{a}{2\gamma^2}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial\gamma}G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\gamma
\end{aligned}
$$

Since f_c^0 is the uniform limit of continuous functions, it is continuous. The limit f_c^0 may not be differentiable, but it is bi-Lipschitz continuous with Lipschitz constants $\left(\underline{f}_\mu, \overline{f}_\mu\right)$ and $\left(\underline{f}_\gamma, \overline{f}_\gamma\right)$. We know therefore that for μ low, $f_c^0(\mu, \overline{\gamma}) < \omega^f$ (remember that ω^f is the mean of the fixed cost distribution), and that for μ high, $f_c^0(\mu, \overline{\gamma}) > \omega^f$. By the Intermediate Value theorem, we know that there exists a point μ_l at which $f_c^0(\mu_l, \overline{\gamma}) = \omega^f$. Since f_c^0 is strictly increasing in γ, we have that $f_c^0\left(\mu_l, \underline{\gamma}\right) < \omega^f$. Using the fact that f_c^0 is bi-Lipschitz continuous, we have the following

51

inequality:

$$f_c^0\left(\mu, \underline{\gamma}\right) \quad \leq \quad f_c^0\left(\mu_l, \underline{\gamma}\right) + \overline{f}_\mu \cdot \left(\mu - \mu_l\right).$$

Define $\mu_h = \mu_l + \dfrac{\omega^f - f_c\left(\mu_l, \underline{\gamma}\right)}{\overline{f}_\mu} > \mu_l$. Then, for any $\mu \in (\mu_l, \mu_h)$:

$$f_c^0\left(\mu, \underline{\gamma}\right) \leq f_c^0\left(\mu_l, \underline{\gamma}\right) + \overline{f}_\mu \cdot \left(\mu - \mu_l\right) < \omega^f < f_c^0\left(\mu, \overline{\gamma}\right).$$

We will now show that the interval (μ_l, μ_h) is a range of values for μ in which we are guaranteed to have two steady-states. Pick any $\mu^* \in (\mu_l, \mu_h)$. Then $f_c^0\left(\mu^*, \underline{\gamma}\right) < \omega^f$ (meaning that $n^0\left(\mu^*, \underline{\gamma}\right) = 0$) and $f_c^0\left(\mu^*, \overline{\gamma}\right) > \omega^f$ (meaning $n^0\left(\mu^*, \overline{\gamma}\right) = 1$). By continuity of f_c^0, we can pick (γ_1, γ_2) with $\underline{\gamma} < \gamma_1 < \gamma_2 < \overline{\gamma}$, such that $f_c^0\left(\mu^*, \gamma_1\right) < \omega^f$ and $f_c^0\left(\mu^*, \gamma_2\right) > \omega^f$. Therefore, $n^0\left(\mu^*, \gamma_1\right) = 0$ and $n^0\left(\mu^*, \gamma_2\right) = 1$. We have:

$$
\begin{aligned}
\varphi_{\mu^*}^{n^0}(\gamma_1) &= \left(\frac{1}{\gamma_1 + \gamma_y + n^0\left(\mu^*, \gamma_1\right)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_1 \\
&= \left(\frac{1}{\gamma_1 + \gamma_y} + \sigma_\theta^2\right)^{-1} - \gamma_1 < \left(\frac{1}{\underline{\gamma} + \gamma_y} + \sigma_\theta^2\right)^{-1} - \underline{\gamma} = 0
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{\mu^*}^{n^0}(\gamma_2) &= \left(\frac{1}{\gamma_2 + \gamma_y + n^0\left(\mu^*, \gamma_2\right)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 \\
&= \left(\frac{1}{\gamma_2 + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 > \left(\frac{1}{\overline{\gamma} + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \overline{\gamma} = 0.
\end{aligned}
$$

Since $n^{\sigma^f}(\mu, \gamma) \underset{\sigma^f \to 0}{\longrightarrow} n^0(\mu, \gamma)$, for σ^f small enough, we will have: $\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_1) < 0$ and $\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_2) > 0$, which implies that there exists at least two locally stable steady-states γ_l^* and γ_h^* $\left(\varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_l^*) = \varphi_{\mu^*}^{n^{\sigma^f}}(\gamma_h^*) = 0\right)$ with $\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \overline{\gamma}$ (one can pick at least 2 locally stable steady-states because $\varphi_{\mu^*}^{n^{\sigma^f}}$ must cross the x-axis from above at least twice). $\qquad\square$

Proposition 4. *The recursive competitive equilibrium is constrained inefficient and the efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$ and a uniform tax. In turn, when γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proof. 1. In the limit case where the number of firms is large enough that the approximation $n = N/\overline{N} = F(f_c)$ is valid, we can write the constrained planner's decision as a choice over the

optimal cutoff $f_c^{eff} \in \mathbb{R} \cup \{-\infty, \infty\}$ under which firms should invest:

$$W(\mu, \gamma) = \max_{f_c^{eff}} \int_{-\infty}^{f_c^{eff}} \left(\mathbb{E}[u(\theta + \varepsilon^x) \mid \mu, \gamma] - \tilde{f} \right) dF(\tilde{f})$$
$$+ \beta \mathbb{E}[W(\mu', \gamma')]$$
$$\text{s.t} \quad \mu' = \frac{\gamma\mu + \gamma_y Y + n\gamma_x X}{\gamma + \gamma_y + n\gamma_x}$$
$$\gamma' = \left(\frac{1}{\gamma + \gamma_y + n\gamma_x} + \frac{1}{\gamma_\theta} \right)^{-1}$$
$$n = F\left(f_c^{eff} \right)$$

with $\theta' = \theta + \varepsilon^\theta, \varepsilon^\theta \sim \mathcal{N}\left(0, \gamma_\theta^{-1}\right), Y = \theta + \varepsilon^y, \varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right)$ and $X = \theta + \varepsilon^X, \varepsilon^X \sim \mathcal{N}\left(0, (n\gamma_x)^{-1}\right)$. The first order condition with respect to the cutoff is

$$F'\left(f_c^{eff}\right) \left(\mathbb{E}[u(\theta + \varepsilon^x) \mid \mu, \gamma] - f_c^{eff} + \beta \frac{d}{dn} \mathbb{E}[W(\mu + s(n, \gamma)\varepsilon, \Gamma(n, \gamma))] \right) = 0,$$

where ε is a unit normal, so that we can derive an expression for the efficient cutoff:

$$f_c^{eff}(\mu, \gamma) = \mathbb{E}[u(\theta + \varepsilon^x) \mid \mu, \gamma] + \beta \frac{d}{dn} \mathbb{E}[W(\mu + s(n, \gamma), \Gamma(n, \gamma))].$$

We show that this optimal cutoff is implementable using beliefs-dependent investment subsidies $\tau(\mu, \gamma)$ and a uniform tax $T(\mu, \gamma)$ levied on all firms at the beginning of the period. Let us write the problem of firms facing these policy instruments:

$$V^\tau(\mu, \gamma, f) = \max\left\{ \mathbb{E}[u(\theta + \varepsilon^x) \mid \mu, \gamma] - f + \tau(\mu, \gamma), \beta \mathbb{E}\left[V^\tau(\mu', \gamma', f')\right] \right\} - T(\mu, \gamma)$$

which yields the individual cutoff rule f_c:

$$f_c^\tau(\mu, \gamma) = \mathbb{E}[u(\theta + \varepsilon^x) \mid \mu, \gamma] + \tau(\mu, \gamma) - \beta \mathbb{E}\left[V^\tau(\mu', \gamma', f')\right].$$

Requiring that the government's budget constraint balances implies

$$\tau(\mu, \gamma) F(f_c^\tau(\mu, \gamma)) = T(\mu, \gamma).$$

To implement the efficient allocation, we must identify the two cutoffs

$$f_c^\tau(\mu, \gamma) = f_c^{eff}(\mu, \gamma)$$
$$\Leftrightarrow \quad \tau(\mu, \gamma) = \underbrace{\beta \frac{d}{dn} \mathbb{E}[W(\mu + s(n, \gamma), \Gamma(n, \gamma))]}_{\text{information externality}} + \underbrace{\beta \mathbb{E}\left[V^\tau(\mu', \gamma', f')\right]}_{\text{option value of waiting}}. \tag{42}$$

Expression (42) is a functional equation in τ because V^τ depends implicitly on τ. To show that

this functional equation has a solution, we define the following mapping T on the set of continuous and bounded functions to itself such that

$$
\begin{aligned}
[T(V)](\mu, \gamma, f) &= \max\left\{\mathbb{E}\left[u\left(\theta + \varepsilon^x\right) \mid \mu, \gamma\right] - f + \tau(\mu, \gamma), \beta \mathbb{E}\left[V\left(\mu', \gamma', f'\right)\right]\right\} \\
&\quad - T(\mu, \gamma) \\
\text{s.t.} \quad \tau(\mu, \gamma) &= \beta \frac{d}{dn} \mathbb{E}\left[W\left(\mu', \gamma'\right)\right] + \beta \mathbb{E}\left[V\left(\mu', \gamma', f'\right)\right] \\
T(\mu, \gamma) &= n^{eff}(\mu, \gamma) \tau(\mu, \gamma) \\
\mu' &= \mu + s\left(n^{eff}(\mu, \gamma), \gamma\right) \varepsilon \\
\gamma' &= \Gamma\left(n^{eff}(\mu, \gamma), \gamma\right).
\end{aligned}
$$

By standard arguments, this mapping defines a contraction. The maximization yields the following decision: invest if and only if

$$
\begin{aligned}
f &\leq \mathbb{E}\left[u\left(\theta + \varepsilon^x\right) \mid \mu, \gamma\right] + \tau(\mu, \gamma) - \beta \mathbb{E}\left[V\left(\mu', \gamma', f'\right)\right] \\
&\leq \mathbb{E}\left[u\left(\theta + \varepsilon^x\right) \mid \mu, \gamma\right] + \beta \frac{d}{dn} \mathbb{E}\left[W\left(\mu', \gamma'\right)\right],
\end{aligned}
$$

which coincides with the efficient cutoff f_c^{eff}. Thus, denoting V^* the only fixed point of this mapping, the investment subsidy $\tau(\mu, \gamma) = \beta \frac{d}{dn} \mathbb{E}_{n^{eff}}\left[W\left(\mu', \gamma'\right)\right] + \beta \mathbb{E}_{n^{eff}}\left[S\left(V^*\right)\left(\mu', \gamma'\right)\right]$ is a solution to the functional equation (42). It implements the efficient cutoff rule and balances the government budget by construction. An explicit expression for the optimal subsidy can be derived by noticing that V^* satisfies

$$
\begin{aligned}
[S(V^*)](\mu, \gamma) &= F\left(f_c^{eff}(\mu, \gamma)\right)\left(\mathbb{E}\left[u\left(\theta + \varepsilon^x\right) \mid \mu, \gamma\right] - \mathbb{E}\left[f \mid f \leq f_c^{eff}(\mu, \gamma)\right]\right) \\
&\quad + \left(1 - F\left(f_c^{eff}(\mu, \gamma)\right)\right) \beta \mathbb{E}\left[S\left(V^*\right)\left(\mu', \gamma'\right)\right],
\end{aligned}
$$

which can be computed from primitives once f_c^{eff} is known.

We have shown that the efficient allocation can be implemented by transfers to investing firms. To complete the proof, we show that these transfers are positive and non-zero in non-trivial cases. More precisely, rewrite the mapping satisfied by these transfers:

$$
\tau(\mu, \gamma) = \beta \underbrace{\frac{d}{dn} \mathbb{E}\left[W\left(\mu', \gamma'\right)\right]}_{\equiv A(\mu, \gamma)} + \beta \underbrace{\mathbb{E}\left\{[S(V^*)]\left(\mu', \gamma'\right)\right\}}_{\equiv B(\mu, \gamma)}.
$$

As long as the efficient allocation is not trivial, i.e. that there exists some (μ, γ, f) at which firms invest (which is guaranteed since f has an unbounded support), term $B(\mu, \gamma)$ is strictly positive for some (μ, γ).

We now prove that A is non-negative. The effect of an increase of n in $\mathbb{E}\left[W\left(\mu', \gamma'\right)\right]$ is proportional to that of an exogenous arrival of information. The following discussion thus focuses on the

54

impact on welfare of an exogenous arrival of information. It is useful for our purpose to rewrite the planner's problem in a sequential way. A strategy for the planner is a collection of cutoff functions $\{f_0, f_1, \ldots, f_t, \ldots\}$ such that for each date t, f_t maps the set of all past histories of signals up to time t, $\{Y_s, X_s\}_{s=0}^t$, to the real line. Pick some date t_0. We are going to show that the exogenous arrival of a signal S of precision γ_S at date t_0 allows the planner to do at least as well as without it, because it can ignore it. Denote $\mathcal{F}_t$ the information set $\{Y_s, X_s\}_{s=0}^t$ of the planner at each date without the exogenous signal, and $\mathcal{F}_t^S$ the information set $\{Y_s, X_s^S\}_{s=0}^t$ of the planner when the arrival of the exogenous signal is known and anticipated. Let $\{f_{c,t}\}$ be any strategy considered by the planner without the exogenous signal. Construct the following strategy for the case with exogenous arrival of information:

$$\forall t < t_0, \qquad f_{c,t}^S \left(\{Y_s, X_s\}_{s=0}^t\right) = f_{c,t} \left(\{Y_s, X_s\}_{s=0}^t\right),$$
$$\forall t \geq t_0, \qquad f_{c,t}^S \left(\{Y_s, X_s\}_{s=0}^t, S\right) = f_{c,t} \left(\{Y_s, X_s\}_{s=0}^t\right),$$

so that the two strategies and the information sets $\mathcal{F}_t$ and $\mathcal{F}_t^S$ coincide up to time $t_0 - 1$. After date t_0, strategy f_c^S deliberately ignores the new information. Therefore, by the law of iterated expectations, the two strategies have the same ex-ante payoffs. Welfare can only be increased with the arrival of new information, hence term $A(\mu, \gamma)$ is non-negative.

We conclude that the symmetric, efficient allocation can be implemented with positive transfers. In non-trivial cases, these transfers are strictly positive, which implies that the decentralized economy without transfers is inefficient.

2. The proof that the efficient allocation is subject to uncertainty traps follows closely that of the decentralized case. Thus, we only state the major steps of the proof:

- The optimal cutoff for the planner is defined by:

$$f_c^{eff} (\mu, \gamma) = \mathbb{E}\left[u\left(\theta + \varepsilon^x\right) \mid \mu, \gamma\right] + \beta \frac{d}{dN} \mathbb{E}\left[W\left(\mu + s(N, \gamma), \Gamma(N, \gamma)\right)\right].$$

  The first step of the proof is to show that $\frac{d}{dN}\mathbb{E}\left[W\left(\mu + s(N, \gamma), \Gamma(N, \gamma)\right)\right]$ is a $O(\gamma_x)$, so that for γ_x low enough f_c^{eff} is strictly increasing in μ and γ with derivatives that can be bounded away from 0;

- In a second step, show that when $\sigma^f \to 0$, then f_c^{eff, σ^f} converges uniformly to some limit $f_c^{eff,0}$ that is bi-Lipschitz continuous, strictly increasing in μ and γ with derivatives bounded away from 0. Thus, we have the pointwise limit:

$$\forall (\mu, \gamma), \qquad n^{eff,\sigma^f} (\mu, \gamma) \to n^{eff,0} (\mu, \gamma) = \mathbb{I}\left(\omega^f \leq f_c^{eff,0} (\mu, \gamma)\right);$$

- Conclude identically to proposition 3 that for σ^f sufficiently small there are at least two locally stable steady-states in the dynamics of γ.

$\square$

E Online Appendix (NOT FOR PUBLICATION)

This online appendix contains the proofs of the two technical lemmas 3 and 4.

First, we prove the technical lemma that establishes the continuity of the cutoff $f_c^{\sigma^f}$ in σ^f.

Lemma 3. *As $\sigma^f \to 0$, the equilibrium cutoff value $f_c^{\sigma^f}$ converges uniformly towards some limit f_c^0:*

$$\sup_{(\mu,\gamma)\in\mathcal{S}} \left| f_c^{\sigma^f}(\mu,\gamma) - f_c^0(\mu,\gamma) \right| \underset{\sigma^f\to 0}{\longrightarrow} 0$$

and the fraction of investing firms converges pointwise to the following limit:

$$\forall (\mu,\gamma), \quad n^{\sigma^f}(\mu,\gamma) = F^{\sigma^f}\left(f_c^{\sigma^f}(\mu,\gamma)\right) \underset{\sigma^f\to 0}{\longrightarrow} n^0(\mu,\gamma) \equiv \mathbb{1}\left(\omega^f \leq f_c^0(\mu,\gamma)\right).$$

Proof. This proof is similar to the argument developed in proposition 2. Since $n = p = F(f_c(\mu,\gamma))$, we use n and p interchangeably from now on and abuse notation in saying that $\mathcal{M}$ is a mapping for $n : \mathcal{N} \longrightarrow \mathcal{N}$. Pick two different variances for the fixed cost σ_1^f and σ_2^f. The notation $\mathcal{T}^{n,\sigma_i^f}$ denotes the mapping $\mathcal{T}$ for the value function G when n is the aggregate number of investing firms perceived by agents and the fixed costs are distributed according to $F^{\sigma_i^f}$.

Outline of the proof: Starting with the same initial aggregate law n, we compare the objects f_c^{n,σ_1^f} and f_c^{n,σ_2^f} after the first iteration of the mappings $\mathcal{M}^{\sigma_1^f}$ and $\mathcal{M}^{\sigma_2^f}$. In a second step, we establish a recursive relationship to compare the same objects after an arbitrary number of iterations. We then conclude that the limits of both contractions $n^{\sigma_i^f} = \lim_{k\to\infty} \left(\mathcal{M}^{\sigma_i^f}\right)^k n$ produce equilibrium cutoffs that are close in the following sense:

$$\| f_c^{n^{\sigma_2^f},\sigma_2^f} - f_c^{n^{\sigma_1^f},\sigma_1^f} \| \leq \overline{A}\left| \sigma_2^f - \sigma_1^f \right|$$

for some strictly positive constant $\overline{A}$, which suffices to establish the result.

Step 1. Start with some functions G and N, identical for both mappings. Denote $G_k^{n,\sigma_i^f} \equiv \left(\mathcal{T}^{n,\sigma_i^f}\right)^k G$. Let me prove by recursion that:

$$\left| \left(G_k^{n,\sigma_2^f} - G_k^{n,\sigma_1^f} \right)(\mu,\gamma,f) \right| \leq \beta \frac{1-\beta^k}{1-\beta} \left| \sigma_2^f - \sigma_1^f \right|.$$

This is trivially true for $k = 0$. Assume it is true for until $k \geq 0$, then:

$$\left| \left(G_{k+1}^{n,\sigma_2^f} - G_{k+1}^{n,\sigma_1^f} \right) (\mu,\gamma,f) \right|$$

$$\leq \beta \left| E \left\{ \left[S^{\sigma_2^f} \left(G_k^{n,\sigma_2^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} - E \left\{ \left[S^{\sigma_1^f} \left(G_n^{N,\sigma_1^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} \right|$$

$$\leq \beta \left| E \left\{ \left[S^{\sigma_2^f} \left(G_k^{n,\sigma_2^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} - E \left\{ \left[S^{\sigma_1^f} \left(G_n^{N,\sigma_2^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} \right|$$

$$+ \beta \left| E \left\{ \left[S^{\sigma_1^f} \left(G_k^{n,\sigma_2^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} - E \left\{ \left[S^{\sigma_1^f} \left(G_k^{n,\sigma_1^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} \right|$$

$$\leq \beta \left| \int \left(G_k^{n,\sigma_2^f} \left(\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma), \omega^f + \sigma_2^f \upsilon \right) \right. \right.$$

$$\left. \left. - G_k^{n,\sigma_2^f} \left(\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma), \omega^f + \sigma_1^f \upsilon \right) \right) d\Phi(\varepsilon) dF^1 \left(\upsilon + \omega^f \right) \right| + \beta \times \beta \frac{1 - \beta^k}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right|$$

$$\leq \beta \int \left| \sigma_2^f - \sigma_1^f \right| |\upsilon| dF^1 \left(\omega^f + \upsilon \right) + \beta^2 \frac{1 - \beta^k}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right| \quad \text{(Lipschitz of constant 1 in } f \text{)}$$

$$\leq \beta \left| \sigma_2^f - \sigma_1^f \right| + \beta^2 \frac{1 - \beta^k}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right| = \beta \frac{1 - \beta^{k+1}}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right|$$

which proves the recursion. Taking the limit $G^{n,\sigma_i^f} = \lim_{k \to \infty} \left(\mathcal{T}^{n,\sigma_i^f} \right)^k G$:

$$\| G^{n,\sigma_2^f} - G^{n,\sigma_1^f} \| \leq \frac{\beta}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right|. \tag{43}$$

Turning to the equilibrium cutoff rule and using the same argument:

$$\left| f_c^{n,\sigma_2^f}(\mu,\gamma) - f_c^{n,\sigma_1^f}(\mu,\gamma) \right|$$

$$= \beta \left| E \left\{ \left[S^{\sigma_2^f} \left(G^{n,\sigma_2^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} - E \left\{ \left[S^{\sigma_1^f} \left(G^{n,\sigma_1^f} \right) \right] (\mu + s(n,\gamma)\varepsilon, \Gamma(n,\gamma)) \right\} \right|$$

$$\leq \frac{\beta}{1 - \beta} \left| \sigma_2^f - \sigma_1^f \right| \tag{44}$$

Let us now consider the number of investing firms n. Denote $n_k^{\sigma_i^f} \equiv \left(\mathcal{M}^{\sigma_i^f} \right)^k n$, starting from the same arbitrary initial n.

$$\left| \left(n_1^{\sigma_2^f} - n_1^{\sigma_1^f} \right) (\mu,\gamma) \right| \leq \left| F^{\sigma_2^f} \left(f_c^{n,\sigma_2^f}(\mu,\gamma) \right) - F^{\sigma_1^f} \left(f_c^{n,\sigma_1^f}(\mu,\gamma) \right) \right|$$

$$\leq \left| F^{\sigma_2^f} \left(f_c^{n,\sigma_2^f}(\mu,\gamma) \right) - F^{\sigma_2^f} \left(f_c^{n,\sigma_1^f}(\mu,\gamma) \right) + F^{\sigma_2^f} \left(f_c^{n,\sigma_1^f}(\mu,\gamma) \right) - F^{\sigma_1^f} \left(f_c^{n,\sigma_1^f}(\mu,\gamma) \right) \right|$$

where we see that $n^{\sigma_2^f}$ may not always be close to $n^{\sigma_1^f}$ under the sup norm. The problem is that the above expression could be close to 1 for a few of points if σ_i^f is low and $f_c^{n,\sigma_2^f} \neq f_c^{n,\sigma_1^f}$. However, we now show that this is not a problem as they will be close on *average*. The only thing we need for the final result is pointwise convergence for n^{σ^f} as $\sigma^f \to 0$.

Step 2. We will now establish a recursive relationship to compare the two objects $f_c^{n_k^{\sigma_1^f}, \sigma_1^f}$ and $f_c^{n_k^{\sigma_2^f}, \sigma_2^f}$. Assume that after k iterations of the mapping $\mathcal{M}$, we have two different functions $n_k^{\sigma_2^f}$ and n_k

$n_k^{\sigma_1^f}$ and that

$$\forall\,(\mu,\gamma)\,,\quad \left| f_c^{n_k^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{n_k^{\sigma_1^f},\sigma_1^f}(\mu,\gamma) \right| \le A_k \left| \sigma_2^f - \sigma_1^f \right|.$$

Let us study the following term:

$$\left| \left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)(\mu,\gamma,f) \right|$$

$$\le \left| \left(G^{n_{k+1}^{\sigma_2^f},\sigma_2^f} - G^{n_{k+1}^{\sigma_2^f},\sigma_1^f} \right)(\mu,\gamma,f) \right| + \left| \left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)(\mu,\gamma,f) \right|$$

$$\le \frac{\beta}{1-\beta} \left| \sigma_2^f - \sigma_1^f \right| + \left| \left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)(\mu,\gamma,f) \right| \tag{45}$$

where we have controlled the first term by the same argument as in (43). We need to study the second term:

$$\left| \left(G^{n_{k+1}^{\sigma_2^f},\sigma_1^f} - G^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)(\mu,\gamma,f) \right|$$

$$= \left| \left(\lim_{m\to\infty} \left(\mathcal{T}^{n_{k+1}^{\sigma_2^f},\sigma_1^f} \right)^m G - \lim_{m\to\infty} \left(\mathcal{T}^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)^m G \right)(\mu,\gamma,f) \right|$$

$$= \left| \left(\lim_{m\to\infty} G_m^{n_{k+1}^{\sigma_2^f},\sigma_1^f} - \lim_{m\to\infty} G_m^{n_{k+1}^{\sigma_1^f},\sigma_1^f} \right)(\mu,\gamma,f) \right|.$$

Starting with the first iteration:

$$\left| \left(G_1^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_1^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right)(\mu, \gamma, f) \right|$$

$$\leq \beta \left| \int \left[G\left(\mu + s\left(n_{k+1}^{\sigma_2^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right.\right.$$
$$\left.\left. -G\left(\mu + s\left(n_{k+1}^{\sigma_1^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] d\Phi\left(\varepsilon \right) dF^{\sigma_1^f}\left(f' \right) \right|$$

$$\leq \beta \left| \int \left[G\left(\mu + s\left(n_{k+1}^{\sigma_2^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right.\right.$$
$$\left. -G\left(\mu + s\left(n_{k+1}^{\sigma_1^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] d\Phi\left(\varepsilon \right) dF^{\sigma_1^f}\left(f' \right)$$
$$+ \int \left[G\left(\mu + s\left(n_{k+1}^{\sigma_1^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right.$$
$$\left.\left. -G\left(\mu + s\left(n_{k+1}^{\sigma_1^f}, \gamma \right)\varepsilon, \Gamma\left(n_{k+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] d\Phi\left(\varepsilon \right) dF^{\sigma_1^f}\left(f' \right) \right|$$

$$\leq \beta \left[\int \overline{G}_\mu \left| \varepsilon \right| \left| s\left(n_{k+1}^{\sigma_2^f}, \gamma \right) - s\left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right| d\Phi\left(\varepsilon \right) dF^{\sigma_1^f}\left(f' \right) \right.$$
$$\left. + \int \overline{G}_\gamma \left| \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right) - \Gamma\left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right| d\Phi\left(\varepsilon \right) dF^{\sigma_1^f}\left(f' \right) \right]$$

$$\leq \beta \left[\overline{G}_\mu \left| s\left(n_{k+1}^{\sigma_2^f}, \gamma \right) - s\left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right| + \overline{G}_\gamma \left| \Gamma\left(n_{k+1}^{\sigma_2^f}, \gamma \right) - \Gamma\left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right| \right]$$

$$\leq \beta \gamma_x \left(\overline{G}_\mu B_s + \overline{G}_\gamma B_\Gamma \right) \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right|$$

$$\leq \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right|$$

where $C = B_s \overline{G}_\mu + B_\Gamma \overline{G}_\gamma$ is a constant similar to the one we used in proposition 2. We now establish recursively that for $m \geq 2$:

$$\left| \left(G_m^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right)(\mu, \gamma, f) \right| \leq \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right|$$
$$+ \beta^2 \frac{1 - \beta^{m-1}}{1 - \beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|$$

where constants C and D are those coming from lemma 4 below. Assuming the relationship is true until $m \geq 2$, we have:

$$\left| \left(G_{m+1}^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_{m+1}^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right)(\mu, \gamma, f) \right| \leq \beta \left| E \left[G_m^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right](\mu', \gamma', f') \right|$$

$$\leq \beta \left| \int \left[G_m^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right. \right.$$

$$\left. \left. - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f}(f') \right|$$

$$\leq \beta \left| \int \left[G_m^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right. \right.$$

$$\left. \left. - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f}(f') \right|$$

$$+ \beta \left| \int \left[G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) \right. \right.$$

$$\left. \left. - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] d\Phi(\varepsilon) \, dF^{\sigma_1^f}(f') \right|$$

$$\leq \beta \int \left| \left(G_m^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right), f' \right) d\Phi(\varepsilon) \, dF^{\sigma_1^f}(f') \right|$$

$$+ \beta \int \left| \frac{\partial s}{\partial n} \varepsilon \frac{\partial G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f}}{\partial \mu} + \frac{\partial \Gamma}{\partial n} \frac{\partial G_m^{n_{k+1}^{\sigma_1^f}, \sigma_1^f}}{\partial \gamma} \right| \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right| d\Phi(\varepsilon) \, dF^{\sigma_1^f}(f')$$

$$\leq \beta \left(\beta C \gamma_x \int \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right) \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \right) \right| d\Phi(\varepsilon) \right.$$

$$\left. + \beta^2 \frac{1 - \beta^{m-1}}{1 - \beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right| \right)$$

$$+ \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right|$$

Using lemma 4, we can control the term:

$$\int \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right) \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \right) \right| d\Phi(\varepsilon) \leq \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

Therefore:

$$\left| \left(G_{m+1}^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G_{m+1}^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right)(\mu, \gamma, f) \right|$$

$$\leq \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right| + \beta^2 \frac{1 - \beta^m}{1 - \beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|$$

which establishes the recursion. Taking the limit as $m \to \infty$:

$$\left| \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right)(\mu, \gamma, f) \right|$$

$$\leq \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right)(\mu, \gamma) \right| + \frac{\beta^2}{1 - \beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|. \tag{46}$$

We see that G may not converge pointwise. However, the expectation of G will, which is what we need for our final result. Going back to equation (45):

$$\left| \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_2^f} - G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right|$$

$$\leq \left| \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_2^f} - G^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| + \left| \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_1^f} - G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right|$$

$$\leq \frac{\beta}{1-\beta} \left| \sigma_2^f - \sigma_1^f \right| + \beta C \gamma_x \left| \left(n_{k+1}^{\sigma_2^f} - n_{k+1}^{\sigma_1^f} \right) (\mu, \gamma) \right| + \frac{\beta^2}{1-\beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

where we have used equations (43) and (46). Let us turn to the cutoff value:

$$\left| f_c^{n_{k+1}^{\sigma_2^f}, \sigma_2^f} (\mu, \gamma) - f_c^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} (\mu, \gamma) \right|$$

$$= \beta \left[\mathbb{E} \left\{ \left[S^{\sigma_2^f} \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_2^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \right) \right\} \right.$$

$$\left. - \mathbb{E} \left\{ \left[S^{\sigma_1^f} \left(G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right) \right\} \right]$$

$$\leq \beta \left| \mathbb{E} \left\{ \left[S^{\sigma_2^f} \left(G^{n_{k+1}^{\sigma_2^f}, \sigma_2^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_2^f}, \gamma \right) \right) \right\} \right.$$

$$\left. - \mathbb{E} \left\{ \left[S^{\sigma_2^f} \left(G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right) \right\} \right|$$

$$+ \beta \left| \mathbb{E} \left\{ \left[S^{\sigma_2^f} \left(G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right) \right\} \right.$$

$$\left. - \mathbb{E} \left\{ \left[S^{\sigma_1^f} \left(G^{n_{k+1}^{\sigma_1^f}, \sigma_1^f} \right) \right] \left(\mu + s \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma \left(n_{k+1}^{\sigma_1^f}, \gamma \right) \right) \right\} \right|$$

$$\leq \beta \left(\frac{\beta}{1-\beta} + \beta C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right|$$

$$+ \beta \left| \sigma_2^f - \sigma_1^f \right|$$

$$\leq \beta \left(\frac{1}{1-\beta} + \frac{\beta}{1-\beta} C \gamma_x \left(A_k D + E \sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right|$$

$$\leq \underbrace{\left[\frac{\beta}{1-\beta} \left(1 + \beta \gamma_x C E \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x C D A_k \right]}_{\equiv A_{k+1}} \left| \sigma_2^f - \sigma_1^f \right|$$

This expression defines a recursive relationship:

$$A_{k+1} = \frac{\beta}{1-\beta} \left(1 + \beta \gamma_x C E \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x C D A_k$$

which converges to a unique limit $\overline{A}$ as long as $\frac{\beta^2}{1-\beta} \gamma_x C D < 1$ which is true if γ_x is chosen sufficiently small. Taking the limit as $k \to \infty$, we have:

$$\left| f_c^{n_{\sigma_2^f}^{\sigma_2^f}, \sigma_2^f} (\mu, \gamma) - f_c^{n_{\sigma_1^f}^{\sigma_1^f}, \sigma_1^f} (\mu, \gamma) \right| = \left| f_c^{\lim_{k\to\infty} n_k^{\sigma_2^f}, \sigma_2^f} (\mu, \gamma) - f_c^{\lim_{k\to\infty} n_k^{\sigma_1^f}, \sigma_1^f} (\mu, \gamma) \right|$$

$$\leq \overline{A} \left| \sigma_2^f - \sigma_1^f \right|.$$

61

This tells us that as $\sigma^f \to 0$, the equilibrium cutoff converges uniformly to some limit:

$$\forall (\mu, \gamma), \quad f_c^{n^{\sigma^f}, \sigma^f} (\mu, \gamma) \to f_c^0 (\mu, \gamma).$$

Turning to the equilibrium entry schedule, n converges pointwise towards the limit:

$$\forall (\mu, \gamma), \quad n^{\sigma^f} (\mu, \gamma) = F^{\sigma^f} \left(f_c^{n^{\sigma^f}, \sigma^f} (\mu, \gamma) \right) \xrightarrow[\sigma^f \to 0]{} n^0 (\mu, \gamma) = \mathbb{1} \left(\omega^f \le f_c^0 (\mu, \gamma) \right).$$

$\square$

Lemma 4. *Suppose two functions f_1 and f_2 are such that $\sup |f_2 (\mu, \gamma) - f_1 (\mu, \gamma)| \le A \left| \sigma_2^f - \sigma_1^f \right|$ for some strictly positive constant A. Assume also that both f_i's are continuously differentiable and that $\frac{\partial f_i}{\partial \mu} > \underline{f}_\mu$. Then, for $n^i = F^{\sigma_i^f} (f_i)$, there exists two strictly positive constants D and E such that for $i = 1, 2$:*

$$\int \left| (n^2 - n^1) \left(\mu + s (n^i, \gamma) \varepsilon, \Gamma (n^i, \gamma) \right) \right| d\Phi (\varepsilon) \le \left(AD + E \sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

Proof. Abusing notation slightly with the convention $s \equiv s \left(n_1^{\sigma_i^f}, \gamma \right)$ and $\gamma^{'} \equiv \Gamma \left(n_1^{\sigma_i^f}, \gamma \right)$:

$$\int \left| (n^2 - n^1) \left(\mu + s (n^i, \gamma) \varepsilon, \Gamma (n^i, \gamma) \right) \right| d\Phi (\varepsilon)$$

$$= \int \left| F^{\sigma_2^f} \left(f_2 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) - F^{\sigma_1^f} \left(f_1 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) \right| d\Phi (\varepsilon)$$

$$\le \underbrace{\int \left| F^{\sigma_2^f} \left(f_2 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) - F^{\sigma_2^f} \left(f_1 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) \right| d\Phi (\varepsilon)}_{\equiv A_1}$$

$$+ \underbrace{\int \left| F^{\sigma_2^f} \left(f_1 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) - F^{\sigma_1^f} \left(f_1 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) \right| d\Phi (\varepsilon)}_{\equiv A_2}$$

Let us take care of the first term:

$$A_1 = \int \left| F^{\sigma_2^f} \left(f_2 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) - F^{\sigma_2^f} \left(f_1 \left(\mu + s\varepsilon, \gamma^{'} \right) \right) \right| d\Phi (\varepsilon)$$

$$\le \int \left[F^{\sigma_2^f} \left(f_2 \left(\mu + s\varepsilon, \gamma^{'} \right) + A \left| \sigma_2^f - \sigma_1^f \right| \right) - F^{\sigma_2^f} \left(f_2 \left(\mu + s\varepsilon, \gamma^{'} \right) - A \left| \sigma_2^f - \sigma_1^f \right| \right) \right] d\Phi (\varepsilon)$$

using equation (44). f_2 is a continuously differentiable, strictly increasing function of μ, so we can

proceed to the change of variable $x = f_2\left(\mu + s\varepsilon, \gamma'\right)$:

$$
\begin{aligned}
A_1 &\leq \int \left[F^{\sigma_2^f}\left(x + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(x - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] \underbrace{\frac{\Phi'\left((f_2)^{-1}(x)\right) dx}{s \cdot (f_2)'\left((f_2)^{-1}(x)\right)}}_{\equiv d\varphi(x)} \\
&\leq \int_{x=-\infty}^{\infty} \int_{f=x-A\left|\sigma_2^f-\sigma_1^f\right|}^{x+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \leq \int_{f=-\infty}^{\infty} \int_{x=f-A\left|\sigma_2^f-\sigma_1^f\right|}^{f+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f} d\varphi(x) \\
&\leq \int_{f=-\infty}^{\infty} \left[\varphi\left(f + A\left|\sigma_2^f - \sigma_1^f\right|\right) - \varphi\left(f - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right] dF^{\sigma_2^f} \\
&\leq \int_{f=-\infty}^{\infty} \left[\varphi'\left(\tilde{f}\right) 2A\left|\sigma_2^f - \sigma_1^f\right|\right] dF^{\sigma_2^f} \quad \text{(mean value theorem)} \\
&\leq 2A\left|\sigma_2^f - \sigma_1^f\right| \int_{f=-\infty}^{\infty} \left|\varphi'\left(\tilde{f}\right)\right| dF^{\sigma_2^f} \\
&\leq A \cdot 2 \frac{\sup|\Phi'|}{\inf\left|s \cdot (f_2)'\right|} \left|\sigma_2^f - \sigma_1^f\right| \equiv AC\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}
$$

where we have used the fact the PDF of a unit normal is bounded, $s \equiv s\left(n_1^{\sigma_i^f}, \gamma\right)$ is uniformly bounded from below and away from 0, and the derivative of f_2 is strictly positive, uniformly bounded away from 0 for γ_x small enough. Notice that the upper bound we derived is uniform: it does not depend on μ, γ, γ_x, etc. Let us control the second term A_2:

$$
\begin{aligned}
A_2 &= \int \left|F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right)\right| d\Phi(\varepsilon) \\
&\leq \int \left|F^{\sigma_2^f}(x) - F^{\sigma_1^f}(x)\right| d\varphi(x) \quad \text{(change of variable } x = f_1\left(\mu + s\varepsilon, \gamma'\right)) \\
&\leq \int \left|\Phi\left(\frac{x - \omega^f}{\sigma_2^f}\right) - \Phi\left(\frac{x - \omega^f}{\sigma_1^f}\right)\right| d\varphi(x) \quad \text{(change of variable } x = \sigma_1^f \sigma_2^f \tilde{x} + \omega^f) \\
&\leq \int \left|\Phi\left(\sigma_1^f \tilde{x}\right) - \Phi\left(\sigma_2^f \tilde{x}\right)\right| \sigma_1^f \sigma_2^f d\varphi\left(\sigma_1^f \sigma_2^f \tilde{x} + \omega^f\right) \\
&\leq \left[\int \left|\Phi'\left(\hat{x}\right) \tilde{x}\right| d\varphi\left(\sigma_1^f \sigma_2^f \tilde{x} + \omega^f\right)\right] \sigma_1^f \sigma_2^f \left|\sigma_2^f - \sigma_1^f\right| \equiv D\sigma_1^f \sigma_2^f \left|\sigma_2^f - \sigma_1^f\right|,
\end{aligned}
$$

which concludes the proof of the lemma. $\qquad\square$