A Market Microstructure Theory
of the Term Structure of Asset Returns

ALBERT S. KYLE, ANNA A. OBIZHAEVA, AND YAJUN WANG[*]

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We formulate a theory of expected returns using a dynamic market microstructure model of speculative trading among oligopolistic imperfectly competitive traders, who agree to disagree about precision of private information. The equilibrium returns depend not only on the parameters used by the market but also the true parameters. When the former parameters are incorrect, the returns are usually predictable. Even when traders apply Bayes law consistently, the incorrect parameters of the market can be a result of the information aggregation process itself. Our structural model for equilibrium returns relates them to the history of dividends and the history of dividend-to-price ratios. For some parameters, the implied returns exhibit short-run momentum and long-run mean-reversion.

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* Kyle: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, akyle@rhsmith.umd.edu. Obizhaeva: New Economics School, Moscow, Russia, obizhaeva@nes.ru. Wang: Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, ywang22@rhsmith.umd.edu.

Introduction

This paper examines empirical implications of a dynamic theoretical model of speculative trading, in which trading is motivated by both differences in private information and differences in beliefs. The derived structural model imposes economic restrictions on the term structure of expected returns but yet it is sufficiently flexible to be consistent with the documented patterns of medium-term momentum and long-term mean-reversion, see Jegadeesh and Titman (1993) as well as DeBondt and Thaler (1985) for earlier references.

We formulate a theory of expected returns in the context of smooth trading model of Kyle, Obizhaeva and Wang (2013). In that dynamic model of speculative trading among oligopolistic traders, traders agree to disagree about the precision of their private information about unobservable growth rate of the risky asset. When there is enough disagreement among traders, the equilibrium exists in which traders trade on their information smoothly, balancing the incentives to slow down trading and reduce market impact costs versus the incentives to speed up and profit from perishable information.

Strictly speaking, it is impossible to make empirical predictions about expected returns based on such a model, because the players in the model might have beliefs about parameters which are incorrect in any arbitrary manner. If traders have incorrect beliefs about model parameters, then traders predictions about expected returns "in the model" are likely to be incorrect. This may explain some asset pricing puzzles, as we illustrate using several stylized examples with a representative agent who has incorrect beliefs.

We therefore have to introduce the "true" subjective beliefs. We suppose there is an external observer or an economist who studies the equilibrium prices. He believes in his own "true" parameters of the model. The equilibrium returns then depend both on the traders' parameters that determine prices and the economist's parameters that define dynamics.

We show that expected returns depend on the history of dividend-to-price ratios and the history of dividend surprises. If the economist agrees with traders on the total precision of signals, then expected returns depend only on the current dividend-to-price ratio and the history of dividend surprises. These implications can be thought of as specific economic restrictions that can be imposed on the present-value approach by Campbell and Shiller (1988).

According to Samuelson (1965) and the efficient market hypothesis popularized by Fama (1970), the prices are expected to follow a martingale. Except for rare coincidences, however, the economist in our framework will usually find anomalies and predictable returns. The intuition is as follows. Each trader's valuation is a martingale under his own beliefs. The evolution of traders' valuations is complicated because they learn about the common value of a growth rate. The market aggregates those correlated valuations into the single market price that, as a result, does not follow a martingale under any beliefs. The average of martingales is usually not a martingale.

It is fashionable to attribute predictability in asset returns to irrational behavior motivated by psychology. This presumes that rational behavior not motivated by psychology would lead to no returns predictability. Our paper shows that predictable returns may result from rational behavior if markets aggregate information in a manner that results in predictability.

Information aggregation can be implemented through a construct of a representative agent. In our framework a representative agent exists. To replicate the price dynamics and ensure consistency with the Bayesian learning, however, one would need to assign him beliefs that are different from beliefs prevailing in the market, even with respect to parameters the traders themselves may agree with each other about. Thus, even in the world where traders behave rationally, it may be necessary to attribute unusual values to the parameters when calibrating the equivalent representative-agent frameworks.

One could find the anomalies being more in a line with behavior finance paradigm. In our framework, however, even though the traders are overconfident about their own signals, they rationally maximize utility functions and correctly apply Bayes law. Fama (1998) criticizes behavior finance for its inability to formulate a consistent alternative to market efficiency that would explain both underreaction and overreaction patterns, which must therefore instead be attributed to a chance.

Every scientific theory has to be formulated within some framework describing the subject of interest and also making it possible to come up with alternatives and derive their predictions. A proposed theory should be described in sufficiently precise terms and predicts sufficiently improbable outcomes in contrast to those implied by alternatives. For example, Muth (1961) describes the theory of rational expectations equilibrium in the context of a simple production economy and contrasts it with alternative theories of adaptive expectations. In a similar spirit, we formulate a theory of expected returns in the context of smooth trading model of Kyle, Obizhaeva and Wang (2013). Our structural approach can be thought of as a constructive synthesis of the efficient market and behavior finance paradigms.

We illustrate our ideas using the smooth trading model developed in Kyle, Obizhaeva and Wang (2013), where trading among traders is generated by overconfidence. There is ample evidence that market participants have heterogeneous beliefs, as reflected, for example, in dispersion of forecasts by professional analysts. However, the proposed methodology can also be applied in the context of other models where trading is motivated either by private values as in Du and Zhu (2013) or endowment shocks as in Vayanos (1999).

The idea of studying the term structure of returns using theoretical modeling was pioneered by Cox, Ingersoll and Ross (1985), who examine the interest rates and bond prices in the context of the general equilibrium model with rational expectations. In their model, identical competitive individuals optimally decide on their consumption path and allocation to risky projects giving their expectations about changing investment opportunities. There are no effects of heterogeneous information or heterogeneous beliefs, which are the main subject of our paper.

Our approach is similar to Xiong and Yan (2010) who discuss how to make empirical predictions about bond returns using the dynamic competitive model. In their model, agents observe the same signal about the long-run mean of the inflation rate but interpret it differently due to their heterogeneous beliefs. The paper primary focuses on the effects of fluctuating relative wealth and shuts down the effects of dynamic learning, since the later can be presumably captured using a representative-agent framework; different from our paper, however, the belief of the constructed representative agent does not follow the Bayes law. In contrast, we shut down the wealth effect by using the CARA utility functions and focus on the learning effect; in addition to heterogeneous beliefs, we also incorporate private information and imperfect competition. Our paper complements the growing literature on equilibrium implications of heterogeneity of market participants.

In modeling the term structure of expected returns, we place a predominant emphasis on the expectations and abstract from modeling other components of the term structure. One of those may be related to differences in risk premium for exposure to various factors at different horizons. It is known, for example, that market beta is sensitive to return interval used to estimate it; market betas of small stocks tend to increase with the horizon, while market betas of large stocks tend to decrease with the horizon, see Levhari and Levy (1977) for the earlier discussion. Another component may be related to liquidity premium, which can be calibrated based on market microstructure invariance proposed by Kyle and Obizhaeva (2013).

This paper is structured as follows. Section I discusses several stylized examples in which incorrect beliefs of a representative agent may explain some known empirical facts in asset pricing. Section II presents a formal model of dynamic information processing by a representative agent with similar implications. Section III explains that incorrect beliefs of a representative agent may arise as a result of information aggregation in the model with heterogeneous agents and derives a structural model for equilibrium returns. Section IV concludes.

I.   Motivating Examples

We first present three simple examples which illustrate several principles which are important in this paper.

- The actual return process depends on two sets of parameters: correct parameters and possibly incorrect parameters used by the market.

- The possibly incorrect parameters used by the market can affect expected return, return volatility, and the entire term structure of expected returns.

- It is usually more appropriate to model financial markets using dynamic steady-state models, because insights of static non-stationary models often can not be easily mapped into real data.

While none of these examples correspond precisely to the model examined in the paper, they are very helpful for understanding the main point of our paper.

Gordon's Growth Model With Geometric Brownian Motion Dividends.

The simplest illustration assumes that investors use a possibly incorrect dividend growth rate when applying Gordon's growth formula to an asset whose dividend follows a geometric Brownian motion. Suppose that dividends follow the geometric motion process

$$(1) \qquad dD(t) = \gamma \cdot D(t) \cdot dt + \sigma \cdot D(t) \cdot dB(t),$$

where γ is the growth rate investors expect and σ measures the volatility of dividends. Suppose that investors require expected return r. Then, the asset's price is determined by Gordon's growth formula

$$(2) \qquad P(t) = \frac{D(t)}{r - \gamma}.$$

The actual percentage return process is

$$(3) \qquad \frac{dP(t) + D(t) \cdot dt}{P(t)} = r \cdot dt + \sigma \cdot dB(t).$$

Investors expect a return of r, which can be decomposed into a return of $r - \gamma$ from dividend yield $D(t)/P(t) \cdot dt$ and expected return of γ from capital gain $dP(t)/P(t)$.

Suppose the true growth rate in equation (1) is $\hat{\gamma}$, not γ. Then, contrary to investors' expectations if $\gamma \neq \hat{\gamma}$, investors obtain an expected return of $r - \gamma + \hat{\gamma}$; the observed dividend yield of $r - \gamma$ remains unchanged, but the unobserved expected return from capital gains changes from γ to $\hat{\gamma}$. Let $\hat{E}_t\{\ldots\}$ denote an expectation operator calculated using information available at time t based on true beliefs. Then, the actual expected return is given by

$$(4) \qquad \hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = (r - \gamma + \hat{\gamma}) \cdot P(t).$$

A higher expected rate of growth rate γ lowers dividend yield and therefore lowers expected returns. Note that the actual expected return $r - \gamma + \hat{\gamma}$ depends on two parameters, the investors' expected growth rate γ and the true expected growth rate $\hat{\gamma}$. In a steady state with unchanging required return r, unchanging actual growth rate $\hat{\gamma}$ and unchanging expected growth rate γ, expected returns are constant $r - \gamma + \hat{\gamma}$, not time-varying.

The volatility of returns σ is the "Black-Scholes volatility".[1] Incorrect expectations about the growth rate of dividends have no effect on returns volatility.

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[1] We refer to a constant standard deviation of log-returns as a "Black-Scholes volatility." In the

In this example, expected returns are constant over time and the volatility of returns is not affected by investors' expectations of the growth rate. In the next example, expected returns vary over time, and the constant standard deviation of dollar returns changes in response to changes in investors' beliefs about parameters governing the dividend process.

Excess Volatility and Mean Reversion Model With Arithmetic AR-1 Dividends.

Suppose investors believe that de-meaned dividends follow an Ornstein-Ühlenbeck process given by

$$(5) \qquad dD(t) = -\alpha \cdot (D(t) - \bar{D}) \cdot dt + \sigma \cdot dB(t),$$

where α is a constant rate of mean reversion, σ measures the volatility of dividends, $\bar{D}$ is the constant steady state mean dividend level, and $B(t)$ is a standardized Brownian motion process (zero mean and unit variance). Let r denote the required return which equals the risk free rate for a zero-net supply asset. Then the asset's price $P(t)$ is

$$(6) \qquad P(t) = \frac{\bar{D}}{r} + \frac{D(t) - \bar{D}}{r + \alpha}.$$

This formula is obtained by applying Gordon's growth formula to the two components $\bar{D}$ and $D(t) - \bar{D}$, with growth rates of zero and $-\alpha$, respectively.

Suppose that investors' beliefs about the mean reversion parameter in equation (5) are possibly incorrect, with a correct value of the mean reversion parameter equal to $\hat{\alpha}$, not α. The actual returns process (in dollars per share) is given by[2]

$$(7) \quad dP(t) + D(t) \cdot dt = r \cdot P(t) \cdot dt + \frac{\alpha - \hat{\alpha}}{r + \alpha} \cdot (D(t) - \bar{D}) \cdot dt + \frac{\sigma}{r + \alpha} \cdot dB(t).$$

Let $\hat{E}_t\{\ldots\}$ denote an expectation operator calculated using information available at time t based on true beliefs. Then, the actual expected return is given

Black-Scholes model of option pricing, both the actual volatility and the implied volatility have the same value σ, where $\sigma \Delta t^{1/2}$ measures the percentage standard deviation of log-returns over a period Δt. When the stock price and therefore returns are observed continuously, the variance of returns σ^2 can be inferred with arbitrarily high accuracy from the realized variance of log-returns $\sum_{n=1}^{N} (\ln[P(t_{n+1})/P(t_n)])^2 / \Delta t$ over an arbitrary time interval Δt with $t_n = t_0 + n\Delta t/N$. The realized variance is a consistent estimator of σ^2 for large N.

[2]For reasons of analytical tractability, this paper uses arithmetic AR-1 processes rather than geometric brownian motion. With arithmetic processes, the standard deviation of dividends over a given time period is constant when measured in units of dollars per share, and it is appropriate to measure expected returns in units of dollars per share per unit of time. With geometric brownian motion, the volatility of dividends over a given time period is constant when measured as a percentage of the amount invested, and it is appropriate to measure expected returns in units of percent of amount invested per unit of time.

by

$$(8) \qquad \hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = r \cdot P(t) + \frac{\alpha - \hat{\alpha}}{r + \alpha} \cdot (D(t) - \bar{D}).$$

Investors obtain their expected return $r \cdot P(t)$ when their beliefs are correct and $\alpha = \hat{\alpha}$; otherwise, investors also obtain an unexpected excess return $\frac{\alpha - \hat{\alpha}}{r + \alpha} \cdot (D(t) - \bar{D})$ which varies randomly over time, depending on whether the investors' mean reversion parameter is too high or too low and depending on whether dividends are above or below their long-run mean.

Investors' beliefs also affect the volatility of returns. The standard deviation of dollar returns per share is

$$(9) \qquad \hat{Var}_t^{1/2} \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = \sigma \cdot (r + \alpha)^{-1}.$$

The volatility of returns depends only on the investors' mean reversion parameter α, not on the true mean reversion parameter $\hat{\alpha}$.

If investors believe that the dividend process is more persistent than it actually is, *i.e.*, $\alpha < \hat{\alpha}$, then there is excess volatility and mean reversion. There is excess volatility because the actual volatility $\sigma \cdot (r + \alpha)^{-1}$ is greater than the correct volatility $\sigma \cdot (r + \hat{\alpha})^{-1}$. There is mean reversion because the expected excess return $\frac{\alpha - \hat{\alpha}}{r + \alpha} \cdot (D(t) - \bar{D})$ is negative (positive) when dividends and therefore prices are above (below) their long term mean. In this example, it can be shown that the entire term structure of expected returns varies over time as well.

One-Period Model With Information Processing.

Information processing is an essential function of financial markets. When investors attribute incorrect precisions to the information flow, this may affect the returns process and lead to returns predictability.

Consider the following one-period example (in the spirit of the multi-period model of Daniel, Hirshleifer and Subrahmanyam (1998)). A risky asset has an unobserved liquidation value denoted v. Investors observe a signal denoted ΔI and believe that the signal has the form $\Delta I = \tau^{1/2} v + z$, where τ is the investors' possibly incorrect belief about a parameter governing the precision of the signal. The random variables v and z are identically and independently distributed as $N(0, 1)$. The initial price P_0 is normalized to zero at time $t = 0$. Upon observation of the signal at time $t = 1$, the investors' expectation of the asset's liquidation value changes to P_1. At time 2, the liquidation value v is realized.

The true value $\hat{\tau}$ of the precision parameter is possibly different from the investors' belief τ. Let $\hat{E}\{\ldots\}$ and $\hat{Var}\{\ldots\}$ be expectation and variance under true beliefs.

The two periods in this simple model are quite different. Assuming no discounting, the expected returns and price volatility over the period from $t = 0$ to $t = 1$

are given by

$$\hat{E}\{P_1 - P_0|\Delta I\} = \frac{\tau^{1/2}}{1+\tau}\Delta I, \qquad \hat{Var}^{1/2}\{P_1 - P_0\} = \frac{\tau^{1/2}(1+\hat{\tau})^{1/2}}{1+\tau}.$$

The expected returns and price volatility over the period from $t = 1$ to $t = 2$ are

$$\hat{E}\{v - P_1|\Delta I\} = \left(\frac{\hat{\tau}^{1/2}}{1+\hat{\tau}} - \frac{\tau^{1/2}}{1+\tau}\right)\Delta I, \qquad \hat{Var}^{1/2}\{v - P_1\} = \frac{((1+\tau-\hat{\tau}^{1/2}\tau^{1/2})^2 + \tau)^{1/2}}{1+\tau}.$$

If investors have correct beliefs, *i.e.*, $\tau = \hat{\tau}$, then expected return for the second period is equal to zero; returns variances during two periods are equal to the true expected volatility of $\hat{\tau} \cdot (1+\hat{\tau})^{-1}$ and $(1+\hat{\tau})^{-1}$, respectively. Otherwise, various patterns in expected returns and volatility are possible depending on a particular choice of parameters.

The predictions are quite different for two periods. For example, the first-period volatility may be lower or higher than the second-period volatility depending on parameters. This makes it difficult to use one-period models to derive realistic implications for continuously operating financial markets. Similar concerns are relevant for any non-stationary model.

Summary of Motivating Examples.

The three motivating examples are all based on modeling market prices as the result of a single representative agent processing information.

The first example shows that overly pessimistic beliefs about the growth rate of dividends lead to a higher expected return, thus providing an explanation for the equity premium puzzle of Mehra and Prescott (1985). If investors' beliefs are too pessimistic in recessions and too optimistic in booms, then risk premium is counter-cyclical, as shown by Campbell and Shiller (1988) and Fama and French (1989).

The second example shows that a belief that mean-reverting dividends are more persistent than the actual rate of mean reversion leads to excess volatility and mean reversion in asset prices, consistent with Shiller (1981).

The third example shows that overconfidence about the precisions of signals can lead to excess volatility and mean reversion. Its intrinsic limitation as a one period model reminds us that dynamic steady-state models are more appropriate tools for studying dynamic properties of returns.

The motivating examples do not illustrate how overconfident processing of private information works in a dynamic context, nor do the examples show how information aggregation in prices results from the dynamic trading decisions of individual market participants.

In the rest of this paper we extend these examples in two steps. First, we construct a model of continuous information processing which shows how predictability of returns arises when prices are set by a representative agent with possibly incorrect beliefs about the precision of privately observed signals.

Second, we use the dynamic smooth trading model of Kyle, Obizhaeva and Wang (2013) as the micro-foundations for modeling the behavior of individual investors and show that the incorrect beliefs of the representative agent may naturally arise in the equilibrium due to specific properties of information aggregation.

II.   Dynamic Model of Information Processing

In this section, we construct a dynamic model in which a representative agent observes "private" signals (not observed by the economist), processes this information according to possibly incorrect beliefs, and sets market prices to make the expected return on the risky asset equal to the risk-free rate. In subsequent sections, we replace the representative agent with heterogeneous agents with different beliefs.

In what follows, we mark with "breves" (" $\breve{}$ ") parameters assigned by the market's representative agent.

Suppose a risky asset continuously pays out dividends $D(t)$, self-liquidating itself over time. Dividends follow a stochastic process with mean-reverting stochastic growth rate $G^*(t)$, constant instantaneous volatility $\sigma_D > 0$, and constant rate of mean reversion $\alpha_D > 0$,

$$(10) \qquad dD(t) := -\alpha_D \cdot D(t) \cdot dt + G^*(t) \cdot dt + \sigma_D \cdot dB_D(t).$$

The growth rate $G^*(t)$ follows an AR-1 process with the mean-reversion $\breve{\alpha}_G$ and volatility $\breve{\sigma}_G$:

$$(11) \qquad dG^*(t) := -\breve{\alpha}_G \cdot G^*(t) \cdot dt + \breve{\sigma}_G \cdot dB_G(t).$$

For simplicity, we assume a zero-net-supply asset discounted at a fixed risk-free rate r. If both the dividend $D(t)$ and $G^*(t)$ were observable, then the price of the asset would equal its fundamental value given by the generalization of Gordon's growth formula,

$$(12) \qquad F(t) = \frac{D(t)}{r + \alpha_D} + \frac{G^*(t)}{(r + \alpha_D)(r + \breve{\alpha}_G)}.$$

The market also gets signals about the growth rate $G^*(t)$ and impounds this information into prices.

Let $\breve{E}_t\{\ldots\}$ and $\breve{Var}_t\{\ldots\}$ denote the market's expectations and variances calculated with respect to information at time t. Then the market price is obtained by substituting the market's estimate of $G^*(t)$ for $G^*(t)$ itself into equation (12):

$$(13) \qquad P(t) = \frac{D(t)}{r + \alpha_D} + \frac{\breve{E}_t\{G^*(t)\}}{(r + \alpha_D)(r + \breve{\alpha}_G)}.$$

This formula generalizes Gordon's growth formula (6) in the second example by replacing the constant dividend mean $\bar{D}$ with a stochastically time-vary dividend level $D(t)$.

The market price aggregates the information content of the divided $D(t)$ and N signals $I_1(t), \ldots, I_N(t)$. We assume that each signal $I_n(t)$ produces a continuous stream of information given by

$$(14) \qquad dI_n(t) := \breve{\tau}_n^{1/2} \cdot \frac{G^*(t)}{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}} \cdot dt + dB_n(t), \quad n = 1, \ldots, N,$$

where $dB_D, dB_G, dB_1, \ldots dB_N$ are independent Brownian motions. Since its drift is proportional to $G^*(t)$, each increment $dI_n(t)$ in the process $I_n(t)$ is a noisy observation of the unobserved growth rate $G^*(t)$. This is a convenient way to model information flow, because the "precision" parameter $\breve{\tau}_n$ measures the informativeness of the signal $dI_n(t)$ as a signal-to-noise ratio describing how fast the information flow generates a signal of a given level of statistical significance. It also corresponds to the R squared of the predictive regression. Kyle, Obizhaeva and Wang (2013) discuss this choice for modeling information in more detail.

The parameter $\breve{\Omega}$ is a scaling coefficient in equation (14). It denotes the steady state error variance of the estimate of $G^*(t)$, scaled in units of the standard deviation of its innovation $\breve{\sigma}_G$:

$$(15) \qquad \breve{\Omega} := \breve{Var}\{(G^*(t) - \breve{G}(t))/\breve{\sigma}_G\},$$

where $\breve{G}(t)$ is the market's estimate of the growth rate as defined in (23). If time is measured in years, $\breve{\Omega} = 4$ means that the estimate of $G^*(t)$ is "behind" the true value of $G^*(t)$ by an amount equivalent to four years of volatility unfolding at rate $\breve{\sigma}_G$ per year. We also assume a steady state in which $\breve{\Omega}$ is a constant.

Dividends contain information about the dividend growth. Define $dI_0(t) := [\alpha_D \cdot D(t) \cdot dt + dD(t)]/\sigma_D$, $dB_0 := dB_D$, and

$$(16) \qquad \breve{\tau}_0 := \breve{\Omega} \cdot \breve{\sigma}_G^2/\sigma_D^2$$

Then, the information $I_0(t)$ about growth rate in the divided stream (10) can be written with notation similar to $I_n(t)$:

$$(17) \qquad dI_0(t) := \breve{\tau}_0^{1/2} \cdot \frac{G^*(t)}{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}} \cdot dt + dB_0(t).$$

Observing the process $I_0(t)$ is informationally equivalent to observing the dividend process. The quantity $\breve{\tau}_0$ measures the precision of the dividend process in units analogous to the units of precision for other signals.

Let $\breve{\tau}$ be the total precision of information equal to the sum of all precision parameters:

$$(18) \qquad \breve{\tau} := \breve{\tau}_0 + \sum_{n=1}^{N} \breve{\tau}_n.$$

Stratonovich-Kalman-Bucy filtering implies that the steady state error variance is given by

$$(19) \qquad \breve{\Omega}^{-1} := 2 \cdot \breve{\alpha}_G + \breve{\tau}.$$

As discussed in Kyle, Obizhaeva and Wang (2013), the history of each information flow $I_n(t)$ can be summarized by a sufficient statistic $\breve{H}_n(t)$ or a signal defined as

$$(20) \qquad \breve{H}_n(t) := \int_{u=-\infty}^{t} e^{-(\breve{\alpha}_G + \breve{\tau}) \cdot (t-u)} \cdot dI_n(u), \quad n = 0, 1, ...N.$$

The importance of each bit of information dI_n about the growth rate decays exponentially at a rate $\breve{\alpha}_G + \breve{\tau}$, which depends on the natural mean-reversion rate of the growth rate and the speed of learning in the economy. If all signals are reflected in the market price, then Stratonovich-Kalman-Bucy filtering implies

$$(21) \qquad \breve{E}_t\{G^*(t)\} := \breve{\sigma}_G \cdot \breve{\Omega}^{1/2} \cdot \left(\sum_{n=0}^{N} \breve{\tau}_n^{1/2} \cdot \breve{H}_n(t) \right).$$

It is natural to assume that the market assigns the same precision to all signals. Let $\breve{\tau}_n = \breve{\tau}_I$ for $n = 1, ..N$. Define the aggregated sufficient statistics $\breve{H}(t)$ as a linear combination of individual statistics

$$(22) \qquad \breve{H}(t) = \breve{\tau}_0^{1/2} \cdot \breve{H}_0(t) + \sum_{n=1}^{N} \breve{\tau}_I^{1/2} \cdot \breve{H}_n(t).$$

Then, the market's estimate of the growth rate $\breve{G}(t)$ is

$$(23) \qquad \breve{G}(t) := \breve{E}\{G^*(t)\} = \breve{\sigma}_G \cdot \breve{\Omega}^{1/2} \cdot \breve{H}(t).$$

The market price can be therefore written as

$$(24) \qquad P(t) = \frac{D(t)}{r + \alpha_D} + \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot \breve{H}(t).$$

This formula generalizes equation (2) in the first example and equation (6) in the second example.

The Economist's Inference in Dynamics Model of Information Processing.

In order to derive implications for returns dynamics, we need to introduce the true parameters into the model. In what follows, we mark with "hats" true parameters and refer to them as beliefs of the economist.

Thus, while the market uses possible incorrect parameter values $\breve{\alpha}_G$, $\breve{\sigma}_G$, and $\breve{\tau}_I$ the economist derives implications for return dynamics using correct parameter values $\hat{\alpha}_G$, $\hat{\sigma}_G$, and $\hat{\tau}_I$.

Assume that the economist otherwise has the same beliefs as the market about the structure of the dividend, dividend growth, and information processes. In particular, we assume that the economist and the market agree about the parameters α_D and σ_D. This assumption makes calculations simpler in what follows. It does not make sense to assume that the economist and the market disagree about the value of σ_D since its value can be inferred with perfect accuracy from observing the dividend process $D(t)$ continuously.

By placing "hats" over the variables in equations (11), (14), (16), (18), (19), (20), (22), and (23) above, we obtain definitions of $\hat{\Omega}$, $\hat{\tau}_0$, $\hat{\tau}$, and $\hat{H}_n(t)$ for $n = 0, 1...N$ which are consistent with the economist's expectation operator $\hat{E}_t\{\ldots\}$. We briefly list those definitions below.

$$(25) \qquad\qquad dG^*(t) = -\hat{\alpha}_G \cdot G^*(t) \cdot dt + \hat{\sigma}_G \cdot dB_G(t).$$

$$(26) \qquad\qquad \hat{\tau}_0 := \hat{\Omega} \cdot \hat{\sigma}_G^2/\sigma_D^2,$$

$$(27) \qquad\qquad \hat{\tau} = \hat{\tau}_0 + N \cdot \hat{\tau}_I,$$

$$(28) \qquad\qquad \hat{\Omega} := \hat{Var}[(G^*(t) - \hat{G}(t))/\hat{\sigma}_G] = (2 \cdot \hat{\alpha}_G + \hat{\tau})^{-1},$$

$$(29) \qquad\qquad dI_n(t) := \hat{\tau}_I^{1/2} \cdot \frac{G^*(t)}{\hat{\sigma}_G \cdot \hat{\Omega}^{1/2}} \cdot dt + dB_n(t), \quad n = 0, \ldots, N,$$

$$(30) \qquad\qquad \hat{H}_n(t) := \int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G + \hat{\tau}) \cdot (t-u)} \cdot dI_n(u), \quad n = 0, 1, ...N.$$

$$(31) \qquad\qquad \hat{H}(t) = \hat{\tau}_0^{1/2} \cdot \hat{H}_0(t) + \sum_{n=1}^{N} \hat{\tau}_I^{1/2} \cdot \hat{H}_n(t).$$

$$(32) \qquad\qquad \hat{G}(t) := \hat{E}\{G^*(t)\} = \hat{\sigma}_G \cdot \hat{\Omega}^{1/2} \cdot \hat{H}(t).$$

Both the market and the economist construct their signals $\hat{H}(t)$ and $\breve{H}(t)$ as linear combinations of all increments in information flow with the weights decaying exponentially. There is the following relationship between their signals, $\hat{H}_n(t)$ and $\breve{H}_n(t)$, $n = 0, 1...N$:

$$(33) \qquad \hat{H}_n(t) = \breve{H}_n(t) + (\breve{\alpha}_G + \breve{\tau} - \hat{\alpha}_G - \hat{\tau}) \cdot \int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G + \hat{\tau}) \cdot (t-u)} \cdot \breve{H}_n(u) \cdot du.$$

If the market has correct beliefs, then $\hat{H}_n(t) = \breve{H}_n(t)$. Otherwise, the relationship between the two variables depends on the entire history of information flow, since the economist disagrees with the market on how quickly information becomes obsolete. For example, the economist may assign higher weights to the information from the distant past if he believes that dividends are more persistent or signals are less precise than the market believes, *i.e.*, when $\breve{\alpha}_G + \breve{\tau} > \hat{\alpha}_G + \hat{\tau}$.

Returns Dynamics in a Dynamic Model of Information Processing.

We derive next returns dynamics under the true beliefs of the economist. Plugging the equilibrium price $P(t)$ from equation (24) and using equation (20) for $d\breve{H}_n(t)$, equation (29) for $dI_n(t)$, and then equation (32) for the economist's estimate $\hat{G}(t)$ yields the returns process

$$(34) \qquad dP(t) + D(t) \cdot dt = r \cdot P(t) \cdot dt + (\breve{a} \cdot \breve{H}(t) + \breve{b} \cdot \hat{H}(t)) \cdot dt + d\hat{B}_r(t),$$

where the coefficients $\breve{a}$ and $\breve{b}$ are given by

$$(35) \qquad\qquad \breve{a} := -\frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot (r + \breve{\alpha}_G + \breve{\tau}),$$

$$(36) \qquad \breve{b} := \frac{\hat{\sigma}_G \cdot \hat{\Omega}^{1/2}}{(r + \alpha_D)} + \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot (\breve{\tau}_0^{1/2} \hat{\tau}_0^{1/2} + N \cdot \breve{\tau}_I^{1/2} \cdot \hat{\tau}_I^{1/2}).$$

This equation is the generalization of equation (3) in the first motivating example and equation (7) in the second motivating example.

Recall that $\hat{E}_t\{\ldots\}$ denote an expectation operator based on true beliefs calculated using information available at time t. The true expected return is given by

$$(37) \qquad\qquad \hat{E}_t\left\{\frac{dP(t) + D(t) \cdot dt}{dt}\right\} = r \cdot P(t) + \breve{a} \cdot \breve{H}(t) + \breve{b} \cdot \hat{H}(t).$$

As in the first and second motivating examples, investors obtain the expected return of $r \cdot P(t)$ when beliefs of the market coincide with the true beliefs, *i.e.*, $\breve{\alpha}_G = \hat{\alpha}_G, \breve{\sigma}_G = \hat{\sigma}_G$, and $\breve{\tau}_I = \hat{\tau}_I$. Otherwise, investors also obtain an unexpected risk premium equal to a linear combination $\breve{a} \cdot \breve{H}(t) + \breve{b} \cdot \hat{H}(t)$, where the signals of the market $\breve{H}(t)$ and the signal of the economist $\hat{H}(t)$ depend in a complicated manner on the entire history of information flow, as shown in equations (20) and (30). The first sufficient statistics $\breve{H}(t)$ can be extracted from current prices and dividends using equation (24). The second sufficient statistics $\hat{H}(t)$ can be extracted from the history of $\breve{H}(t)$ using equations (22), (31), and (33).

The uncertainty term $d\hat{B}_r(t)$ in equation (34) is defined as

$$(38)$$
$$d\hat{B}_r(t) := \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \left(\breve{\tau}_0^{1/2} \cdot dB_0^*(t) + \breve{\tau}_I^{1/2} \cdot N \cdot d\bar{B}^*(t)\right) + \frac{\sigma_D}{(r + \alpha_D)} \cdot dB_0^*(t).$$

The processes $d\bar{B}^*(t)$ and $dB_0^*(t)$ are defined as

$$(39) \qquad d\bar{B}^*(t) := \hat{\tau}_I^{1/2} \cdot (\hat{\sigma}_G \cdot \hat{\Omega}^{1/2})^{-1} \cdot (G^*(t) - \hat{G}(t)) \cdot dt + \sum_{n=1}^{N} dB_n(t)/N$$

$$(40) \qquad dB_0^*(t) := \hat{\tau}_0^{1/2} \cdot (\hat{\sigma}_G \cdot \hat{\Omega}^{1/2})^{-1} \cdot (G^*(t) - \hat{G}(t)) \cdot dt + dB_0(t)$$

are the Brownian motions under the true beliefs. Note that the variance of $dB_0^*(t)$ is equal to one, but the variance of $d\bar{B}^*(t)$ is equal to $1/N$. It can be shown that when the market has correct beliefs, i.e., $\breve{\alpha}_G = \hat{\alpha}_G$, $\breve{\sigma}_G = \hat{\sigma}_G$, and $\breve{\tau}_I = \hat{\tau}_I$, the economist will find that prices adjusted for the interest expense follow a random walk and its actual volatility coincides with its true fundamental volatility. Otherwise, volatility is a constant, which may be higher or lower than the fundamental volatility depending on the true parameters and the parameters used by the market.

III.   Micro-Foundation of Dynamic Model with Information Processing

There is returns predictability when the market's beliefs differ from the true beliefs, as illustrated in the motivating examples and the model in section II. We next use the smooth trading model of Kyle, Obizhaeva and Wang (2013) to illustrate how the incorrect beliefs of the market can simply arise in the equilibrium even when investors apply the Bayes law correctly. Predictable returns may result from rational behavior of investors if the market aggregates information in a manner that results in predictability.

We first review the smooth trading model. Suppose there are N traders. Each trader n chooses a consumption intensity $c_n(t)$ and trading rate $X_n(t, .)$ to maximize an expected constant-absolute-risk-aversion (CARA) utility function. Let $U(c_n(s)) := -e^{-A \cdot c_n(s)}$ be an exponential utility function with a constant absolute risk aversion parameter A. Letting $\rho > 0$ denote a time preference parameter, trader n solves the maximization problem

$$(41) \qquad \max_{\{c_n(t), X_n(t,.)\}} E_t^n \left\{ \int_{u=t}^{\infty} e^{-\rho(u-t)} \cdot U(c_n(u)) \cdot du \right\},$$

where $M_n(t)$ is the money account in dollars and $S_n(t)$ is the inventories in shares. These variables are described by the following processes with continuous paths:

$$dM_n(t) = (r \cdot M_n(t) + S_n(t) \cdot D(t) - c_n(t) - P(x_t) \cdot X_n(t, P(t))) \cdot dt,$$

$$dS_n(t) = X_n(t, P(t)) \cdot dt,$$

Here $E_t^n\{\ldots\}$ denotes the expectation of trader n calculated with respect to information at time t. In this notation, the superscript n indicates that the expectation

is taken with respect to the beliefs of trader n. The subscript t indicates that the expectation is taken with respect to trader n's information set at time t. Each trader is assumed to trade "smoothly" and explicitly take into account the effect of his trading on the price of a risky asset $P(x)$, where $x = dS_n(t)/dt$.

Each trader n observes a continuous stream of private information $I_n(t)$ about the unobservable growth rate $G^*(t)$:

$$(42) \qquad dG^*(t) := -\alpha_G \cdot G^*(t) \cdot dt + \sigma_G \cdot dB_G(t),$$

$$(43) \qquad dI_n(t) := \tau_n^{1/2} \cdot \frac{G^*(t)}{\sigma_G \cdot \Omega^{1/2}} \cdot dt + dB_n(t).$$

Since the equilibrium price reveals the average signal in the symmetric model, each trader infers the average of other traders' private signals from the market price. Each trader also infers information $I_0(t)$ about the growth rate from the dividend stream,

$$(44) \qquad dI_0(t) := \tau_0^{1/2} \cdot \frac{G^*(t)}{\sigma_G \cdot \Omega^{1/2}} \cdot dt + dB_0(t),$$

where $dB_0 = dB_D, dB_G, dB_1, ...dB_N$ are independent Brownian motions.

Traders agree on the precision τ_0 of public information and agree to disagree about their interpretations of private information. Agreement to disagree is the mechanism that generates trading in the model. It is a common knowledge that each trader believes his own signal has high precision τ_H while signals of the others have low precision τ_L. Symmetry implies that traders agree on the total precision

$$(45) \qquad \tau := \tau_0 + \tau_H + (N - 1) \cdot \tau_L.$$

In equations (43)-(45), we define the precision of public information τ_0 and the error variance Ω as

$$(46) \qquad \tau_0 := \Omega \cdot \sigma_G^2/\sigma_D^2,$$

$$(47) \qquad \Omega := Var[(G^*(t) - G_n(t))/\sigma_G] = (2 \cdot \alpha_G + \tau)^{-1},$$

where $G_n(t) := E_t^n\{G^*(t)\}$ is trader n's estimate of the growth rate. Due to symmetry, τ_0 and Ω are the same for all traders. The parameters α_G, σ_G, τ, τ_H, and τ_L, without "breves" or "hats", describe the beliefs of traders in the model.

The information set includes the history of public signal $I_0(t)$, the history of private signals $I_n(t)$ and the history of private signals of the others inferred from the market prices. The trader n's estimate $G_n(t)$ can be conveniently written as the weighted sum of signals $H_0(t)$, $H_n(t)$, and $H_{-n}(t)$ that summarize all pubic and private information:

$$(48) \qquad H_n(t) := \int_{u=-\infty}^{t} e^{-(\alpha_G+\tau)\cdot(t-u)} \cdot dI_n(u), \quad n = 0, 1, ...N,$$

$$(49) \qquad H_{-n}(t) := \frac{1}{N-1} \sum_{m=1,..,N, m \neq n} H_m(t).$$

$$(50) \quad G_n(t) := \sigma_G \cdot \Omega^{1/2} \cdot \left(\tau_0^{1/2} \cdot H_0(t) + \tau_H^{1/2} \cdot H_n(t) + (N-1) \cdot \tau_L^{1/2} \cdot H_{-n}(t) \right).$$

When forming his signal, each trader assigns a larger weight τ_H to his own signal and smaller weight τ_L to signals of the others. Each piece of information decays exponentially at a rate $\alpha_G + \tau$.

Kyle, Obizhaeva and Wang (2013) find that, if there is enough disagreement, there is a symmetric linear equilibrium of a simple and intuitive form. Each trader calculates a target inventory proportional to his risk tolerance and proportional to the difference between his own valuation and the average valuation of other traders. Each trader correctly believes that the price is a linear function of the average valuations of other traders, his own inventory, and the derivative of his own inventory. Since trading a nontrivial quantity over a very short period of time has transitory price impact proportional to the derivative a trader's inventory, each trader trades smoothly; demand schedules and market clearing quantities are defined in terms of derivatives of inventories, not levels of inventories. In the equilibrium, each trader adjusts his inventory towards the target level gradually, demanding liquidity by trading on his own private information and providing liquidity to other traders by trading against their information. The rate of partial adjustment is determined by a trade-off between the half-life of private information and price resilience.

In this paper, we are more concerned with equilibrium prices and returns rather than quantities traded. The equilibrium price instantly and fully reveals information. Define the average of the N traders' expected growth rates by

$$(51) \qquad \bar{G}(t) := \frac{1}{N} \sum_{n=1}^{N} G_n(t),$$

the equilibrium price is then given by

$$(52) \qquad P^*(t) = \frac{D(t)}{r + \alpha_D} + C_G \cdot \frac{\bar{G}(t)}{(r + \alpha_D)(r + \alpha_G)}.$$

With one important exception, this formula is similar to the Gordon's growth model formula in equation (2) in the first example, equation (6) in the second example, and equation (24) in the model above.

The exception is the endogenously determined coefficient C_G, which is equal to one in Gordon's growth formula but is not equal to one in equation (52). Based on numerical calculations, Kyle, Obizhaeva and Wang (2013) find that constant C_G is always less than one; this "dampening effect" makes the market price less sensitive to changes in the average growth rate forecasts' of investors than if the average expected growth rate were plugged into Gordon's growth formula. The

intuition for the dampening effect is as follows. Each trader is trading against the others, planning to unwind positions in the future when other traders discover that their information was incorrect. As the degree of disagreement is reduced to the level barely large enough to sustain trading, i.e., $\tau_H^{1/2}/\tau_L^{1/2}$ converges from above to $2 + 2/(N-2)$, market depth decreases, trading opportunities shrink, trading volume decreases, and C_G converges to one.

In the next section, we examine whether there exists a set of beliefs about model parameters, attributable to a hypothetical representative agent, such that the market price in equation (52) is consistent with these beliefs in the sense that the expected return under the representative agent's beliefs is always equal to the risk-free rate.

Incorrect Beliefs of Representative Agent as Result of Information Aggregation

The representative agent's beliefs about parameter values are different from traders' beliefs in a non-trivial way.

Common-sense intuition suggests that if all traders in the market agree about the value of a parameter such as α_G or σ_G (and this value is common knowledge), then the representative agent will have the same beliefs about this parameter. Common-sense intuition also suggests that if agents disagree about the value of a parameter, then the representative agent's belief about this parameter will be equal to some appropriate weighted average across traders. For example, this intuition, combined with the symmetry of the equilibrium, suggests that the representative agent will assign to each trader's signal the same precision, equal to some weighted average of precisions τ_H and τ_L.

The next theorem shows, however, that beliefs of the representative agent may differ from the beliefs suggested by reasonable common-sense intuition. It can therefore be misleading, even incorrect, to impute the beliefs of the representative agent to traders in a market. Specific interactions among individual traders in a dynamic game-theoretic context can make their average beliefs quite different from the beliefs of a representative agent consistent with equilibrium prices.

THEOREM 1: *For any sets of parameters in the smooth trading model, the information aggregation by imperfectly competitive strategic overconfident traders is consistent with the information processing by the representative agent. The beliefs of the representative agent are defined by the three parameters: growth-rate persistency $\breve{\alpha}_G > \alpha_G$, growth-rate volatility $\breve{\sigma}_G$, and the same precision $\breve{\tau}_I$ that he assigns to each of N signals,*

$$\breve{\tau}_I = \tau_p \cdot \frac{C_G \cdot (r + \alpha_G + \tau)}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)},$$

$$\breve{\alpha}_G = \alpha_G + \frac{r + \alpha_G}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)} \cdot (\tau - C_G \cdot (\tau_0 + N \cdot \tau_p)),$$

$$\breve{\sigma}_G = \sigma_G \cdot \left[\frac{C_G \cdot (r + \alpha_G + \tau)}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)} \cdot \left(1 + \frac{\breve{\alpha}_G - \alpha_G}{2 \cdot \alpha_G + \tau}\right) \right]^{1/2},$$

where $\tau_p^{1/2} := (\tau_H^{1/2} + (N-1)\tau_L^{1/2})/N$ and a constant $C_G \leq 1$. The representative agent agrees with traders on the other parameters α_D, σ_D, r, ρ, and A.

Proof. The outline of the proof is as follows. At any point of time, the representative agent would need to have beliefs such that the equilibrium price

$$(53) \quad P(t) = \frac{D(t)}{r + \alpha_D} + C_G \cdot \frac{\sigma_G \cdot \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot \left(\tau_0^{1/2} \cdot H_0(t) + \tau_p^{1/2} \cdot \sum_{n=1,..N} H_n(t)\right)$$

coincides with his estimate of the fundamental value

$$(54) \quad \breve{F}(t) = \frac{D(t)}{r + \alpha_D} + \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot \left(\breve{\tau}_0^{1/2} \cdot \breve{H}_0(t) + \breve{\tau}_I^{1/2} \cdot \sum_{n=1,..N} \breve{H}_n(t)\right).$$

The fundamental value $\breve{F}(t)$ is the version of Gordon's formula given the estimate $\breve{G}(t)$ of a growth rate in equation (23).

First, the history of signals $\breve{H}_n(t)$ in equation (20) has to coincide with the history of signals $H_n(t)$ in equation (48). This implies the following restriction:

$$\breve{\alpha}_G + \breve{\tau} = \alpha_G + \tau.$$

Second, the coefficients of random variables in the two equations (53) and (54) have to match. This leads to the other two restrictions:

$$\frac{C_G \cdot \sigma_G \cdot \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot \tau_0^{1/2} = \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot \breve{\tau}_0^{1/2},$$

$$\frac{C_G \cdot \sigma_G \cdot \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot \tau_p^{1/2} = \frac{\breve{\sigma}_G \cdot \breve{\Omega}^{1/2}}{(r + \alpha_D)(r + \breve{\alpha}_G)} \cdot \breve{\tau}_I^{1/2}.$$

The definition of $\breve{\tau}_0$ and equation (19) yield the last restriction,

$$\breve{\sigma}_G = \sigma_D \cdot \breve{\tau}_0^{1/2} \cdot (2 \cdot \breve{\alpha}_G + \breve{\tau})^{1/2}.$$

The solution of the system is the set of three parameters $\breve{\alpha}_G, \breve{\sigma}_G,$ and $\breve{\tau}_I$ describing beliefs of the representative agent and stated in the theorem as well as the expression for $\breve{\tau}_0$:

$$\breve{\tau}_0 = \tau_0 \cdot \frac{C_G \cdot (r + \alpha_G + \tau)}{r + \alpha_G + C_G \cdot (\tau_0 + N \cdot \tau_p)}.$$

Since $N \cdot \tau_p < \tau_H + (N-1)\tau_L$, we have $\breve{\alpha}_G > \alpha_G$.

The theorem implies that the imputed beliefs of the representative agent will usually be quite different from the typical beliefs of traders in the model. Even

for his beliefs about parameters α_G and σ_G concerning which the consensus among traders exists, researchers may need to assign values different from the consensus ones.

The beliefs of the representative agent will need to be complicated. The mean reversion parameter $\breve{\alpha}_G$ of the representative agent has to be larger than the mean reversion parameter α_G of the market. The volatility parameter $\breve{\sigma}_G$ of the representative agent can be either higher or lower than the market's dividend growth volatility σ_G, depending on particular parameters. Guessing those beliefs would be impossible without solving the model.

This result has a simple intuition. In the dynamic model, the beliefs about precisions play two roles. First, beliefs determine the weights with which the representative agent aggregates incoming information into his current estimate of a growth rate in equation (31); these weights are proportional to the square roots of precisions. Second, beliefs determine the speed with which he thinks his signal deteriorates; this speed is proportional to precisions themselves. In other words, the precisions determine price resilience and the square roots of precisions determine price volatility; both have to match for the representative agent to exist. If $\breve{\alpha}_G = \alpha_G$ and $\breve{\sigma}_G = \sigma_G$, there are no symmetric beliefs that can simultaneously match both current price level and its dynamics.

Returns Dynamics in the Smooth Trading Model.

We discuss next the returns dynamics in the context of the smooth trading model. Actual expected returns are generally different from the risk free rate. The distribution of returns depend on both the parameters traders use and the true parameters.

Suppose the economist believes that the true parameters have values different from parameters assigned by traders in the smooth trading model. Let $\hat{\alpha}_G$, $\hat{\sigma}_G$, and $\hat{\tau}_I$ denote the true parameter values used by the economist. By analogy with equation (31), the economist constructs his signal $\hat{H}(t)$ as

$$(55) \qquad \hat{H}(t) = \hat{\tau}_0^{1/2} \cdot \hat{H}_0(t) + \hat{\tau}_I^{1/2} \cdot \sum_{n=1}^{N} \hat{H}_n(t).$$

To derive returns dynamics, we use a shortcut.

First, we construct returns for the representative agent for the smooth trading model. The representative agent extracts the signal $\breve{H}(t)$ from information flow. We get that signal by plugging the beliefs $\breve{\alpha}_G, \breve{\sigma}_G$, and $\breve{\tau}_I$ from Theorem 1 into equation (22) and then taking into account that the representative agent's signals $\breve{H}_n(t)$ by definition have to coincide with trader's signals $H_n(t)$ for $n = 0, .. N$. Thus, the signal $\breve{H}(t)$ is given by

$$(56) \qquad \breve{H}(t) = \breve{\tau}_0^{1/2} \cdot H_0(t) + \breve{\tau}_I^{1/2} \cdot \sum_{n=1}^{N} H_n(t).$$

Second, we use the insights from section II on how to derive returns dynamics in the model where a single investor processes all information and incorporates it into prices. Replacing this single investor with the constructed representative agent, we use equation (34) to derive the actual dynamics of the returns process. Thus, the expected returns are given by

$$(57) \qquad \hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = r \cdot P(t) + \breve{a} \cdot \breve{H}(t) + \breve{b} \cdot \hat{H}(t).$$

where parameters $\breve{a}$ and $\breve{b}$ are defined in equations (35) and (36). The variables $\breve{H}(t)$ and $\hat{H}(t)$ are sufficient statistics for describing the returns process, calculated by the representative agent and the economist, respectively.

Third, it is convenient to replace "breve"-variables related to the representative agent with the original parameters assigned by traders in the smooth trading model. Define the aggregate weighted average signal $H(t)$ of traders as

$$(58) \qquad H(t) = \tau_0^{1/2} \cdot H_0(t) + \tau_p^{1/2} \cdot \sum_{n=1}^{N} H_n(t).$$

The expected return is a linear combination of the average signal $H(t)$ of traders and the signal $\hat{H}(t)$ of economist. Both signals summarize information available up to time t but put different weights on the past information. The signal $\breve{H}(t)$ of the representative agent in equation (57) can therefore be replaced by the average signal $H(t)$ of traders. Thus, the expected returns are given by

$$(59) \qquad \hat{E}_t \left\{ \frac{dP(t) + D(t) \cdot dt}{dt} \right\} = r \cdot P(t) + a \cdot H(t) + b \cdot \hat{H}(t),$$

where the coefficients a and b are defined in terms of parameters in the model as

$$(60) \qquad a := -\frac{\sigma_G \cdot C_G \cdot \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot (\alpha_G + r + \tau),$$

$$(61) \qquad b := \frac{\hat{\sigma}_G \cdot \hat{\Omega}^{1/2}}{r + \alpha_D} + \frac{\sigma_G \cdot C_G \cdot \Omega^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \cdot (\tau_0^{1/2} \hat{\tau}_0^{1/2} + \tau_p^{1/2} \cdot N \cdot \hat{\tau}_I^{1/2}).$$

This equation is a generalization of equation (3) in the first motivating example, equation (7) in the second motivating example, and equation (34) in the dynamic model. It implies a complicated path-dependent and auto-correlated returns process.

As before, expected returns are equal to the risk-free rate $r \cdot P(t)$ only if the beliefs of the representative agent happen to coincide with the true beliefs. As Theorem 1 suggests, however, the beliefs of the representative agent are usually different from typical beliefs of traders in the smooth trading model and thus most

likely they are different from the true beliefs. Unexpected excess returns thus are time varying, depending in a complicated manner on the entire history of past signals.

The variables $H(t)$ and $\hat{H}(t)$ are ultimately related to the history of dividends and prices. The next theorem states the returns dynamics explicitly in terms of the history of observable market data.

THEOREM 2: *In the smooth trading model, the economist with true beliefs assigns precision $\hat{\tau}_0$ to public information, $\hat{\tau}_I$ to each of N sources of private information; the total true precision is $\hat{\tau} = \hat{\tau}_0 + N \cdot \hat{\tau}_I$, whereas the total precision according to beliefs of traders is $\tau = \tau_0 + \tau_H + (N-1) \cdot \tau_L$. Thus, equilibrium returns dynamics can be expressed as a linear combination of past dividends and prices:*

$$dP(t) + D(t) \cdot dt = r \cdot P(t) \cdot dt + \alpha_1 \cdot \left(P(t) - \frac{D(t)}{r + \alpha_D} \right) \cdot dt$$

$$+ \alpha_2 \cdot \left[\int_{u=-\infty}^{t} \left(P(u) - \frac{D(u)}{r + \alpha_D} \right) \cdot e^{-(\hat{\alpha}_G + \hat{\tau})(t-u)} \cdot du \right] \cdot dt$$

(62)
$$+ \alpha_3 \cdot \left[\int_{u=-\infty}^{t} e^{-(\hat{\alpha}_G + \hat{\tau}) \cdot (t-u)} \cdot dI_0(u) \right] \cdot dt + d\hat{B}_r^*(t),$$

where constants α_1, α_2, and α_3 are defined by

$$\alpha_1 := (a + b \cdot \hat{\tau}_I^{1/2}/\tau_p^{1/2}) \cdot (r + \alpha_D) \cdot (r + \alpha_G)/(C_G \cdot \sigma_G \cdot \Omega^{1/2}),$$

$$\alpha_2 := b \cdot \hat{\tau}_I^{1/2}/\tau_p^{1/2} \cdot (\alpha_G + \tau - \hat{\alpha}_G - \hat{\tau}) \cdot (r + \alpha_D) \cdot (r + \alpha_G)/(C_G \cdot \sigma_G \cdot \Omega^{1/2}),$$

$$\alpha_3 := b \cdot (\hat{\tau}_0^{1/2} - \hat{\tau}_I^{1/2} \cdot \tau_0^{1/2}/\tau_p^{1/2}).$$

Parameters a and b are defined in equations (60) and (61).

The formula has a simple intuition. First, investors obtain the unconditional expected return of $r \cdot P(t)$. Second, investors obtain unexpected excess returns linearly related to the deviation of the current price $P(t)$ from the unconditional level $D/(r + \alpha_D)$. Third, investors obtain unexpected excess returns linearly related to the deviations of prices from their unconditional expected value and the dividends surprises dI_0 in the past. The importance of each past component decays exponentially at a rate $\hat{\alpha}_G + \hat{\tau}$.

The specification suggests that it is the entire history of the dividend-to-price ratios and dividends—rather than only their current values—that may need to be included as explanatory variables into the return-forecast regressions such as in Campbell and Shiller (1988) and Cochrane (2008). Similarly to Campbell and Kyle (1993), our structural model implies economic restrictions of the parameters describing that relationship.

REMARK 1: *When traders have correct beliefs about the mean reversion rate α_G and total precision τ, i.e., $\alpha_G + \tau = \hat{\alpha}_G + \hat{\tau}$, then $\alpha_2 = 0$, which implies that returns depend only on the current deviation of the price from its unconditional mean and the history of dividends surprises.*

REMARK 2: *Suppose $\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$. When economist and traders agree on the total precision, $\hat{\tau} = \tau$, then $\alpha_1 > 0$, $\alpha_2 = 0$, and $\alpha_3 < 0$.*

These inequalities can be proved by noticing that $\hat{\Omega} = \Omega$ and $\hat{\tau}_0 = \tau_0$ and then showing $\hat{\tau}_I > \tau_p$. This implies that prices above (below) their unconditional levels predict high (low) returns in the short run. At the same time, positive (negative) surprises about dividends predict negative (positive) returns in the short run.

The Term Structure of Returns in Dynamic Smooth Trading Model.

In this section, we derive the term structure of returns under the beliefs of an economist who assigns precision $\hat{\tau}_0$ to the public signal and precision $\hat{\tau}_I$ to each private signal; the total precision is $\hat{\tau} = \hat{\tau}_0 + N \cdot \hat{\tau}_I$. The traders and the economist generally have different forecasts of expected returns.

Let $R(t, t + T)$ denote the cumulative holding period mark-to-market cash flow per share on a fully levered investment into the risky asset from time t to time $t + T$.

$$(63) \qquad R(t, t + T) = \int_{u=t}^{t+T} (dP(u) + D(u) \cdot du - r \cdot P(u) \cdot du).$$

From equation (59), we get

$$(64) \qquad R(t, t + T) = \int_{u=t}^{t+T} \left(a \cdot H(u) + b \cdot \hat{H}(u) \right) \cdot du + \int_{u=t}^{t+T} d\hat{B}_r^*(u).$$

The term structure of returns $R(t, t + T)$ at time t can be further simplified to a linear combination of the average signal $H(t)$ of traders and the signal $\hat{H}(t)$ extracted from prices and dividends by the economist. Using the definitions of $H(t)$ and $\hat{H}(t)$ in equations (58) and (55) as well as equations (29), (30), (32), and (33), we can write a continuous 2-vector stochastic process $y(t) = [H(t), \hat{H}(t)]'$ as satisfying the following linear stochastic differential equation:

$$(65) \qquad dy(t) = K \cdot y(t) \cdot dt + C \cdot dZ(t).$$

where K is a 2×2 matrix and C is a 2×2 matrix:

$$K = \begin{pmatrix} -\alpha_G - \tau & \tau_0^{1/2} \cdot \hat{\tau}_0^{1/2} + N \cdot \hat{\tau}_I^{1/2} \cdot \tau_p^{1/2} \\ 0 & -\hat{\alpha}_G \end{pmatrix}.$$

$$C = \begin{pmatrix} \tau_0^{1/2} & N \cdot \tau_p^{1/2} \\ \hat{\tau}_0^{1/2} & N \cdot \hat{\tau}_I^{1/2} \end{pmatrix}.$$

From the perspective of the economist, the vector $dZ(t) = [dB_0^*(t), d\bar{B}^*(t)]'$ is a 2×1-dimensional Brownian motion, where $dB_0^*(t)$ is a Brownian motion with variance of one defined in equation (40) and $d\bar{B}^*(t)$ is a Brownian motion with variance $1/N$ defined in equation (39).

Using results about linear continuous-time stochastic processes, we can represent the process $y(t) = [H(t), \hat{H}(t)]'$ in an integral form as

$$(66) \qquad y(s) = e^{K \cdot (s-t)} \cdot y(t) + \int_{u=t}^{s} e^{K \cdot (s-u)} \cdot C \cdot dZ(u).$$

It can be also shown that the exponential 2×2 matrix $e^{K \cdot t}$ is given by

$$e^{K \cdot t} = \begin{pmatrix} e^{-(\alpha_G + \tau) \cdot t} & \frac{\tau_0^{1/2} \cdot \hat{\tau}_0^{1/2} + N \cdot \hat{\tau}_I^{1/2} \cdot \tau_p^{1/2}}{\tau + \alpha_G - \hat{\alpha}_G} \cdot \left(e^{-\hat{\alpha}_G \cdot t} - e^{-(\alpha_G + \tau) \cdot t} \right) \\ 0 & e^{-\hat{\alpha}_G \cdot t} \end{pmatrix}.$$

Plugging $e^{K \cdot t}$ back into equation (66), we obtain recursive formulas for the stochastic vector $y(s) = [H(s), \hat{H}(s)]'$ as a function of $y(t) = [H(t), \hat{H}(t)]'$. Using this result, we can express the term structure of returns as a linear function of the vector $y(t)$.

THEOREM 3: *The term structure of expected returns can be represented as a linear combination of the average traders' sufficient statistics $H(t)$ and the economist's sufficient statistics $\hat{H}(t)$, inferred from past prices and dividends:*

$$(67) \qquad R(t, t+T) = \beta_1(T) \cdot H(t) + \beta_2(T) \cdot \hat{H}(t) + \bar{B}(t, t+T),$$

where time-varying coefficients $\beta_1(T)$ and $\beta_2(T)$ are defined as

$$\beta_1(T) = \frac{a}{\alpha_G + \tau} \left(1 - e^{-(\alpha_G + \tau)T} \right)$$

$$\beta_2(T) = b \cdot \frac{1 - e^{-\hat{\alpha}_G T}}{\hat{\alpha}_G} + a \cdot \frac{\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \hat{\tau}_I^{1/2} \tau_p^{1/2}}{\hat{\alpha}_G \cdot (\alpha_G + \tau)} \cdot \left(1 + \frac{\hat{\alpha}_G e^{-(\alpha_G + \tau)T} - (\alpha_G + \tau) e^{-\hat{\alpha}_G T}}{\tau + \alpha_G - \hat{\alpha}_G} \right).$$

A random variable $\bar{B}(t, t+T) := \int_{s=t}^{t+T} \int_{u=s}^{t+T} [a, b] e^{K(u-s)} C du \cdot dZ(s) + \int_{s=t}^{t+T} d\hat{B}_r^*(s)$ and constants a and b are defined in equations (60) and (61).

Several forces define the term structure of returns. The term structure depends on the beliefs of traders represented by $H(t)$ and the beliefs of the economist represented by $\hat{H}(t)$. The beliefs of traders anchor current prices; the beliefs of the economist anchor long-run fundamentals.

Prices tend to exhibit momentum due to the dampening effects on prices from the endogenous Keynsian beauty contest among traders. When the economist and traders disagree about the true parameters of the model, however, a wider range

of patterns may arise. Some of those patterns are consistent with the empirical evidence, *i.e.*, the momentum in the short run and the mean-reversion in the long run. We illustrate those patterns with several numerical examples.

We consider several combinations of $H(t)$ and $\hat{H}(t)$. The unconditional means of $H(t)$ and $\hat{H}(t)$ are zero. In order to generate the term structure of returns graphically for different cases of $H(t)$ and $\hat{H}(t)$, we first derive the steady-state unconditional variance-covariance matrix $Q = ((q_{11}, q_{12}), (q_{12}, q_{22}))$ of vector $y(t)$. It can be shown that

$$(68) \qquad q_{11} = \frac{\tau_0 + N \cdot \tau_p}{2(\alpha_G + \tau)} + \frac{(2\hat{\alpha}_G + \hat{\tau}) \cdot (\tau_0^{1/2} \hat{\tau}_0^{1/2} + N \cdot \hat{\tau}_I^{1/2} \cdot \tau_p^{1/2})^2}{2\hat{\alpha}_G \cdot (\alpha_G + \hat{\alpha}_G + \tau) \cdot (\alpha_G + \tau)},$$

$$q_{12} = \frac{(2\hat{\alpha}_G + \hat{\tau}) \cdot (\tau_0^{1/2} \cdot \hat{\tau}_0^{1/2} + N \cdot \hat{\tau}_I^{1/2} \cdot \tau_p^{1/2})}{2\hat{\alpha}_G \cdot (\alpha_G + \tau + \hat{\alpha}_G)},$$

$$q_{22} = \frac{\hat{\tau}}{2\hat{\alpha}_G}.$$

The distribution of $\hat{H}(t)$ conditional on $H(t)$ is described by the first two moments

$$E\{\hat{H}(t)|H(t)\} = q_{12}/q_{11} \cdot H(t), \; Var\{\hat{H}(t)|H(t)\} = q_{22} - q_{12}^2/q_{11}.$$

To be in a line with economically relevant ranges, we consider a one-standard deviation event $H(t) = q_{11}^{1/2}$ and k-standard deviation events $\hat{H}(t)$ conditional on $H(t)$,

$$\hat{H}(t) = q_{12}/q_{11} \cdot H(t) + k \cdot (q_{22} - q_{12}^2/q_{11})^{1/2}, \; k = -2, -1, 0, 1, 2,$$

where q_{11}, q_{12} and q_{22} defined in equation (68). Since the patterns for negative $H(t)$ are symmetric to the patterns for positive $H(t)$, we only focus on cases with positive $H(t)$.

Figure 1 illustrates the case when both the economist and traders agree on the total precision of information flow, $\hat{\tau} = \tau$. They also agree on the parameters of the model, $\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$. The parameters are $r = 0.01, A = 1, \alpha_D = 0.1, \alpha_G = 0.2, \sigma_D = 0.5, \sigma_G = 0.1, \hat{\tau}_0 = \tau_0 = 0.016, \tau_L = 0.016, \tau_H = 0.16$, and $N = 100$; this implies $\hat{\tau} = \tau = 1.9$ assuming $\hat{H}(t)$ coincides with $H(t)$ in this case, both are assumed to be equal to one-standard deviation away from the mean, i.e., $\hat{H}(t) = H(t) = q_{11}^{1/2}$. The figure depicts the cumulative returns $R(t, t + T)$ for different horizons T.

The upward sloping term structure indicates the momentum in returns. It is explained by the dampening effect of the Keynsian beauty contest. The cumulative returns increase monotonically and level up, as the profit opportunities disappear with time.

Figure 2 illustrates the case when traders and the economist agree on the parameters of the model, $\hat{\alpha}_G = \alpha_G$ and $\hat{\sigma}_G = \sigma_G$, but disagree about the total precision
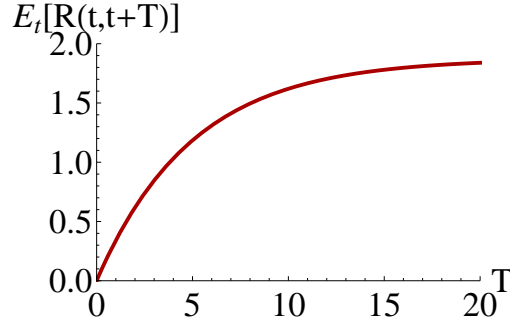
FIGURE 1. THE TERM STRUCTURE OF RETURN WHEN $\tau = \hat{\tau}, \hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$.

in the information flow. Traders are absolutely overconfident and $\tau > \hat{\tau}$. The parameters are the same as before, except $\hat{\tau} = 1.3$ and $\tau = 1.9$. The left subplot and the right subplot depict the term structure for the case when $H(t)$ is one-standard deviation from its unconditional mean, $i.e.$, $H(t) = q_{11}^{1/2}$ and the case when $H(t)$ is two-standard deviation from its unconditional mean, $i.e.$, $H(t) = 2 \cdot q_{11}^{1/2}$, respectively. Several deviations of conditional distribution of $\hat{H}(t)$ are considered for both cases. In each subplot, $\hat{H}(t)$ is k-standard deviation away from its conditional mean, from above to bottom $k = 2, 1, 0, -1$, and -2.
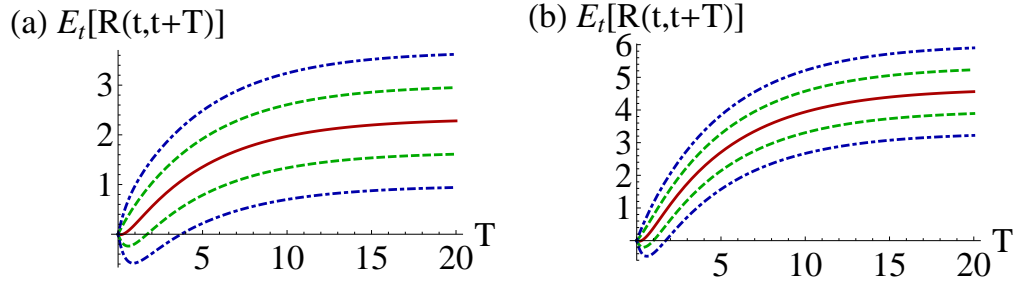

FIGURE 2. THE TERM STRUCTURE OF RETURN WHEN $\tau > \hat{\tau}, \hat{\alpha}_G = \alpha_G$, AND $\hat{\sigma}_G = \sigma_G$ FOR DIFFERENT $H(t)$ AND $\hat{H}(t)$.

The figures imply that the momentum effect continues to be dominating. The only exception are the cases when signals of the economist are low relative to the signal of traders as inferred from the price. The returns then exhibit a slight mean-reversion in the short run before the momentum effects start dominating in the long run.

Figure 3 illustrates a more general case when the economist and traders disagree about the total precision of information and also parameters of the model.

The term structure then depends on the particular assumptions about the mean-reversion rate $\hat{\alpha}_G$ and the volatility $\hat{\sigma}_G$. We assume $\tau = 1.9 > \hat{\tau} = 1.3$. We also assumed $\hat{\alpha}_G = 0.3 > \alpha_G = 0.2$ and $\hat{\sigma}_G = 0.08 < \sigma_G = 0.10$ in Case (a), and $\hat{\alpha}_G = 1 > \alpha_G = 0.2$ and $\hat{\sigma}_G = 0.5 > \sigma_G = 0.1$ in Case (b). As in Figure 2, the left subplot and the right subplot depict the term structure for the case when $H(t)$ is one-standard deviation and two-standard deviation away from its unconditional mean, respectively. In each subplot, $\hat{H}(t)$ is k-standard deviation away from its conditional mean, from above to bottom $k = 2, 1, 0, -1,$ and -2.
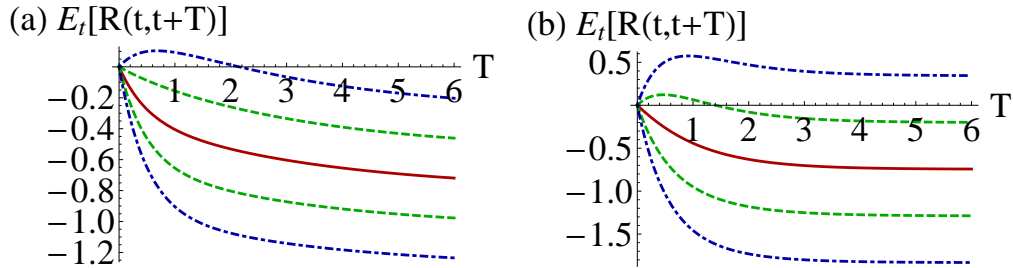

FIGURE 3. THE TERM STRUCTURE OF RETURN WHEN $\tau > \hat{\tau}, \hat{\alpha}_G \neq \alpha_G$, AND $\hat{\sigma}_G \neq \sigma_G$ FOR DIFFERENT $H(t)$ AND $\hat{H}(t)$.

The figure depicts more realistic patterns. When the economist has a more bullish signal relative to traders, the price exhibits momentum in the short run and mean-reversion in the long run.

In general, the term structure of returns exhibits different patterns depending on the parameters. It can be shown that the graph always starts from zero and converges to a constant as the horizon increases. It can be also proved that the derivative of $R(t, t + T)$ with respect to T does not change its sign more than once. There are therefore four possible patterns: only momentum, only mean-reversion, first mean-reversion and then momentum, first momentum and then mean-reversion. It is interesting to think about reasonable values of parameters and examine whether a calibrated model can generate empirically realistic patterns of expected returns.

Excess Volatility in Dynamic Smooth Trading Model.

Our model makes it possible to illustrate some issues concerning the the concept of excess volatility, which is often associated with the concept of market efficiency. The idea that prices are equal to discounted expected cash flows implies that the efficient prices have to satisfy some model-free variance-bound tests. The literature on these tests started with the work of Shiller (1979), LeRoy and Porter (1981), and Shiller (1981).

As discussed by LeRoy (1989), volatility tests are based on the idea that prices are equal to the expected value of the sum of discounted dividends,

$$(69) \qquad P(t) = \hat{E}_t \left\{ \sum_{j=1}^{\infty} \rho^j \cdot D(t+j) \right\}.$$

Given that $Var(E(\tilde{x}|\tilde{y})) < Var(\tilde{x})$ for any random variables $\tilde{x}$ and $\tilde{y}$, prices should satisfy the following variance-bounds test:

$$(70) \qquad \hat{Var}\{P\} = \hat{Var} \left\{ \hat{E}_t \{ \sum_{j=1}^{\infty} \rho^j \cdot D(t+j) \} \right\} \leq \hat{Var} \left\{ \sum_{j=1}^{\infty} \rho^j \cdot D(t+j) \right\}.$$

In efficient markets, prices are supposed to be less volatile than the ex-post realized present values that they should forecast, regardless of information on which traders condition their expectations.

Empirically, however, volatility of asset prices appear to systematically exceed volatility of fundamentals and the market prices appear to be inconsistent with variance-bounds restrictions. Thus, as Shiller (1981) suggests, the market is inefficient and may be even irrational.

Our model illustrates why the connection between market efficiency and excessive volatility is subtle in models with heterogeneous investors. The equilibrium price is formed on the basis of the estimates of market participants of the risky asset's value, a common value in the model. The price depends on the average of expectations of traders,

$$(71) \qquad P(t) = \frac{1}{N} \sum_{n=1}^{N} E_t^n \left\{ \sum_{j=1}^{\infty} \rho^j \cdot D(t+j) \right\} \neq \hat{E}_t \left\{ \sum_{j=1}^{\infty} \rho^j \cdot D(t+j) \right\},$$

where $E_t^n \{\ldots\}$ is the expectation of trader n. The average of expectations is usually not equal the true expectation. Moreover, according to the economist who analyzes the market, traders' estimates are correlated with each other. These problems make the argument underlying variance-bound tests somewhat misleading, because it does not properly take into account the averaging of traders' expectations and potentially non-zero covariances of those expectations.

More formally, the uncertainty term of the process $dP(t) + D(t) \cdot dt$ in equation (62) is defined by analogy with equation (38). The term is given by

$$d\hat{B}_r^*(t) := \frac{\sigma_G \cdot C_G \cdot \Omega^{1/2}}{(r+\alpha_D)(r+\alpha_G)} \left(\tau_0^{1/2} \cdot dB_0^*(t) + \tau_p^{1/2} \cdot N \cdot d\bar{B}^*(t) \right) + \frac{\sigma_D}{(r+\alpha_D)} \cdot dB_0^*(t),$$

where $dB_0^*(t)$ and $d\bar{B}^*(t)$ are as defined in (39) and (40).

The returns volatility depends only on what traders think about the precision of signals, but not on the true beliefs of the economist. The instantaneous volatility

of returns $dP(t) + D(t) \cdot dt$ is equal to

$$\hat{Var}\left\{ d\hat{B}_r^*(t) \right\} = \left[\frac{\sigma_D}{r + \alpha_D} + \frac{\sigma_G \cdot \Omega^{1/2} \cdot C_G \cdot \tau_0^{1/2}}{(r + \alpha_D)(r + \alpha_G)} \right]^2 + \frac{(\sigma_G \cdot \Omega^{1/2} \cdot C_G)^2 \cdot N \cdot \tau_p}{(r + \alpha_D)^2 (r + \alpha_G)^2}.$$

In rare cases when parameters are such that the representative agent turns out to have true beliefs, the returns volatility coincides with the volatility of fundamentals $dF(t) + D(t) \cdot dt$, where $F(t)$ is defined by the Gordon's growth formula but with no dampening effect. It is more likely, however, that the beliefs of the representative agent are different from the beliefs of economist. In this case, the relationship between fundamental volatility and returns volatility becomes more complicated.

The interpretation of the empirical literature on excess returns volatility is even less straightforward. Since it is impossible to calculate instantaneous volatility of returns, researchers usually examine volatility of returns sampled at some frequencies, for example, by looking at daily, monthly, or annual returns. The relationship between these empirical estimates of volatility and fundamental volatility may be even more complicated, because one would need to take into account a non-zero drift term in equation (62). This illustrates why one has to exercise caution when defining market efficiency as the concept related to excess volatility.

IV.   Conclusion

Our framework provides a convenient operational laboratory for testing theories about financial markets. We use the smooth trading model to lay out issues involved, but the principles are more general. Also, in the model where everybody disagrees with each other, it is more natural to think about economists arguing about their theories as well. Any preferred theory of expected returns and potential alternatives can be formulated by specifying particular sets of parameters of the model. Each theory will then imply specific predictions about returns dynamics, which can be formally tested against each other using available data.

The implied term structure usually exhibits complicated patterns of momentum and mean-reversion. The calibration of the model and studying whether it may generate quantitatively realistic patterns of the term structure is an interesting issue for future research.

We have focused on developing tools for testing theories about returns. The smooth trading model, however, allows us to extend analysis to other market variables. It is possible to generate predictions concerning sizes of positions as well as turnover rates and then test alternatives using the data on institutional holdings and trades.

The methods developed have applications beyond those considered in the paper. The same framework can be used to derive predictions concerning the term structures of expected volume and expected volatility, which received less attention in the literature. The proposed theories can then be judged based on their predictions concerning jointly determined term structures of returns, volume, and

volatility. This approach would provide an internally consistent structural benchmark for empirical studies of joint dynamics of those variables. It can also be used for thinking about other anomalies such as excessive volatility, IPO underpricing, and post earnings announcement drift.

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