Dynamic Ex Post Equilibrium, Welfare, and Optimal Trading Frequency in Double Auctions*

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Abstract

We characterize a dynamic ex post equilibrium in a sequence of uniform-price double auctions. Bidders start with private inventories, receive over time a sequence of private signals, have interdependent and linearly decreasing marginal values, and trade with demand schedules. In our ex post equilibrium, each bidder's strategy remains optimal even if he would observe the concurrent and historical private information of other bidders; therefore, the ex post equilibrium is robust to distributions of signals and inventories. The equilibrium prices aggregate dispersed private information, and the equilibrium allocations converge to the efficient allocation exponentially over time. The socially optimal trading frequency is low for scheduled arrivals of information but is high for stochastic arrivals of information.

Keywords: dynamic ex post equilibrium, trading frequency, welfare, double auction
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1 Introduction

Dynamic trading of divisible assets is important in many markets. For example, the exchange trading of equities, futures and options is typically organized as continuous double auctions. Other notable examples include the periodic auctions of treasury securities and commodities such as milk powder, iron ore, and electricity. Strategic trading in these markets involves private information regarding the asset value, inventories, or both. Analyzing the trading behavior in these markets helps us better understand information aggregation, allocative efficiency, and market design.

In this paper we analyze strategic trading in a dynamic market, as well as the associated welfare and optimal trading frequency. In our model, a finite number of agents, whom we call "bidders," trade a divisible asset in an infinite sequence of uniform-price double auctions, held at discrete time intervals. Each bidder receives over time a sequence of private signals that form a martingale, and each bidder's value for owning the traded asset is a weighted average of his most recent signal and other bidders' most recent signals. Since other bidders' signals are unobservable, this type of "interdependent values" capture adverse selection in financial markets, as well as in goods markets where winning bidders subsequently resell part of the assets. In addition to private signals, bidders also start with private inventories of the asset and have linearly decreasing marginal values for owning it. Time-varying valuations and private inventories create the gains from trade. In each double auction, bidders submit demand schedules (i.e., a set of limit orders) and pay for their allocations at the market-clearing price. All bidders maximize their time-discounted utilities and internalize the price impact of their trades.

Our main result is a dynamic and stationary "ex post equilibrium"—an equilibrium in which a bidder's trading strategy depends only on his private information (i.e., his current signal and inventory) but remains optimal even if he learns the concurrent and historical private information of all other bidders.[1] In other words, the ex post equilibrium is regret-free (hence the "ex post" notion) and is a strictly stronger equilibrium notion than Bayesian equilibrium. The ex post equilibrium is robust because a bidder's equilibrium strategy does not depend on the distribution assumptions regarding other bidders' private signals and inventories. Wilson (1987)

[1]The equilibrium is Bayesian optimal with respect to *future* signals because these are yet to realize.

highlights this type of robustness as a desirable feature for models of auctions and trading (the "Wilson criterion"). In addition, the ex post equilibrium notion greatly simplifies the equilibrium analysis because the calculation of a bidder's optimal strategy does not involve the often-intractable conditional distributions of others bidders' private information. In our proposed ex post equilibrium, a bidder's optimal demand in each double auction is a linear function of his most recent signal, the price, his private inventory, and the total asset supply in the market. The equilibrium price in each auction is a multiple of the average of the most recent signals, adjusted for the decreasing marginal valuation for holding the average inventory.

The dynamic ex post equilibrium proves to be a useful tool to answer welfare questions. In the dynamic ex post equilibrium, we show that the equilibrium allocations of assets across bidders after each auction are not fully efficient, but they converge exponentially over time to the efficient allocation. Exponential convergence suggests that a sequence of double auctions is a simple and effective mechanism to quickly achieve allocative efficiency, even if the number of bidders is small.

We further exploit our ex post equilibrium to investigate the impact of trading frequency on social welfare. The social value of increasing trading speed has recently received much regulatory, industry and academic attention. Trading frequency is defined in our model as the number of double auctions per unit of clock time. We show that although the speed of convergence to efficiency increases with trading frequency, continuous trading does not lead to immediate convergence to efficient allocation due to bidders' price impact and adverse selection.

Moreover, we show that depending on the nature of new information, the socially optimal trading frequency can be very high or very low. For scheduled information arrivals, a slow (batch) market tends to be optimal; this is because a longer delay until the next trading opportunity serves as a commitment device to encourage aggressive trading immediately. By contrast, for stochastic information arrivals, a fast (continuous) market tends to be optimal; this is because a shorter delay implies a shorter waiting time between new information arrival and the first trading opportunity to reallocate the asset. Since information in reality is likely to combine scheduled and stochastic components, the optimal trading frequency in practice needs not have a clear-cut answer. While we do not attempt to prescribe the best market design in general, we do present a relevant tradeoff associated with increasing trading speed: the benefit of faster reaction to unpredictable information versus the cost of less active

trading in each trading round. We believe that understanding this salient tradeoff is a key component for answering the important question of optimal trading speed.

Related literature. Our results contribute to two broad branches of literature: dynamic trading under asymmetric information and ex post equilibrium.

Dynamic trading under asymmetric information. Our results are related to dynamic models of informed trading, most notably Kyle (1985) and Glosten and Milgrom (1985), as well as many extensions.[2] A common theme in this literature is that certain investors are informed of the asset value, whereas others are "noise traders" who trade for exogenous reasons. By contrast, we explicitly model the trading motives of all agents; doing away with noise traders makes welfare implications more transparent in our setting and enables us to study optimal trading frequency. Classic models of rational expectations equilibrium (REE), starting from Grossman and Stiglitz (1980) and Grossman (1976, 1981), also generalize to dynamic markets (see, for example, He and Wang 1995 and Biais, Bossaerts, and Spatt 2010, among others). While agents in these models are assumed to take prices as given, bidders in our model fully internalize the impacts of their trades on the equilibrium price.

A separate literature analyzes trading by demand schedules, with the focus mostly on information aggregation and endogenous "demand reduction" (i.e., trading less than the efficient level to avoid price impact). Most of such models are static and study Bayesian equilibria, as in Kyle (1989), Vives (2011), Rostek and Weretka (2012), and Babus and Kondor (2012). Rostek and Weretka (2011) study dynamic trading and allocative trading motive with multiple assets, but they assume commonly observable fundamental information. In comparison, our ex post equilibrium combines dynamic trading and asymmetric information, while retaining analytical tractability and closed-form solutions.

Our analysis on welfare and optimal trading frequency is most closely related to Vayanos (1999), who studies dynamic trading by investors who receive periodic inventory shocks. The key difference is that Vayanos assumes public information of asset fundamentals, whereas we allow interdependent valuations and adverse selection. Moreover, while Vayanos (1999) shows that slower trading tends to be optimal under private inventory information, we show that this conclusion holds only under scheduled information arrivals. If, instead, information arrives at stochastic times,

[2]Recent dynamic extensions of Kyle (1985) include Foster and Viswanathan (1996), Back, Cao, and Willard (2000), and Ostrovsky (2011), among many others.

continuous trading can be optimal because it allows bidders to react immediately to new information.

Our dynamic trading model is also related to the continuous-time model of Kyle, Obizhaeva, and Wang (2013), in which agents have common values but "agree to disagree" on the precision of their signals. Because trading in their model happens in continuous time, Kyle, Obizhaeva, and Wang (2013) do not address the question of optimal trading frequency. (Their main application is non-martingale price dynamics, such as price spikes and reversals.) By contrast, we focus on market design and explicitly characterize the effect of trading frequency on welfare.

Ex post equilibrium. Our definition of dynamic ("periodic") ex post equilibrium is adopted from Bergemann and Valimaki (2010), who study dynamic ex post implementation, that is, designing a mechanism such that the efficient allocations are obtained by a dynamic ex post equilibrium.[3] In contrast, we fix the double auctions mechanism and study its dynamic ex post equilibrium.

Our dynamic ex post equilibrium is similar in spirit to the dynamic ex post equilibria in Hörner and Lovo (2009), Fudenberg and Yamamoto (2011), and Hörner, Lovo, and Tomala (2012). A major distinction is that the ex post equilibria in these three studies rely on dynamic punishments to be sustained and require the discount factors to be close to 1, whereas our dynamic ex post equilibrium is stationary and imposes no restriction on the discount factor.

Our results complement those of Perry and Reny (2005), who construct an ex post equilibrium in a multi-unit ascending-price auction with interdependent values. In their ascending-price auction, bidders' private information is revealed as the auction proceeds, whereas in our model, no private information is revealed until the auction ends. In addition, while Perry and Reny focus on designing an auction format that ex post implements the efficient outcome, we focus on the standard uniform-price double auction and show that multiple rounds of double auctions achieve exponential convergence to efficiency.

In static settings, the ex post equilibrium condition originates from the "uniform incentive compatible" condition of Holmström and Myerson (1983). Klemperer and Meyer (1989) pioneer the study of "supply function equilibria" that are ex post optimal given *supply shocks* in settings where bidders have symmetric information

[3]Other work in the ex post implementation literature includes Crémer and McLean (1985), Bergemann and Morris (2005), and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006), among others.

regarding the marginal values of the asset. Separately, Ausubel (2004) proposes an ascending-price multi-unit auction and characterizes an equilibrium in which truthful bidding is ex post optimal if bidders have purely *private* values.

2 Dynamic Trading in Double Auctions

In this section we study dynamic trading with interdependent values and arrivals of new information. In the next section we analyze the effect of trading frequency on social welfare.

2.1 Model

This subsection describes the dynamic trading model. Discussions of modeling features and assumptions are deferred to Section 2.1.1.

Time is continuous. Trading happens at clock times $\tau \in \{0, \Delta, 2\Delta, 3\Delta, \ldots\}$, where $\Delta > 0$ is the length of clock time between consecutive rounds of trading. The smaller is Δ, the higher is the frequency of trading. The exact trading mechanism is described shortly. We will refer to each trading round as a "period," indexed by $t \in \{0, 1, 2, \ldots\}$, so period-$t$ trading occurs at clock time $t\Delta$. The discount rate per unit of clock time is $r > 0$.

There is one divisible asset traded on the market. The asset can be a commodity, a financial security, or a derivative contract. There are $n > 2$ traders, who we refer to as "bidders." At clock time 0 but before the first round of trading, each bidder i starts with an inventory $z_{i,0}$ of the asset. The initial inventory $z_{i,0}$ is bidder i's private information, but the total inventory $Z = \sum_{i=1}^{n} z_{i,0}$ is a constant and is common knowledge. For example, the total supply of stocks and bonds is public information, and the total net supply of a derivative contract is zero.

In addition to the private inventories, each bidder i receives a private "signal" $s_{i,\tau} \in \mathbb{R}$ at clock time τ. The sequence of signals, $\{s_{i,\tau}\}_{\tau \geq 0}$, follows a continuous-time martingale (conditional on period-0 inventories). That is, for every i and $\tau' > \tau \geq 0$,

$$\mathbb{E}[s_{i,\tau'} \mid \{s_{j,\tau''}\}_{1 \leq j \leq n, 0 \leq \tau'' \leq \tau}, \{z_{j,0}\}_{1 \leq j \leq n}] = s_{i,\tau}. \tag{1}$$

The conditional variables in (1) include the histories of other bidders' signals and initial inventories because, as will become clear shortly, our equilibrium is optimal

with respect to such private information. A simple application of the law of iterated expectation shows that (1) implies that a bidder's signal is also a martingale conditional only on his private information:

$$\mathbb{E}[s_{i,\tau'} \mid \{s_{i,\tau''}\}_{0\leq\tau''\leq\tau}, z_{i,0}] = s_{i,\tau}. \tag{2}$$

We can interpret the martingales as conditional expectations (of some future payoff) that are updated over time. Under the martingale assumption, each bidder i's current signal $s_{i,\tau}$ is the best estimate of his future signals. As long as the martingale property is satisfied, the exact detail of the signal processes is inconsequential to our equilibrium analysis.[4]

The trading mechanism is uniform-price double auctions. In each period $t \geq 0$, a new double auction is held to reallocate the asset among the bidders. (That is, there is no external supply.) Each bidder can buy or sell the asset, and we use the convention that a positive quantity $q_{i,t\Delta} \geq 0$ denotes buying $q_{i,t\Delta}$ units of the asset, and a negative $q_{i,t\Delta} < 0$ denotes selling $-q_{i,t\Delta}$ units. In the period-t auction, bidder i starts with an inventory of $z_{i,t\Delta}$ and submits a demand schedule $x_{i,t\Delta}(p) : \mathbb{R} \to \mathbb{R}$. A demand schedule is essentially a set of limit orders, as in the exchange trading of equities, futures and options. The auctioneer, which can be a human or (more commonly today) a computer algorithm, determines the market-clearing price $p^*_{t\Delta}$ by

$$\sum_{i=1}^{n} x_{i,t\Delta}(p^*_{t\Delta}) = 0. \tag{3}$$

In the equilibrium we state shortly, there exists a unique market clearing price $p^*_{t\Delta}$ in each trading period.[5] Given the price $p^*_{t\Delta}$, bidder i receives $q_{i,t\Delta} = x_{i,t\Delta}(p^*_{t\Delta})$ units of the asset at the price of $p^*_{t\Delta}$. That is, all transactions in the same trading period happen at the same market-clearing price. Inventories therefore evolve according to

$$z_{i,(t+1)\Delta} = z_{i,t\Delta} + q_{i,t\Delta}. \tag{4}$$

[4]For example, future signals can arrive continuously and follow a diffusion process; or, they can arrive in discrete, irregular intervals, in which case the signal process exhibits "jumps." Each bidder's signal process can have arbitrary autocorrelation and conditional variance, and any pair of signal processes, $\{s_{i,\tau}\}_{\tau\geq 0}$ and $\{s_{j,\tau}\}_{\tau\geq 0}$, for $i \neq j$, can have arbitrary conditional covariance.

[5]We specify the off-equilibrium behavior as follows. If no market clearing price $p^*_{t\Delta}$ exists, each bidder gets zero quantity $q_{i,t\Delta} = 0$ from that auction; if multiple market clearing prices exist, one is selected arbitrarily.

Since bidder i's initial inventory is his private information, so is his inventory history.

After describing the information structure and trading protocol, we now turn to the preferences. Specifically, after acquiring the quantity $q_{i,t\Delta}$ in period t but before the start of period $t+1$, bidder i receives a "flow utility" (not counting the price) of

$$v_{i,t\Delta}(z_{i,t\Delta}+q_{i,t\Delta})-\frac{\lambda}{2}(z_{i,t\Delta}+q_{i,t\Delta})^2 \tag{5}$$

per unit of clock time, where

$$v_{i,t\Delta}\equiv\alpha\, s_{i,t\Delta}+(1-\alpha)\,\frac{1}{n-1}\sum_{j\neq i}s_{j,t\Delta} \tag{6}$$

is bidder i's value for owning an infinitesimal amount of asset in period t, and $\alpha\in(0,1]$ and $\lambda>0$ are constants known to all bidders. The simple specifications (5) and (6) capture a number of realistic features of dynamic trading, as we further discuss in Section 2.1.1. In particular, Equation (6) is a standard way of modeling interdependence in bidders' valuations (see, for example, Perry and Reny 2005 and Bergemann and Morris 2009).

Given (5), bidder i's period-t utility is the integral of time-discounted flow utility less the one-off payment of asset transaction, i.e.,

$$\begin{aligned}
&U(q_{i,t\Delta},p^*_{t\Delta};v_{i,t\Delta},z_{i,t\Delta})\\
&=\int_{\tau=0}^{\Delta}e^{-\tau r}\left(v_{i,t\Delta}(z_{i,t\Delta}+q_{i,t\Delta})-\frac{\lambda}{2}(z_{i,t\Delta}+q_{i,t\Delta})^2\right)d\tau-p^*_{t\Delta}\,q_{i,t\Delta}\\
&=\frac{1-e^{-r\Delta}}{r}\left(v_{i,t\Delta}(z_{i,t\Delta}+q_{i,t\Delta})-\frac{\lambda}{2}(z_{i,t\Delta}+q_{i,t\Delta})^2\right)-p^*_{t\Delta}\,q_{i,t\Delta}.
\end{aligned} \tag{7}$$

Bidder i's overall utility (or "continuation value") in all future trading periods, evaluated at the clock time $t\Delta$, is

$$\begin{aligned}
V_{i,t\Delta}&=\sum_{t'=t}^{\infty}e^{-r(t'-t)\Delta}U(q_{i,t'\Delta},p^*_{t'\Delta};v_{i,t'\Delta},z_{i,t'\Delta})\\
&=U(q_{i,t\Delta},p^*_{t\Delta};v_{i,t\Delta},z_{i,t\Delta})+e^{-r\Delta}V_{i,(t+1)\Delta}.
\end{aligned} \tag{8}$$

We emphasize that in period t before the new auction is held, bidder i's information consists of the paths of his signals $\{s_{i,\tau}\}_{0\leq\tau\leq t\Delta}$ and of his inventories $\{z_{i,t'\Delta}\}_{0\leq t'\leq t}$,

as well as his submitted demand schedules $\{x_{i,t'\Delta}(p)\}_{0\le t'<t}$. For notational simplicity, we denote bidder i's information set at the beginning of period t by

$$H_{i,t\Delta} = \{\{s_{i,\tau}\}_{0\le\tau\le t\Delta}, \{z_{i,t'\Delta}\}_{0\le t'\le t}, \{x_{i,t'\Delta}(p)\}_{0\le t'<t}\}\,. \tag{9}$$

Notice that by the identity $z_{i,(t'+1)\Delta} - z_{i,t'\Delta} = q_{i,t'\Delta} = x_{i,t'\Delta}(p^*_{t'\Delta})$, a bidder can infer the previous price path $\{p^*_{t'\Delta}\}_{t'<t}$ from his information set $H_{i,t\Delta}$. Bidder i's period-t strategy, $x_{i,t\Delta} = x_{i,t\Delta}(p; H_{i,t\Delta})$, is measurable with respect to $H_{i,t\Delta}$. This completes the description of our model.

2.1.1 Discussion of Model Assumptions

Before describing our equilibrium concept and solution, we discuss the motivation and interpretation of our model, including the valuation form (6) and the utility form (5) and (8).

First, values $\{v_{i,t\Delta}\}$ are interdependent; that is, bidder i's period-t value for owning the asset, $v_{i,t\Delta}$, depends on the most recent information possessed by all bidders.[6] This type of interdependent values captures adverse selection. Since each bidder assigns a potentially different weight on his own signal than on another bidder's signal, our model essentially has a private component of valuations. A key benefit of having this private component is that we do not rely on "noise traders" to generate trades. The economic implications of allocative efficiency and welfare are therefore more transparent in our setting.

Second, bidders have linearly decreasing marginal values for owning the asset, which leads to the linear-quadratic form of (5). The quadratic term in (5) can be viewed as a reduced-form specification for (unmodeled) risk-aversion or collateral costs (with λ being the variance of the price or return of the asset), or as a first order approximation of nonlinear marginal values (which are hard to analyze in a dynamic model). The linear-quadratic utility (5) is also used in the static models by Biais, Martimort, and Rochet (2000), Vives (2011), Rostek and Weretka (2012), Malamud

[6]It is natural to have values depend on most recent signals, as information generally improves over time (and thus a later signal subsumes an earlier one). In principal, a new signal may arrive between two trading clock times $t\Delta$ and $(t+1)\Delta$. Given the martingale property, however,

$$\mathbb{E}[v_{i,\tau} \mid \{s_{j,\tau'}\}_{1\le j\le n,\tau'\le t\Delta}] = v_{i,t\Delta}$$

for all $\tau \in (t\Delta, (t+1)\Delta)$. Thus, the specification of flow utility is almost without loss of generality.

and Rostek (2013), among others.

Third, a flow utility is a natural way to capture the benefit of holding an asset for a (potentially short) period of time. In practice, the flow utility can be the dividend of a stock, the accrued interest of a bond, the "convenience yield" of a commodity, or the potential benefit of pledging the asset as collateral to obtain financing.[7] A flow utility for owning the asset is also standard in models of over-the-counter markets (see, for example, Duffie, Garleanu, and Pedersen 2005). We emphasize, however, that our results also hold if the marginal values in the flow utility of the asset were received "once," instead of continually over time.[8]

Fourth, the general form of our model does *not* impose restrictions on the price $p_{t\Delta}$, the pre-auction marginal value $v_{i,t\Delta} - \lambda z_{i,t\Delta}$, or post-auction marginal value $v_{i,t\Delta} - \lambda(z_{i,t\Delta} + q_{i,t\Delta})$. In particular, we do not restrict these prices and marginal values to be positive. Indeed, the market prices of many financial and commodity derivatives—including forwards, futures and swaps—are zero upon inception and can become arbitrarily negative as market conditions change over time. A unilateral break (or "free disposal") of loss-making derivatives contracts constitutes a default and leads to the loss of posted margin and reputation. For practical purposes, therefore, it is realistic to think of these derivative contracts as having unlimited liabilities (until default occurs). It is for these applications that we do not impose free disposal as a necessary element of our model. In reality, it is not uncommon for investors to pay

[7] In particular, the collateral benefit of the asset is relevant intraday at a relatively high frequency. In the U.S. repo markets, for example, financial institutions routinely pledge Treasury securities as collateral to obtain financing; in this process, the same Treasury security can be pledged multiple times per day.

[8] For example, an alternative utility function is that each unit of asset yields its marginal value only once, at the time of its acquisition. That is, given a path of quantities $\{q_{i,t\Delta}\}_{t=0}^{\infty}$ and prices $\{p_{t\Delta}\}_{t=0}^{\infty}$, the utility from transacting at these quantities and prices is:

$$\tilde{V}_{i,0} = \sum_{t=0}^{\infty} e^{-rt\Delta} \int_{q=z_{i,t\Delta}}^{z_{i,t\Delta}+q_{i,t\Delta}} (v_{i,t\Delta} - \lambda q - p_{t\Delta})\, dq. \tag{10}$$

One can easily show that, up to some normalizing constants, $\tilde{V}_{i,0}$ is the same as the sum of flow utilities in (8):

$$\begin{aligned} &\mathbb{E}\left[v_{i,0} z_{i,0} - \frac{\lambda}{2}(z_{i,0})^2 \; + \; \tilde{V}_{i,0} \;\middle|\; s_{i,0}, z_{i,0} \right] \qquad (11) \\ &= r \cdot \mathbb{E}\left[\sum_{t=0}^{\infty} \frac{1 - e^{-r\Delta}}{r} e^{-rt\Delta} \left(v_{i,t\Delta}(z_{i,t\Delta} + q_{i,t\Delta}) - \frac{\lambda}{2}(z_{i,t\Delta} + q_{i,t\Delta})^2 \right) - \frac{e^{-rt\Delta}}{r} p_{t\Delta} q_{i,t\Delta} \;\middle|\; s_{i,0}, z_{i,0} \right], \end{aligned}$$

and likewise for any continuation value.

others (e.g. broker-dealers and market makers) to dispose of loss-making derivative positions, presumably because the negative marginal values for holding these positions exceed (in absolute value) the negative price for selling them.[9]

2.2 Equilibrium Characterization

In this dynamic market we use a variant of the notion of "periodic ex post equilibrium" introduced by Bergemann and Valimaki (2010). In this notion of equilibrium, for any period t each bidder's strategy is "ex post optimal" with respect to other bidders' information sets up to period t, but is Bayesian optimal with respect to signals in the future. This equilibrium is "ex post" because, in the absence of new information immediately after the period-t auction, each bidder has no regret. We make this equilibrium notion precise in the following definition.

Definition 1. A **periodic ex post equilibrium** consists of the strategy profile $\{x_{j,t\Delta}\}_{1\leq j\leq n, t\geq 0}$ such that for every bidder i and for every path of his information set $H_{i,t\Delta}$, bidder i has no incentive to deviate from $\{x_{i,t'\Delta}\}_{t'\geq t}$ even if he learns the profile of other bidders' information set. That is, for every alternative strategy $\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t}$ and every profile of other bidders' information sets $\{H_{j,t\Delta}\}_{j\neq i}$,

$$\begin{aligned}&\mathbb{E}[V_{i,t\Delta}(\{x_{i,t'\Delta}\}_{t'\geq t}, \{x_{j,t'\Delta}\}_{j\neq i, t'\geq t}) \mid H_{i,t\Delta}, \{H_{j,t\Delta}\}_{j\neq i}]\\ \geq &\mathbb{E}[V_{i,t\Delta}(\{\tilde{x}_{i,t'\Delta}\}_{t'\geq t}, \{x_{j,t'\Delta}\}_{j\neq i, t'\geq t}) \mid H_{i,t\Delta}, \{H_{j,t\Delta}\}_{j\neq i}],\end{aligned} \tag{12}$$

where the conditional expectations are taken over all possible realizations of future signals $\{s_{j,\tau}\}_{1\leq j\leq n, \tau > t\Delta}$.

A key feature of this equilibrium definition is that each bidder i's strategy is optimal for all realizations of other bidders' information sets, $\{H_{j,t\Delta}\}_{j\neq i}$, even though $\{H_{j,t\Delta}\}_{j\neq i}$ is unobservable to bidder i. The ex post equilibrium notion greatly simplifies the analysis of dynamic trading. By contrast, in a conventional perfect Bayesian equilibrium, bidder i needs to form conditional beliefs about $\{H_{j,t\Delta}\}_{j\neq i}$, which can quickly become intractable along the equilibrium path; and off the equilibrium path

[9] On the other hand, assets that have limited liabilities, such as stocks and bonds, should have nonnegative prices and marginal values. Such restrictions are satisfied if λ is a sufficiently small relative to the support of $\{v_{i,t\Delta}\}$. More precisely, it is easy to show that if $v_{i,t\Delta} - \lambda z_{i,t\Delta} \geq 0$ for every bidder i, then in the equilibrium of Proposition 1 every bidder i acquires a quantity $q_{i,t\Delta}$ such that $v_{i,t\Delta} - \lambda(z_{i,t\Delta} + q_{i,t\Delta}) \geq 0$ as well, which implies $p^*_{t\Delta} \geq 0$.

the conditional beliefs can be sensitive to the specific equilibrium refinement argument.

We now present a periodic ex post equilibrium in the dynamic market. We focus on a stationary equilibrium, in which a bidder's strategy only depends on his current signal $s_{i,t\Delta}$ and current level of inventory $z_{i,t\Delta}$, but does not depend explicitly on t.

Proposition 1. *Suppose that $n\alpha > 2$. In the market with dynamic trading, there exists a stationary periodic ex post equilibrium in which bidder i submits the demand schedule*

$$x_{i,t\Delta}(p; s_{i,t\Delta}, z_{i,t\Delta}) = a\left(s_{i,t\Delta} - rp - \frac{\lambda(n-1)}{n\alpha - 1} z_{i,t\Delta} + \frac{\lambda(1-\alpha)}{n\alpha - 1} Z\right), \tag{13}$$

where

$$a = \frac{n\alpha - 1}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right) > 0. \tag{14}$$

The period-t equilibrium price is

$$p^*_{t\Delta} = \frac{1}{r}\left(\frac{1}{n}\sum_{i=1}^{n} s_{i,t\Delta} - \frac{\lambda}{n} Z\right). \tag{15}$$

Proof. See Section 2.2.2. □

Before discussing the equilibrium construction, we make a few observations on its properties. First, the market-clearing price $p^*_{t\Delta}$ aggregates the most recent signals $\{s_{i,t\Delta}\}$. The adjustment term $\lambda Z/n$ results from the declines in marginal value for holding the average inventory. As an asset purchased in period t gives a bidder a stream of flow utilities during the clock time $\tau \in (t\Delta, \infty)$, the multiplier $1/r$ in (15) reflects the present value of a perpetuity,[10] $\int_{\tau=0}^{\infty} e^{-r\tau}\, d\tau = 1/r$. Moreover, since the signal processes are martingales, so are the equilibrium prices $\{p^*_{t\Delta}\}_{t\geq 0}$.

Second, although bidders learn from $p^*_{t\Delta}$ the average signal $\sum_i s_{i,t\Delta}/n$ in period t, they do not learn *individual* inventories or signals. Thus, information does not become symmetric after each round of trading. Moreover, because new information

[10] We can do away with the $1/r$ factor if we use the utility normalization in Equation (10); see footnote 8.

may arrive by the clock time $(t+1)\Delta$ of the next auction, a period-$(t+1)$ strategy that depends explicitly on the lagged price $p^*_{t\Delta}$ is generally not ex post optimal.

Third, in a pure private-value setting, i.e. if $\alpha = 1$, bidders' strategies are independent of the total inventory Z, and the equilibrium of Proposition 1 remains an equilibrium if bidders face uncertainty regarding Z. This feature is reminiscent to Klemperer and Meyer (1989), who characterize supply-function equilibria that are ex post optimal given demand shocks. In their static model, however, bidders's marginal values are common knowledge. Similarly, in a static setting with a commonly known asset value, Ausubel, Cramton, Pycia, Rostek, and Weretka (2011) characterize an ex post equilibrium with uncertain supply.

Finally, for our construction of the periodic ex post equilibrium, it is important that each bidder i puts the same additive weight $(1-\alpha)/(n-1)$ on other bidders' signals, as in (6), and that the marginal value declines in quantity at the same rate $\lambda > 0$ for every bidder. A dynamic model with asymmetric parameters is hard to solve and is left for future research.[11] The condition $n\alpha > 2$ guarantees that the second-order condition holds. Intuitively, if the weight α on one's own signal were too small, a bidder would rely too much on other bidders' signals, and his demand would be increasing in price (i.e., upward-sloping).

We illustrate the intuition of Proposition 1 in two ways: in Section 2.2.1 we explicitly construct the equilibrium for the static special case, and in Section 2.2.2 we sketch the key steps of the full proof of Proposition 1, leaving some details to Section A.1.

2.2.1 The Static Special Case of Proposition 1 with $\Delta = \infty$

Corollary 1. *Suppose that $n\alpha > 2$ and $\Delta = \infty$. There is only one trading round at time zero. At time zero, each bidder i submits the demand schedule*

$$x_{i,0}(p; s_{i,0}, z_{i,0}) = \frac{n\alpha - 2}{\lambda(n-1)}(s_{i,0} - rp) - \frac{n\alpha - 2}{n\alpha - 1} z_{i,0} + \frac{(1-\alpha)(n\alpha - 2)}{(n-1)(n\alpha - 1)} Z, \tag{16}$$

[11] For pure private values (i.e., $\alpha = 1$) and in a static setting (i.e., $\Delta = \infty$), we can solve an ex post equilibrium for heterogeneous λ's across bidders. This result is presented in the online appendix of this paper.

and the equilibrium price is

$$p_0^* = \frac{1}{r}\left(\frac{1}{n}\sum_{i=1}^{n} s_{i,0} - \frac{\lambda}{n}Z\right). \tag{17}$$

Corollary 1 is obtained simply by taking the limit of Proposition 1 as $\Delta \to \infty$. However, to better illustrate the intuition of Proposition 1, we now provide an explicit construction of the equilibrium in Corollary 1.

Equilibrium construction of Corollary 1. Given $\Delta = \infty$, bidders never have a second chance to trade after the first auction. Thus, we only need to calculate the strategies in the first trading period, $t = 0$.

We conjecture a strategy profile $(x_{1,0}, \ldots, x_{n,0})$. Given that all other bidders use this strategy profile and for a fixed profile of signals $(s_{1,0}, \ldots, s_{n,0})$ and inventories $(z_{1,0}, \ldots, z_{n,0})$, the expected utility of bidder i, at the price of p_0 and conditional on the period-0 signals and inventories, is:

$$\begin{aligned}\Pi_{i,0}(p_0) \equiv \frac{1}{r}\Bigg(&\left(\alpha s_{i,0} + \beta\sum_{j\neq i} s_{j,0}\right)\left(z_{i,0} - \sum_{j\neq i} x_{j,0}(p_0)\right) - \frac{\lambda}{2}\left(z_{i,0} - \sum_{j\neq i} x_{j,0}(p_0)\right)^2\Bigg) \\ &- p_0\left(-\sum_{j\neq i} x_{j,0}(p_0)\right),\end{aligned} \tag{18}$$

where for simplicity of notation we define

$$\beta \equiv \frac{1-\alpha}{n-1}. \tag{19}$$

We can see that in (18) bidder i is effectively selecting an optimal price p_0 given the residual demand $-\sum_{j\neq i} x_{j,0}(p_0)$. (In particular, bidder i can guarantee himself his outside option from not trading by selecting a price p_0 such that $-\sum_{j\neq i} x_{j,0}(p_0) = 0$.) This optimization against residual demand schedule is also present in the construction of Bayesian equilibrium in Kyle (1989), Vives (2011), and Rostek and Weretka (2012). Taking the first-order condition of $\Pi_{i,0}(p_0)$ at the market clearing price $p_0 = p_0^*$, we

have, for all i,

$$0 = \Pi'_{i,0}(p_0^*) = -x_{i,0}(p_0^*) + \frac{1}{r}\left(\alpha s_{i,0} + \beta \sum_{j\neq i} s_{j,0} - r p_0^* - \lambda(z_{i,0} + x_{i,0}(p_0^*))\right)\left(-\sum_{j\neq i} \frac{\partial x_{j,0}}{\partial p}(p_0^*)\right). \tag{20}$$

We refer to the above equation as the ex post first order condition.

We further conjecture that the equilibrium strategy is linear and symmetric:

$$x_{j,0}(p; s_{j,0}, z_{j,0}) = a s_{j,0} - bp + d z_{j,0} + fZ, \tag{21}$$

where a, b, d and f are constants. Given that all bidders $j \neq i$ use the strategy (21) in this conjectured equilibrium, we can rewrite bidder j's signal in terms of the price and his demand and inventory as follows:

$$\sum_{j\neq i} s_{j,0} = \sum_{j\neq i} \frac{x_{j,0}(p_0^*) + b p_0^* - d z_{j,0} - fZ}{a} = \frac{1}{a}\big(-x_{i,0}(p_0^*) + (n-1)(b p_0^* - fZ) - d(Z - z_{i,0})\big), \tag{22}$$

where we have used the market clearing condition and the condition that all inventories add up to Z. Intuitively, in the above equation bidder i infers $\sum_{j\neq i} s_{j,0}$ from the price, his inventory, and his equilibrium demand. Substituting (22) into bidder i's first order condition (20) and rearranging, we have

$$\begin{aligned} x_{i,0}(p_0^*) &= \frac{(n-1)b}{r} \cdot \frac{\alpha s_{i,0} - \left[r - \frac{\beta}{a}(n-1)b\right] p_0^* - \left[\lambda - \frac{\beta}{a} d\right] z_{i,0} - \frac{\beta}{a}\left[(n-1)f + d\right] Z}{1 + \lambda \frac{(n-1)b}{r} + \frac{\beta}{a}\frac{(n-1)b}{r}} \\ &\equiv a s_{i,0} - b p_0^* + d z_{i,0} + fZ. \end{aligned}$$

Matching the coefficients and using the normalization that $\alpha + (n-1)\beta = 1$, we solve

$$a = \frac{n\alpha - 2}{\lambda(n-1)}, \quad b = r \cdot \frac{n\alpha - 2}{\lambda(n-1)}, \quad d = -\frac{n\alpha - 2}{n\alpha - 1}, \quad f = \frac{(1-\alpha)(n\alpha - 2)}{(n-1)(n\alpha - 1)}. \tag{23}$$

It is easy to verify that under this linear strategy,

$$\Pi''_{i,0}(p_0) = -(n-1)b + \frac{(n-1)b}{r}(-r - \lambda(n-1)b) \tag{24}$$

for every p_0. So $\Pi''_{i,0}(p_0) < 0$ if $n\alpha > 2$. Thus, we have a periodic ex post equilibrium

for $\Delta = \infty$.

The static special case in Corollary 1 is similar to equilibria in the static trading models of Vives (2011) without the non-strategic aggregate buyer, and of Rostek and Weretka (2012) with a symmetric valuation covariance. Bidders in these models also value the asset at a weighted average of signals and submit demand schedules. The key difference is that we study ex post equilibrium in a dynamic model, whereas Vives (2011) and Rostek and Weretka (2012) study Bayesian equilibria in a static model.

Even in the static setting, the ex post equilibrium notion proves to be useful. For example, we prove that the equilibrium in Corollary 1 is the unique ex post equilibrium in the class of continuously differentiable strategies. (This uniqueness result is provided in an online appendix to preserve space.) Intuitively, the ex post optimality condition can rule out most Bayesian equilibria because beliefs—which often give rise to the multiplicity of Bayesian equilibria—are irrelevant for ex post equilibrium. By contrast, Vives (2011) and Rostek and Weretka (2012) obtain uniqueness of equilibrium only in the class of linear and symmetric strategies.

2.2.2 Proof Outline for Proposition 1

We now sketch the key steps for the construction of the periodic ex post equilibrium of Proposition 1. A challenge in analyzing dynamic trading is that transactions in any period affects bidder's inventory, which in turn affects his strategy in all subsequent periods.

We conjecture that bidders use the following stationary and symmetric strategy:

$$x_{j,t\Delta}(p; s_{j,t\Delta}, z_{j,t\Delta}) = as_{j,t\Delta} - bp + dz_{j,t\Delta} + fZ. \tag{25}$$

This conjecture implies the market-clearing prices of

$$p^*_{t\Delta} = \frac{a}{nb}\sum_{j=1}^{n} s_{j,t\Delta} + \frac{d+nf}{nb}Z. \tag{26}$$

Because of the stationarity of the equilibrium, it is without loss of generality to examine bidders' incentives at period 0. Fix a profile of period-0 signals $(s_{1,0}, \ldots, s_{n,0})$ and inventories $(z_{1,0}, \ldots, z_{n,0})$. We use the single-deviation principle to construct (25) that forms a periodic ex post equilibrium: under the conjecture that other bidders $(j \neq i)$ use strategy (25) in every period, and that bidder i uses strategy (25) in period

1 and onwards, we verfiy that bidder i has no incentive to deviate from strategy (25) in period 0 only.

If bidder i uses alternative strategies in period 0, he faces the residual demand $-\sum_{j\neq i} x_{j,0}(p_0)$ and is effectively choosing a price p_0 and getting $x_{i,0}(p_0) = -\sum_{j\neq i} x_{j,0}(p_0)$ as before. Therefore, by differentiating the conditional expectation of the total utility (8) with respect to p_0 and evaluating it at $p_0 = p_0^*$, we obtain the ex post first order condition in period 0:

$$\mathbb{E}\Bigg[(n-1)b\cdot\left(\frac{1-e^{-r\Delta}}{r}\sum_{t=0}^{\infty}e^{-rt\Delta}\frac{\partial(z_{i,t\Delta}+x_{i,t\Delta})}{\partial x_{i,0}}(v_{i,t\Delta}-\lambda(z_{i,t\Delta}+x_{i,t\Delta}))-\sum_{t=0}^{\infty}e^{-rt\Delta}\frac{\partial x_{i,t\Delta}}{\partial x_{i,0}}\,p^*_{t\Delta}\right)$$
$$-\sum_{t=0}^{\infty}e^{-rt\Delta}\,x_{i,t\Delta}\,\frac{\partial p^*_{t\Delta}}{\partial p_0}\;\Bigg|\;\{s_{i,0},z_{i,0}\},\{s_{j,0},z_{j,0}\}_{j\neq i}\Bigg]=0, \tag{27}$$

where we write $x_{i,t\Delta} = x_{i,t\Delta}(p^*_{t\Delta}; s_{i,t\Delta}, z_{i,t\Delta})$ for the strategy $x_{i,t\Delta}(\,\cdot\,)$ defined in (25), $z_{i,(t+1)\Delta} = z_{i,t\Delta} + x_{i,t\Delta}$, and the conditional expectation $\mathbb{E}$ is over the future signals $\{s_{j,\tau}\}_{1\leq j\leq n,\tau>0}$. By focusing on the ex post first order condition, we need not specify bidder i's beliefs about $\{s_{j,0}, z_{j,0}\}_{j\neq i}$; by the symmetry and stationarity of the strategy (25), we need not specify the belief of any bidder j in any period t. The ex post optimality makes solving for the equilibrium strategy tractable in this dynamic game.

Since bidders follow the conjectured strategy in (25) from period 1 and onwards, we have the following evolution of inventories: in any period $t \geq 1$,

$$\begin{aligned} z_{i,t\Delta}+x_{i,t\Delta} =&(as_{i,t\Delta}-bp^*_{t\Delta}+fZ)+(1+d)(as_{i,(t-1)\Delta}-bp^*_{(t-1)\Delta}+fZ)\\ &+\cdots+(1+d)^{t-1}(as_{i,\Delta}-bp^*_{\Delta}+fZ)+(1+d)^t(x_{i,0}+z_{i,0}). \end{aligned} \tag{28}$$

The evolutions of prices and inventories, Equation (26) and (28), reveal that by changing the demand/price in period 0, bidder i has the following effects on inventories and prices in period $t \geq 1$:

$$\frac{\partial(z_{i,t\Delta}+x_{i,t\Delta})}{\partial x_{i,0}} = (1+d)^t, \tag{29}$$

$$\frac{\partial x_{i,t\Delta}}{\partial x_{i,0}} = (1+d)^{t-1}d, \tag{30}$$

$$\frac{\partial p^*_{t\Delta}}{\partial p_0} = \frac{\partial p^*_{t\Delta}}{\partial x_{i,0}} = 0. \tag{31}$$

It turns out that the equilibrium value of d satisfies $-1 < d < 0$. This means that $x_{i,0}$ has a diminishing effect on future inventories and that there is a less than one-to-one substitution between the acquisition today $x_{i,0}$ and the future acquisition $x_{i,t\Delta}$. These are natural consequences of the marginal value of the asset declining with its quantity.

Given the above equations and the martingale property of signals, the ex post first order condition in (27) simplifies to:

$$(n-1)b\left[\frac{1-e^{-r\Delta}}{r}\left(\sum_{t=0}^{\infty}e^{-rt\Delta}(1+d)^t\left(v_{i,0}-\lambda(\mathbb{E}[z_{i,t\Delta}+x_{i,t\Delta}\mid\{s_{j,0},z_{j,0}\}_{j=1}^n])\right)\right)\right.$$
$$\left.-p_0^*-\sum_{t=1}^{\infty}e^{-rt\Delta}(1+d)^{t-1}d\,p_0^*\right]-x_{i,0}=0, \tag{32}$$

where we have (cf. Equation (28)):

$$\begin{aligned}&\mathbb{E}[z_{i,t\Delta}+x_{i,t\Delta}\mid\{s_{j,0},z_{j,0}\}_{j=1}^n]\\&=(as_{i,0}-bp_0^*+fZ)\left(\frac{1}{-d}-\frac{(1+d)^t}{-d}\right)+(1+d)^t(x_{i,0}+z_{i,0}).\end{aligned} \tag{33}$$

Given (33), the first order condition (32) can be written as a linear combination of $v_{i,0}$, $z_{i,0}$, p_0^* and $x_{i,0}$. As in Section 2.2.1, we can infer the sum of others' signals in $v_{i,0}$ from $x_{i,0}$, p_0^* and $z_{i,0}$ (Equation (22)), and then match the coefficients with those in our conjecture in (25) to solve for the periodic ex post equilibrium. We leave the details to Section A.1.

2.3 Efficiency

We now study the allocative efficiency (or inefficiency) in the equilibrium of Proposition 1. As a benchmark, we consider the social planner's problem. In this dynamic market, the planner who reallocates the asset according to the current information

would solve the following problem in each period t:

$$\{q^e_{i,t\Delta}\} = \text{argmax}_{\{q_{i,t\Delta}\}} \sum_{i=1}^{n} \left(v_{i,t\Delta}\left(z_{i,t\Delta} + q_{i,t\Delta}\right) - \frac{\lambda}{2}(z_{i,t\Delta} + q_{i,t\Delta})^2 \right) \quad (34)$$
$$\text{subject to: } \sum_{i=1}^{n} q_{i,t\Delta} = 0,$$

where $z_{i,t\Delta}$ is bidder i's inventory before period t's reallocation, and $z_{i,t\Delta} + q^e_{i,t\Delta}$ is his efficient allocation in period t. We can easily derive that

$$z^e_{i,t\Delta} \equiv z_{i,t\Delta} + q^e_{i,t\Delta} = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_{i,t\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,t\Delta} \right) + \frac{1}{n} Z. \quad (35)$$

That is, in each period the efficient allocation to bidder i is equal to $1/n$ of the total asset supply Z plus a constant multiple of the difference between bidder i's signal and the average signal. In particular, the efficient allocation $z^e_{i,t\Delta}$ is independent of the pre-existing inventory $z_{i,t\Delta}$.

By contrast, the equilibrium of Proposition 1 does not achieve allocative efficiency immediately. This feature is known as "demand reduction" in the literature on static divisible auctions (see Ausubel, Cramton, Pycia, Rostek, and Weretka 2011). According to the periodic ex post equilibrium strategy, the post-auction allocation to bidder i in period t is

$$\begin{aligned} &z_{i,t\Delta} + x_{i,t\Delta}(p^*_{t\Delta}; s_{i,t\Delta}, z_{i,t\Delta}) \\ &= a\left(s_{i,t\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,t\Delta} \right) + \frac{1}{n} Z + (1+d)\left(z_{i,t\Delta} - \frac{1}{n} Z \right), \end{aligned} \quad (36)$$

where the constant $a > 0$ is given in Proposition 1, and

$$d \equiv -a\frac{\lambda(n-1)}{n\alpha - 1} = -1 + \frac{1}{2e^{-r\Delta}} \left(\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}} - (n\alpha - 1)(1 - e^{-r\Delta}) \right), \quad (37)$$

is the coefficient of $z_{i,t\Delta}$ in the equilibrium demand schedule (13).

It is straightforward to show that $a < \frac{n\alpha-1}{\lambda(n-1)}$; thus, relative to the efficient allocation in (35), post-auction allocations in the periodic ex post equilibrium "under-react" to the cross-bidder dispersion in signals. This under-reaction is a natural consequence

of adverse selection and of bidders' price impact. Moreover, we can also verify that $1 + d \in (0, 1)$; thus, a bidder who starts period-t trading with an above-average inventory only partially "liquidate" his excess inventory in period t. This partial adjustment, along with the under-reaction in signals, leads to allocative inefficiency. Nonetheless, as we show in the following proposition, there is exponential convergence to efficient allocation over time in the periodic ex post equilibrium. Let us denote by $\{z^*_{i,t\Delta}\}$ the path of inventories obtained by the periodic ex post equilibrium $x_{i,t\Delta}$ in Proposition 1: $z^*_{i,0} = z_{i,0}$, and for any $t \geq 0$,

$$z^*_{i,(t+1)\Delta} = z^*_{i,t\Delta} + x_{i,t\Delta}(p^*_{t\Delta}; s_{i,t\Delta}, z^*_{i,t\Delta}). \tag{38}$$

Proposition 2. *Given any $0 \leq \underline{t} \leq \bar{t}$, if $s_{i,t\Delta} = s_{i,\underline{t}\Delta}$ for all i and all $t \in \{\underline{t}, \underline{t}+1, \ldots, \bar{t}\}$, then the equilibrium inventories $z^*_{i,t\Delta}$ satisfy: for every i,*

$$z^*_{i,t\Delta} - z^e_{i,\underline{t}\Delta} = (1+d)^{t-\underline{t}}(z^*_{i,\underline{t}\Delta} - z^e_{i,\underline{t}\Delta}), \quad \forall t \in \{\underline{t}+1, \underline{t}+2, \ldots, \bar{t}+1\}, \tag{39}$$

where $\{z^e_{i,\underline{t}\Delta}\}$ is the the period-$\underline{t}$ efficient allocation defined in (35), and $d \in (-1, 0)$ is the coefficient of $z_{i,t\Delta}$ in the equilibrium strategy (13).

Moreover, we define the rate of convergence to efficiency per unit of clock time to be $-\log[(1+d)^{1/\Delta}]$. [12] *This convergence rate is increasing with the number n of bidders, the weight α of the private components in bidders' valuations, the discount rate r, and the clock-time frequency of trading $1/\Delta$.*

Proof. We first prove the convergence to efficient allocation. Conditional on the signals staying the same from period $\underline{t}$ to $\bar{t}$, the efficient allocation in each of these periods is also the same and is given by

$$z^e_{i,\underline{t}\Delta} = \frac{n\alpha - 1}{\lambda(n-1)}\left(s_{i,\underline{t}\Delta} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\underline{t}\Delta}\right) + \frac{1}{n}Z. \tag{40}$$

[12] The number $R = -\log[(1+d)^{1/\Delta}]$ is the rate of convergence per clock time, because for any clock times $\underline{\tau} < \bar{\tau}$ that are multiple of Δ, if no new signals arrive between $\underline{\tau}$ and $\bar{\tau}$, then we have (cf. Equation (39))

$$z^*_{i,\bar{\tau}} - z^e_{i,\underline{\tau}} = e^{-R(\bar{\tau}-\underline{\tau})}(z^*_{i,\underline{\tau}} - z^e_{i,\underline{\tau}}).$$

We rewrite the equilibrium strategy (13) as

$$x_{i,\underline{t}\Delta}(p; s_{i,\underline{t}\Delta}, z_{i,\underline{t}\Delta}) = as_{i,\underline{t}\Delta} - bp + dz_{i,\underline{t}\Delta} + fZ. \tag{41}$$

It is easy to verify that at the efficient allocation $z^e_{i,\underline{t}\Delta}$, bidder i trades zero unit in equilibrium:

$$x_{i,\underline{t}\Delta}(p^*_{\underline{t}\Delta}; s_{i,\underline{t}\Delta}, z^e_{i,\underline{t}\Delta}) = 0. \tag{42}$$

That is, for every $i \in \{1, 2, \ldots, n\}$,

$$z^e_{i,\underline{t}\Delta} = \frac{as_{i,\underline{t}\Delta} - bp^*_{i,\underline{t}\Delta} + fZ}{-d}. \tag{43}$$

By definition, we have

$$\begin{aligned} z^*_{i,(\underline{t}+1)\Delta} &= z^*_{i,\underline{t}\Delta} + x_{i,\underline{t}\Delta}(p^*_{i,\underline{t}\Delta}; s_{i,\underline{t}\Delta}, z^*_{i,\underline{t}\Delta}) \\ &= as_{i,\underline{t}\Delta} - bp^*_{i,\underline{t}\Delta} + (1+d)z^*_{i,\underline{t}\Delta} + fZ \\ &= (-d)z^e_{i,\underline{t}\Delta} + (1+d)z^*_{i,\underline{t}\Delta}, \end{aligned}$$

where the last equality follows from (43). This proves (39) for $\bar{t} = \underline{t}$. The case of $\bar{t} > \underline{t}$ follows by induction. The comparative statics regarding convergence speed are proved in Section A.2. □

Proposition 2 reveals that a sequence of double auctions is an effective mechanism to dynamically achieve allocative efficiency. Since $0 < 1 + d < 1$ in (39), inventory allocations under the periodic ex post equilibrium converge *exponentially* over time to the efficient allocation, as determined by the most recent signals. Once new signals arrive, the efficient allocation changes accordingly, and allocations under the periodic ex post equilibrium start to converge toward the new efficient level. This convergence result complements Rustichini, Satterthwaite, and Williams (1994), Cripps and Swinkels (2006), and Reny and Perry (2006), among others, who show that allocations in a one-shot double auction converge, at a polynomial rate, to the efficient level as the number of bidders increases.

As the proof of Proposition 2 makes clear, the exponential convergence result is driven by (i) the linearity and the stationarity of the equilibrium strategy, and (ii) at the efficient inventory level, the equilibrium strategy buys and sells zero additional unit. Because of the price impact of his limit orders, a bidder's equilibrium demand

$x_{i,t\Delta}$ in any given period moves less than one-for-one with respect to his inventory $z_{i,t\Delta}$. This less than one-for-one movement implies that in equilibrium the efficient inventory level is reached not immediately but gradually over time. It also implies that the convergence is monotone; the equilibrium inventory path does not oscillate between "overshooting" and "undershooting" the efficient inventory level.

The intuition for the comparative statics of Proposition 2 is simple. A larger n makes bidders more competitive, and a larger r makes them more impatient. Both effects encourage aggressive bidding and speed up convergence. The effect of α is slightly more subtle. Intuitively, the interdependence of valuations, represented by $1-\alpha$, creates adverse selection for the bidders. To protect themselves from trading losses, bidders reduce their demand or supply relative to the fully competitive market. The higher is α, the more bidders care about the private components of their valuations, and the less they worry about adverse selection. Therefore, a higher α implies more aggressive bidding and faster convergence to the efficient allocation. Finally, a higher trading frequency increases the convergence speed in clock time, even though it makes bidders more patient and thus less aggressive in each trading period. The last comparative static suggests that for purely allocative purpose, a higher trading frequency is always better, as it leads to a faster convergence to the efficient allocation in clock time. It does *not* say, however, that a higher trading frequency always leads to a higher level of social welfare. We turn to the welfare question in Section 3.

2.4 Limiting Equilibrium with High Trading Frequency

In this subsection we examine the limit of the equilibrium in Proposition 1 as $\Delta \to 0$, that is, as trading becomes continuous in clock time.

Proposition 3. *Suppose that $n\alpha > 2$. As $\Delta \to 0$, the equilibrium of Proposition 1 converges to the following periodic ex post equilibrium:*

1. *Bidder i's equilibrium strategy is represented by a process $\{x^{\infty}_{i,\tau}\}_{\tau\in\mathbb{R}^+}$. At the clock time τ, $x^{\infty}_{i,\tau}$ specifies bidder i's* rate *of order submission and is defined by*

$$x^{\infty}_{i,\tau}(p; s_{i,\tau}, z_{i,\tau}) = a^{\infty}\left(s_{i,\tau} - rp - \frac{\lambda(n-1)}{n\alpha-1}z_{i,\tau} + \frac{\lambda(1-\alpha)}{n\alpha-1}Z\right), \tag{44}$$

where

$$a^{\infty} = \frac{(n\alpha-1)(n\alpha-2)r}{2\lambda(n-1)}. \tag{45}$$

Given a clock time $T > 0$, in equilibrium the total amount of trading by bidder i in the clock-time interval $[0, T]$ is

$$z^*_{i,T} - z_{i,0} = \int_{\tau=0}^{T} x^{\infty}_{i,\tau}(p^*_\tau; s_{i,\tau}, z^*_{i,\tau})\, d\tau. \tag{46}$$

2. *The equilibrium price at any clock time τ is*

$$p^*_\tau = \frac{1}{r}\left(\frac{1}{n}\sum_{i=1}^{n} s_{i,\tau} - \frac{\lambda}{n}Z\right). \tag{47}$$

3. *Given any $0 \leq \underline{\tau} < \overline{\tau}$, if $s_{i,\tau} = s_{i,\underline{\tau}}$ for all i and all $\tau \in [\underline{\tau}, \overline{\tau}]$, then the equilibrium inventories $z^*_{i,\tau}$ in this interval satisfy:*

$$z^*_{i,\tau} - z^e_{i,\underline{\tau}} = e^{-\frac{1}{2}r(n\alpha-2)(\tau-\underline{\tau})}\left(z^*_{i,\underline{\tau}} - z^e_{i,\underline{\tau}}\right), \tag{48}$$

where

$$z^e_{i,\underline{\tau}} = \frac{n\alpha - 1}{\lambda(n-1)}\left(s_{i,\underline{\tau}} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\underline{\tau}}\right) + \frac{1}{n}Z \tag{49}$$

is the the efficient allocation at clock time $\underline{\tau}$ (cf. Equation (34)).

Proof. The proof follows by directly calculating the limit of Proposition 1 as $\Delta \to 0$ using L'Hopitâl's rule. □

Proposition 3 reveals that even if trading occurs continuously, in equilibrium the efficient allocation is not reached instantaneously. The delay comes from bidders' price impact and the associated demand reduction. Although submitting aggressive orders allows a bidder to achieve his desired allocation sooner, aggressive bidding also moves the price against the bidder and increases his trading cost. Facing this tradeoff, each bidders uses a finite rate of order submission in the limit. Consistent with Proposition 2, the rate of convergence to efficiency in Proposition 3, $r(n\alpha-2)/2$, is increasing in the number of bidders n, the discount rate r, and the weight α of the private components in bidders' valuations.

3 Welfare and Optimal Trading Frequency

In this section we study the effect of trading frequency on welfare and characterize the optimal trading frequency. We show that the optimal trading frequency depends critically on the nature of new information (i.e., the signals). If new information arrives at deterministic and scheduled intervals, then slow, batch trading (i.e., a large Δ) is optimal. If new information arrives stochastically according to a Poisson process, then fast, continuous trading (i.e., a small Δ) is optimal. Throughout this section we assume that $n\alpha > 2$ and conduct the analysis based on the periodic ex post equilibrium of Proposition 1.

We define the equilibrium welfare as the expectation of the integral of time-discounted flow utilities, summed over all bidders:

$$W(\Delta) = \mathbb{E}\left[\sum_{i=1}^{n} \int_{\tau=0}^{\infty} e^{-r\tau} \left(v_i z_{i,\tau}^* - \frac{\lambda}{2}(z_{i,\tau}^*)^2\right) d\tau\right], \tag{50}$$

where $\{z_{i,\tau}^*\}_{\tau \geq 0}$ is the path of equilibrium inventories implied by Proposition 1: $z_{i,0}^* = z_{i,0}$, and $z_{i,\tau}^* = z_{i,t\Delta}^* + x_{i,t\Delta}(p_{t\Delta}^*; s_{i,t\Delta}, z_{i,t\Delta}^*)$ for every integer $t \geq 0$ and every $\tau \in (t\Delta, (t+1)\Delta]$, where $x_{i,t\Delta}$ is the equilibrium strategy in Proposition 1. The inventory path is discontinuous as it "jumps" after trading in each period.

We let bidder i's time-τ efficient allocation be (cf. Equation (34)):

$$z_{i,\tau}^e = \frac{n\alpha - 1}{\lambda(n-1)} \left(s_{i,\tau} - \frac{1}{n}\sum_{j=1}^{n} s_{j,\tau}\right) + \frac{1}{n} Z. \tag{51}$$

Since the signals are martingales, $\{z_{i,\tau}^e\}_{\tau \geq 0}$ also forms a martingale (adapted to the signals $\{s_{j,\tau}\}_{1 \leq j \leq n, \tau \geq 0}$) for each i. Note that while the equilibrium allocation only changes when trading occurs, the efficient allocation changes whenever new signals arrive (which may or may not coincide with trading time).

We denote by Δ^* the optimal trading interval that maximizes the welfare $W(\Delta)$. The optimal trading frequency is then $1/\Delta^*$.

3.1 Scheduled Arrivals of New Information

We first consider scheduled information arrivals. In particular, we suppose that new private information arrive at regularly spaced clock times $\{0, \gamma, 2\gamma, \ldots\}$, where $\gamma >$

0. Formally, this means that for every bidder i, $s_{i,\tau}$ is constant on the interval $[k\gamma, (k+1)\gamma)$ for every integer $k \geq 0$.

To rule out trivialities, we impose a non-degeneracy condition: the initial inventories are not always efficient given the initial signals or some future signals. That is, either $\mathbb{E}[(z_{i,0} - z^e_{i,0})^2] > 0$ for some bidder i, or $\mathbb{E}[(z^e_{i,l\gamma} - z^e_{i,(l-1)\gamma})^2] > 0$ for some integer $l \geq 1$ and bidder i. If this non-degeneracy condition were not satisfied, there would be no trade in equilibrium, and trading frequency would have no effect on welfare.

Proposition 4. *Assume that new information arrives at clock times* $\{0, \gamma, 2\gamma, \ldots\}$ *and that the above non-degeneracy condition holds. Then* $W(\Delta) < W(\gamma)$ *for any* $\Delta < \gamma$*. That is, the optimal* $\Delta^* \geq \gamma$*.*

Proof. See Section A.3. □

Proposition 4 says that if new information repeatedly arrives at scheduled times (e.g., macroeconomic data releases or corporate earnings announcements), then the optimal trading frequency cannot be higher than the frequency of information arrivals. The intuition for this result is simple. For a large Δ, bidders have to wait for a long time before the next round of trading. So they bid aggressively (and hence mitigate the "demand reduction") whenever they have the opportunity to trade, which leads to a relatively efficient allocation early on. In other words, a large Δ serves as a commitment device to encourage aggressive trading immediately. If Δ is small, bidders know that they can trade again soon. Consequently, they bid less aggressively in each round of trading and end up holding relatively inefficient allocations in early rounds. We show that if $\Delta < \gamma$, then a larger Δ leads to a higher welfare. [13]

As Δ increases beyond γ, the bidders face a tradeoff: a large $\Delta > \gamma$ gives the benefit of a commitment device, but incurs the cost that bidders cannot react quickly to new information. Generally, the optimal Δ^* in this case would depend on the detail of the signal processes and would not be solvable in closed-form. Nonetheless, we know that the optimal Δ^* cannot be lower than γ.

[13] In Proposition 4, we have implicitly assumed that the first round of trading always starts at clock time zero, immediately after the arrival of time-zero signals. This assumption is without loss of generality: we can show that the equilibrium welfare from starting at time 0 and trading at frequency $1/\gamma$ always dominates the equilibrium welfare from starting at some time $\tau_0 > 0$ and trading at some frequency $1/\Delta \geq 1/\gamma$; the proof is similar to that in Section A.3.1 and is available upon request. Intuitively, there is no reason to delay trading after the information arrival because in each trading period the equilibrium strategy always gives a weakly higher utility than that from not trading.

Note that Proposition 4 is true for *any* martingale of signals that arrive at clock times spaced by γ, regardless of how the volatility of signals changes over time. In particular, this means that it is not possible to put a finite upper bound on Δ^* without making additional assumptions. For example, if the signals do not change over time (i.e., no new information arrives after time 0), then we effectively have $\gamma = \infty$, and hence $\Delta^* = \infty$ (i.e., it is optimal to trade only once at time zero).

3.2 Stochastic Arrivals of New Information

We now turn to stochastic arrivals of information. While there are many possible specifications for stochastic information arrivals, our objective is not to calculate the optimal trading frequency for all of them. Instead, our objective is to demonstrate that a moderate change from scheduled to stochastic information can dramatically change the optimal trading frequency. We use a simple yet natural Poisson process to make this point.

In particular, we suppose that new and private signals of all bidders arrives according to a homogeneous Poisson process with intensity $\mu > 0$: within a time interval τ, there are, in expectation, $\tau\mu$ arrivals of new signals. Clearly, μ is the analogue of $1/\gamma$ from Section 3.1. Moreover, conditional on the event that new signals $\{s_{i,\tau}\}_{i=1}^n$ arrive at time τ, an event we denote by $\mathcal{A}(\tau)$, the joint distribution of the increments $\{s_{i,\tau} - s_{i,\tau-}\}_{i=1}^n$ is independent of τ. Thus, the increment in each bidder i's efficient allocation has a constant conditional variance:

$$\sigma_i^2 \equiv \mathbb{E}[(z_{i,\tau}^e - z_{i,\tau-}^e)^2 \mid \mathcal{A}(\tau)], \tag{52}$$

which we assume to be positive.

Finally, we suppose that the initial inventories are efficient given the information in period 0, that is, $z_{i,0} = z_{i,0}^e$ for every bidder i, where the efficient allocation is given by (34). We interpret the period-0 information as existing before period 0, and the efficiency of the initial inventories as the result of trading activities before period 0. We make this assumption to isolate the trading motives associated with the arrivals of new information, rather than the existing information.

Proposition 5. *Given the above assumptions on stochastic arrivals of new information, the welfare $W(\Delta)$ is strictly decreasing in Δ, and the optimal $\Delta^* = 0$.*

Proof. See Section A.3. □

Proposition 5 suggests that faster trading is better if the arrival times of new information are stochastic and unpredictable. This is because more frequent trading enables bidders to react sooner after a new information arrival, which dominates the cost of lower bidding aggressiveness in the subsequent rounds of trading. As a result, a continuous market (with $\Delta^* = 0$) is optimal.

3.3 Discussion

Our focus on socially optimal trading frequency is most closely related to Vayanos (1999), who studies a dynamic market in which the asset fundamental value (dividend) is public information, but agents receive periodic inventory shocks. He finds that if the inventory shocks are private information and are small, then a lower trading frequency is always better for welfare. This result by Vayanos and our Proposition 4 share the common intuition and mechanism that traders tend to reduce their demand now if they can trade again soon; this *strategic* effect favors a low trading frequency.

Nonetheless, our analysis offers several new insights that are not in Vayanos (1999). First, our model allows interdependent values and hence partly captures adverse selection. We show that the intuition on strategic trading and demand reduction presented by Vayanos (1999) also holds under adverse selection. Moreover, the higher is adverse selection (the lower is α), the more severe is the demand reduction, and hence the slower is the convergence of the equilibrium allocations to efficiency (Proposition 2). Proposition 4 also naturally links the frequency of information arrivals to the optimal frequency of trading for scheduled information arrivals. Modeling details aside, our Proposition 4 can be viewed as a generalization of Vayanos' corresponding result to interdependent values and scheduled arrivals of new information.

Second, and perhaps more importantly, we characterize natural conditions under which the "lower trading frequency improves welfare" result of Vayanos (1999) is overturned. Specifically, if new information arrives at stochastic times, then too low a trading frequency prevents bidders from reacting to new information quickly, thus reducing welfare. We show that this latter *information* effect can dominate the strategic effect, implying that continuous trading can be optimal (Proposition 5). Our model makes this intuition salient and transparent by decoupling the clock times of trading from the clock times of information arrival.

Third, we develop a new solution method of dynamic ex post equilibrium. The exact equilibrium notion is more than a technical nuance; it is relevant for answering welfare and policy questions. For example, while the private-inventory equilibrium of Vayanos (1999) indicates a low optimal trading frequency, his public-inventory equilibrium—selected among multiple equilibria by a trembling-hand-perfect equilibrium refinement—argues for a higher trading frequency. By contrast, our ex post equilibrium of Proposition 1 holds under private as well as public information (of signals and inventories); its implications for socially optimal trading frequency are the same regardless of the information asymmetry. Therefore, answers to important policy questions such as optimal trading speed can depend on equilibrium selection. We argue that the ex post equilibrium refinement is particularly suitable for welfare analysis because of its robustness.

Our results on optimal trading frequency are also related to Fuchs and Skrzypacz (2013), who consider a lemons market with many competitive one-time buyers and a single seller who has private information. In their model, private information arrives only once (at time 0), and continuous trading can always be improved, in terms of welfare, by an "early closure" of market. By contrast, we show that continuous trading can be optimal if information arrivals are stochastic.

Our approach complements the small but growing theory literature that focuses on *differential* trading speed among agents (see, for example, Foucault, Hombert, and Rosu 2012, Pagnotta and Philippon 2012, and Biais, Foucault, and Moinas 2012). In these papers, certain traders are faster than others, and their welfare question is whether investments in superior trading technology is socially wasteful. By contrast, bidders in our model have the same trading speed, and our welfare criterion is allocative efficiency.

4 Concluding Remarks

In this paper we characterize a dynamic and stationary ex post equilibrium in a sequence of uniform-price double auction with interdependent values. In this ex post equilibrium, a bidder's strategy depends only on his own private information, but it remains optimal even after observing the concurrent and historical private information of other bidders. This ex post equilibrium aggregates the most recent private information dispersed across bidders, and is robust to the probability distributions of

signals and inventories.

Our ex post equilibrium greatly simplifies the analysis of dynamic trading and allows us to derive, in a tractable way, interesting implications regarding welfare and optimal trading frequency. Although the equilibrium allocations of assets among bidders are not fully efficient after each auction, they converge exponentially over time to the efficient level. Therefore, a sequence of double auctions is a simple and effective mechanism to quickly achieve allocative efficiency. Moreover, while convergence speed is increasing in trading frequency, continuous trading does not lead to immediate convergence to the efficient allocation. We further show that the socially optimal trading frequency is low for scheduled information arrivals, but is high for stochastic information arrivals. Our results suggest that the nature of information is a critical consideration for answering the policy question of optimal trading speed.

The ex post equilibrium allows us to derive a number of additional results, which we delegate to the online appendix. One such result is the uniqueness of ex post equilibrium in a static special case (that is, for $\Delta = \infty$). Under mild conditions, we prove that the ex post optimality criterion is so strong that only one equilibrium survives it, and that equilibrium has linear demand schedules and is given by Corollary 1. Intuitively, uniqueness is possible in our model because beliefs—which often lead to multiple Bayesian equilibria—are irrelevant for ex post optimality. Moreover, our model makes no price-taking assumption that is typical in models of rational expectations equilibria (REE).[14] While we do not know whether our dynamic equilibrium is unique, the uniqueness result in the static special case suggests that the ex post optimality is nonetheless a useful equilibrium selection criterion for uniform-price auctions, which often admit a continuum of Bayesian-Nash equilibria (Wilson 1979).

We also apply the ex post equilibrium to other markets, so far in the static special

[14]The price-taking behavior of agents is important for existing proofs of the uniqueness of REE. Ausubel (1990) demonstrates the uniqueness of partially-revealing REE under certain conditions. DeMarzo and Skiadas (1998) show the uniqueness of fully-revealing REE in an economy that nests the Grossman (1976) model. Pálvölgyi and Venter (2011) and Breon-Drish (2012) prove the uniqueness of REE for continuous price function in the Grossman and Stiglitz (1980) model. Agents in these models are price-takers. Separately, Rochet and Vila (1994) prove that the equilibrium in the Kyle (1985) model is unique if the single informed trader observes the demand from noise traders; Back (1992) proves the uniqueness of equilibrium in continuous-time model where the single informed trader can infer the flow of noise trades by observing the price flow. Our ex post equilibrium differs in that we have multiple strategic bidders with dispersed information, and none of them observe the private information of others.

case ($\Delta = \infty$). For example, a simultaneous auction of multiple assets also admits an ex post equilibrium, which can be naturally applied to "program trading" of multiple stocks on the NYSE and to "default management auctions" of derivative portfolios run by clearinghouses. In the special case of pure private values, we also solve an ex post equilibrium in which bidders have different rates of decreasing marginal valuations (different λ's). These results are presented in the online appendix to preserve space. Separately, Du and Zhu (2012) use the ex post equilibrium to analyze price discovery in the settlement auctions of credit default swaps, in which bidders' preexisting derivatives contracts distort their bidding incentives for the underlying assets. While dynamic ex post equilibria in those markets are not analyzed in this paper, we expect that the methodology can also be applied there. We leave these markets for future research.

A Appendix: Proofs

A.1 Proof of Proposition 1

Here we continue the construction of the periodic ex post equilibrium in Section 2.2.2. Averaging Equation (32) across all bidders and using the fact that $\sum_{i=1}^{n} x_{i,t\Delta} = 0$ and $\sum_{i=1}^{n} z_{i,t\Delta} = Z$, we get:

$$p_0^* = \frac{1}{r}\left(\bar{s}_0 - \frac{\lambda}{n}Z\right), \tag{53}$$

where

$$\bar{s}_0 \equiv \frac{1}{n}\sum_{i=1}^{n} s_{i,0}.$$

Therefore, in (25) we must have

$$b = ra, \quad \frac{a\lambda}{n} + \frac{d}{n} + f = 0. \tag{54}$$

Substituting (33), (53) and (54) into the first-order condition (32), we have:

$$\begin{aligned}(n-1)(1-e^{-r\Delta})a\Bigg[&\frac{1}{1-e^{-r\Delta}(1+d)}\left(v_{i,0} - \bar{s}_0 + \frac{\lambda}{n}Z\right) \\ &- \sum_{t=1}^{\infty}\lambda e^{-rt\Delta}(1+d)^t\left(\frac{1}{-d} - \frac{(1+d)^t}{-d}\right)\left(a(s_{i,0}-\bar{s}_0) - \frac{d}{n}Z\right) \\ &- \frac{\lambda}{1-e^{-r\Delta}(1+d)^2}(x_{i,0}+z_{i,0})\Bigg] - x_{i,0} = 0.\end{aligned} \tag{55}$$

Rearranging the terms gives:

$$\begin{aligned}&\left(1 + \frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right)x_{i,0} \\ =\,&(n-1)(1-e^{-r\Delta})a\Bigg[\frac{1}{1-e^{-r\Delta}(1+d)}\left(\alpha - \frac{1-\alpha}{n-1}\right)(s_{i,0}-\bar{s}_0) \\ &- \frac{\lambda e^{-r\Delta}(1+d)}{(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})}a(s_{i,0}-\bar{s}_0) \\ &- \frac{\lambda}{1-e^{-r\Delta}(1+d)^2}z_{i,0} + \frac{1}{1-e^{-r\Delta}(1+d)^2}\frac{\lambda}{n}Z\Bigg].\end{aligned} \tag{56}$$

On the other hand, Equation (54) simplifies the conjectured strategy (25) to

$$x_{i,0} = a(s_{i,0} - \bar{s}_0) + dz_{i,0} - \frac{d}{n}Z.$$

Matching the coefficients in the above expression with those in (56), we obtain two equations for a and d:

$$\begin{aligned}
&\left(1 + \frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right) = \frac{(1-e^{-r\Delta})(n\alpha-1)}{1-e^{-r\Delta}(1+d)} - \frac{(n-1)(1-e^{-r\Delta})\lambda e^{-r\Delta}(1+d)a}{(1-(1+d)e^{-r\Delta})(1-(1+d)^2e^{-r\Delta})},\\
&\left(1 + \frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}\right) d = -\frac{(n-1)(1-e^{-r\Delta})a\lambda}{1-e^{-r\Delta}(1+d)^2}.
\end{aligned} \tag{57}$$

There are two solutions to the above system of equations. One of them leads to unbounded inventories,[15] so we drop it. The other solution leads to converging inventories and is given by

$$\begin{aligned}
a &= \frac{n\alpha-1}{2(n-1)e^{-r\Delta}\lambda}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right),\\
d &= -\frac{1}{2e^{-r\Delta}}\left((n\alpha-1)(1-e^{-r\Delta}) + 2e^{-r\Delta} - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}\right),
\end{aligned} \tag{58}$$

where under the condition of $n\alpha > 2$ we have $a > 0$ and $-1 < d < 0$. From (54), we have $b = ra > 0$ and $f = -d/n - a\lambda/n$.

Finally, we verify the second-order condition. Under the linear strategy in (25) with $b > 0$, differentiating the first-order condition (27) with respect to p_0 gives

$$(n-1)b\frac{1-e^{-r\Delta}}{r}\left(-\lambda(n-1)b\sum_{t=0}^{\infty}e^{-rt\Delta}(1+d)^{2t} - 1\right) - (n-1)b < 0. \tag{59}$$

This completes the construction of the stationary periodic ex post equilibrium.

[15] This solution to (57) has the property of $(1+d)e^{-r\Delta} < -1$, which leads to an unbounded path of inventories (cf. Equation (28)) and utilities, even after discounting.

A.2 Proof of comparative statics in Proposition 2

We write

$$1+d=\frac{1}{2e^{-r\Delta}}\left(\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2+4e^{-r\Delta}}-(n\alpha-1)(1-e^{-r\Delta})\right). \tag{60}$$

The comparative statics with respect to n, α and r follow by differentiating $1+d$ with respectively n, α and r and straightforward calculations.

For the comparative statics with respect to Δ, we find that

$$\frac{\partial(\log(1+d)/\Delta)}{\partial\Delta}=-\frac{1}{\Delta^2}\left(r\Delta\frac{\eta\sqrt{\eta^2(e^{r\Delta}-1)^2+4e^{r\Delta}}-\eta^2(e^{r\Delta}-1)-2}{\sqrt{\eta^2(1-e^{-r\Delta})^2+4e^{-r\Delta}}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2+4e^{r\Delta}}-\eta(e^{r\Delta}-1)\right)}\right.$$
$$\left.+\log\left(\frac{1}{2}\left(\sqrt{\eta^2(e^{r\Delta}-1)^2+4e^{r\Delta}}-\eta(e^{r\Delta}-1)\right)\right)\right),$$

where we let $\eta\equiv n\alpha-1$. Given $\eta>1$, it is easy to show that the two terms in the right-hand side of the above equation are both positive, which implies our conclusion.

A.3 Proofs of Proposition 4 and 5

We first establish some general properties of the equilibrium welfare, before specializing to scheduled (Section A.3.1) and stochastic (Section A.3.2) arrivals of new information.

The following lemma relates the amount of inefficiency associated with an inventory allocation to the square distance between that allocation and the efficient allocation:

Lemma 1. *Let $\{z_i^e\}$ be the efficient allocation given valuations $\{v_i\}$:*

$$\{z_i^e\}=\text{argmax}_{\{z_i'\}}\ \sum_{i=1}^n\left(v_i\left(z_i'\right)-\frac{\lambda}{2}(z_i')^2\right)\ \textit{subject to:}\ \sum_{i=1}^n z_i'=Z.$$

For any profile of inventories $(z_1,z_2,\ldots,z_n)$ satisfying $\sum_{i=1}^n z_i=Z$, we have:

$$\sum_{i=1}^n\left(v_iz_i^e-\frac{\lambda}{2}(z_i^e)^2\right)-\sum_{i=1}^n\left(v_iz_i-\frac{\lambda}{2}(z_i)^2\right)=\frac{\lambda}{2}\sum_{i=1}^n(z_i-z_i^e)^2. \tag{61}$$

Proof. First, by the definition of efficient allocation, we have $v_i - \lambda z_i^e = \nu$ for every i, where ν is the Lagrange multiplier for the constraint $\sum_{i=1}^n z_i^e = Z$ in the maximization problem of $\{z_i^e\}$.

Since $(z_i)^2 = (z_i^e)^2 + 2z_i^e(z_i - z_i^e) + (z_i - z_i^e)^2$, we have:

$$\sum_{i=1}^n \left(v_i z_i - \frac{\lambda}{2}(z_i)^2 \right) = \sum_{i=1}^n \left(v_i z_i^e - \frac{\lambda}{2}(z_i^e)^2 \right) + \sum_{i=1}^n (v_i - \lambda z_i^e)(z_i - z_i^e) - \frac{\lambda}{2}\sum_{i=1}^n (z_i - z_i^e)^2. \tag{62}$$

The middle term in the right-hand side of (62) is zero because $v_i - \lambda z_i^e = \nu$ and $\sum_{i=1}^n (z_i - z_i^e) = Z - Z = 0$. This proves the lemma. □

By Lemma 1 we write $W(\Delta)$ as:

$$W(\Delta) = \mathbb{E}\left[\sum_{i=1}^n \int_{\tau=0}^\infty e^{-r\tau} \left(v_{i,\tau} z_{i,\tau}^e - \frac{\lambda}{2}(z_{i,\tau}^e)^2 \right) d\tau \right] - X(\Delta), \tag{63}$$

where

$$X(\Delta) = \mathbb{E}\left[\frac{\lambda}{2} \sum_{i=1}^n \int_{\tau=0}^\infty e^{-r\tau} (z_{i,\tau}^e - z_{i,\tau}^*)^2 \, d\tau \right] \tag{64}$$

is the amount of inefficiency associated with the equilibrium path of inventories. Since the efficient allocations depend only on the signal processes but not on trading frequency, the optimal trading frequency is determined by the comparative statics of $X(\Delta)$ with respect to Δ.

Lemma 2. *Suppose that $\tau \in (t\Delta, (t+1)\Delta)$. Then we have*

$$\mathbb{E}[(z_{i,\tau}^* - z_{i,\tau}^e)^2] = \mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] + \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2]. \tag{65}$$

Proof. Recall that $z_{i,\tau}^* = z_{i,(t+1)\Delta}^*$ for $\tau \in (t\Delta, (t+1)\Delta)$ because trading does not happen in $(t\Delta, (t+1)\Delta)$. Thus, we can rewrite, for any $\tau \in (t\Delta, (t+1)\Delta)$,

$$\mathbb{E}[(z_{i,\tau}^* - z_{i,\tau}^e)^2] = \mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)^2] + \mathbb{E}[(z_{i,t\Delta}^e - z_{i,\tau}^e)^2] + \mathbb{E}[2(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)(z_{i,t\Delta}^e - z_{i,\tau}^e)]. \tag{66}$$

Since $z_{i,(t+1)\Delta}^*$ is measurable with respect to $\{H_{i,t\Delta}\}_{i=1}^n$, $\tau > t\Delta$, and $\{z_{i,\tau'}^e\}_{\tau' \geq 0}$ is a martingale, we have $\mathbb{E}[(z_{i,(t+1)\Delta}^* - z_{i,t\Delta}^e)(z_{i,t\Delta}^e - z_{i,\tau}^e)] = 0$ by the law of iterated expectations. □

By Lemma 2, we can further decompose $X(\Delta)$ into two terms:

$$X(\Delta) = X_1(\Delta) + X_2(\Delta), \tag{67}$$

where

$$X_1(\Delta) = \frac{1 - e^{-r\Delta}}{r} \cdot \frac{\lambda}{2} \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-rt\Delta} \mathbb{E}[(z^*_{i,(t+1)\Delta} - z^e_{i,t\Delta})^2] \tag{68}$$

and

$$X_2(\Delta) = \frac{\lambda}{2} \sum_{i=1}^{n} \sum_{t=0}^{\infty} \int_{\tau=t\Delta}^{(t+1)\Delta} e^{-r\tau} \mathbb{E}[(z^e_{i,t\Delta} - z^e_{i,\tau})^2] \, d\tau. \tag{69}$$

$X_2(\Delta)$ is in terms of the efficient inventories (and hence of the signals) and is thus easy to analyze. In contrast, $X_1(\Delta)$ depends on the equilibrium inventories. The next lemma simplifies the equilibrium inventory terms in $X_1(\Delta)$.

Lemma 3.

$$\mathbb{E}[(z^*_{i,(t+1)\Delta} - z^e_{i,t\Delta})^2] = (1+d)^{2(t+1)} \mathbb{E}[(z^*_{i,0} - z^e_{i,0})^2] + \sum_{t'=0}^{t-1} (1+d)^{2(t-t')} \mathbb{E}[(z^e_{i,(t'+1)\Delta} - z^e_{i,t'\Delta})^2]$$

Proof. From Proposition 2, we have

$$z^*_{i,(t+1)\Delta} - z^e_{i,t\Delta} = (1+d)(z^*_{i,t\Delta} - z^e_{i,t\Delta}). \tag{70}$$

Therefore,

$$\begin{aligned} \mathbb{E}[(z^*_{i,(t+1)\Delta} - z^e_{i,t\Delta})^2] &= (1+d)^2 \mathbb{E}[(z^*_{i,t\Delta} - z^e_{i,t\Delta})^2] && (71) \\ &= (1+d)^2 \mathbb{E}[(z^*_{i,t\Delta} - z^e_{i,(t-1)\Delta})^2] + (1+d)^2 \mathbb{E}[(z^e_{i,t\Delta} - z^e_{i,(t-1)\Delta})^2] \\ &\quad + 2(1+d)^2 \mathbb{E}[(z^*_{i,t\Delta} - z^e_{i,(t-1)\Delta})(z^e_{i,t\Delta} - z^e_{i,(t-1)\Delta})]. && (72) \end{aligned}$$

Because $z^*_{i,t\Delta}$ is measurable with respect to $\{H_{i,(t-1)\Delta}\}_{i=1}^n$ and $\{z^e_{i,\tau}\}_{\tau \geq 0}$ is a martingale, $\mathbb{E}[(z^*_{i,t\Delta} - z^e_{i,(t-1)\Delta})(z^e_{i,t\Delta} - z^e_{i,(t-1)\Delta})] = 0$ by the law of iterated expectations. The rest follows by induction. □

Finally, Lemma 4 expresses $X_1(\Delta)$ in terms of the efficient inventories, similar to $X_2(\Delta)$.

Lemma 4.

$$X_1(\Delta) = \frac{\lambda(1+d)}{2r(n\alpha-1)}\left(\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}-z^e_{i,0})^2] + \sum_{i=1}^{n}\sum_{t=0}^{\infty} e^{-r(t+1)\Delta}\mathbb{E}[(z^e_{i,(t+1)\Delta}-z^e_{i,t\Delta})^2]\right). \tag{73}$$

Proof. By Lemma 3, we have

$$\begin{aligned}
&X_1(\Delta) \qquad (74)\\
=&\frac{\lambda(1-e^{-r\Delta})}{2r}\sum_{i=1}^{n}\sum_{t=0}^{\infty} e^{-rt\Delta}\left((1+d)^{2(t+1)}\mathbb{E}[(z^*_{i,0}-z^e_{i,0})^2] + \sum_{t'=0}^{t-1}(1+d)^{2(t-t')}\mathbb{E}[(z^e_{i,(t'+1)\Delta}-z^e_{i,t'\Delta})^2]\right)\\
=&\frac{\lambda}{2r}\frac{(1-e^{-r\Delta})(1+d)^2}{1-(1+d)^2e^{-r\Delta}}\sum_{i=1}^{n}\mathbb{E}[(z^*_{i,0}-z^e_{i,0})^2]\\
&+\frac{1-e^{-r\Delta}}{r}\frac{\lambda}{2}\sum_{i=1}^{n}\sum_{t'=0}^{\infty}\mathbb{E}[(z^e_{i,(t'+1)\Delta}-z^e_{i,t'\Delta})^2]\sum_{t=t'+1}^{\infty} e^{-rt\Delta}(1+d)^{2(t-t')}\\
=&\frac{\lambda}{2r}\frac{(1-e^{-r\Delta})(1+d)^2}{1-(1+d)^2e^{-r\Delta}}\sum_{i=1}^{n}\mathbb{E}[(z^*_{i,0}-z^e_{i,0})^2]\\
&+\frac{\lambda}{2r}\frac{(1-e^{-r\Delta})(1+d)^2}{1-(1+d)^2e^{-r\Delta}}\sum_{i=1}^{n}\sum_{t'=0}^{\infty}\mathbb{E}[(z^e_{i,(t'+1)\Delta}-z^e_{i,t'\Delta})^2]e^{-r(t'+1)\Delta}.
\end{aligned}$$

We can simplify the constant in the above equations by direct calculation:

$$\begin{aligned}
&e^{-r\Delta}(1+d)^2 \qquad (75)\\
=&\frac{2(n\alpha-1)^2(1-e^{-r\Delta})^2+4e^{-r\Delta}-2(n\alpha-1)(1-e^{-r\Delta})\sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2+4e^{-r\Delta}}}{4e^{-r\Delta}}\\
=&1-(n\alpha-1)(1-e^{-r\Delta})(1+d),
\end{aligned}$$

which implies:

$$\frac{(1-e^{-r\Delta})(1+d)^2}{1-(1+d)^2e^{-r\Delta}} = \frac{1+d}{n\alpha-1}. \tag{76}$$

Finally, by construction: $z^*_{i,0} = z_{i,0}$ for every bidder i. □

With these lemmas we are ready to prove Proposition 4 and Proposition 5.

A.3.1 Proof of Proposition 4

For any $\tau > 0$, we let $\bar{t}(\tau) = \min\{t \geq 0 : t \in \mathbb{Z}, t\Delta \geq \tau\}$. That is, if new signals arrive at the clock time τ, then $\bar{t}(\tau)\Delta$ is the clock time of the next trading period (including time τ).

For any $\Delta \leq \gamma$, by the assumption of Proposition 4 there is at most one new signal profile arrival in each interval $[t\Delta, (t+1)\Delta)$. Thus, we only need to count the changes in efficient allocation between period $\bar{t}((l-1)\gamma)$ and $\bar{t}(l\gamma)$, for $l \in \mathbb{Z}_+$. Using this fact, we can rewrite $X_1(\Delta)$ and $X_2(\Delta)$ as:

$$\begin{aligned} X_1(\Delta) &= \frac{\lambda(1+d)}{2r(n\alpha-1)}\left(\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}-z^e_{i,0})^2] + \sum_{i=1}^{n}\sum_{l=1}^{\infty} e^{-r\bar{t}(l\gamma)\Delta}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]\right) \\ &= \frac{\lambda(1+d)}{2r(n\alpha-1)}\left(\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}-z^e_{i,0})^2] + \sum_{i=1}^{n}\sum_{l=1}^{\infty} e^{-rl\gamma}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]\right) \\ &\quad - \frac{\lambda(1+d)}{2(n\alpha-1)}\sum_{i=1}^{n}\sum_{l=1}^{\infty}\frac{e^{-rl\gamma}-e^{-r\bar{t}(l\gamma)\Delta}}{r}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]. \end{aligned} \tag{77}$$

and

$$\begin{aligned} X_2(\Delta) &= \frac{\lambda}{2}\sum_{i=1}^{n}\sum_{t=0}^{\infty}\int_{\tau=t\Delta}^{(t+1)\Delta} e^{-r\tau}\mathbb{E}[(z^e_{i,t\Delta}-z^e_{i,\tau})^2]\,d\tau \\ &= \frac{\lambda}{2}\sum_{i=1}^{n}\sum_{l=1}^{\infty}\frac{e^{-rl\gamma}-e^{-r\bar{t}(l\gamma)\Delta}}{r}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]. \end{aligned} \tag{78}$$

Note that all the expectations in the expressions of $X_1(\Delta)$ and $X_2(\Delta)$ do not depend on Δ. To make clear the dependence of d on Δ, we now write $d = d(\Delta)$. Since $(1+d(\Delta))/(n\alpha-1) < 1$, we have for any $\Delta < \gamma$:

$$\begin{aligned} X(\Delta) &= X_1(\Delta) + X_2(\Delta) \\ &> \frac{\lambda(1+d(\Delta))}{2r(n\alpha-1)}\left(\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}-z^e_{i,0})^2] + \sum_{i=1}^{n}\sum_{l=1}^{\infty} e^{-rl\gamma}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]\right) \\ &> \frac{\lambda(1+d(\gamma))}{2r(n\alpha-1)}\left(\sum_{i=1}^{n}\mathbb{E}[(z_{i,0}-z^e_{i,0})^2] + \sum_{i=1}^{n}\sum_{l=1}^{\infty} e^{-rl\gamma}\mathbb{E}[(z^e_{i,l\gamma}-z^e_{i,(l-1)\gamma})^2]\right) \\ &= X(\gamma), \end{aligned} \tag{79}$$

where the last inequality holds because $d(\Delta)$ decreases with Δ (which can be verified by taking derivative $d'(\Delta)$) and where the last equality holds because $\bar{t}(l\gamma)\Delta = l\gamma$ if $\gamma = \Delta$.

Therefore, we have $W(\Delta) < W(\gamma)$ for any $\Delta < \gamma$. This proves Proposition 4.

A.3.2 Proof of Proposition 5

Since signals arrive according to a Poisson process with the intensity μ, we have

$$\mathbb{E}[(z^e_{i,\tau} - z^e_{i,t\Delta})^2] = (\tau - t\Delta)\mu\sigma_i^2, \quad \tau \in [t\Delta, (t+1)\Delta), \tag{80}$$

$$\mathbb{E}[(z^e_{i,(t+1)\Delta} - z^e_{i,t\Delta})^2] = \Delta\mu\sigma_i^2. \tag{81}$$

Substituting the above two expressions into (69) and (73), and using the assumption that $z_{i,0} = z^e_{i,0}$, we have

$$X_1(\Delta) = \frac{\lambda(1+d)}{2r(n\alpha-1)} \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-r(t+1)\Delta} \Delta\mu\sigma_i^2 = \frac{\lambda(1+d)}{2r(n\alpha-1)} \frac{\Delta e^{-r\Delta}}{1-e^{-r\Delta}} \sum_{i=1}^{n} \mu\sigma_i^2 \tag{82}$$

and

$$X_2(\Delta) = \frac{\lambda}{2} \sum_{i=1}^{n} \sum_{t=0}^{\infty} e^{-rt\Delta} \int_{\tau=0}^{\Delta} e^{-r\tau} \tau\mu\sigma_i^2 \, d\tau = -\frac{\lambda}{2r} \frac{\Delta e^{-r\Delta}}{1-e^{-r\Delta}} \sum_{i=1}^{n} \mu\sigma_i^2 + \frac{\lambda}{2r^2} \sum_{i=1}^{n} \mu\sigma_i^2. \tag{83}$$

Therefore,

$$X(\Delta) = X_1(\Delta) + X_2(\Delta) = \frac{\lambda}{2r^2} \sum_{i=1}^{n} \mu\sigma_i^2 - \frac{\lambda}{2r} \left(1 - \frac{1+d}{n\alpha-1}\right) \frac{\Delta e^{-r\Delta}}{1-e^{-r\Delta}} \sum_{i=1}^{n} \mu\sigma_i^2. \tag{84}$$

We can write

$$\begin{aligned} &\left(1 - \frac{1+d}{n\alpha-1}\right) \frac{\Delta e^{-r\Delta}}{1-e^{-r\Delta}} \\ &= \frac{(n\alpha-1)(1+e^{-r\Delta}) - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2(n\alpha-1)e^{-r\Delta/2}} \cdot \frac{\Delta e^{-r\Delta/2}}{1-e^{-r\Delta}}. \end{aligned} \tag{85}$$

It is easy to take derivatives to show that both

$$\frac{(n\alpha-1)(1+e^{-r\Delta}) - \sqrt{(n\alpha-1)^2(1-e^{-r\Delta})^2 + 4e^{-r\Delta}}}{2(n\alpha-1)e^{-r\Delta/2}} \tag{86}$$

and

$$\frac{\Delta e^{-r\Delta/2}}{1-e^{-r\Delta}}. \tag{87}$$

are strictly decreasing in Δ.

Therefore, $X(\Delta)$ increases in Δ, which means that $W(\Delta)$ decreases with Δ and the optimal $\Delta^* = 0$. This proves Proposition 5.

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