Stable Matching in Large Economies*

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Abstract

Complementarities of preferences have been known to jeopardize the stability of two-sided matching markets, yet they are a pervasive feature in many matching markets. We revisit the stability issue with such preferences in a large market. Workers have preferences over firms while firms have preferences over distributions of workers and may exhibit complementarity. We demonstrate that if each firm's choice changes continuously as the set of available workers changes, then there exists a stable matching even with complementarity. Building on this result, we show that there exists an approximately stable matching in any large finite economy. We apply our analysis to show the existence of stable matchings in probabilistic and time-share matching models with a finite number of firms and workers.

1 Introduction

Since the celebrated work by Gale and Shapley (1962), matching theory has taken a center stage in market design and more broadly, economic theory. In particular, its successful application in medical matching and school choice has fundamentally changed how these

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markets are organized. A key desideratum in the design of such matching markets is "stability"—that the mechanism admits no incentives for its participants to "block" the suggested matching. Stability is crucial for long-term sustainability of a market; unstable matching would be undermined by the parties side-contracting around it either during or after a market.[1] When one side of the market is under centralized control, as with school choice, blocking by a pair of agents on both sides is less of a concern; but even this case, stability is desirable from a fairness standpoint, as it would eliminate justified envy—envy that cannot be explained away by the preferences of the agents on the other side. In the school choice application, if schools' preferences rest on the test score or other priority that a student feels entitled to, no justified envy appears to be a necessary requirement.

Unfortunately, a stable matching exists only under limited market conditions. It is well known that the existence of a stable matching is not generally guaranteed unless the preferences of participants, say firms, are substitutable.[2] In other words, complementarity can lead to the nonexistence of a stable matching.

This is a serious limitation on the applicability of centralized matching mechanisms, since complementarities of preferences are a pervasive feature of many matching markets. Firms often seek to hire workers with complementary skills. For instance, in professional athletic leagues, teams demand athletes that complement one another in skills as well as in the positions they play. Some public schools in New York City seek diversity of their student bodies in their skill levels. US colleges tend to exhibit a desire to assemble a class that is complementary and diverse in terms of their aptitudes, life backgrounds, and demographics.

Unless we can get a handle on complementarities, we would not know how to organize such markets, and the applicability of centralized matching will remain severely limited. The limitation is particularly pertinent for many decentralized markets that may potentially benefit from centralization. College admissions and graduate admissions are obvious examples. Decentralized matching leaves much to be desired in terms of efficiencies and fairness, and of the yield management burden put on the institutions (see Che and Koh (2013)). Despite the potential benefit from centralizing these markets, the exact benefit as

[1]Table 1 in Roth (2002) shows that unstable matching algorithms tend to die out while stable ones survive the test of time.

[2]Substitutability here means that a firm's demand for a worker never grows with more workers being available. More precisely, if a firm does not wish to hire a worker from a set of workers, then it never prefers to hire that worker from a bigger (in the sense of set inclusion) set of workers. Existence of a stable matching under substitutable preferences is established by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2010).

well as the method of centralized matching remains unclear, given the instability that may arise from complementary preferences of the participants.

This paper takes a step toward accommodating complementarities and other forms of general preferences. The general impossibility means, however, that the notion of stability needs to be weakened in some way. Our approach is to consider a large market. Specifically, we consider a market which consists of a large number of workers/students on one side and a finite number of firms/colleges with large capacities on the other, and ask whether stability can be achieved in an "asymptotic" sense—i.e., whether participants' incentives for blocking disappears as the size of workers and firms' capacities grow large. Large markets we envision approximate college admissions and labor markets. Our stability notion also preserves the motivation behind the original notion of stability: as long as the incentive for blocking is sufficiently weak, the instability and fairness concerns will not be so serious as to jeopardize the mechanism.

We first consider a continuum model in which there are a finite number of firms and a continuum of workers. Each worker desires to match with at most one firm. Firms have preferences over groups of workers, and importantly, their preferences may exhibit complementarities. A matching is a distribution of workers across firms. The model generalizes Azevedo and Leshno (2011) who assume responsive preferences (a special case of substitutable preferences) for the firms.

Our main result is that there exists a stable matching if firms' preferences exhibit continuity—that is, the set of workers chosen by each firm varies continuously as the set of workers available to that firm changes. This result is quite general since continuity is satisfied by a rich class of preferences including those exhibiting complementarities.[3] The existence of a stable matching follows from two results: (i) a stable matching can be characterized as a fixed point of a suitably defined mapping over a functional space, and (ii) such a fixed point exists given the continuity assumption. The construction of our fixed point mapping differs from the existing matching literature such as Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006), among others. The existence of a fixed point is established by using the Kakutani-Fan-Glicksberg fixed point theorem—a generalization of Kakutani's fixed point theorem to functional spaces—which appears to be new to the matching literature.

When firm preferences satisfy substitutability (but not necessarily continuity), we show that the set of stable matchings is nonempty and forms a complete lattice. In particu-

[3]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, desiring to hire all types in equal size (or density).

lar, there exist worker-optimal (firm-pessimal) and firm-optimal (worker-pessimal) stable matchings. A version of the rural hospital theorem also holds given an appropriate version of the law of aggregate demand. While these results are well-known and can thus be expected from the existing matching theory on finite markets, we also provide a novel condition that generalizes the full support assumption of Azevedo and Leshno (2011) and guarantees the uniqueness of stable matching under substitutable preferences.

Building on our analysis on the continuum model, we show that there is a sense in which it serves as a legitimate approximation of large finite economies. More specifically we demonstrate that, for any large finite economy that is sufficiently close to our continuum economy (in terms of the distribution of worker types and firms' preferences), there exists an **approximately stable** matching in the sense that the incentives for blocking is arbitrarily small.

Although the basic model assumes that firm preferences are strict, our framework can be extended to allow for indifferences in the firms' preferences. Accommodating indifferences is particularly important in the school choice context, in which preferences are given by coarse priorities that put many students in the same priority class. To accommodate this extension, we represent a firm's preference as a choice correspondence (as opposed to a function). We then extend both the fixed point characterization (via a correspondence defined on a functional space) and the proof of the existence.

Equipped with this generalization, we can extend the "fractional" matching models to allow for general preferences. These models study how schools/firms and students/workers can share time or match probabilistically in a stable manner (see Alkan and Gale (2003), Sotomayor (1999), and Kesten and Ünver (2009), among others). Our continuum model lends itself to studying such a probabilistic/time share environment; we can simply interpret types in different subsets within the type space as probabilistic/time units belonging to alternative (finite) workers. Our novel contribution is to allow for more general preferences including complementarities as well as indifferences. Accommodating indifferences is important in school choice design, and complementarities are also relevant since some schools (as those in NYC) seek diversity in their student bodies. We also establish existence of a strongly stable matching suggested by Kesten and Ünver (2009), in this more general environment.

Relationship with the Literature

The present paper is connected with several strands of literature. Most importantly, it is related to the growing literature on matching and market design. Since the seminal contributions by Gale and Shapley (1962) and Roth (1984), stability has been recognized as the most compelling solution concept in matching markets.[4] As argued and demonstrated by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2010) in various situations, the substitutability condition is necessary and sufficient for guaranteeing the existence of a stable matching when the number of agents is finite. Our paper contributes to this line of studies by showing that substitutability is not needed for the existence of a stable matching once there is a continuum of agents on one side of the market, and moreover, there exists an approximately stable matching in large finite markets.

Our study was inspired by a recent research on matching with a continuum of agents by Azevedo and Leshno (2011).[5] As in our paper, they assume that there are a finite number of firms and a continuum of workers and, among other things, show the existence and uniqueness of a stable matching in that setting. The crucial difference of their work from ours is that they assume firms have responsive preferences (which is a special case of substitutability). One of our contributions is that, while almost universally assumed in the literature, restrictions on preferences such as responsiveness or even substitutability are unnecessary for guaranteeing the existence of a stable matching in the continuum markets. Also, the uniqueness result of Azevedo and Leshno (2011) is obtained as a special case of our uniqueness result under substitutable, not necessarily responsive, preferences.

An independent study by Azevedo and Hatfield (2012) also analyzes matching with a continuum of agents.[6] Like the current paper, their study finds that a stable matching exists even when not all agents have substitutable preferences. There are a number of notable differences between their study and ours, however. First, they consider a large number (more precisely, continuum) of firms each employing a finite number of workers, so they consider a continuum of agents on both sides of the market. By contrast, we consider a finite number of firms each employing a large number (continuum) of workers. These two models provide complementary approaches for studying large markets. In the school choice context, for example, in many school districts, there are usually a small number of

[4] See Roth (1991) and Kagel and Roth (2000) for empirical and experimental evidence on the importance of stability in labor markets, and Abdulkadiroğlu and Sonmez (2003) for the interpretation of stability as a fairness concept in school choice.

[5] Also related, although formally different, are various recent studies on large matching markets, such as Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2008), Kojima and Manea (2008), Manea (2009), Che and Kojima (2010), and Lee (2012).

[6] Although less related, our study also has some analogy with Azevedo, Weyl and White (2012). They show the existence of competitive equilibrium in an exchange economy with continuum agents and indivisible objects.

schools each admitting hundreds of students, which fits well with our modeling approach. But in a big school district such as New York City, the number of schools is also large, so their model offers a reasonably good approximation. Second, Azevedo and Hatfield (2012) assume that there are finite number of firm and worker types. This enables them to use the Brouwer's fixed point theorem to characterize the stable matching. We put no restriction on the firms' preferences and on the number of workers' types. The general preferences required a topological fixed point theorem from functional analysis. This type of mathematics has never been applied to discrete two-sided matching literature to our knowledge, and we view the introduction of these tools to the matching literature as one of our methodological contributions. Our model also has an advantage of subsuming the previous work by Azevedo and Leshno (2011) as well as many others mentioned above, which assume a continuum of worker types. Finally, they also consider many-to-many matchings, although our applications to time-share and probabilistic matching models allow for many-to-many matching. And they also consider matching with contracts, while our study focuses on the case with a fixed term of contract, as assumed in the standard matching literature.

Our methodological contribution is related to another recent advance in matching theory based on the monotone method. In the one-to-one matching context, Adachi (2000) defines a certain operator whose fixed points are equivalent to stable matchings. His work has been generalized in many directions by such papers as Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield and Kominers (2010). We also define an operator whose fixed points are equivalent to stable matchings. A crucial difference is, however, that these previous studies impose restrictions on preferences (e.g., responsiveness or substitutability) so that the operator is monotone, which enables one to apply Tarski's fixed point theorem to show existence of stable matchings. By contrast, we do not impose responsiveness or substitutability restrictions and instead rely on the continuum of workers, along with continuity of firms' preferences, to guarantee continuity of the operator (in an appropriately chosen topology). That approach allows us to use (a generalization of) Kakutani fixed point theorem, a more familiar tool in traditional economic theory such as the existence proofs of general equilibrium and the Nash equilibrium in mixed strategies.

The current paper is also related with the literature on the matching with couples. Like a firm in our model, a couple can be seen as a joint decision maker with complementary preferences. Roth (1984) and unpublished work by Sotomayor show that there does not necessarily exist a stable matching if there exists a couple. Klaus and Klijn (2005) provide a condition to guarantee the existence of stable matchings. A more recent work by Kojima, Pathak and Roth (2013) presents conditions under which the probability that a stable matching exists even in the presence of couples converges to one as the market becomes infinitely large, and similar conditions have been further analyzed by Ashlagi, Braverman and Hassidim (2011). Pycia (2012) and Echenique and Yenmez (2007) study many-to-one matching with complementarity as well as peer effect. Our paper is different from these studies in various respects, but it complements these papers by formalizing a sense in which finding a stable matching becomes easier in a large market even in the presence of complementarities.

The remainder of this paper is organized as follows. Section 2 provides an example that illustrates the main contribution of our paper. Section 3 describes a matching model in the continuum economy. Section 4 establishes the existence of a stable matching under general, continuous preferences and also under substitutable preferences. In Section 5, we use this existence result to show that an approximately stable matching can be found in any large finite economy. In Section 6, we extend our analysis to the case where firms may have multi-valued choice mappings (that is, choice correspondences), and apply it to time share/probabilistic matching models.

2 Illustrative Example

Before proceeding, we illustrate the main contribution of our paper using an example. We first illustrate how complementary preferences may lead to non-existence of a stable matching when there are a finite number of agents. To this end, suppose that there are two firms f and f' and two workers θ and θ'. The agents have the following preferences:

$$\theta : f \succ f';$$
$$\theta' : f' \succ f;$$
$$f : \{\theta, \theta'\} \succ \emptyset;$$
$$f' : \{\theta\} \succ \{\theta'\} \succ \emptyset.$$

That is, worker θ prefers f to f', and worker θ' prefers f' to f; firm f prefers employing both workers to employing no one, which the firm in turn prefers to employing only one of them; and firm f' prefers worker θ to θ', which it in turn prefers to employing neither. Firm f has a "complementary" preference, and this creates instability. To see this, recall stability requires that there be no blocking coalition. Due to f's complementary preference, it must employ either both workers or neither in any stable matching. The former case is unstable since worker θ' prefers firm f' to firm f, and f' prefers θ' to being unmatched, thus they can block the matching. The latter is also unstable since, in such a case, f' will only hire θ, leaving θ' unemployed; and this outcome will be blocked by f forming a coalition with θ and θ', benefiting all members of the coalition.

Can stability be restored if the market becomes large? As long as the market remains finite, the answer is no. To see this, consider a scaled-up version of the above model: there are q workers of type θ and q workers of type θ', and they have the same preferences as above. Firm f' prefers type-θ workers to type-θ' workers, and wishes to hire in that order but at most up to q workers. Firm f has a complementary preference for hiring exactly identical numbers of type θ and θ' workers (with no capacity limit). Formally, if x and x' are the numbers of available workers of types θ and θ', respectively, then firm f would choose $\min\{x, x'\}$ workers of each type.

As long as q is odd (including the original economy with $q = 1$), there exists no stable matching.[7] If firm f hires more than $q/2$ workers of each type, then firm f' has a vacant position, so f' can block with a θ' worker who prefers f' to f. If f hires fewer than $q/2$ workers of each type, then some workers will remain unmatched (since f' hires at most q workers). If a θ worker is unmatched, then f' will form a blocking coalition with that worker. If a θ' worker is unmatched, then firm f will form a blocking coalition by adding that worker along with a θ' worker (possibly matched with f'). Consequently, "exact" stability is not guaranteed even in the large market. Nevertheless, one may hope to achieve approximate stability. This is indeed the case with the above example; the "magnitude" of instability diminishes as the economy grows large. To see this, let q be odd and consider a matching in which f hires $\frac{q+1}{2}$ workers of each type while f' hires $\frac{q-1}{2}$ workers of each type. This matching is unstable because f' has one vacant position it wants to fill and a worker of type θ' who is matched to f prefers f'. However, note that this is the only possible block of this matching, and it involves only one worker. As the economy grows large, if the additional single worker becomes insignificant for firm f' relative to its entire workforce—and this is what the continuity of a firm's preference captures—, then the payoff consequence of forming such a block must also become insignificant, suggesting that the instability problem becomes insignificant as well.

This can be seen most clearly in the limit of the above economy. Suppose there is a unit mass of workers, half of whom (in Lebesgue measure, say) are of type θ and the other half are of type θ'. Their preferences are the same as before. And suppose firm f wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of type-θ and type-θ'

[7] Here we sketch the argument, which is in Appendix A.1 in fuller form. When q is even, a matching in which each firm hires $\frac{q}{2}$ of each type of workers is stable.

workers, respectively. Firm f' can hire at most $\frac{1}{2}$, and prefers to fill as much of this quota as possible with type-θ workers and fill the remaining quota with type-θ' workers. In this economy, there is a (unique) stable matching in which each firm hires exactly one half of workers of each type. To see this, note that any blocking coalition involving firm f requires taking away a positive, and identical, measure of type-θ' and type-θ workers from firm f', which is impossible since type-θ' workers will object to it. Also, any blocking coalition involving firm f' requires taking away a positive measure type-θ workers away from firm f and replacing the same measure of type-θ' workers in its workforce, which is impossible since type-θ workers will object to it. Our analysis below will show that the continuity of firms' preferences, to be defined more clearly, is responsible for guaranteeing existence of a stable matching in the continuum economy and approximate stability in the large finite economies in this example.

3 Model of a Continuum Economy

There exist a finite set $F = \{f_1, \ldots, f_n\}$ of firms and a unit mass of workers. Let $\emptyset$ be the null firm, representing the workers' option of not being matched with any firm, and define $\tilde{F} := F \cup \{\emptyset\}$. The workers are identified with types $\theta \in \Theta$, where Θ is a compact metric space. Let Σ denote a Borel σ-algebra of space Θ. Let $\overline{\mathcal{X}}$ be the set of all nonnegative measures such that for any $X \in \overline{\mathcal{X}}$, $X(\Theta) \leq 1$. Assume that the population of entire workers is distributed according to a finite, nonnegative (Borel) measure $G \in \overline{\mathcal{X}}$ on (Θ, Σ). That is, for any $E \in \Sigma$, $G(E)$ is the measure of workers belonging to E. Assume $G(\Theta) = 1$ for normalization.

Any subset of the population, or **subpopulation**, is represented by a nonnegative measure X on (Θ, Σ) such that $X(E) \leq G(E)$ for all $E \in \Sigma$. Let $\mathcal{X} \subset \overline{\mathcal{X}}$ denote the set of all subpopulations. We further say that a nonnegative measure $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leq X(E)$ for all $E \in \Sigma$. We use $\mathcal{X}_X$ to denote the set of all subpopulations of X.

Given the order $\sqsubset$, for any $X, X' \in \mathcal{X}$, we define $X \vee X'$ (join) and $X \wedge X'$ (meet) to be the supremum and infimum of X and X', respectively.[8] That $X \vee X'$ and $X \wedge X'$ are

[8] That is, $X \vee X'$ for instance is the smallest measure of which both X and X' are subpopulations. One can show that for all $E \in \Sigma$,

$$(X \vee X')(E) = \sup_{D \in \Sigma} X(E \cap D) + X'(E \cap D^c),$$

which is a special case of Lemma 1 in Appendix A.4.

well-defined, i.e. they are also measures belonging to $\mathcal{X}$, follows from the following lemma, whose proof is in the subsection A.2 of Appendix.

Lemma 1. *The partially ordered set* $(\mathcal{X}, \sqsubset)$ *is a complete lattice.*

The meet and join of X and X' in $\mathcal{X}$ can be illustrated conveniently via an example. Suppose $\Theta = [0,1]$ and the measure G admits a density g for all $\theta \in [0,1]$. In this case, it easily follows that for $X, X' \sqsubset G$, their densities x and x' are well defined and $(X \vee X')$ and $(X \wedge X')$ admit densities y and z defined by $y(\theta) = \max\{x(\theta), x'(\theta)\}$ and $z(\theta) = \min\{x(\theta), x'(\theta)\}$ for all θ, respectively.

Consider the space of all (signed) measures (of bounded variation) on (Θ, Σ). We endow this space with a weak* topology and its subspace $\mathcal{X}$ with the relative topology. Given a sequence of measures $\{X_k\}$ and a measure X on (Θ, Σ), we write $X_k \xrightarrow{w^*} X$ to indicate that $\{X_k\}$ converges to X under weak* topology, and simply say that $(X_k)_k$ **weakly converges** to X.[9]

We now describe preferences on both sides of the economy. Each worker is assumed to have a strict preference over $\tilde{F}$, while a generic worker preference is denoted by P. We write $f \succ_P f'$ to indicate that f is strictly preferred to f' according to P. Let $\mathcal{P}$ denote the (finite) set of all possible worker preferences. For each $P \in \mathcal{P}$, let $\Theta_P \subset \Theta$ denote the set of all worker types whose preference is given by P, and assume that Θ_P is measurable and $G(\partial\Theta_P) = 0$ ($\partial\Theta_P$ denotes the boundary of Θ_P). Since all worker types have strict preferences, Θ can be partitioned into the sets in $\mathcal{P}_\Theta \equiv \{\Theta_P : P \in \mathcal{P}\}$.

A firm f's **choice function** is a mapping $C_f : \mathcal{X} \to \mathcal{X}$ such that $C_f(X)$ is a subpopulation of X for any $X \in \mathcal{X}$ and satisfies the following **revealed preference** property: for any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \sqsubset X'$, then $C_f(X') = C_f(X)$.[10] Note we are assuming that the firm's demand is unique given any set of available workers. In Section 6.1, we consider a generalization of the model in which the firm's choice is not unique. Let $R_f : \mathcal{X} \to \mathcal{X}$ be a **rejection function** defined by $R_f(X) := X - C_f(X)$. By convention, we let $C_\emptyset(X) = X, \forall X \in \mathcal{X}$, meaning that $R_\emptyset(X)(E) = 0$ for all $X \in \mathcal{X}$ and $E \subset \Sigma$.

[9] We use the term "weak convergence" since it is common in statistics and mathematics, though weak* convergence is a more appropriate term from the perspective of function analysis. As is well known, $X_k \xrightarrow{w^*} X$ if and only if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all bounded continuous function h. See Theorem 12 in Appendix A.3 to see some other implications of this convergence.

[10] This property must hold if the choice is made by a firm optimizing with a well-defined preference relation. See for instance Hatfield and Milgrom (2005), Fleiner (2003), and Alkan and Gale (2003) for some implicit or explicit use of the revealed preference property in matching theory literature. Recently, Aygün and Sönmez (2012) have clarified the role of this property in the context of matching with contracts.

Let $C^f : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in \tilde{F}$ be given by

$$C^f(X)(E) := \max\left\{G(E) - \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\succ_P f} X_{f'}(\Theta_P \cap E), 0\right\}, \forall E \in \Sigma. \tag{1}$$

For any profile $X = (X_f)$ of workers with offers from alternative firms, $C^f(X)$ returns the workers whose best offer is from firm f. In other words, $C^f(X)$ corresponds to the workers willing to match with firm f at X. Note that the function $C^f(X)$ does not depend on X_f. A continuum matching model is summarized as a tuple $(G, F, \mathcal{P}_\Theta, C_F)$.

A **matching** is $M = (M_f)_{f\in\tilde{F}}$ such that $M_f \in \mathcal{X}$ for all $f \in \tilde{F}$" and $\sum_{f\in\tilde{F}} M_f = G$.

Definition 1. *A matching $M = (M_f)_{f\in\tilde{F}}$ is* ***stable*** *if*

1. *For all $P \in \mathcal{P}$ and $E \in \Sigma$, we have $M_f(\Theta_P \cap E) = 0$ for any f satisfying $\varnothing \succ_P f$; and for each $f \in F$, $M_f = C_f(M_f)$, and*

2. *There exist no $f \in F$ and $M'_f \in \mathcal{X}, M'_f \neq M_f$ such that*

$$M'_f = C_f(M'_f \vee M_f) \sqsubset C^f(M'_f \vee M_f, M_{-f}). \tag{2}$$

Condition 1 of this definition means that each matched worker prefers the matching over being unmatched, and that each firm never wishes to drop a subset of its matched workers. Condition 2 requires that there be no firm and a distribution of workers who are currently unmatched and want to be matched to one another. When Condition 2 is violated by f and M'_f, we say that f and M'_f *block* M.

Remark 1. We say that a matching M is **group stable** if Condition 1 of Definition 1 holds and, in addition,

2'. *There exist no $F' \subseteq F$ and $M'_{F'} \in \mathcal{X}^{|F'|}, M'_{F'} \neq M_{F'}$ such that $M'_f = C_f(M'_f \vee M_f) \sqsubset C^f(M'_{F'} \vee M_{F'}, M_{F\setminus F'})$ for all $f \in F'$.*[11]

[11]Recall from earlier definition of C^f in (1) that

$$C^f(M'_{F'} \vee M_{F'}, M_{F\setminus F'})(E) := \max\{0, G(E) - \sum_{P\in\mathcal{P}} \sum_{f'\in F':f'\succ_P f} \max\{M_{f'}(\Theta_P \cap E), M'_{f'}(\Theta_P \cap E)\} - \sum_{f''\in\tilde{F}\setminus F':f''\succ_P f} M_{f''}(\Theta_P \cap E)\}, \forall E \in \Sigma.$$

To see why the last condition Assumption 1 is sensible, note that any worker in a blocking coalition would object to being matched with a firm less preferred to the firms that they were originally matched with. This implies that for each firm $f' \in F'$ such that $f' \succ_P f$ where $\theta \in \Theta_P$, the size of type θ workers equal to $\max\{M_{f'}(\Theta_P \cap E), M'_{f'}(\Theta_P \cap E)\}$ is unavailable to f and should thus be excluded from the set of workers available to f when forming a blocking coalition, which gives us the above expression.

This definition is a strengthening of our stability concept, as it requires that the matching be immune to blocks by coalitions potentially involving multiple firms. Such stability concepts with coalitional blocks are analyzed by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2010), among others. Clearly any group stable matching is stable, because if Condition 2 of stability is violated by a firm f and M'_f, then Condition 2' of group stability is violated by a singleton set $F' = \{f\}$ and $M'_{\{f\}}$. Conversely, since each worker is matched to at most one firm in our model, a stable matching is also group stable. To see this point, note that if Condition 2' of group stability is violated by $F' \subseteq F$ and $M'_{F'}$, then Condition 2 of stability is violated by f and M'_f such that $f \in F'$ and $M'_f \neq M_f$, because $M'_f = C_f(M'_f \vee M_f)$ by assumption and moreover $M'_f \sqsubset C^f(M'_{F'} \vee M_{F'}, M_{F \setminus F'}) \sqsubset C^f(M'_f \vee M_f, M_{-f})$, where the first inclusion relation results from assumption about $M'_{F'}$ and the second from the definition of C^f.

Remark 2. Our model adopts the approach that takes firms' choice functions as a primitive, which gives us some flexibility in describing their preferences, in particular preferences over the alternatives that are not chosen. This approach is also adopted by other studies in matching theory, which include Alkan and Gale (2003) and Aygün and Sönmez (2012) among others. An alternative, albeit more restrictive, approach would be to assume that each firm is endowed with a complete, continuous preference relation over $\mathcal{X}$. Maximization with such a preference will result in an upper hemicontinuous choice correspondence defined over $\mathcal{X}$.[12] Assuming a unique optimal choice will then give us a continuous choice function.

4 Existence in The Continuum Economy

Let $\mathcal{M}$ denote the set of stable matchings. Our characterization of stable matchings involves relating each matching in $\mathcal{M}$ to a fixed point of a certain operator defined over $\mathcal{X}^{n+1}$. To this end, define a map $\Phi_f : \mathcal{X}^{n+1} \to \mathcal{X}$, where for each $X = (X_f)_{f \in \tilde{F}} \in \mathcal{X}^{n+1}$,

$$\Phi_f(X) = G - R_f(X_f). \tag{3}$$

Let $\Phi := (\Phi_f)_{f \in \tilde{F}}$.

To understand this mappings, we say firm f is **available to** $X_f \in \mathcal{X}$ if f is willing to match with each of these workers, and workers X'_f are **available to** firm f if each of these workers has no firm better than f available to him/her. (Note X'_f also includes those who

[12] This also relies on the fact that the set of alternatives $\mathcal{X}$ is compact, a fact we establish in the proof of Theorem 7.

are available to firms worse than f.) Suppose a profile $X = (X_f)$ of workers are available to the firms. The mapping Φ_f then returns as output another profile $X' = \Phi(X)$, where X'_f represents the workers who are selected by f from the available workers X_f *plus* those who are available only to firms strictly preferred to f according to their preferences. That is, Φ_f returns the workers to whom firm f or an even better firm is available.

To define another map, for any $P \in \mathcal{P}$ and $f \in \tilde{F}$, let $f_-^P \in \tilde{F}$ denote an **immediate predecessor** of f (if any) according to preference P: that is, $f_-^P \succ_P f$ *and there is no* $f' \in \tilde{F}$ *such that* $f_-^P \succ_P f' \succ_P f$. Then, define $\Psi_f : \mathcal{X}^{n+1} \to \mathcal{X}$ such that for any $E \in \Sigma$,

$$\Psi_f(X)(E) = G(E) - \sum_{P \in \mathcal{P}} X_{f_-^P}(\Theta_P \cap E), \tag{4}$$

with the convention that $X_{f_-^P}(\Theta_P \cap E) = 0$ if f is the most preferred according to P. Let $\Psi := (\Psi_f)_{f \in \tilde{F}}$. To gain intuition of this mapping, let $X = (X_f)$ be the output of Φ; i.e., X_f are the workers to whom f or a more preferred firm is available. The mapping Ψ_f takes as input such a profile $X = (X_f)$ and returns as output all workers *except for* those who have firm f_-^P or even better firm available to them.[13] Clearly, these latter workers (i.e., those excluded) account for all those who are unavailable to firm f. Since Ψ excludes these workers, Ψ returns as output those workers who are available to f. The particular map also serves a more technical purpose toward an existence of a fixed point, which is used to show the existence of a stable matching (Theorem 2). Importantly, Ψ departs from the standard mapping used in the literature.

Define $T : \mathcal{X}^{n+1} \to \mathcal{X}^{n+1}$ by $T = \Psi \circ \Phi$.

Theorem 1. *A matching M is stable if and only if there is a fixed point $X = T(X)$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = C^f(M), \forall f \in \tilde{F}$.*

Proof. This result follows a corollary of Theorem 11. For details, see Appendix A.3. ∎

The following example illustrates how stable matching is related to a fixed point of the mapping T.

Example 1. Suppose that there are four firms, f_1, f_2, f_3, f_4, and the workers are distributed uniformly on $\Theta = [0, 1]$. All workers prefer f_1 to f_2 to f_3 to f_4. Every firm has responsive

[13] This fact suggests that we could equivalently use a mapping $\tilde{\Psi}_f(X)(E) := G(E) - \sup_{f \in \tilde{F}}\{X_f\}(\Theta_P \cap E)$, where sup is taken with $\sqsubset$ order over $\{X_f\}$. While the fixed point set based on this as a component mapping in place of Ψ also characterizes the set of stable matchings, this perhaps more intuitive mapping is not amenable to the standard fixed point theorem since the "sup" operation turns out not to preserve continuity.

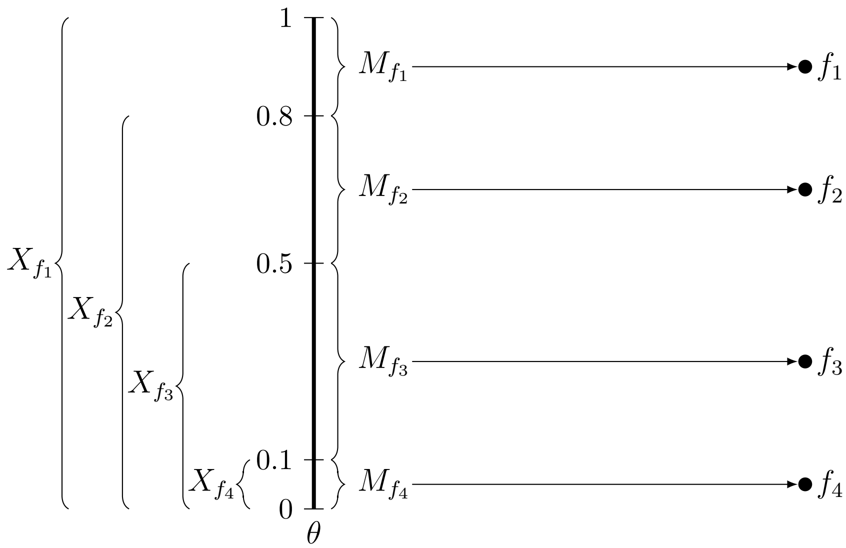

Figure 1: Fixed Point X and Stable Matching M

preferences preferring workers with larger θ. The capacities of f_1, f_2, f_3, and f_4 are 0.2, 0.3, 0.4, and 0.1, respectively.

In this economy, it is clear that there exists a unique stable matching, such that f_1 is matched with all workers in $[0.8, 1]$, f_2 is matched with $[0.5, 0.8)$, f_3 is matched with $[0.1, 0.5)$, and f_4 is matched with $[0, 0.1)$. Figure 1 graphically presents the matching. Denote this stable matching by M. Consider the profile of X as defined in the statement of Theorem 1: For each f, X_f are the workers available to f under matching M. Then, X_{f_1}, X_{f_2}, X_{f_3}, and X_{f_4} correspond to (the uniform distribution over) intervals $[0, 1], [0, 0.8), [0, 0.5)$, and $[0, 0.1)$, respectively.

Now let us verify that, in this environment, X is a fixed point of the operator $T = \Psi \circ \Phi$. Note first that Φ returns for each firm f the workers to whom f is available, or all those workers firm f does not reject from X_f. Specifically, this corresponds to the interval remaining after the truncation of the lower subinterval rejected by each firm. Hence, Φ_{f_1}, Φ_{f_2}, Φ_{f_3}, and Φ_{f_4} correspond to subintervals $[0.8, 1], [0.5, 1], [0.1, 1]$ and $[0, 1]$, respectively.

Applying Ψ to $\Phi(X)$ amounts to truncating for each firm f the workers who have firms better than f available to them. For instance, $T_{f_3}(X) = \Psi_{f_3}(\Phi(X))$ is obtained by truncating the subinterval $\Phi_{f_2}(X) = [0.5, 1]$ (the types to whom f_1 and f_2 are available) from the whole interval $\Theta = [0, 1]$. The resulting worker types are $[0, 0.5)$, coinciding precisely with X_{f_3}. Similarly, one can verify $X = T(X)$.

The fixed-point characterization in this theorem is similar in spirit to previous studies such as Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006). However, the proof is quite different and necessitates a new method of proof. Part of the reason is that, as mentioned above, the continuous type space requires the construction of the mapping T to be distinct from the ones in the previous literature.

4.1 General Preferences

We now introduce a condition on the firms' preferences that ensures existence of stable matchings.

Definition 2. *The firm f's preference is* ***continuous*** *if, for any sequence $(X_k)_k$ and X in $\mathcal{X}$ such that $X_k \xrightarrow{w^*} X$, it holds that $C_f(X_k) \xrightarrow{w^*} C_f(X)$.*

As suggested by the name, continuity of the firm's preferences means that the firm's choice changes continuously with the distribution of workers available to it. Under this assumption, we obtain a general existence result as follows:

Theorem 2. *If each firm's preference is continuous, then there exists a stable matching.*

Proof. This result follows as a corollary of Theorem 7. For details, see Appendix A.3. ∎

To get some intuition, note that in a continuum economy, the distributions of workers matched to firms can change continuously. Therefore, a matching can occur in a way that balances out the number of workers demanded by different firms even with complementary preferences. This helps eliminate any blocks that typically lead to non-existence of stable matchings. To formalize this intuition, we first demonstrate that continuity of firms' preferences implies that mapping T is also continuous. We also verify that $\mathcal{X}$ is a compact set. Continuity of T and compactness of $\mathcal{X}$ allow us to apply the Kakutani-Fan-Glicksberg fixed-point theorem to guarantee that T has a fixed point. Then, the existence of a stable matching follows from Theorem 1, which shows the equivalence between the set of stable matchings and the set of the fixed points of T.

Many complementary preferences are compatible with continuous preferences and thus with existence of a stable matching. Recall the continuum economy example toward the end of Section 2. In that example, firm f has Leontief type preferences, as it wants to hire equal measures of workers of types θ and θ' (so, in particular, the firm wants to hire workers of type θ only if workers of type θ' are also available, and vice versa). However, we have observed in Section 2 that there exists a stable matching in that example. As the firm's preferences are clearly continuous in that example, the existence of a stable matching in that example is implied by Theorem 2.

Stable matching may fail to exist even in the continuum economy unless all firms have continuous preferences, as the following example illustrates.

Example 2. Consider the following economy, which is modified from the opening example in Section 2. There are worker types θ and θ' (each with measure 1/2) and firms f and f'. Firm f wants to hire only measure 1/2 of each worker types together, and would like to be unmatched otherwise; meanwhile, the preference of firm f is continuous and responsive with capacity of measure 1/2 such that it likes θ better than θ', and θ' better than a vacant position. Then obviously C_f violates continuity while $C_{f'}$ does not. As before, we assume

$$\theta : f \succ f';$$
$$\theta' : f' \succ f.$$

We can see that no stable matching exists here, by considering the following two cases:

1. Suppose f is matched to measure 1/2 of each type of workers. For such a matching, none of the capacity of f' is filled. Thus such a matching is blocked by f' and type θ' workers (note that every worker of type θ' is currently matched to f, so prefers f').

2. Suppose f hires no worker. Then, the only candidate for a stable matching is one in which f' is matched to θ up to capacity 1/2 (otherwise, f' and a measure of unmatched workers of type θ would block the matching). Then, since f is most preferred by all θ workers, and θ' workers prefer f to $\emptyset$, the matching is blocked by a coalition of measure 1/2 of type θ workers, measure 1/2 of type θ' workers, and f.

The continuity assumption is important for existence of a stable matching, as this example shows that nonexistence can happen even if only one firm f has a discontinuous choice function. This example might also suggest that non-existence is a rule rather than an exception, given that it shows a way in which a non-existence example for a finite economy can be translated into a non-existence example for a continuum economy by assuming appropriate "lumpiness" of firm preferences, that is, by having firms want to hire only in (multiples of) a certain minimum quantity. However, note that this kind of lumpiness may be a strong assumption in continuum economies. This is in a sharp contrast to finite economies, where lumpiness is a natural consequence of the feasibility constraint, that is, that each worker is indivisible.

4.2 Substitutable Preferences

One of the most well-known classes of preferences studied extensively in the matching theory literature are substitutable preferences. Stable matching has been known to exist with such preferences. We show that the standard set of results including existence of stable matchings are extended in a natural fashion to our continuum economy. Since the arguments establishing these results are by now fairly standard, we shall be brief in our treatment of the case. One relatively novel issue, though, is the question of uniqueness of a stable matching. Azevedo and Leshno (2011) show that multiplicity of stable matchings disappear in the large economy if firms have rich preferences over workers or if their quotas are generic. This striking result is obtained with the restricted preference domain of "responsive" preferences. We provide a condition for uniqueness of a stable matching under general substitutable preferences.

Definition 3. *Firm f's preference is **substitutable** if $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.*

In words, substitutability means that a firm rejects more of any given worker types when facing a bigger set of workers. Clearly, the kind of complementary preferences illustrated in the previous section are excluded by this assumption. Yet, the substitutable preferences are not a special case of the preferences considered in Section 4.1 either, since continuity of preferences need not be satisfied here.

The lattice-theoretic tool is crucial for many standard results, so we begin with a partially ordered set $(\mathcal{X}, \sqsubset)$. As $(\mathcal{X}, \sqsubset)$ is a complete lattice (Lemma 1), so is the partially ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$, where $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ if $X_f \sqsubset X'_f$ for all $f \in \tilde{F}$.

The order defined by $\sqsubset$ can be used to order matchings from the perspective of the firms and the workers. Let M and M' be two matchings, and say $M' \succeq_f M$ if there exist $X'_f \sqsupset X_f$ such that $M_f = C(X_f)$ and $M'_f = C_f(X'_f)$ and say $M' \succeq_F M$ if $M' \succeq_f M$ for every $f \in F''$. Likewise, we say that $M' \succeq_\Theta M$ if for each $f \in \tilde{F}$ and $P \in \mathcal{P}$, $\sum_{f'' \in \tilde{F}: f'' \succeq_P f} M'_{f''}(\Theta_P \cap E) \geq \sum_{f'' \in \tilde{F}: f'' \succeq_P f} M_{f''}(\Theta_P \cap E), \forall E \in \Sigma$. We say that a stable matching $\overline{M}$ is **firm-optimal** (resp., **firm-pessimal**) if $\overline{M} \succeq_F M$ (resp., $\overline{M} \preceq_F M$) for every stable matching M. A matching $\underline{M}$ is **worker-optimal** (resp., **worker-pessimal**) if $\underline{M} \succeq_\Theta M$ (resp., $\underline{M} \preceq_\Theta M$) for every stable matching M.

Again, by Theorem 1, the fixed points of the map $T = \Psi \circ \Phi$ characterize the stable matchings. Since we do not assume continuity of the choice mappings, however, Theorem 2 does not apply. Instead, as proven by the next theorem, substitutability of the firms' preferences implies that the map $T = \Psi \circ \Phi$ is monotone increasing with respect to the partial order $\sqsubset_{\tilde{F}}$. Hence, Tarski's fixed point theorem yields the following result.

Theorem 3. *When the firms' preferences are substitutable, (i) the set $\mathcal{X}^*$ of fixed points of T is nonempty, and $(\mathcal{X}^*, \sqsubset_{\tilde{F}})$ is a complete lattice; and (ii) there exists a firm-optimal (and worker-pessimal) stable matching $\overline{M} = (C_f(\overline{X}_f))_{f \in F}$, where $\overline{X} = \sup_{\sqsubset_F} \mathcal{X}^*$, and a firm-pessimal (and worker-optimal) stable matching $\underline{M} = (C_f(\underline{X}_f))_{f \in F}$, where $\underline{X} = \inf_{\sqsubset_F} \mathcal{X}^*$.*

Proof. See Section A.4. ∎

As has been noted by Hatfield and Milgrom (2005), the algorithm finding the fixed point corresponds to the Gale and Shapley's deferred acceptance algorithm, although the algorithm may not terminate in finite rounds in our continuum model.

Consider an additional restriction on the preferences.

Definition 4. *Firm f's preference exhibits the **law of aggregate demand** if for any $X, X' \in \mathcal{X}$, with $X \sqsubset X'$, $C_f(X)(\Theta) \leq C_f(X')(\Theta)$.*[14]

This property simply ensures that a firm demands more workers (in terms of cardinality) when more workers (in terms of set inclusion) becomes available. This property is needed to obtain the next two results.

Theorem 4 (Rural hospital theorem)**.** *If firms' preferences exhibit substitutability and the law of aggregate demand, then for any stable matching M, we have $M_f(\Theta) = \overline{M}_f(\Theta)$ for each $f \in F$ and $M_\emptyset = \overline{M}_\emptyset$.*

Proof. See Section A.4. ∎

In words, across all stable matchings, the measure of workers matched with each firm $f \in F$ is the same while the set of unmatched workers is identical.

We next introduce a condition that would ensure uniqueness of a stable matching. The condition refers to some new notation. Recall that for any X, X', $X \vee X'$ is the supremum of X and X'. For any matching M and subset F' of firms, let $M^f_{F'}$ be a subpopulation of workers, defined by

$$M^f_{F'}(E) := \sum_{P \in \mathcal{P}} \sum_{f' : f \succ_P f', f' \notin F'} M_{f'}(\Theta_P \cap E) \text{ for each } E \in \Sigma,$$

who are matched outside firms F' and available to firm f under M.

Definition 5 (Rich preferences)**.** *The firms' preferences are **rich** if the following holds. Let M and $\hat{M}$ be two individually rational matchings such that $M_\emptyset = \hat{M}_\emptyset$, $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ for each $f \in F$, and the set $\bar{F} := \{f' \in F | M_{f'} \neq C_{f'}(\hat{M}_{f'} \vee M_{f'})\}$ is nonempty. Then, there exists $f^* \in F$ such that $M_{f^*} \neq C_{f^*}((M_{f^*} + \hat{M}^{f^*}_{\bar{F}}) \wedge G)$.*[15]

[14] This property is an adaptation of the same property to our continuum economy that appears in the literature such as Hatfield and Milgrom (2005), Alkan (2002), and Fleiner (2003).

[15] Recall that $(M_{f^*} + \hat{M}^{f^*}_{\bar{F}}) \wedge G$ is the infimum of $M_{f^*} + \hat{M}^{f^*}_{\bar{F}}$ and G, where $M_{f^*} + \hat{M}^{f^*}_{\bar{F}}$ is defined as $M_{f^*}(E) + \hat{M}^{f^*}_{\bar{F}}(E)$ for all $E \in \Sigma$.

In words, the condition is explained as follows. Suppose there are any two individually rational matchings M and $\hat{M}$ such that all firms prefer $\hat{M}$ to M, strictly so for at least one firm. Then, at matching M, there must exist a firm that would be happy to match with some workers that are not hired by the firms F' that are strictly better off under $\hat{M}$ and are willing to match with that firm under $\hat{M}$.

Theorem 5. *If firms' preferences are rich and substitutable and exhibit the law of aggregate demand, then the set of stable matchings is unique.*

Proof. See Section A.4. ∎

We now show that the uniqueness result of Azevedo and Leshno (2011) under responsive preferences follows from Theorem 5, by establishing that the rich preference condition is implied by the full support condition of Azevedo and Leshno (2011). In fact, we establish a more general result that is applicable even when firm preferences violate responsiveness due to affirmative action or diversity constraint. To do so, let us assume that a worker type is described by a pair (t, θ). The first element, t, is the "type" as used in Abdulkadiroğlu and Sonmez (2003), which describes certain characteristics of a worker such as ethnicity, gender, and socio-economic status: let us call this "ethnic type t", and let a finite set T be the set of ethnic types. The second element, θ, describes all other characteristics of the worker, including the worker's preferences. Let $s_f(t, \theta) \in [0, 1]$ be the score of a worker of type (t, θ) at firm f. We assume that firms have affirmative action constraints, but their preferences are otherwise responsive. That is, the firm f will accept from the highest-score workers up to capacity, but every time either its total capacity or ethnic-type capacity for an ethnic type t is reached, then no more worker of ethnic type t is chosen.

The full support condition for the responsive preferences is then stated as follows.

Definition 6 (Full Support)**.** *Firms' preferences have **full support** if for each preference $P \in \mathcal{P}$, any ethnic type $t \in T$, and for any non-empty open cube $S \subset [0, 1]^n$, the worker types*

$$\Theta_P(t, S) := \{\theta \in \Theta_P \mid (s_f(t, \theta))_{f \in F} \in S\}$$

have a positive measure; i.e., $G(\Theta_P(t, S)) > 0$.[16]

Lemma 2. *If firms have responsive preferences with affirmative action constraints that satisfy the full support condition, then the preferences are rich.*

Proof. See Section A.4. ∎

[16]Note that the type space is assumed to have a Cartesian product structure.

Our full support condition boils down to the full support condition of Azevedo and Leshno (2011) without affirmative action constraint, if T is assumed to be a singleton set. Lemma 2 and Theorem 5 imply that if the full support condition holds, then there exists a unique stable matching even under the affirmative action constraints.

Lastly, the following example demonstrates that if the firm preferences violate the law of aggregate demand, then uniqueness of a stable matching does not necessarily hold even if the firm preferences are rich.

Example 3. Consider a continuum economy with worker types θ_1 and θ_2 (each with measure 1/2) and firms f_1 and f_2. Preferences are as follows:

1. Firm f_1 wants to hire as many workers of type θ_2 as possible if no worker of type θ_1 is available, but if any positive measure of type θ_1 workers is available, then f wants to hire only type θ_1 workers and no type θ_2 workers at all, and f wants to hire only up to measure 1/3 of type θ_1 workers.

2. The preference of firm f_2 is symmetric, changing the roles of worker types θ_1 and θ_2. More specifically, Firm f_2 wants to hire as many workers of type θ_1 as possible if no worker of type θ_2 is available, but if any positive measure of type θ_2 workers is available, then f wants to hire only type θ_2 workers and no type θ_1 workers at all, and f wants to hire only up to measure 1/3 of type θ_2 workers.

3. Worker preferences are as follows:

$$\theta_1 : f_2 \succ f_1,$$
$$\theta_2 : f_1 \succ f_2.$$

Clearly, the firm preferences are substitutable. Also it is clear that firm preferences violate the law of aggregate demand because, for instance, the choice of f_1 from measure 1/2 of θ_2 is to hire all of them, but even adding a measure $\epsilon < 1/2$ of type θ_1 workers rejects all θ_2 workers. There are two stable matchings as shown below:

$$\overline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{3}\theta_1 & \frac{1}{3}\theta_2 \end{pmatrix},$$
$$\underline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta_2 & \frac{1}{2}\theta_1 \end{pmatrix},$$

where the notation is such that, for instance, measure 1/3 of type θ_1 workers are matched to f_1 and measure 1/3 of type θ_2 workers are matched to f_2 at matching $\overline{M}$, for instance.

Note that $\overline{M}$ is the firm-optimal stable matching, while $\underline{M}$ is the firm-pessimal (and worker-optimal) stable matching.

We now show that the firm preferences above satisfy the richness condition. First, note that M in the hypothesis of the richness condition cannot be $\overline{M}$. This is because there is no $\hat{M}$ such that a firm $f' \in \{f_1, f_2\}$ satisfies $\overline{M}_{f'} \neq C_f(\overline{M}_{f'} \vee \hat{M}_{f'})$, as both firms are matched to their most preferred choices at $\overline{M}$. So, without loss of generality, assume $M_{f_1} \neq \overline{M}_{f_1}$. Now, consider the following cases.

1. Suppose that $M_{\o}(\theta_1) \neq 0$. Then, since $M_{\o}(\theta_1) = \hat{M}_{\o}(\theta_1)$ by the hypothesis, we have $\hat{M}_{\o}(\theta_1) \neq 0$, so $\hat{M}_{F}^{f_1}(\theta_1) \neq 0$ as well. This and the assumption $M_{f_1} \neq \overline{M}_{f_1}$ imply that $M_{f_1} \neq C_{f_1}(M_{f_1} + \hat{M}_{F}^{f_1})$, obtaining the desired conclusion.

2. So suppose $M_{\o}(\theta_1) = 0$. This is possible only if $M_{f_2}(\theta_1) \neq 0$ given firm f_1's preferences (such that f_1 does not want to hire more than $1/3$ of θ_1 in any individually rational matching) and the assumption that M is individually rational. In particular, this implies $M_{f_2} \neq \overline{M}_{f_2}$.

    (a) Now, if $M_{\o}(\theta_2) \neq 0$, then (because we have just seen $M_{f_2} \neq \overline{M}_{f_2}$) the argument identical to Case 1 shows $M_{f_2} \neq C_{f_2}(M_{f_2} + \hat{M}_{F}^{f_2})$, obtaining the desired conclusion.

    (b) So assume $M_{\o}(\theta_2) = 0$. This assumption, the maintained assumption $M_{\o}(\theta_1) = 0$, and the individual rationality of M imply that $M = \underline{M}$. In particular, there exists no other individually rational $\hat{M} \neq M$ such that $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$, so the richness condition is vacuously satisfied in this case.

The above cases show that the firm preferences in this example satisfy the richness condition.

5 Approximate Stability in Finite Economies

As we have seen in the illustrative example of Section 2, no matter how large the economy is, as long as it is finite, there does not necessarily exist a stable matching. This motivates us to look for an approximately stable matching in a large finite economy. In this section, we build on the existence of a stable matching in the continuum economy to demonstrate that an approximate stability can be achieved if the economy is finite but sufficiently large.

In order to analyze economies of finite sizes, we consider a sequence of economies $(\Gamma^q)_q$ indexed by a positive integer q. In each economy Γ^q, there is a set of n firms $f_1, \ldots, f_n$, which is fixed across all q. There are also q workers, each with a type in Θ. The worker distribution is normalized with the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(E)$ represents the number of workers with type in E divided by q. Any subpopulation $X^q \sqsubseteq G^q$ is then a discrete distribution over types that is similarly normalized. Note that G^q, and thus X^q, belongs to $\overline{\mathcal{X}}$, though it does not have to be an element of $\mathcal{X}$, i.e. subpopulation of G.

In order to formalize the approximate stability concept, we describe each firm f's preference such that the firm evaluates the set of workers it is matched with in Γ^q, using f's preferences as in the continuum economy, but with the distribution of workers normalized by the economy's size. To do so, we first endow firms with cardinal utility functions over distributions of workers. Let $u_f : \overline{\mathcal{X}} \to \mathbb{R}$ denote the continuous utility function of firm f, with $u_f(X)$ being the firm's utility from matching with a subpopulation of workers $X \in \overline{\mathcal{X}}$.[17] That there is a single utility function for each firm defined on the space $\overline{\mathcal{X}}$, which includes worker distributions in any finite economy as well as continuum economy, means that the preferences of the firms remain constant (or consistent) over a sequence of economy $(\Gamma^q)_q$ and its limit Γ. Each firm's choice function in Γ^q is modified to choose the most preferred subpopulation among the discrete distributions — the chosen subpopulation should be a step function with each step being a multiple of $\frac{1}{q}$ that maximizes utility u_f of f.

Given any $\epsilon > 0$, we say that a matching is ϵ-stable if, for any block, the utility gain for any blocking firm is less than ϵ. Formally,

Definition 7. *A matching M in an economy Γ_q is ϵ-stable if, for any f and M'_f that block M, $u_f(M'_f) < u_f(M_f) + \epsilon$.*

Let us say that a sequence of economies $(\Gamma^q)_q$ converges to a continuum economy Γ if the measure G^q of worker types converges weakly to the measure G of the continuum economy, that is, $G^q \xrightarrow{w^*} G$.

Theorem 6. *Fix any $\epsilon > 0$ and a sequence of economies $(\Gamma^q)_q$ that converges to a continuum economy. For any sufficiently large q, there exists an ϵ-stable matching in Γ^q.*

Proof. See Appendix A.5. ∎

[17] To guarantee the existence of such a utility function, we may assume as in Remark 2 that each firm is endowed with a complete, continuous preference relation. Then, since the set of alternatives, $\overline{\mathcal{X}}$, is a compact metric space, such a preference can be represented by a continuous utility function according to the Debreu representation theorem (Debreu, 1954).

Below we revisit the example in Section 2 to give a concrete example of an approximately stable matching.

Example 4. Recall the finite economy described in Section 2.[18] If the index q is odd, then a stable matching does not exist. Let us consider the following matching: firm f matches with $\frac{q+1}{2}$ workers of each type and firm f' matches with all remaining workers, i.e. $\frac{q-1}{2}$ workers of each type. Given this matching, it is straightforward to see that there is no blocking coalition involving firm f. Also, the only blocking coalition involving firm f' entails taking only a single worker of type θ' away from firm f. If q is large, then this deviation will only result in a small utility gain for firm f', so the above matching is ϵ-stable. In fact, any finite matching converging to the stable matching of the continuum economy, found in the example of Section 2, will suffice for our purpose.

6 Indifferences and Time Share/Probabilistic Matching Models

In this section, we extend our analysis to the case in which a firm's choice from a distribution of available workers is not necessarily unique. In other words, we allow a firm's choice to be a correspondence (multi-valued function). There are at least three motivations for this generalization. First, there is a continuum of workers in our environment, and in such a situation it is natural to assume that a firm may be indifferent between some distributions and choose more than one distribution as most preferred. Second, indifferences appear to be inherent in some applications. In school choice, for instance, schools are often required by law to regard many students to have the same priority,[19] in which case the choice is multi-valued. Lastly, as will be seen in Subsection 6.2, our model turns out to have a connection with probabilistic and time share matching models. In that model, a distribution of workers corresponds to time shares/probabilities with which workers are matched to firms, and indifferences naturally arise between distributions that represent the same matching in terms of time shares/probabilities.

[18] With slight abuse of notation, this example assumes that there are a total of $2q$ workers (q workers of θ and θ' each) rather than q. Of course, this is done for purely expositional purposes.

[19] In the public school choice program in Boston, for instance, a student's priority at a school is based only on coarse criteria such as the student's residence and whether her sibling is currently enrolled at that school. Consequently, at each school, many students are equipped with the same priority (Abdulkadiroğlu et al., 2005).

6.1 Stable Matching with Choice Correspondence

Let $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **choice correspondence**: i.e., for any $X \in \mathcal{X}$, $C_f(X) \subset \mathcal{X}$ is the set of subpopulations of X that are the most preferred by f among all subpopulations of X. By convention, we let $C_\emptyset(X) = \{X\}, \forall X \in \mathcal{X}$. Then, let $R_f(X) := X - C_f(X)$, or equivalently $R_f(X) = \{Y \in \mathcal{X} \,|\, Y = X - X' \text{ for some } X' \in C_f(X)\}$. We assume that for any $X \in \mathcal{X}$, $C_f(X)$ is nonempty. Assume further that $C_f(\cdot)$ satisfies the revealed preference property: For any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \cap \mathcal{X}_{X'} \neq \emptyset$, then $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$ (recall $\mathcal{X}_{X'}$ is the set of subpopulations of X'). Define a function $C^f : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in F$ to be the same as the one in (1).

Definition 8. *A matching M is **stable** if*

1. *For all $P \in \mathcal{P}$ and $E \in \Sigma$, we have $M_f(\Theta_P \cap E) = 0$ for any f satisfying $\emptyset \succ_P f$; and for each $f \in F$, $M_f \in C_f(M_f)$, and*

2. *There exist no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f})$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$.*

Clearly, this definition is a generalization of Definition 1 to the case of choice correspondences. The existence of a stable matching then follows from imposing appropriate conditions on the firms' choice correspondences that allow for the existence of a fixed-point for a correspondence operator defined similarly to the mapping T in Section 4.[20]

The notion of continuous preferences is naturally extended to the following:

Definition 9. *The firm f's choice correspondence C_f is **upper hemicontinuous** if, whenever any sequence $(X^k)_{k=1}^\infty$ in $\mathcal{X}$ and sequence $(\tilde{X}^k)_{k=1}^\infty$ in $\mathcal{X}$ weakly converge to some X and $\tilde{X}$, respectively, such that $\tilde{X}^k \in C_f(X^k)$, then $\tilde{X} \in C_f(X)$.*

Theorem 7. *Suppose that for each $f \in F$, C_f is convex-, closed-valued, and upper hemicontinuous. Then, there exists a stable matching.*

Proof. See Appendix A.3. ∎

This result shows that our main result — that there exists a stable matching when there are a continuum of workers — does not hinge on the restrictive assumption that each firm's choice is unique. On the contrary, this result holds for a wide range of specifications that allow for indifferences and choice correspondences. As will be seen in the next section, this generalization turns out to be useful when analyzing a model of probabilistic and time-share matchings, which may appear to be unrelated to our model at a first glance.

[20] For details, refer to Appendix A.3.

6.2 Probabilistic and Time-Share Matching Models

As mentioned above, our model turns out to have a connection with probabilistic and time share matching models. In both models, there is a *finite* set of workers with a set of firms remaining to be finite. In the probabilistic matching model, a matching between firms and workers corresponds to the probabilities that they are matched to one another, while, in the time-share model, it corresponds to the time that they spend together. Probabilistic matching is often used in allocation problems without money such as school choice, while time-share models have been proposed as a solution to labor matching markets in which part-time jobs are available. We will follow the time share interpretation for the remainder of this Section.

In the time-share (or probabilistic) matching model, there are a finite set $\Theta = \{\theta_1, ..., \theta_m\}$ of workers endowed with time, and as before a finite set $F = \{f_1, ..., f_n\}$ of firms who are looking to hire these workers for nonnegative shares of their time. As before, $\tilde{F} := F \cup \{\o\}$. We let $\succ_\theta$ and $\succeq_\theta$ denote the strict and weak preferences of each worker θ, respectively. Assume without loss of generality that each individual worker has a time endowment normalized to $\frac{1}{m}$. A **matching** in the time-share model is an $(n+1)\times m$ matrix $Z = [z_{f\theta}]_{f\in\tilde{F},\theta\in\Theta}$ such that $z_{f\theta} \geq 0$ and $\sum_{f\in\tilde{F}} z_{f\theta} = \frac{1}{m}$ for each θ. Throughout, vectors $z_f = (z_{f\theta})_{\theta\in\Theta}$ and $z_\theta = (z_{f\theta})_{f\in\tilde{F}}$ denote firm f's and worker θ's time shares, respectively.

Define

$$C_\theta^f(Z) := \max\left\{\frac{1}{m} - \sum_{f'\in\tilde{F}: f'\succ_\theta f} z_{f'\theta}, 0\right\},$$

which is the share of time of worker θ that is not taken by the firms preferred to f and thus is available for f. Then, we let $C^f(Z) = (C_\theta^f(Z))_{\theta\in\Theta}$.

Next each firm f's preference is described as follows. We allow a firm to be indifferent among a set of workers. To allow for indifference, we let a partition $\{S_f^1, ..., S_f^{K_f}\}$ of Θ (with $S^i \cap S^j = \emptyset$ and $\cup_{i=1}^{K_f} S_f^k = \Theta$) denote the set of indifference classes. The partition means that firm f is indifferent to redistributing the total time contracted with workers within each group S_f^k. Let $I_f = \{1, ..., K_f\}$ be the associated index set. For any $z \in [0, \frac{1}{m}]^m$, we let $\mathbb{P}_f(z) := (\sum_{\theta\in S_f^j} z_\theta)_{j\in I_f}$ denote the vector of total times that firm f contracts with alternative indifference classes. The firm f's preference is then represented by a choice correspondence C_f such that for any $z \in [0, \frac{1}{m}]^m$ and $y \in C_f(z)$, $y_\theta \in [0, z_{f\theta}], \forall\theta \in \Theta$ (i.e. $y \sqsubset z$), and also $y' \in C_f(z)$ if and only if $\mathbb{P}_f(y) = \mathbb{P}_f(y')$. (Let $C_\o(z) = \{z\}$.) In other words, each firm f has uniquely optimal choice up to the time share allocation within each indifference class. It then follows that C_f induces a function, $\Gamma_f : [0, \frac{1}{m}]^m \to \mathbb{R}_+^{K_f}$, given by $\Gamma_f(z) = \mathbb{P}_f(y)$ for some $y \in C_f(z)$. In words, the k-th component of $\Gamma_f(z)$, denoted

$\Gamma_f^k(z)$, $k \in I_f$, indicates firm f's demand for total time share from (indifference) class k of workers, given z. Given our earlier assumption, $\Gamma_f(z)$ is well (i.e., uniquely) defined.

A time share model described so far can be translated into a continuum matching model by simply setting $G(\{\theta\}) = \frac{1}{m}$ for each $\theta \in \Theta$ and $\Theta_P = \{\theta \in \Theta \mid \succ_\theta = \succ_P\}$ for each $P \in \mathcal{P}$. So we denote a time share model as $(G, F, \mathcal{P}_\Theta, C_F)$. Observe that when the space Θ is finite, the boundary of any subset is a null set,[21] so our assumption that $G(\partial\Theta_P) = 0$ for each $P \in \mathcal{P}$ is satisfied.

Notice that our time share model has the same structure as the *fractional matching model* that is also known as *random or aggregate matching model* in the literature.[22]

By adapting to the time share model, we state the notion of stability as follows:

Definition 10. *A matching Z in a time share model is **stable** if (i) for any θ and $f \in F$ satisfying $\emptyset \succ_\theta f$, $z_{f\theta} = 0$; (ii) $z_f \in C_f(z_f)$ for every f; and (iii) there exist no $f \in F$ and z'_f such that $z'_f \sqsubset C^f(z'_f \vee z_f, z_{-f})$, $z'_f \in C_f(z'_f \vee z_f)$, and $z_f \notin C_f(z'_f \vee z_f)$.*[23]

Notice that the stability in the time share model corresponds to ex ante stability, if one interprets the time share as probability share. That is, if any worker (or student) were to shift shares from a less preferred to a more preferred firm (school), the latter must necessarily be worse off. In this sense, stability, in particular condition (iii), involves the notion of fairness or no justified envy. What this condition reduces to when preferences are all responsive (as is the case in the existing literature) is that whenever a worker w enjoys a positive share with a firm f, then all workers preferred by f more than w have zero share with firms they prefer less than f.

Remark 3. *We say that a matching Z in a time share model is **group stable** if (i) and (ii) of Definition 10 hold and (iii) there exist no $F' \subset F$ and $Z' \neq Z$ such that $z'_f \leq C^f(z'_{F'} \vee z_{F'}, z_{\tilde{F}\setminus F'})$, $z'_f \in C_f(z'_f \vee z_f)$ for all $f \in F'$, and $z_f \notin C_f(z'_f \vee z_f)$ for some $f \in F'$. This definition is a strengthening of our stability concept, requiring that the matching be immune to blocks by coalitions potentially involving multiple firms. Following the argument presented earlier for our basic continuum model, one can show that stability and group stability are equivalent in our time share model.*

[21] To see this, consider any set $E \subset \Theta$. Then, $\partial E = \overline{E} \cap \overline{E^c}$, where $\overline{E}$ and $\overline{E^c}$ are the closure of E and its complement, respectively. Since the space is finite, we have $\overline{E} = E$ and $\overline{E^c} = E^c$, so $\overline{E} \cap \overline{E^c} = E \cap E^c = \emptyset$.

[22] This literature includes Hylland and Zeckhauser (1979), Roth, Rothblum and Vande Vate (1993), Alkan and Gale (2003), Baïou and Balinski (2000), Echenique et al. (2013), and Kesten and Ünver (2009) among others.

[23] As usual, the operator $\vee$ here gives the componentwise maximum of two vectors.

The existence of stable matching in the time share model follows from Theorems 7 in a straightforward manner.

Theorem 8. *Suppose that Γ_f is continuous for each $f \in F$. Then, there exists a stable matching in the time share model.*

Proof. By Theorem 7, it suffices to show that C_f is convex- and closed-valued, and upper hemicontinuous.

For any given $z \in [0, \frac{1}{m}]^m$, consider any $z', z'' \in C_f(z)$. Then, for any $\lambda \in [0, 1]$,

$$\mathbb{P}_f(\lambda z' + (1-\lambda)z'') = \lambda \mathbb{P}_f(z') + (1-\lambda)\mathbb{P}_f(z'') = \Gamma_f(z),$$

where the first equality holds due to the linearity of $\mathbb{P}_f$, and the second equality due to the assumption that $z', z'' \in C_f(z)$ so $\mathbb{P}_f(z') = \mathbb{P}_f(z'') = \Gamma_f(z)$. This string of equalities implies that $\lambda z' + (1-\lambda)z'' \in C_f(z)$, so C_f is convex-valued.

To show the upper hemicontinuity, consider two sequences $(z^k)_{k=1}^{\infty}$ and $(\tilde{z}^k)_{k=1}^{\infty}$ converging to some z and $\tilde{z}$ in $[0, \frac{1}{m}]^m$, respectively, such that for each k, $\tilde{z}^k \in C_f(z^k)$, i.e. $\mathbb{P}_f(\tilde{z}^k) = \Gamma_f(z^k)$. Since $\mathbb{P}_f$ and Γ_f are all continuous, we have $\mathbb{P}_f(\tilde{z}) = \lim_{k\to\infty} \mathbb{P}_f(\tilde{z}^k) = \lim_{k\to\infty} \Gamma_f(z^k) = \Gamma_f(z)$, which means that $\tilde{z} \in C_f(z)$, establishing the upper hemicontinuity of C_f. The proof of closed-valuedness is similar and hence omitted. ▌

As mentioned before, a matching can be interpreted as a profile of probability shares. In particular, for probabilistic matching, say in the school choice context, stability involves the notion of fairness or no justified envy in an ex ante sense. In such an environment with indifference of school preferences/priority, the following stronger notion of fairness, proposed by Kesten and Ünver (2009), is of interest.

Definition 11. *A matching Z in the time share model is* ***strongly stable*** *if (i) it is stable and (ii) for any $f \in F$ and any $\theta, \theta' \in S_f^k$, if $z_{f\theta} < z_{f\theta'}$, then $\sum_{f' \in \tilde{F}: f' \prec_\theta f} z_{f'\theta} = 0$.*

In words, strong stability means the following. Suppose a worker θ is in the same priority class as θ' with respect to firm f. If θ enjoys a strictly lower share with f than θ' does, then it must be that θ has no share with any firm she prefers strictly less than f. In that sense, the workers in the same priority class should not be discriminated against one another. This is an additional requirement not implied by the stability alone (which requires fairness across workers that a firm is not indifferent between).

We now show the existence of a strongly stable matching by introducing another choice correspondence. To do so, let us first define a new correspondence: for any $z \in [0, \frac{1}{m}]^m$,

$$B_f(z) := \{z' \sqsubset z \,|\, \text{ for each } k, \text{ there is some } \alpha^k \in [0, \tfrac{1}{m}] \text{ such that } z'_{f\theta} = \min\{z_{f\theta}, \alpha^k\}, \forall \theta \in S_f^k\}.$$

We then modify each firm f's choice correspondence C_f into

$$\tilde{C}_f(z) = C_f(z) \cap B_f(z). \tag{5}$$

We also let $\tilde{C}_\varnothing = C_\varnothing$. The correspondence $B_f(z)$ requires each firm to hire all workers in each indifference class k for the same amount of time α^k, except for workers whose available time $z_{f\theta}$ falls short of α^k (and for any worker whose available time is smaller than α^k, the entire time available from her is chosen).[24] Then, we can show the following result:

Lemma 3. *Consider the correspondence $\tilde{C}_f$ defined in (5). For any $z \in [0, \frac{1}{m}]^m$, $\tilde{C}_f(z)$ is a singleton set. Also, $\tilde{C}_f$ satisfies the revealed preference property.*

Proof. See Section A.6. ∎

Using this choice correspondence, we establish the equivalence between the set of strongly stable matchings in the original time share model and the set of stable matchings in a modified time share model:

Theorem 9. *A matching Z in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$ is strongly stable if and only if it is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ with $\tilde{C}_f$ defined in (5).*

Proof. See Section A.6. ∎

Given this equivalence result, we are now ready to show the existence of a strongly stable matching.

Theorem 10. *Suppose that Γ_f is continuous. Then, there exists a strongly stable matching in the time share model.*

Proof. See Section A.6. ∎

This result generalizes the existence result of a strongly stable matching in the school choice context as studied by Kesten and Ünver (2009). They consider a school choice problem in which schools may regard multiple students as having the same priority. In that environment, they define probabilistic matchings that satisfy our strong stability property (which they call strong ex ante stability), and show their existence, in the environment in which schools have responsive preferences with ties. Our contribution here lies in formalizing a strongly stable matching and establishing its existence with general preferences that may violate responsiveness or even substitutability. Our result could be useful for

[24] Note that for any z, $B_f(z)$ is nonempty since $z \in B_f(z)$, which is true since $\min\{z_\theta, \frac{1}{m}\} = z_\theta, \forall \theta \in S_f^k$.

school choice environments in which the schools may need balanced student body in terms of gender or ethnicity, income, or skill levels. For example, in New York City, the so-called Education Option (EdOpt) school programs are required to assign 16 percent, 68 percent, and another 16 percent of the seats to the top performers, middle performers, and the lower performers, respectively (Abdulkadiroğlu, Pathak and Roth, 2005). Strong stability will ensure ex ante fairness in the sense that it not only implements the desired mix of students but also will ensure fairness in the treatment of students relative to the schools' objective.

A Appendix

A.1 Analysis of the Example in Section 2

Let r be the number of workers with each of the two types who are matched to f. We consider the following cases:

1. Suppose $r > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus such a matching is blocked by f' and a type θ' worker who is currently matched to f.

2. Suppose $r < q/2$. Consider the following cases.

   (a) Suppose that there exists a type θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

   (b) Suppose that there exists no type θ worker who is unmatched. This implies that there exists a type θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $r < q/2$, the matching is blocked by a coalition of a type θ worker, a type θ' worker, and f.

A.2 Proof of Lemma 1

For any subset $\mathcal{Y} \subset \mathcal{X}$, define

$$\overline{Y}(E) := \sup\{\sum_i Y_i(E_i) \,|\, \{E_i\} \text{ is a finite partition of } E \text{ in } \Sigma \text{ and } \{Y_i\} \text{ is a finite collection of measures in } \mathcal{Y}, \forall i\}, \forall E.$$

and $\underline{Y}$ analogously (by replacing "sup" with "inf"). We prove the lemma by showing that $\overline{Y} = \sup \mathcal{Y} \in \mathcal{Y}$ and $\underline{Y} = \inf \mathcal{Y} \in \mathcal{X}$.

First of all, note that $\overline{Y}$ and $\underline{Y}$ are monotonic, i.e. for any $E \subset D$, we have $\overline{Y}(D) \geq \overline{Y}(E)$ and $\underline{Y}(D) \geq \underline{Y}(E)$, whose proof is straightforward and thus omitted.

We next show that $\overline{Y}$ and $\underline{Y}$ are measures. We only prove the countable additivity of $\overline{Y}$, since the other properties are straightforward to prove and also since a similar argument applies to $\underline{Y}$. To this end, consider any countable collection $\{E_i\}$ of disjoint sets in Σ and let $D = \cup E_i$. We need to show that $\overline{Y}(D) = \sum_i \overline{Y}(E_i)$. For doing so, consider any finite partition $\{D_i\}$ of D and any finite collection of measures $\{Y_i\}$. Letting $E_{ij} = E_i \cap D_j$, for any i, the collection $\{E_{ij}\}_j$ is a finite partition of E_i in Σ. Thus, we have

$$\sum_i Y_i(D_i) = \sum_i \sum_j Y_i(E_{ij}) \leq \sum_i \overline{Y}(E_i).$$

Since this inequality holds for any finite partition $\{D_i\}$ of D and collection $\{Y_i\}$, we must have $\overline{Y}(D) \leq \sum_i \overline{Y}(E_i)$. To show that the reverse inequality also holds, for each E_i, we consider any finite partition $\{E_{ij}\}_j$ of E_i in Σ and collection of measures $\{Y_{ij}\}_j$ in $\mathcal{Y}$. We prove that $\overline{Y}(D) \geq \sum_i \sum_j Y_{ij}(E_{ij})$, which would imply $\overline{Y}(D) \geq \sum_i \overline{Y}_{ij}(E_i)$ as desired since the partition $\{E_{ij}\}_j$ and collection $\{Y_{ij}\}_j$ are arbitrarily chosen for each i. Suppose not for contradiction. Then, we must have $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij})$ for some k. Letting $E := \cup_{i=1}^k (\cup_j E_{ij})$, this implies $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij}) \leq \overline{Y}(E)$, where the second inequality holds by the definitioin of $\overline{Y}$. This contradicts with the monotonicity of $\overline{Y}$ since $E \subset D$.

We now show that $\overline{Y}$ and $\underline{Y}$ are the supremum and infimum of $\mathcal{Y}$, respectively. It is straightforward to check that for any $Y \in \mathcal{Y}$, $Y \sqsubset \overline{Y}$ and $\underline{Y} \sqsubset Y$. Consider any $X, X' \in \mathcal{X}$ such that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$ and $X' \sqsubset Y$. We show that $\overline{Y} \sqsubset X$ and $X' \sqsubset \underline{Y}$. First, if $\overline{Y} \not\sqsubset X$ to the contrary, then there must be some $E \in \Sigma$ such that $\overline{Y}(E) > X(E)$. This means there are a finite partition $\{E_i\}$ of E and a collection of measures $\{Y_i\}$ in $\mathcal{Y}$ such that $\overline{Y}(E) \geq \sum Y_i(E_i) > X(E) = \sum X(E_i)$. Thus, for at least one i, we have $Y_i(E_i) > X(E_i)$, which contradicts the assumption that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$. An analogous argument can be used to show $X' \sqsubset \underline{Y}$.

A.3 Proof of Theorems 1, 2 and 7

Since Theorem 1 and 2 follow as corollaries from their counterparts in the correspondence case, we begin our analysis by introducing a correspondence version of the fixed point mapping T. To do so, we define Φ_f and Ψ_f to be the same as in (3) and (4), except that C_f and thus R_f are now correspondences. Let $\Phi := (\Phi_f)_{f\in\tilde{F}}$ and $\Psi := (\Psi_f)_{f\in\tilde{F}}$. Now consider a composite map $T := \Psi \circ \Phi$. Note that Ψ is a function but Φ is a correspondence, so T is also a correspondence.

Then we obtain the characterization of stable matching as fixed points of T, of which Theorem 1 is a special case.

Theorem 11. *A matching M is stable if and only if there is a fixed point $X \in T(X)$ such that $M_f \in C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = C^f(M), \forall f \in \tilde{F}$.*

Proof. **("Only if" part):** Suppose M is a stable matching. Define $X = (X_f)$ as

$$\begin{aligned} X_f(E) = C^f(M)(E) &= G(E) - \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E) \\ &= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma. \end{aligned} \tag{6}$$

Let $\bar{X} \in \mathcal{X}^{n+1}$ be given by

$$\bar{X}_f(E) = \sum_{P \in \mathcal{P}} \left(\sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap E) \right), \forall E \in \Sigma.$$

Since M is a matching, we have

$$\begin{aligned} X_f(E) + \bar{X}_f(E) &= \sum_{P \in \mathcal{P}} \left(M_f(\Theta_P \cap E) + \sum_{f' \in \tilde{F}} M_{f'}(\Theta_P \cap E) \right) \\ &= M_f(E) + \sum_{f' \in \tilde{F}} \sum_{P \in \mathcal{P}} M_{f'}(\Theta_P \cap E) = M_f(E) + \sum_{f' \in \tilde{F}} M_{f'}(E) = M_f(E) + G(E), \end{aligned}$$

so $G - X_f + M_f = \bar{X}_f$ for all $f \in \tilde{F}$.

We now prove that X is a fixed point of T. First, for any $E \in \Sigma$, we have

$$\begin{aligned} C^f(X_f, M_{-f})(E) &= G(E) - \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E) \\ &= \sum_{P \in \mathcal{P}} \left(G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E) \right) \\ &= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = X_f(E), \end{aligned}$$

where the first equality follows from the definition of C^f along with the fact that C^f does not depend on its first argument, and the last equality follows from the definition of X. Given this and the fact that $M_f \sqsubset X_f$, we must have $M_f \in C_f(X_f)$. If not, then $M_f \notin C_f(X_f)$ and there is some $M'_f \in C_f(X_f)$. By the revealed preference and the fact that $(M'_f \vee M_f) \sqsubset X_f$, we have $M_f \notin C_f(M'_f \vee M_f)$ and $M'_f \in C_f(M'_f \vee M_f)$, which means that M is unstable, and we have a contradiction. It thus follows that $X_f - M_f \in R_f(X_f)$ and hence $\bar{X}_f = G - (X_f - M_f) \in \Phi_f(X)$ for each $f \in \tilde{F}$, or $\bar{X} \in \Phi(X)$.

Note now that for all $f \in \tilde{F}$ and $E \in \Sigma$,

$$\Psi_f(\bar{X}_f)(E) = G(E) - \sum_{P \in \mathcal{P}} \bar{X}_{f_-^P}(\Theta_P \cap E) = G(E) - \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \succ_P f} M_f(\Theta_P \cap E)$$
$$= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_f(\Theta_P \cap E) = X_f(E).$$

Thus, $X = \Psi(\bar{X}) \in \Psi(\Phi(X)) = T(X)$ so X is a fixed point of T.

("If" part): Let $X \in \mathcal{X}^{n+1}$ be a fixed point of T. Then, there exists $X' \in \mathcal{X}^{n+1}$ such that for each $f \in \tilde{F}$, $X_f = \Psi_f(X')$ and $X'_f \in \Phi_f(X)$. By definition of Φ, there exists $M_f \in C_f(X)$ such that $X'_f = G - (X_f - M_f)$. We first show that for all $f \in \tilde{F}$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_f(\Theta_P \cap E) = G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E). \tag{7}$$

It is straightforward to see from (4) that if f is ranked first according to P, then $X'_{f_-^P}(\Theta_P \cap E) = 0$, so $X_f(\Theta_P \cap E) = \Psi_f(X')(\Theta_P \cap E) = G(\Theta_P \cap E)$. Hence, (7) holds for such f.

Consider now any $f \in \tilde{F}$ and assume that (7) holds true for f_-^P, so $X_{f_-^P}(\Theta_P \cap E) = G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f_-^P} M_{f'}(\Theta_P \cap E)$. We show that (7) also holds for f. To do so, observe

$$X'_{f_-^P}(\Theta_P \cap E) = G(\Theta_P \cap E) - \Big(X_{f_-^P}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E)\Big)$$
$$= \sum_{f' \in \tilde{F}: f' \succ_P f_-^P} M_{f'}(\Theta_P \cap E) + M_{f_-^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E), \tag{8}$$

where the first equality follows from the fact that $X'_f = G - (X_f - M_f)$ and the second equality follows from the induction hypothesis. Thus, using (8), we have

$$X_f(\Theta_P \cap E) = \Psi_f(X')(\Theta_P \cap E) = G(\Theta_P \cap E) - X'_{f_-^P}(\Theta_P \cap E)$$
$$= G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E),$$

as was to be shown.

Now recall $M_\emptyset \in C_\emptyset(X_\emptyset) = \{X_\emptyset\}$, and set $f = \emptyset$ in (7) to obtain for any $E \in \Sigma$

$$G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P \emptyset} M_{f'}(\Theta_P \cap E) = X_\emptyset(\Theta_P \cap E) = M_\emptyset(\Theta_P \cap E),$$

or

$$G(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \succeq_P \emptyset} M_{f'}(\Theta_P \cap E). \tag{9}$$

Then, for $f \in F$ such that $\emptyset = f_-^P$, (7) and (9) imply

$$X_f(\Theta_P \cap E) = G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succeq_P \emptyset} M_{f'}(\Theta_P \cap E) = 0$$

so $M_f(\Theta_P \cap E) = 0$ for such f since $M_f \in C_f(X_f)$ so $M_f \sqsubset X_f$. A similar argument can be used to show that $M_f(\Theta_P \cap E) = 0$ for all other f such that $\emptyset \succ_P f$. Combine this with (9) to observe that for all $E \in \Sigma$, we have $G(\Theta_P \cap E) = \sum_{f' \in \tilde{F}} M_{f'}(\Theta_P \cap E)$, which means $\sum_{f \in \tilde{F}} M_f(E) = G(E)$ for all $E \in \Sigma$, i.e. M is a matching. Also, the first part of Condition 1 in Definition 8 has also been shown. The second part (i.e. $M_f \in C_f(M_f)$) follows from the revealed preference property since $M_f \sqsubset X_f$ and $M_f \in C_f(X_f)$.

It only remains to check Condition 2 of Definition 8. Suppose for contradiction it fails. Then, there exists f and M'_f such that

$$M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f}),\ M'_f \in C_f(M'_f \vee M_f) \text{ and } M_f \notin C_f(M'_f \vee M_f). \tag{10}$$

We first observe that for all $f \in F$, $\succ \in \mathcal{P}$, and $E \in \Sigma$,

$$C^f(M'_f \vee M_f, M_{-f})(\Theta_P \cap E) = G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E) = X_f(\Theta_P \cap E),$$

where the first equality follows from the definition of C^f and the fact that it does not depend on the first argument, and the second equality follows from (7). Then, summing the LHS and RHS of the above equation across all $P \in \mathcal{P}$ yields $C^f(M'_f \vee M_f, M_{-f})(E) = X_f(E)$ for all $E \in \Sigma$, as desired. Thus, we have $M'_f \sqsubset C^f(M'_f \vee M_f, M_{-f}) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (10). We have thus proven that M is stable.

Lastly, $X_f = C^f(M)$ follows from (7) since for all $E \in \Sigma$

$$\begin{aligned} X_f(E) = \sum_{P \in \mathcal{P}} X_f(\Theta_P \cap E) &= \sum_{P \in \mathcal{P}} \left(G(\Theta_P \cap E) - \sum_{f' \in \tilde{F}: f' \succ_P f} M_{f'}(\Theta_P \cap E) \right) \\ &= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E). \blacksquare \end{aligned}$$

We now prove Theorem 7, from which Theorem 2 follows as corollary since, if T is a single-valued mapping, then the convex- and close-valuedness hold trivially while the upper hemicontinuity is equivalent to the continuity. To prove existence, by Theorem 11, it suffices to show that the mapping T has a fixed point. To this end, we establish a series of Lemmas.

Lemma 4. *The set $\mathcal{X}$ is convex, and Φ and Ψ are convex valued.*

Proof. Convexity of $\mathcal{X}$ follows trivially. To prove that Φ is convex valued, consider any $X \in \mathcal{X}$ and $X', X'' \in \Phi_f(X)$. It means that there are some $Y', Y'' \in C_f(X)$ satisfying $X' = G - (X - Y')$ and $X'' = G - (X - Y'')$. Then, the convexity of $C_f(X)$ implies that for any $\lambda \in [0, 1]$, $\lambda Y' + (1-\lambda)Y'' \in C_f(X)$ so $\lambda X' + (1-\lambda)X'' = G - (X - (\lambda Y' + (1-\lambda)Y'')) \in \Phi(X)$. Ψ is a function and, as such, convex-valued in a trivial sense. ▮

We now establish the compactness of $\mathcal{X}$ and the upper-hemi continuity of Φ and Ψ. Recall that $\mathcal{X}$ is endowed with weak* topology. The notion of convergence in this topology, i.e. weak convergence, can be stated as follows: A sequence of measures $(X_k)_k$ in $\mathcal{X}$ weakly converges to a measure $X \in \mathcal{X}$, written as $X_k \xrightarrow{w^*} X$, if and only if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all $h \in C(\Theta)$, where $C(\Theta)$ is the space of all continuous functions defined on Θ.[25] The next result provides some conditions that are equivalent to weak convergence.[26]

Theorem 12. *Let X and $\{X_k\}$ be finite measures on Σ. The following conditions are equivalent:*

(a) $X_k \xrightarrow{w^*} X$;

(b) $\int_\Theta h dX_k \to \int_\Theta h dX$ *for all $h \in C_u(\Theta)$, where $C_u(\Theta)$ is the space of all uniformly continuous functions defined on Θ.*

(c) $\liminf_k X_k(A) \geq X(A)$ *for every open set $A \subset \Theta$, and $X_k(\Theta) \to X(\Theta)$;*

(d) $\limsup_k X_k(A) \leq X(A)$ *for every closed set $A \subset \Theta$, and $X_k(\Theta) \to X(\Theta)$;*

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[25] Note that $C(\Theta)$ is a Banach space endowed with the sup (or max) norm, i.e., for any $h \in C(\Theta)$, $\|h\| = \max_{\theta\in\Theta} |h(\theta)|$ while $\mathcal{X}$ is endowed with the dual norm, i.e., for any $X \in \mathcal{X}$,

$$\|X\| = \sup\left\{\left|\int_\Theta h dX\right| : h \in C(\Theta) \text{ and } \|h\| \leq 1\right\} = |X(\Theta)|,$$

where the last equality holds since the supremum is achieved by setting $h \equiv 1$.

[26] This theorem is a modified version of "Portmanteau Theorem" that is modified to deal with any finite (i.e. not necessarily probability) measures. Refer to 4.5.1 Theorem of Ash (1977) for this result, for instance.

(e) $X_k(A) \to X(A)$ *for every set* $A \in \Sigma$ *such that* $X(\partial A) = 0$ *(*∂A *denotes the boundary of* A*).*

Lemma 5. *The space* $\mathcal{X}$ *is compact. Also, for any* $X \in \mathcal{X}$*, the set of its subpopulations is compact.*

Proof. Let $C(\Theta)^*$ denote the dual space of $C(\Theta)$ and note that $C(\Theta)^*$ is the space of all (signed) measures on (Θ, Σ), given Θ is a compact metric space.[27] Then, by Alaoglu's Theorem,[28] the closed unit ball of $C(\Theta)^*$, denoted U^*, is weak* compact. Clearly, $\mathcal{X}$ is a subspace of U^* since for any $X \in \mathcal{X}$, $\|X\| = X(\Theta) \leq G(\Theta) = 1$. The compactness of $\mathcal{X}$ will thus follow if $\mathcal{X}$ is shown to be a closed set. For this, we prove that for any sequence $\{X_k\}$ in $\mathcal{X}$ and $X \in C(\Theta)^*$ such that $X_k \xrightarrow{w^*} X$, we must have $X \in \mathcal{X}$. To prove this, we make the following observation: every finite (Borel) measure X on the metric space Θ is normal,[29] which means among others that for any set $E \in \Sigma$,

$$X(E) = \inf\{X(O) : E \subset O \in \Sigma \text{ and } O \text{ open}\}. \tag{11}$$

In order to prove $X \in \mathcal{X}$, we need to show that $X(E) \leq G(E)$ for any $E \in \Sigma$. Consider first any open set $O \in \Sigma$ such that $E \subset O$. Then, since $X_k \in \mathcal{X}$ for each k, we must have $X_k(O) \leq G(O)$ for each k, which, combined with (c) of Theorem 12 above, implies that $X(O) \leq \liminf_k X_k(O) \leq G(O)$. Given (11), this inequality means that $X(E) \leq G(E)$, as desired.

The proof of the second statement is analogous and thus omitted. ▮

Lemma 6. *The mapping* $T = \Phi \circ \Psi$ *is a upper-hemi continuous correspondence from* $\mathcal{X}^{n+1}$ *to itself, and is closed valued.*

Proof. We first show that Φ is upper-hemi continuous. To this end, consider any sequence $(X^k)_{k=1}^{\infty}$ in $\mathcal{X}^{n+1}$ and $(\tilde{X}^k)_{k=1}^{\infty}$ in $\mathcal{X}$ weakly converging to some X and $\tilde{X}$, respectively, such that $\tilde{X}^k \in \Phi_f(X^k)$ for each k. Then, for each k, there is some $Y^k \in C_f(X^k)$ satisfying $\tilde{X}^k = G-(X_f^k - Y^k)$ or $Y^k = \tilde{X}^k + X_f^k - G$. Hence, Y^k converges to $Y = \tilde{X} + X_f - G$. By the upper hemicontinuity and compact-valuedness of C_f,[30] $Y \in C_f(X)$ so $\tilde{X} = G - (X_f - Y) \in \Phi(X)$, as desired. The proof of closedness is similar and thus omitted.

[27] More precisely, $C(\Theta)^*$ is isometrically isomorphic to the space of all signed measures on (Θ, Σ). This is a version of Riesz Representation Theorem (refer to 14.15 Corollary of Aliprantis and Border (2006) for instance).

[28] Refer to 6.21 Alaoglu's Theorem of Aliprantis and Border (2006) for instance.

[29] Refer to 12.5 Theorem of Aliprantis and Border (2006).

[30] C_f is compact-valued since for each $X \in \mathcal{X}$, $C_f(X)$ is a closed subset of compact set $\mathcal{X}$, and thus is compact.

Next, we show that Ψ is a continuous mapping from $\mathcal{X}^{n+1}$ to itself. To this end, for any $X \in \mathcal{X}$, let X_f^P be a restriction of X to Θ_P, i.e. $X_f^P(E) = X_f(\Theta_P \cap E)$ for each $E \in \Sigma$, and note that X_f^P is also a measure (as can be easily verified). Then, the RHS of (4) is a finite linear combination of signed measures and thus is a measure itself. Also, one can easily verify that for any $X \in \mathcal{X}^{n+1}$, $\Psi_f(X) \sqsubset G$ and thus $\Psi_f(X) \in \mathcal{X}$. So, Ψ is a mapping from $\mathcal{X}^{n+1}$ to itself.

To establish the continuity of Ψ, consider a sequence $(X_k)_k$ and X in $\mathcal{X}^{n+1}$ such that X_k weakly converges to X. We first prove the following claim.

Claim 1. *A sequence $(X_k^P)_k$ also weakly converges to X^P, where X_k^P and X^P are a restriction of X_k and X to Θ_P, respectively.*

Proof. Due to Theorem 12, it suffices to show that X^P and $(X_k^P)_k$ satisfy the condition (c) of Theorem 12. To do so, consider any open set $O \subset \Theta$. Then, letting Θ_P° denote the interior of Θ_P,

$$\begin{aligned}\liminf_k X_k^P(O) &= \liminf_k X_k(O \cap \Theta_P^\circ) + X_k(O \cap \partial\Theta_P) \\ &= \liminf_k X_k(O \cap \Theta_P^\circ) \geq X(O \cap \Theta_P^\circ) = X^P(O),\end{aligned}$$

where the second equality follows from the fact that $X_k(O \cap \partial\Theta_P) \leq X_k(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$, the lone inequality from $X_k \xrightarrow{w^*} X$, (c) of Theorem 12, and the fact that $O \cap \partial\Theta_P^\circ$ is an open set, and the last equality from repeating the first two equalities with X instead X_k. Also, we have

$$X_k^P(\Theta) = X_k(\Theta_P) \to X(\Theta_P) = X^P(\Theta),$$

where the convergence is due to $X_k \xrightarrow{w^*} X$, (e) of Theorem 12, and the fact that $X(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$. Thus, the two requirements in condition (c) of Theorem 12 are satisfied, so $X_k^P \xrightarrow{w^*} X^P$, as desired. ∎

Now let $X_{k,f}$ denote the component in X_k that corresponds to f. Then, for any $h \in C(\Theta)$,

$$\begin{aligned}\int hd\Psi_f(X_k) &= \int hd(G - \sum_{P \in \mathcal{P}} X_{k,f_-^P}^P) \\ &= \int hdG - \sum_{P \in \mathcal{P}} \int hdX_{k,f_-^P}^P \\ &\to \int hdG - \sum_{P \in \mathcal{P}} \int hdX_{f_-^P}^P \\ &= \int hd(G - \sum_{P \in \mathcal{P}} X_{f_-^P}^P) = \int hd\Psi_f(X),\end{aligned}$$

where the convergence occurs since, by Claim 1, $X^P_{k,f} \xrightarrow{w^*} X^P_f$ for each $f \in \tilde{F}$; the second and third equalities hold since the mapping $(h, X) \mapsto \langle h, X\rangle = \int h dX$ is linear with respect to X.[31] The above convergence means that $\Psi_f(X_k) \xrightarrow{w^*} \Psi_f(X)$, so Ψ is continuous. ▮

Proof of Theorem 7. Since both Ψ and Φ are convex-valued and upper hemicontinuous, so is their composition $T = \Psi \circ \Phi$.[32] Also, it is straightforward to show that T is closed-valued. We can thus invoke Kakutani-Fan-Glicksberg's fixed point theorem to conclude that the mapping T, which is defined on a convex set $\mathcal{X}^{n+1}$, has a nonempty set of fixed points.[33] ▮

A.4 Substitutable Preferences Case

Proof of Theorem 3. We first prove part (i). It follows from Tarski's fixed point theorem if we show that T is monotonic in $\sqsubset_{\tilde{F}}$. To see this, let $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ both in $\mathcal{X}^n$. First, substitutability of firms' preferences (and the definition of $R_\varnothing \equiv 0$) implies that $R_f(X_f) \sqsubset R_f(X'_f)$ for $f \in \tilde{F}$, which yields $Y_f := \Phi(X_f) = G - R_f(X_f) \sqsupset G - R_f(X'_f) = \Phi(X'_f) =: Y'_f$ for each $f \in \tilde{F}$ or $Y_{\tilde{F}} \sqsupset_{\tilde{F}} Y'_{\tilde{F}}$. Next, for each $f \in \tilde{F}$ and $E \in \Sigma$,

$$\Psi_f(Y_{\tilde{F}})(E) = G(E) - \sum_{P \in \mathcal{P}} Y_{f^P_-}(\Theta_P \cap E) \leq G(E) - \sum_{P \in \mathcal{P}} Y'_{f^P_-}(\Theta_P \cap E) = \Psi_f(Y'_{\tilde{F}})(E),$$

which implies that $\Psi(Y_{\tilde{F}}) \sqsubset \Psi(Y'_{\tilde{F}})$, or $T(X_{\tilde{F}}) \sqsubset_{\tilde{F}} T(X'_{\tilde{F}})$.

We next prove part (ii). The stable matching $\overline{M}$ is firm-optimal since $\overline{X} \sqsupset X$ for all $X \in \mathcal{X}^*$. To see it is worker-pessimal, fix any stable matching M. Then, by Theorem 1, $M = (C_f(X))_{f \in \tilde{F}}$ for some $X \in \mathcal{X}^*$ and

$$X_f(E) = C^f(M) = \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \not\succ_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

[31] In fact, the mapping $\langle h, X\rangle$ is bilinear, i.e. linear with respect to both h and X. Refer to p. 211 of Aliprantis and Border (2006) for instance.

[32] For the upper hemicontinuity, refer to 16.23 Theorem in Aliprantis and Border (2006), for instance, which says that a composition of two upper hemicontinuous correspondences is upper hemicontinuous.

[33] For Kakutani-Fan-Glicksberg's fixed point theorem, refer to 16.12 Closed Graph Theorem and 16.51 Corollary in Aliprantis and Border (2006).

The latter means that for each $P \in \mathcal{P}$ and $E \in \Sigma$,

$$\begin{aligned}\Phi_f(X)(\Theta_P \cap E) &= G(\Theta_P \cap E) - R_f(X_f)(\Theta_P \cap E) \\ &= G(\Theta_P \cap E) - (X_f(\Theta_P \cap E) - M_f(\Theta_P \cap E)) \\ &= G(\Theta_P \cap E) - \left(\sum_{f' \in \tilde{F} : f' \not\prec_P f} M_{f'}(\Theta_P \cap E) - M_f(\Theta_P \cap E) \right) \\ &= G(\Theta_P \cap E) - \sum_{f' \in \tilde{F} : f' \prec_P f} M_{f'}(\Theta_P \cap E) = \sum_{f'' \in \tilde{F} : f'' \succeq_P f} M_{f''}(\Theta_P \cap E).\end{aligned}$$

The same equality holds for $\overline{X}$. The worker-pessimality of $\overline{M}$ then follows since

$$\begin{aligned}\sum_{f'' \in \tilde{F} : f'' \succeq_P f} \overline{M}_{f''}(\Theta_P \cap E) =& \Phi_f(\overline{X})(\Theta_P \cap E) \\ \leq& \Phi_f(X)(\Theta_P \cap E) = \sum_{f'' \in \tilde{F} : f'' \succeq_P f} M_{f''}(\Theta_P \cap E) \qquad (12)\end{aligned}$$

for all $P \in \mathcal{P}$, $E \in \Sigma$, and $f \in \tilde{F}$. ▌

Proof of Theorem 4. Let M be a stable matching. Then, there exists $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for each $f \in F$. Since $X \sqsubset \overline{X}$, by the law of aggregate demand, we have

$$\overline{M}_f(\Theta) = C_f(\overline{X}_f)(\Theta) \geq C_f(X_f)(\Theta) = M_f(\Theta), \forall f \in F. \qquad (13)$$

Next since $\overline{M}$ is worker pessimal, (12) holds for any $f \in \tilde{F}$. Let $w_P := \emptyset_-^P$ be the immediate predecessor of $\emptyset$ (i.e., the worst individually rational firm) for types in Θ_P. Then, setting $f = w_P$ in (12), we obtain

$$\begin{aligned}\sum_{f'' \in F} \overline{M}_{f''}(\Theta_P \cap E) &= \sum_{f'' \in \tilde{F} : f'' \succeq_P w_P} \overline{M}_{f''}(\Theta_P \cap E) \\ &\leq \sum_{f'' \in \tilde{F} : f'' \succeq_P w_P} M_{f''}(\Theta_P \cap E) = \sum_{f'' \in F} M_{f''}(\Theta_P \cap E),\end{aligned}$$

or, by the countable additivity of $\overline{M}_{f''}$ and $M_{f''}$,

$$\sum_{f'' \in F} \overline{M}_{f''}(E) \leq \sum_{f'' \in F} M_{f''}(E). \qquad (14)$$

Since this inequality must hold with $E = \Theta$, combining it with (13) implies that $M_f(\Theta) = \overline{M}(\Theta)$ for all $f \in F$, as desired.

Further, we must have $\sum_{f\in F}\overline{M}_f=\sum_{f\in F}M_f$ for all θ. To prove this, suppose otherwise. Then, by (14), we must have $\sum_{f''\in F}\overline{M}_{f''}(E)<\sum_{f''\in F}M_{f''}(E)$ for some $E\in\Sigma$. Also, by (14), $\sum_{f''\in F}\overline{M}_{f''}(E^c)\leq\sum_{f''\in F}M_{f''}(E^c)$. Combining these two inequalities and using the countable additivity of $\overline{M}$ and M, we obtain $\sum_{f''\in F}\overline{M}_{f''}(\Theta)<\sum_{f''\in F}M_{f''}(\Theta)$, which contradicts with (13). ▌

Proof of Theorem 5. Suppose otherwise. Then there exists a stable matching M that differs from the firm-optimal stable matching $\overline{M}$. Let X and $\overline{X}$ be respectively such that $M_f=C_f(X_f)$, $\overline{M}_f=C_f(\overline{X}_f)$ and $X_f\sqsubset\overline{X}_f$, for each $f\in F$.

First of all, by Theorem 4, $\sum_{f\in F}M_f=\sum_{f\in F}\overline{M}_f$. Next, since $X_f\sqsubset\overline{X}_f$ for each $f\in F$, we have $\overline{M}_f\vee M_f\sqsubset\overline{X}_f$. Revealed preference then implies that, for each $f\in F$,

$$\overline{M}_f=C_f(\overline{M}_f\vee M_f).$$

Moreover, since $M\neq\overline{M}$, the set $\bar{F}:=\{f'\in F|M_{f'}\neq C_{f'}(\bar{M}_{f'}\vee M_{f'})\}$ is nonempty. But then by the rich preferences, there exists $f^*\in F$ such that

$$M_{f^*}\neq C_{f^*}(M_{f^*}+\overline{M}_{\bar{F}}^{f^*}).$$

For each $f\in F\backslash\bar{F}$, $M_f=\overline{M}_f$, by definition of $\bar{F}$, and Theorem 4 guarantees that $M_{ø}=\overline{M}_{ø}$. Consequently, we have that

$$M_{\bar{F}}^{f^*}=\int\sum_{f\in\tilde{F}\backslash\bar{F}}m_f(s)\cdot\mathbf{1}_{\{f\prec_s f^*\}}ds=\int\sum_{f\in\tilde{F}\backslash\bar{F}}\overline{m}_f(s)\cdot\mathbf{1}_{\{f\prec_s f^*\}}ds=\overline{M}_{\bar{F}}^{f^*}.$$

It then follows that

$$M_{f^*}\neq C_{f^*}(M_{f^*}+M_{\bar{F}}^{f^*}).\tag{15}$$

Letting $\hat{M}_{f^*}:=C_{f^*}(M_{f^*}+M_{\bar{F}}^{f^*})$, we have

$$\hat{M}_{f^*}\sqsubset M_{f^*}+M_{\bar{F}}^{f^*}\sqsubset C^{f^*}(M_{f^*}\vee\hat{M}_{f^*},M_{-f^*}),$$

since all workers in $M_{f^*}+M_{\bar{F}}^{f^*}$ (at least weakly) prefer f^* to their current match partners. Since $\hat{M}_{f^*}\sqsubset M_{f^*}\vee\hat{M}_{f^*}\sqsubset M_{f^*}+M_{\bar{F}}^{f^*}$, by the revealed preference condition, we have

$$\hat{M}_{f^*}=C_{f^*}(M_{f^*}\vee\hat{M}_{f^*})\sqsubset C^{f^*}(M_{f^*}\vee\hat{M}_{f^*},M_{-f^*}).$$

Since $\hat{M}_{f^*}\neq M_{f^*}$ by (15), we have a contradiction to the stability of M. ▌

Proof of Lemma 2. Fix any two matchings M and $\hat{M}$ such that $\sum_{f\in F}M_f=\sum_{f\in F}\hat{M}_f$, $\hat{M}_f=C_f(\hat{M}_f\vee M_f)$ for each $f\in F$, and $\bar{F}:=\{f'\in F|M_{f'}\neq C_{f'}(\hat{M}_{f'}\vee M_{f'})\}$ is nonempty. Let $p_f^t = \inf\{s_f(t,\theta)|m_f^t(\theta) > 0\}$ and $\hat{p}_f^t = \inf\{s_f(t,\theta)|\hat{m}_f^t(\theta) > 0\}$ be the lowest scores among the workers of ethnic type t who are matched with firm f under M and $\hat{M}$, respectively.[34] (These are interpreted as the cutoff scores for firm f, one for each ethnic type t, under the two matchings.)

Because the choice function of each f satisfies the law of aggregate demand (by inspection of the choice rule), $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f = C_f(M_f)$ imply $\sum_{t\in T} M_f^t(1) \leq \sum_{t\in T} \hat{M}_f^t(1)$ for each $f \in F$. Hence, together with $\sum_{f\in F} M_f = \sum_{f\in F} \hat{M}_f$, we conclude that $\sum_{t\in T} M_f^t(1) = \sum_{t\in T} \hat{M}_f^t(1)$ for each $f \in F$. Then, for each $f \in \bar{F}$, there must be some t such that we have $p_f^t < \hat{p}_f^t$. To see this, suppose to the contrary that $p_f^t \geq \hat{p}_f^t$ for all t. Since $\sum_{t\in T} M_f^t(1) = \sum_{t\in T} \hat{M}_f^t(1)$ and $M_f \neq \hat{M}_f$, there exists t such that for a positive measure of θ, $s_f(t,\theta) > p_f^t \geq \hat{p}_f^t$ and $m_f^t(\theta) > \hat{m}_f^t(\theta)$. This means that M contains more preferred workers, which contradicts the fact $\hat{M}_f = C_f(\hat{M}_f \vee M_f) \neq M_f$.

Fix any $f \in \bar{F}$ and choose $t \in T$ such that $p_f^t < \hat{p}_f^t$, and let

$$\tilde{\Theta}_f^t := \{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, p_f^t < s_f(t,\theta) < \hat{p}_f^t \text{ and } s_{f'}(t,\theta) < \hat{p}_{f'}^t \, \forall f' \in \bar{F} \setminus \{f\}\}$$

be a set of ethnic type t workers who prefer f to all other firms and have scores that will make them employable at f under M but not under $\hat{M}$ and not employable at all other firms in $\bar{F}$ under $\hat{M}$. Let $\tilde{M}^t$ denote its distribution. The full support assumption implies that $\sum_{t\in T} \tilde{M}^t(1) > 0$. We show that these workers are not employed by any firm in $\bar{F}$ under either $\hat{M}$ or M. It is easy to see that these workers are not employed by any firm in $\bar{F}$ under $\hat{M}$ since their scores are below the cutoffs of these firms at $\hat{M}$. Since $\sum_{f\in F} M_f = \sum_{f\in F} \hat{M}_f$, and since $M_f = \hat{M}_f$ for each $f \in F \setminus \bar{F}$, we must have $\sum_{f\in \bar{F}} M_f = \sum_{f\in \bar{F}} \hat{M}_f$. It thus follows that these workers are not employed by firms in $\bar{F}$ under matching M either.

Next, note that the above argument implies $\tilde{M} \sqsubset \hat{M}_{\bar{F}}^f$. Since $\hat{p}_f^t > p_f^t$, firm f will wish to replace some of its workers with these workers under M. Hence, $M_f \neq C_f(M_f + \hat{M}_{\bar{F}}^f)$, so the rich preferences property follows. ▮

A.5 Proof of Theorem 6

Let Γ be the limit continuum economy which the sequence $(\Gamma^q)_q$ converges to. For any population G, fix a sequence (G^q) of finite-economy populations such that $G^q \xrightarrow{w^*} G$. Let $\Theta^q = \{\theta_1^q, \theta_2^q, \ldots, \theta_{\bar{q}}^q\} \subset \Theta$ be the support for G^q.[35] We say that X^q is **feasible in** Γ^q

[34] $m_f^t(\theta)$ denotes the measure of type (t,θ) workers matched to f at matching M, and $M_f^t(\theta) = \int_{s=0}^{\theta} m_f^t(s)$.

[35] Note that we allow for possibility that there are more than one worker of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

if, for each $\theta \in \Theta^q$, $X^q(\theta)$ is a multiple of $1/q$ and $X^q \sqsubset G^q$. We first prove a couple of preliminary results.

Lemma 7. *For any $r > 0$, there is a finite number of open balls, $B_1, \ldots, B_L$ that have radius smaller than r with a boundary of zero measure (i.e. $G(\partial B_\ell) = 0, \forall \ell$) and cover Θ.*

Proof. Let $B(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| < r\}$ and $S(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| = r\}$, where $\|\cdot\|$ is a metric for the space Θ. For all $\theta \in \Theta$ and $r > 0$, there must be some $r_\theta \in (0, r)$ such that $G(S(\theta, r_\theta)) = 0$,[36] which means that $\partial B(\theta, r_\theta) = S(\theta, r_\theta)$ has a zero measure. Consider now a collection $\{B(\theta, r_\theta) \,|\, \theta \in \Theta\}$ of open balls that covers Θ. The compactness of Θ then guarantees the existence of a finite cover, as desired. ∎

Lemma 8. *Suppose $G^q \xrightarrow{w^*} G$ and $X \sqsubset G$. Then, there exists a sequence $\{X^q\}$ such that X^q is feasible in Γ^q and $X^q \xrightarrow{w^*} X$.*

Proof. Consider a decreasing sequence $(\epsilon_k)_k$ of real numbers converging to 0. Then, according to Lemma 7, we can find a finite cover $\{B_\ell^k\}_{\ell=1,\ldots,L_k}$ for each k such that for each ℓ, B_ℓ^k has a radius smaller than ϵ_k and $G(\partial B_\ell^k) = 0$. For each k, define $A_1^k = B_1^k$ and $A_\ell^k = B_\ell^k \setminus (\cup_{\ell'=1}^{\ell-1} B_{\ell'}^k)$ for each $\ell \geq 2$. So, for each k, $\{A_\ell^k\}$ constitutes a partition of Θ. It is straightforward to see that $G(\partial A_\ell^k) = 0, \forall \ell$, since $G(\partial B_\ell^k) = 0, \forall \ell$. Given this and $G^q \xrightarrow{w^*} G$, condition (e) of Theorem 12 implies that for each k, there exists sufficiently large q, denoted q_k, such that for all $q \geq q_k$

$$\frac{1}{q} < \frac{\epsilon_k}{L_k} \text{ and } |G(A_\ell^k) - G^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}, \forall \ell = 1, \ldots, L_k. \tag{16}$$

Let us choose $(q_k)_k$ to be a sequence that strictly increases with k.

Now we can construct X^q as follows: (i) $X^q \sqsubset G^q$ and (ii) for each k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$,

$$X^q(A_\ell^k) = \max\left\{ \frac{m}{q} \,\middle|\, m \in \mathbb{N} \text{ and } \frac{m}{q} \leq \min\{X(A_\ell^k), G^q(A_\ell^k)\} \right\} \text{ for each } \ell = 1, \ldots, L_k.$$

We now show that for all k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$, we have

$$|X(A_\ell^k) - X^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}. \tag{17}$$

[36]To see this, note first that $B(\theta, r) = \cup_{\tilde{r} \in [0,r)} S(\theta, \tilde{r})$ and $G(B(\theta, r)) < \infty$. Then, $G(S(\theta, \tilde{r})) > 0$ for at most countably many $\tilde{r}$'s, since otherwise the set $R_n \equiv \{\tilde{r} \in [0, r) \,|\, G(S(\theta, \tilde{r})) \geq 1/n\}$ has to be infinite for at least one n, which yields $G(B(\theta, r)) \geq G(\cup_{\tilde{r} \in R_n} S(\theta, \tilde{r})) \geq \frac{\infty}{n}$, a contradiction.

To see this, consider first the case $X(A_\ell^k) < G^q(A_\ell^k)$. Then, by definition of X^q and (16), we have $0 \leq X(A_\ell^k) - X^q(A_\ell^k) < \frac{1}{q} < \frac{\epsilon_k}{L_k}$. In case $X(A_\ell^k) \geq G^q(A_\ell^k)$, we have $X^q(A_\ell^k) = G^q(A_\ell^k) \leq X(A_\ell^k) \leq G(A_\ell^k)$, which implies by (16)

$$|X(A_\ell^k) - X^q(A_\ell^k)| \leq |G(A_\ell^k) - G^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}.$$

We are now ready to prove that $X^q \xrightarrow{w^*} X$. We do so by invoking (b) of Theorem 12, according to which $X^q \xrightarrow{w^*} X$ if and only if $|\int hdX^q - \int hdX| \to 0$ as $q \to \infty$, for any uniformly continuous function $h \in C_u(\Theta)$.

Hence, to begin, fix any $h \in C_u(\Theta)$, and fix any $\epsilon > 0$. Next we define for each k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$

$$\bar{h}_\ell^{q,k} \equiv \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)h(\theta)}{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)} = \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)h(\theta)}{X^q(A_\ell^k)}$$

if $X^q(A_\ell^k) > 0$, and if $X^q(A_\ell^k) = 0$, then define $\bar{h}_\ell^{q,k} \equiv h(\theta)$ for some arbitrarily chosen $\theta \in A_\ell^k$.

Note $C_u(\Theta)$ is endowed with the sup norm $\|\cdot\|_\infty$ and $\|h\|_\infty$ is finite for any $h \in C_u(\Theta)$. Hence, there exists $K \in \mathbb{N}$ sufficiently large such that, for all $k > K$ and $q \in \{q_k, \ldots, q_{k+1} - 1\}$, we have $\|h\|_\infty \epsilon_k < \epsilon/2$, and

$$\sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) < \frac{\epsilon}{2}, \forall \ell = 1, \ldots, L_k, \tag{18}$$

which is possible since h is uniformly continuous, $A_\ell^k \subset B_\ell^k$, and B_ℓ^k has a radius smaller than ϵ_k with ϵ_k converging to 0 as $k \to \infty$.

Then, for any $q > Q := q_K$, there exists $k > K$ with $q \in \{q_k, \ldots, q_{k+1} - 1\}$ such that

$$\begin{aligned}
&\left| \int h dX^q - \int h dX \right| \\
=& \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X^q(A_\ell^k) - \int h dX \right| \\
\leq& \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} (X^q(A_\ell^k) - X(A_\ell^k)) \right| + \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X(A_\ell^k) - \int h dX \right| \\
\leq& \sum_{\ell=1}^{L_k} \|h\|_\infty |X^q(A_\ell^k) - X(A_\ell^k)| + \left| \sum_{\ell=1}^{L_k} \int \bar{h}_\ell^{q,k} \mathbb{1}_{A_\ell^k} dX - \sum_{\ell=1}^{L_k} \int h \mathbb{1}_{A_\ell^k} dX \right| \\
\leq& \|h\|_\infty \epsilon_k + \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) \\
\leq& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned} \tag{19}$$

where the third inequality follows from (16) while the fourth from (18). ▮

By Theorem 2, there exists a stable matching M in the continuum economy. Now, construct M^q by the following procedure.

1. Define $M^q_{f_1}$ as the X^q in Lemma 8 with respect to $X := M_{f_1}$.
2. Define $M^q_{f_2}$ as the X^q in Lemma 8 with respect to $X := M_{f_2}$, while replacing G with $G - M_{f_1}$, and G^q with $G^q - M^q_{f_1}$. (This is possible since $G^q - M^q_{f_1} \xrightarrow{w^*} G - M_{f_1}$.)
3. Generally, for any $k \in \{1, 2, \ldots, n\}$, inductively define $M^q_{f_k}$ as the X^q in Lemma 8 with respect to $X := M_{f_k}$, while replacing G with $G - \sum_{k'<k} M_{f_{k'}}$, and G^q with $G^q - \sum_{k'<k} M^q_{f_{k'}}$.

Noting that the number of firms is finite, we have $M^q \xrightarrow{w^*} M$. Thus, by continuity of firms' utility functions, for any $f \in F$ and $\epsilon > 0$,

$$u_f(M^q_f) > u_f(M_f) - \frac{\epsilon}{2}, \tag{20}$$

for any sufficiently large q. Let $C^f(M^q)$ be the subpopulation of workers in economy Γ^q who weakly prefer f to their match in M^q.[37] Since $M^q \xrightarrow{w^*} M$, we have $C^f(M^q) \xrightarrow{w^*} C^f(M)$.[38]

[37] To be precise, $C^f(M^q)$ is given as in (1) with G and X being replaced by G^q and M^q, respectively.

[38] This convergence can be shown using an argument similar to that which we have used to establish the continuity of Ψ in the proof of Lemma 6.

Let $\tilde{M}^q_f = C_f(C^f(M^q))$. In words, $\tilde{M}^q_f$ is the most profitable block of M^q for f in the continuum economy, that is, the optimal deviation in a situation where the current matching is M^q, but the firm can deviate to any subpopulation, not just a discrete distribution. Then the above-mentioned property that $C^f(M^q) \xrightarrow{w^*} C^f(M)$ and continuity of C_f imply that $\tilde{M}^q_f \xrightarrow{w^*} M_f$. Thus, by continuity of firms' utility functions,

$$u_f(\tilde{M}^q_f) < u_f(M_f) + \frac{\epsilon}{2}, \tag{21}$$

for any sufficiently large q. Let M'_f be the most profitable block of M^q for f in economy Γ^q. Then M'_f is the optimal deviation facing the same population G^q and matching M^q as when computing $\tilde{M}_f$ but with an additional constraint that the deviation is feasible in Γ^q (multiples of $1/q$), so $u_f(M'_f) \leq u_f(\tilde{M}_f)$. This and inequality (21) imply

$$u_f(M'_f) < u_f(M_f) + \frac{\epsilon}{2}. \tag{22}$$

Combining inequalities (20) and (22), we obtain $u_f(M'_f) < u_f(M^q_f) + \epsilon$, completing the proof.

A.6 Proofs for Section 6.2

Proof of Lemma 3. We first establish that for $z \in [0, \frac{1}{m}]^m$, $\tilde{C}_f(z)$ is a singleton set. To do so, for any $z \in [0, \frac{1}{m}]^m$ and $\alpha \in [0, \frac{1}{m}]$, define $\zeta^k_f(\alpha) := \sum_{\theta \in S^k_f} \min\{z_\theta, \alpha\}$. From now, we assume $C_f(z) \neq \{z\}$, since, in case $C_f(z) = \{z\}$, we have $\tilde{C}_f(z) = \{z\}$. We show that there exists a unique $\hat{\alpha}^k$ satisfying $\zeta^k_f(\hat{\alpha}^k) = \Gamma^k_f(z)$, which means that $\tilde{C}_f(z)$ is a singleton set. First, we must have $\hat{\alpha}^k < \max_{\theta \in S^k_f} z_\theta$ since otherwise $\zeta^k_f(\hat{\alpha}^k) = \sum_{\theta \in S^k_f} z_\theta > \Gamma^k_f(z)$ (which follows from the assumption that $C_f(z) \neq \{z\}$ and thus, for any $z' \in C_f(z)$, $z' \lneq z$). Next, $\zeta^k_f(\cdot)$ is strictly increasing in the range $[0, \max_{\theta \in S^k_f} z_\theta)$. Then, the continuity of ζ^k_f, along with the fact that $\zeta^k_f(0) = 0$ and $\zeta^k_f(\max_{\theta \in S^k_f} z_\theta) > \Gamma^k_f(z)$, implies that there is a unique $\alpha \in [0, \max_{\theta \in S^k_f} z_\theta)$ satisfying $\zeta^k_f(\alpha) = \Gamma^k_f(z)$.

To show the revealed preference property, consider any $z, z', z'' \in [0, \frac{1}{m}]^m$ such that $\tilde{C}_f(z) = \{z'\}$ and $z' \sqsubset z'' \sqsubset z$. Since we already know that $C_f(\cdot)$ satisfies the revealed preference property, we have $z' \in C_f(z'')$. It suffices to show that $z' \in B_f(z'')$, since it means $\tilde{C}_f(z'') = \{z'\}$, from which the revealed preference property follows. To do so, note that $z' \in B_f(z)$ means that $z'_\theta = \min\{z_\theta, \alpha^k\}$ for each k and $\theta \in S^k_f$. Then, since $z_\theta \geq z''_\theta \geq z'_\theta$ and $\alpha^k \geq z'_\theta$, we have

$$z'_\theta = \min\{z_\theta, \alpha^k\} \geq \min\{z''_\theta, \alpha^k\} \geq z'_\theta,$$

so $z'_\theta = \min\{z''_\theta, \alpha^k\}$ as desired. ▮

Proof of Theorem 9. **("only if" part)** Consider a strongly stable matching Z in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$. We show that Z is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. First, it is straightforward to see that Z satisfies Condition (i) of Definition 10. To show that $z_f \in \tilde{C}_f(z_f)$ for each $f \in F$ (i.e. Condition (ii) of Definition 10), note first that we have $z_f \in C_f(z_f)$, since Z is strongly stable and thus stable. That $z_f \in B_f(z_f)$ follows from setting $\alpha^k = \frac{1}{m}$, since then $z_{f\theta} \leq \frac{1}{m} = \alpha^k, \forall \theta \in S_f^k$ so $\min\{z_{f\theta}, \alpha^k\} = z_{f\theta}, \forall \theta \in S_f^k$.

To show Condition (iii) of Definition 10, we first prove the following claim:

Claim 2. *Fix any strongly stable matching Z. For each firm $f \in F$, there is some $\alpha^k \geq 0$ for each $k \in \{1, \ldots, K_f\}$ such that $z_{f\theta} = \min\{C_\theta^f(Z), \alpha^k\}, \forall \theta \in S_f^k$.*

Proof. It suffices to show that for any k and $\theta, \theta' \in S_f^k$, if $z_{f\theta} < C_\theta^f(Z)$ and $z_{f\theta'} < C_{\theta'}^f(Z)$, then $z_{f\theta} = z_{f\theta'}$. Note first that for any stable matching Z,

$$C_\theta^f(Z) = \frac{1}{m} - \sum_{f' \in \tilde{F}: f' \succ_\theta f} z_{f'\theta} = \sum_{f' \in \tilde{F}: f' \preceq_\theta f} z_{f'\theta} = z_{f\theta} + \sum_{f' \in \tilde{F}: f' \prec_\theta f} z_{f'\theta}.$$

If $z_{f\theta} < C_\theta^f(Z)$ and $z_{f\theta'} < C_{\theta'}^f(Z)$, then we have $\sum_{f' \in \tilde{F}: f' \prec_\theta f} z_{f'\theta} > 0$ and $\sum_{f' \in \tilde{F}: f' \prec_{\theta'} f} z_{f'\theta'} > 0$. Given this, the strong stability implies that $z_{f\theta} = z_{f\theta'}$. ▮

Since Z is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, we have $z_f \in C_f(C^f(Z))$ for each $f \in F$. Then, for any $z'_f \in C^f(z'_f \vee z_f, z_{-f}) = C^f(Z)$, we have $z_f \in C_f(z'_f \vee z_f)$ due to the fact that $z_f \in C_f(C^f(Z))$, $(z'_f \vee z_f) \sqsubset C^f(Z)$, and C_f satisfies the revealed preference. Condition (iii) will then hold if $z_f \in B_f(z'_f \vee z_f)$, since it means $z_f \in C_f(z'_f \vee z_f) \cap B_f(z'_f \vee z_f) = \tilde{C}_f(z'_f \vee z_f)$. For this, for each k, we choose α^k as in Claim 2 and show that for all $\theta \in S_f^k$, $z_{f\theta} = \min\{\max\{z'_{f\theta}, z_{f\theta}\}, \alpha^k\}$. Let us first consider the case where $z_{f\theta} = C_\theta^f(Z) < \alpha^k$. Since $z'_{f\theta} \leq C_\theta^f(Z)$, we have $\max\{z'_{f\theta}, z_{f\theta}\} = z_{f\theta} < \alpha^k$ and thus $z_{f\theta} = \min\{\max\{z'_{f\theta}, z_{f\theta}\}, \alpha^k\}$. For the other case where $z_{f\theta} = \alpha^k \leq C_\theta^f(Z)$, observe that $z_{f\theta} = \alpha^k = \min\{\max\{z'_{f\theta}, z_{f\theta}\}, \alpha^k\}$ since $\max\{z_{f\theta'}, z_{f\theta}\} \geq z_{f\theta} = \alpha^k$.

***(*"if" part)** Consider a stable matching $Z = (z_f)_{f \in \tilde{F}}$ in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. To show that Z is strongly stable in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$, we first show that it is stable (i.e. Condition (i) of Definition 11). It is straightforward, thus omitted, to check Conditions (i) and (ii) of Definition 10. To check Condition (iii), suppose to the contrary that there is a blocking pair f and z'_f, which means that $z'_f \sqsubset C^f(z'_f \vee z_f, z_{-f})$, $z'_f \in C_f(z'_f \vee z_f)$, and $z_f \notin C_f(z'_f \vee z_f)$. Given this, by Lemma 3, there exists $\tilde{z}_f$ such that $\tilde{C}_f(z'_f \vee z_f) = \{\tilde{z}_f\}$. First, by the revealed preference property of $\tilde{C}_f$ and the fact that $\tilde{z}_f \sqsubset (\tilde{z}_f \vee z_f) \sqsubset (z'_f \vee z_f)$, we have $\tilde{z}_f \in \tilde{C}_f(\tilde{z}_f \vee z_f)$ and $z_f \notin \tilde{C}_f(\tilde{z}_f \vee z_f)$. Second, since $z_f \sqsubset C^f(z_f, z_{-f}) = C^f(\tilde{z}_f \vee z_f, z_{-f})$ and $z'_f \sqsubset C^f(z'_f \vee z_f, z_{-f}) = C^f(\tilde{z}_f \vee z_f, z_{-f})$, we have $\tilde{z}_f \sqsubset (z'_f \vee z_f) \sqsubset C^f(\tilde{z}_f \vee z_f, z_{-f})$. In sum, f and $\tilde{z}_f$ form a blocking pair in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, which is a contradiction.

To show that Condition (ii) of Definition 11 also holds, suppose not. Then, there must be some f, k, and $\theta, \theta' \in S_f^k$ for whom $z_{f\theta} < z_{f\theta'}$ and $\sum_{f':f' \prec_\theta f} z_{f'\theta} > 0$. Fixing any such f, k, and θ, let f' be any firm such that $f' \prec_\theta f$ and $z_{f'\theta} > 0$. Letting $\Theta' = \arg\max_{\tilde{\theta} \in S_f^k} z_{f\tilde{\theta}}$, note that $\theta \notin \Theta'$. We define a matching for firm f as follows: for each $\tilde{\theta}$,

$$z'_{f\tilde{\theta}} = \begin{cases} z_{f\tilde{\theta}} + \epsilon z_{f'\tilde{\theta}} & \text{if } \tilde{\theta} = \theta \\ \epsilon' z_{f\tilde{\theta}} & \text{if } \tilde{\theta} \in \Theta' \\ z_{f\tilde{\theta}} & \text{otherwise} \end{cases},$$

where $\epsilon, \epsilon' \in (0, 1)$ are chosen to satisfy $z'_{f\tilde{\theta}} < z'_{f\theta'}, \forall \theta' \in \Theta', \tilde{\theta} \notin \Theta'$ and

$$z'_{f\theta} + \sum_{\theta' \in T'} z'_{f\theta'} = z_{f\theta} + \sum_{\theta' \in \Theta'} z_{f\theta'}. \tag{23}$$

Note first that $z'_f \sqsubset C^f(Z) = C^f(z'_f \vee z_f, z_{-f})$, since $f' \prec_\theta f$ and thus $z'_f \sqsubset (z_f + z_{f'} 1_{\{\tilde{\theta}=\theta\}}) \sqsubset C^f(Z)$. Let us show next that $z'_f \in \tilde{C}_f(z'_f \vee z_f, z_{-f})$. The fact that $z_f \in \tilde{C}_f(C^f(Z))$ (due to the stability of Z) and $z_f \vee z'_f \sqsubset C^f(Z)$, implies $z_f \in \tilde{C}_f(z_f \vee z'_f)$ by the revealed preference. This means $z_f \in C_f(z_f \vee z'_f)$, which can be combined with (23) to yield $\mathbb{P}_f(z'_f) = \mathbb{P}_f(z_f)$ and thus $z'_f \in C_f(z_f \vee z'_f)$. To show $z'_f \in B_f(z_f \vee z'_f)$, we set $\alpha^k = \max_{\tilde{\theta} \in S_f^k} z'_{f\tilde{\theta}}$ and observe that for all $\theta' \in \Theta'$, $\min\{(z_f \vee z'_f)_{\theta'}, \alpha^k\} = \min\{z_{f\theta'}, \alpha^k\} = \alpha^k = z'_{f\theta'}$ while for all $\tilde{\theta} \notin \Theta'$, $\min\{(z_f \vee z'_f)_{\tilde{\theta}}, \alpha^k\} = \min\{z'_{f\tilde{\theta}}, \alpha^k\} = z'_{f\tilde{\theta}}$, which implies $z'_f \in B_f(z_f \vee z'_f)$. Therefore, $z'_f \in \tilde{C}_f(z_f \vee z'_f)$.

We now draw a contradiction by showing that $z_f \notin B_f(z_f \vee z'_f)$ and thus $z_f \notin \tilde{C}_f(z_f \vee z'_f)$. This is shown by arguing that there is no α^k such that $z_{f\tilde{\theta}} = \min\{(z_f \vee z'_f)_{\tilde{\theta}}, \alpha^k\}, \forall \tilde{\theta} \in S_f^k$. If $\alpha^k < \max_{\tilde{\theta} \in S_f^k} z_{f\tilde{\theta}}$, then, for $\theta' \in \Theta'$,

$$\min\{(z_f \vee z'_f)_{\theta'}, \alpha^k\} = \min\{z_{f\theta'}, \alpha^k\} = \alpha^k < z_{f\theta'},$$

a contradiction. If $\alpha^k \geq \max_{\tilde{\theta} \in S_f^k} z_{f\tilde{\theta}}$, then

$$\min\{(z_f \vee z'_f)_\theta, \alpha^k\} = \min\{z'_{f\theta}, \alpha^k\} = z'_{f\theta} > z_{f\theta},$$

again a contradiction. ▌

Proof of Theorem 10. By Theorem 9 and the Kakutani-Fan-Glicksberg fixed point theorem, it suffices to show that $\tilde{C}_f$ defined in (5) is closed- and convex-valued, and upper hemicontinuous.

The convexity and close-valuedness of $\tilde{C}_f(z)$ for any $z \in [0, \frac{1}{m}]^m$ follow directly from the fact that $\tilde{C}_f(z)$ is a singleton set. In the rest of the proof, we prove the upper hemicontinuity.

By 16.25 Theorem of Aliprantis and Border (2006), the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous. Since we already know that $C_f(\cdot)$ is closed- and compact-valued, and upper hemicontinuous, we only need to prove that $B_f(\cdot)$ is upper hemicontinuous (given that its closed-valuedness has been proved). To do so, consider sequences $(z^\ell)_{\ell=1}^\infty$ and $(\tilde{z}^\ell)_{\ell=1}^\infty$ with $\tilde{z}^\ell \in B_f(z^\ell), \forall \ell$, converging to z and $\tilde{z}$, respectively. So, for each $k \in K_f$, there is a sequence $(\alpha_\ell^k)_{\ell=1}^\infty$ such that $\tilde{z}_\theta^\ell = \min\{z_\theta^\ell, \alpha_\ell^k\}, \forall \theta \in S_f^k$. For each k, let α^k be a limit to which a subsequence of the sequence $(\alpha_\ell^k)_{\ell=1}^\infty$ converges. We claim that $\tilde{z}_\theta = \min\{z_\theta, \alpha^k\}, \forall \theta \in S_f^k$. If, for instance, $\tilde{z}_\theta > \min\{z_\theta, \alpha^k\}$, then one can find sufficiently large ℓ to make $\tilde{z}_\theta^\ell$, z_θ^ℓ, and α_ℓ^k close to $\tilde{z}_\theta$, z_θ, and α^k, respectively, so that $\tilde{z}_\theta^\ell > \min\{z_\theta^\ell, \alpha_\ell^k\}$, which is a contradiction. The same argument applies to the case with $\tilde{z}_\theta < \min\{z_\theta, \alpha^k\}$. ▌

References

Abdulkadiroğlu, Atila, and Tayfun Sonmez. 2003. "School Choice: A Mechanism Design Approach." *American Economic Review*, 93: 729–747.

Abdulkadiroğlu, Atila, Parag A. Pathak, Alvin E. Roth, and Tayfun Sönmez. 2005. "The Boston Public School Match." *American Economic Review Papers and Proceedings*, 95: 368–372.

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth. 2005. "The New York City High School Match." *American Economic Review Papers and Proceedings*, 95: 364–367.

Adachi, H. 2000. "On a characterization of stable matchings." *Economics Letters*, 68(1): 43–49.

Aliprantis, Charalambos D., and Kim C. Border. 2006. *Infinite Dimensional Analysis: A Hitchhiker's Guide.* Springer.

Alkan, Ahmet. 2002. "A class of multipartner matching markets with a strong lattice structure." *Economic Theory*, 19(4): 737–746.

Alkan, Ahmet, and David Gale. 2003. "Stable Schedule Matching Under Revealed Preferences." *Journal of Economic Theory*, 112: 289–306.

Ashlagi, Itai, Mark Braverman, and Avinatan Hassidim. 2011. "Stability in large matching markets with complementarities." Discussion paper, Mimeo.

Ash, Robert B. 1977. *Real Analysis and Probability.* Academic Press.

Aygün, Orhan, and Tayfun Sönmez. 2012. "The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions." Unpublished mimeo.

Azevedo, Eduardo M, and Jacob D Leshno. 2011. "A supply and demand framework for two-sided matching markets." *Unpublished mimeo, Harvard Business School.*

Azevedo, Eduardo M, and John William Hatfield. 2012. "Complementarity and Multidimensional Heterogeneity in Matching Markets." mimeo.

Azevedo, Eduardo M, E Glan Weyl, and Alexander White. 2012. "Walrasian Equilibrium in Large, Quasilinear Markets." *Theoretical Economics.* forthcoming.

Baïou, Mourad, and Michel Balinski. 2000. "Many-to-many matching: stable polyandrous polygamy (or polygomous polyandry)." *Discrete Applied Mathematics*, 101: 1–12.

Che, Yeon-Koo, and Fuhito Kojima. 2010. "Asymptotic equivalence of probabilistic serial and random priority mechanisms." *Econometrica*, 78(5): 1625–1672.

Che, Yeon-Koo, and Youngwoo Koh. 2013. "Decentralized College Admissions." Unpublished mimeo.

Debreu, G. 1954. "Representation of a preference ordering by a numerical function." *Decision processes*, 159–165.

Echenique, Federico, and Jorge Oviedo. 2004. "Core Many-to-One Matchings by Fixed Point Methods." *Journal of Economic Theory*, 115: 358–376.

Echenique, Federico, and Jorge Oviedo. 2006. "A theory of stability in many-to-many matching." *Theoretical Economics*, 1: 233–273.

Echenique, Federico, and M Bumin Yenmez. 2007. "A solution to matching with preferences over colleagues." *Games and Economic Behavior*, 59(1): 46–71.

Echenique, Federico, Sangmok Lee, Matthew Shum, and M Bumin Yenmez. 2013. "The revealed preference theory of stable and extremal stable matchings." *Econometrica*, 81(1): 153–171.

Fleiner, Thomas. 2003. "A Fixed-Point Approach to Stable Matchings and Some Applications." *Mathematics of Operations Research*, 28: 103–126.

Gale, David, and Lloyd S. Shapley. 1962. "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69: 9–15.

Hatfield, John William, and Fuhito Kojima. 2008. "Matching with Contracts: Comment." *American Economic Review*, 98: 1189–1194.

Hatfield, John William, and Paul Milgrom. 2005. "Matching with Contracts." *American Economic Review*, 95: 913–935.

Hatfield, J.W., and S.D. Kominers. 2010. "Contract design and stability in matching markets." mimeo.

Hylland, Aanund, and Richard Zeckhauser. 1979. "The Efficient Allocation of Individuals to Positions." *Journal of Political Economy*, 87: 293–314.

Immorlica, Nicole, and Mohammad Mahdian. 2005. "Marriage, Honesty, and Stability." *SODA 2005*, 53–62.

Kagel, John H, and Alvin E Roth. 2000. "The dynamics of reorganization in matching markets: A laboratory experiment motivated by a natural experiment." *The Quarterly Journal of Economics*, 115(1): 201–235.

Kesten, O., and U. Ünver. 2009. "A theory of school choice lotteries." mimeo, Boston College and Carnegie Mellon University.

Klaus, Bettina, and Flip Klijn. 2005. "Stable matchings and preferences of couples." *Journal of Economic Theory*, 121(1): 75–106.

Kojima, Fuhito, and Mihai Manea. 2008. "Incentives in the Probabilistic Serial Mechanism." forthcoming, *Journal of Economic Theory*.

Kojima, Fuhito, and Parag A. Pathak. 2008. "Incentives and Stability in Large Two-Sided Matching Markets." forthcoming, *American Economic Review*.

Kojima, Fuhito, Parag A Pathak, and Alvin E Roth. 2013. "Matching with couples: Stability and incentives in large markets." *forthcoming, Quarterly Journal of Economics*.

Lee, SangMok. 2012. "Incentive compatibility of large centralized matching markets." *URL http://www. hss. caltech. edu/˜ sangmok/research. html*, 2.

Manea, Mihai. 2009. "Asymptotic Ordinal Inefficiency of Random Serial Dictatorship." forthcoming, *Theoretical Economics*.

Ostrovsky, Michael. 2008. “Stability in supply chain networks.” *The American Economic Review*, 897–923.

Pycia, Marek. 2012. “Stability and preference alignment in matching and coalition formation.” *Econometrica*, 80(1): 323–362.

Roth, Alvin E. 1984. “The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory.” *Journal of Political Economy*, 92: 991–1016.

Roth, Alvin E. 1991. “A natural experiment in the organization of entry-level labor markets: regional markets for new physicians and surgeons in the United Kingdom.” *The American economic review*, 415–440.

Roth, Alvin E. 2002. “The economist as engineer: Game theory, experimentation, and computation as tools for design economics.” *Econometrica*, 70: 1341–1378.

Roth, Alvin E., and Elliot Peranson. 1999. “The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design.” *American Economic Review*, 89: 748–780.

Roth, Alvin E, Uriel G Rothblum, and John H Vande Vate. 1993. “Stable matchings, optimal assignments, and linear programming.” *Mathematics of Operations Research*, 18(4): 803–828.

Sönmez, T., and M.U. Ünver. 2010. “Course Bidding at Business Schools.” *International Economic Review*, 51(1): 99–123.

Sotomayor, M. 1999. “Three remarks on the many-to-many stable matching problem.” *Mathematical social sciences*, 38(1): 55–70.