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Rethinking Optimal Currency Areas*

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ABSTRACT

The classic optimal currency area criterion is that countries with more correlated shocks are better candidates to form a union. We show that when countries have credibility problems this simple criterion must be changed: Symmetric countries gain credibility when joining the union only when the shocks affecting credibility are not highly correlated. Our analysis provides a amended optimal currency area criterion that we argue is more relevant than the classic one. We illustrate our argument both for a reduced form model and for a relatively standard sticky-price general equilibrium model. We argue that our new criterion should lead to a rethinking of the massive amount of empirical work on optimal currency areas.
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The traditional case for flexible exchange rates and against a monetary union with a single currency dates back to at least Friedman (1953) and Mundell (1961). (See Dellas and Tavlas (2009) for a survey.) The argument is that with flexible exchange rates countries can tailor their monetary policy to respond to their idiosyncratic shocks while in a monetary union countries cannot. This inability to set monetary policy independently is a major cost of a monetary union. Moreover, this cost is larger the greater is the variability of country-specific shocks. This traditional case implicitly assumes that countries had no credibility problems. Here we argue that when countries face substantial credibility problems, the loss of monetary independence can be a major benefit of joining a monetary union. Indeed, this benefit can increase with the variability of country-specific shocks and can lead a monetary union to be preferred to flexible exchange rates.

Some early work also considered credibility problems but with a very different institutional arrangement for how policies are set than the one considered here. For example, Friedman (1973) argued that for some countries it may be optimal to forswear a flexible exchange rate system and go beyond a monetary union all the way to "dollarization" in which a country simply abandons its currency. Specifically, Friedman argued that for a country with severe credibility problems it may be optimal to give up its currency and adopt the currency of another country, called an *anchor* currency.

> The surest way to avoid using inflation as a deliberate method of taxation is to unify the country's currency with the currency of some other country or countries. In this case, the country would not have any monetary policy of its own. It would, as it were, tie its monetary policy to the kite of the monetary policy of another country—preferably a more developed, larger, and relatively stable country.

This latter view of Friedman has been formalized by Alesina and Barro (2002). We interpret this work as making the case for *dollarization* for countries with credibility problems. A key institutional assumption of this work is that after dollarization the monetary policy of the country with the anchor currency is completely unaffected by the presence of another country that uses the same currency. This assumption seems particularly applicable when a small country, such as Ecuador, adopts the currency of a large country, such as the United

States, and forswears all explicit and implicit influence on the monetary policies followed by the large country. This assumption seems far less applicable to situations in which groups of countries come together in a monetary union and set up institutional arrangements in which they jointly decide on monetary policy.

In monetary unions, such as the European Monetary Union, monetary policy is made jointly by representatives of all countries in the union. A key distinction between our work on monetary unions and the existing work on dollarization is that in our work upon forming a union the members jointly decide on monetary policy in a way that takes account of the impact of policy on all members. Such a policy-making process raises the possibility that the union as a whole will suffer from the same type of credibility problems that the individual members face on their own. This possibility is especially acute if countries are symmetric with respect to their credibility problems. The question we address is how can symmetric countries increase their credibility by forming a union in which they jointly decide on monetary policy?

We analyze these issues in two models. The first is a reduced form model along the lines of Kydland and Prescott (1977) and Barro and Gordon (1983). The second is a simple sticky price model in the spirit of Gali and Monacelli (2005) and Farhi and Werning (2013).

Consider first the reduced form model. The monetary authority's objective is to minimize the deviations of unemployment from its natural level and the deviations of inflation from zero. There are two types of shocks to the natural level of unemployment: *ex-ante* shocks that are realized before price setters set their prices and *ex-post* shocks that are realized after. Both shocks have aggregate and idiosyncratic components.

Under commitment, the model is consistent with the standard Friedman–Mundell argument in favor of flexible exchange rates. With commitment the monetary authority finds it not optimal to respond to any ex-ante shocks. The reason is that if it does, the price setters will offset this response in their choice of prices and the net result will be no change in unemployment and an undesirable increase in the variability of inflation. The monetary authority, however, does find it optimal to respond to ex-post shocks since by doing so the authority can make the variability of unemployment lower. In a union the inability to respond to the idiosyncratic component of ex-post shocks raises the variability of unemployment and lowers welfare relative to a system of flexible exchange rates.

Our new result occurs when monetary authorities lack commitment. Here the role of ex ante shocks is critical. After the private agents set their prices, the monetary authority is tempted to engineer a surprise inflation that depends on the level of these shocks in order to reduce unemployment. In equilibrium, the private agents accurately forecast this policy and undo the effects of monetary policy on unemployment. Hence, the net effect of these forces is that, in equilibrium, the ex ante shocks leads the monetary authority to simply increase undesirable inflation variability.

In a union, in contrast, the monetary authority is unable to respond to the idiosyncratic component of the ex ante shocks. Hence, in equilibrium the variability of undesirable inflation is lower than under flexible exchange rates. Thus, in this model entering a monetary union is essentially a commitment device to less variable undesirable inflation that arises from reacting to ex ante shocks. This force tends to raise the value of the union relative to flexible exchange rates. Of course, in a union the monetary authority is also unable to respond to ex-post shocks, even though it is desirable to do so for the standard Friedman-Mundell reasons. This force tends to lower the value of the union relative to flexible exchange rates. Overall, if the variability of ex ante shocks is sufficiently large relative to that of ex post shocks then the credibility-enhancing benefits of the monetary union outweigh the standard Friedman–Mundell flexibility costs and the union is preferred to flexible exchange rates.

We then turn to a general equilibrium monetary model that is related to those of Obstfeld and Rogoff (1995), Gali and Monacelli (2005), and especially Farhi and Werning (2013). The economy consists of a continuum of ex-ante identical countries, each of which uses labor to produce traded and nontraded goods. The only shocks in the model are to the production of nontraded goods: the production function for each of the nontraded goods producers is subject to country-specific shocks and aggregate shocks to *productivity shocks* and to shocks to the the elasticity of substitution between the varieties of nontraded goods which we would refer to as *markup shocks*. To keep the analysis simple we purposefully abstract from the standard sources of gains from a monetary union, namely the reduction in transactions costs in trade. By doing so we highlight our main result: when countries have credibility problems, the inability of monetary policy to respond to idiosyncratic shocks in a monetary union may be a benefit rather than a cost.

The model features two key frictions. The first is that nontraded goods have sticky prices and are produced by monopolistically competitive firms. In each period, nontraded goods firms set their prices after the markup shocks are realized, but before either productivity shocks are realized or monetary policy has been set. These firms set their prices as a markup over their expected marginal costs and hence distort downward the production of nontraded goods. This distortion gives the monetary authority an incentive to engineer a surprise inflation so as to diminish the effective markup and increase the production of nontraded goods. This incentive is stronger the larger is the value of the markup shock. In contrast, the traded goods sector have flexible prices and are produced by competitive firms and hence have no such distortions.

The second friction is that purchases of traded goods must be made with money brought into the period while the purchase of nontraded goods are made with credit. This feature of the model generates costs for both surprise inflation and for expected inflation. (Other work that have used a similar device includes Svensson (1983), Nicolini (1998), and Albanesi, Chari, and Christiano (2003).) In the model a surprise inflation inefficiently lowers the consumption of traded goods ex post while an expected inflation distorts the consumption of the goods purchased with money–the traded goods–by raising the costs of purchasing them. (Notice that the presence of this second friction implies that in an equilibrium without commitment the monetary authority balances the benefits of surprise inflation against these costs and this friction leads to an interior solution for inflation.)

In our model, if the monetary authorities have no credibility issues then the standard Friedman-Mundell argument for flexible exchange rates holds. Specifically, when the monetary authority can commit to its policy, the flexible exchange rates regime is always preferable to the monetary union. Here membership in the monetary union simply restricts a policy instrument and adds an extra constraint to the Ramsey problem. This constraint binds whenever productivity shocks have an idiosyncratic component and leads welfare in the union to be lower then welfare under flexible exchange rates.

The economics behind the Friedman–Mundell effects is straightforward. When the idiosyncratic productivity of nontraded goods in a country is high, efficiency requires reducing the relative price of nontraded goods. Since nontraded goods prices are sticky, under flexible

exchange rates this relative price reduction can be accomplished by an increase in the price of traded goods—a devaluation. In a monetary union no such devaluation can occur. Of course, it is possible to increase the price of all traded goods in the union by engineering a union-wide increase in the price of traded goods, but such an increase is not optimal in response to an idiosyncratic shock in one country. Hence, the monetary union restricts the ability of monetary policy to ensure efficient adjustment to idiosyncratic productivity shocks. The ex ante cost of this restriction is greater the larger is the volatility of idiosyncratic productivity shocks. In sum, under commitment our model is consistent with the standard argument for flexible exchange rates: flexible exchange rates helps to minimize the distortions imposed by sticky prices as suggested by Friedman (1953) and formalized by Gali and Monacelli (2005).

The more interesting analysis is what happens when countries have credibility issues. We model lack of commitment by considering a monetary authority that sets its policy in a Markovian fashion. Our novel result is that when countries have credibility problems, the standard Friedman–Mundell logic can be overturned. In particular, when the idiosyncratic component of the markup shocks is sufficiently high, countries can gain from giving up their monetary independence when moving from a system of flexible exchange rates to a monetary union. The key idea is that giving up the ability to target policy to country-specific markup shocks can raise credibility and hence raise welfare as long these credibility gains outweigh the standard Friedman–Mundell costs of being unable to target country-specific productivity shocks.

To understand the credibility gains in a union, consider what happens under flexible exchange rates after the realization of a high markup shock. Under flexible exchange rates this high shock increases the temptation of the monetary authority to generate a surprise inflation to reduce the monopoly distortion in the non-traded sector. In equilibrium this temptation is frustrated by the behavior of the sticky price firm: upon seeing a high markup shock, the nontraded goods firms anticipate the monetary authority's action and simply increase their price. By so doing these firms undo the real effects of the monetary policy. Hence, in equilibrium, the increase in the monetary authority's temptation due to the shock results only in a higher and more volatile inflation. Such inflation is welfare reducing because it generates a distortion in the tradable good sector: the high inflation increases the effective cost of traded

goods and introduces a wedge between the marginal rate of substitution between labor and consumption of traded goods and the marginal rate of transformation between these same goods.

In contrast, in a monetary union, the union-wide monetary authority reacts only to union-wide variation in the markup shock. This inability to react to idiosyncratic markup shocks results in a lower volatility in the distortions in the traded good sector and thus, by itself, leads to higher welfare in the union. Of course, even here the inability to react to idiosyncratic productivity shocks, by itself, leads to lower welfare in the union. Overall, the Markov equilibrium in a monetary union has more volatile distortions in the non-traded sector and lower in the traded sector relative to the Markov equilibrium under flexible exchange rates. As long at the variability in idiosyncratic markup shocks is sufficiently large relative to the idiosyncratic volatility of productivity shock, the credibility gains from a monetary union outweigh the standard costs, and a monetary union is preferred to a system of flexible exchange rates.

Our model of a monetary union differs from some in the literature. We assume that countries that join the union stay in the union. In our setup as long as countries that join a union cannot leave the one until the end of the current period our analysis is unchanged. This assumption mimics that in Fuchs and Lippi (2006).

1. A Reduced Form Model

In each period t, an i.i.d. aggregate shock $z_t = (z_{1t}, z_{2t}) \in Z$ is drawn and each of a continuum of countries draws a vector of idiosyncratic shocks $v_t = (v_{1t}, v_{2t}) \in V$ which are i.i.d. both over time and across countries. The probability of aggregate shocks is $f(z_{1t}, z_{2t}) = f^1(z_{1t})f^2(z_{2t})$ and the probability of the idiosyncratic shocks is $g(v_{1t}, v_{2t}) = g^1(v_{1t})g^2(v_{2t})$. Here Z and V are finite sets. We let $s_t = (s_{1t}, s_{2t})$ with $s_{it} = (z_{it}, v_{it})$ and let $h(s_t) = h^1(s_{1t})h^2(s_{2t})$ with $h^i(s_{it}) = f^i(z_{it})g^i(s_{it})$. There are two shocks that, for concreteness only, we label a *mark-up* shock, $\theta(s_{1t})$, and a *productivity* shock, $A(s_{2t})$. These shocks will roughly correspond to the markup shocks and productivity shocks in the general equilibrium model that follows. Here these labels are purely for convenience and to set the stage for the equilibrium model. As will become evident, in this model the key distinction between them

is that the shock we labeled the markup shock is an *ex-ante* shock that occurs before private agents make their decisions and the shock we labeled a productivity shock is an *ex-post* shock that occurs after private agents make their decisions. We normalize the unconditional mean of the productivity shock to be zero.

The timing within the period is as follows: the mark-up shock is realized, the sticky price p_t is chosen by private agents, the productivity shock is realized and then the policy π_t is chosen. Let $s^t = (s_1, \ldots, s_t)$ denote the history of idiosyncratic and aggregate shocks of a given country and let $h_t(s^t) = h(s_1) \ldots h(s_t)$ denote the unconditional probability of these shocks.

The objective function of the the monetary authority is defined over the deviations $U(p, \pi, s)$ from the natural level of unemployment $\theta(s_1) - A(s_2)$ where

$$(1) \qquad U(p, \pi, s) = \theta(s_1) - A(s_2) + p - \pi$$

and inflation π. The objective is to minimize a weighted average of the square of these unemployment deviations and the square of the deviation of inflation π from zero. We capture this idea by assuming that the period t objective function of the monetary authority given the realization s_t is

$$(2) \qquad R(p, \pi, s) = -\frac{1}{2}\left[U(p, \pi, s)^2 + \kappa \pi^2\right].$$

The monetary authority discounts the future by β so that expected discounted value of utility is

$$(3) \qquad \sum_t \sum_{s^t} \beta^t h_t(s^t) R(p_t(s^{t-1}, s_{1t}), \pi_t(s^t), s_t)$$

The only action of private agents is to choose the price p_t. We model these private agents as choosing p_t equal to the expected level of the policy so that

$$(4) \qquad p_t(s^{t-1}, s_{1t}) = \sum_{s^t} g_t(s^t | s^{t-1}, s_{1t}) \pi_t(s^t).$$

(It will turn out that (4) mimics the optimality condition of the sticky price firms in the general equilibrium model.)

We model a *monetary union* by imposing the restriction that the policy must be the same for all union members at any point in time so that the union-wide policy cannot vary

with the idiosyncratic shocks of individual countries. This restriction implies that in a union the policy must depend only on the history of aggregate shocks, $\pi_t(z^t)$. The objective function for the monetary authority in the union is an equally weighted sum of utility of each country in the union. This objective function also is equal to the ex ante welfare of any single country. (Note that if we break the union into groups of countries of positive measure and allow for different weights on the utilities of members of each group the outcomes will be the same as those under the objective function (3). The reason is that within each group the fractions of members that experience specific histories of idiosyncratic shocks is the same.)

We model a system of *flexible exchange rates* by allowing each country to freely choose its own policy and to let that policy react to its own history of idiosyncratic shocks. Under flexible exchange rates we represent policy by $\pi_t(s^t)$ which represents the policy of any country with shock history s^t. Here (3) represents the welfare of any individual country. In sum, there are two differences between a monetary union and flexible exchange rates: In a monetary union policies are common across countries and are set at the union-level while under flexible exchange rates policies can differ across countries and are set at the country level.

A. Optimal Policy with Commitment

Consider first a system of flexible exchange rates in which each monetary authority can commit once-and-for-all to its policy $\{\pi_t(s^t)\}$. Here it suffices to consider each monetary authority in isolation solving the problem of maximizing (3) subject to (4). Given the repeated nature of the monetary authority's problem we can reduce it to the following problem for any period:

$$(5) \qquad V^R = \max_{\{p(s_1),\pi(s)\}} \sum_s h(s) R(p(s_1), \pi(s), s)$$

subject to

$$(6) \qquad p(s_1) = \sum_s g(s|s_1)\pi(s)$$

As we show in Appendix A, under (2) the Ramsey policy that solves (5) is given by

$$(7) \qquad \pi^R(s) = -\frac{A(s_2)}{1+\kappa}.$$

The price associated with the Ramsey policy is $p(s_1) = 0$ for all s_1. Substituting these prices and policies back into the objective function (5) gives that

$$(8) \qquad V^R = -\frac{1}{2}\left[E\theta^2 + \left(\frac{\kappa}{1+\kappa}\right)EA^2\right]$$

Notice from (7) that the monetary authority optimally responds to the ex post shock $A(s_2)$ but does not respond to the ex ante shock $\theta(s_1)$.

To gain some intuition for this result suppose that the monetary authority contemplates using the linear rule

$$(9) \qquad \pi(s) = a\theta(s_1) + bA(s_2).$$

The monetary authority realizes how its policies will affect the best responses of the private agents so that the monetary authority realizes that private agents will use the price setting rule

$$(10) \qquad p(s_1) = \sum_s g(s|s_1)\left[a\theta(s_1) + bA(s_2)\right] = a\theta(s_1)$$

where we have used our normalization that mean productivity is zero. Under these rules the unemployment term in the objective function

$$(11) \qquad \theta(s_1) - A(s_2) + p(s_1) - \pi(s) = \theta(s_1) - A(s_2) - bA(s_2).$$

Hence, unemployment is unaffected by the monetary authority's response to the ex ante shock, since a doesn't enter (11), but inflation term is now more variable via the coefficient b in (9). So by responding to an ex ante shock the monetary authority simply adds unwanted variance to inflation. Now consider the value of responding to the ex post shock $A(s_2)$. From (11) it is clear the raising inflation when productivity is low by setting b negative will help reduce the variability of unemployment, but this reduction in variability in unemployment comes at the cost of making inflation more variable. To see how the monetary authority balances these benefits and costs substitute the policies of the monetary authority and the best responses of private agents into the objective function to get

$$\max_b -\frac{1}{2}\sum_s h(s)\left[(\theta(s_1) - (1+b)A(s_2))^2 + \kappa(bA(s_2))^2\right].$$

The first order condition is

$$\sum_s h(s)A(s_2)\left[(\theta(s_1) - (1 + (1+\kappa)b)A(s_2))\right] = 0$$

Since A has mean zero and the A and θ are independent this simplifies to

$$(1 + (1+\kappa)b)\sum_s h(s)A(s_2)^2 = 0$$

and hence the optimal choice of b is $-1/(1+\kappa)$. In sum, responding to ex-ante shock does nothing to reduce the variability of unemployment and simply adds undesirable variance in inflation. In contrast, it is optimal to respond to ex post shocks in a way that balances the benefits of reducing the variability in unemployment with the costs of adding variability to inflation.

Consider now the problem for the monetary union. The monetary authority chooses $\{\pi_t(z^t)\}$ to maximize (3), the equally weighted sum of ex-ante utility of the continuum of countries in the union, subject to (4). Given the repeated nature of the union-wide monetary authority's problem we can reduce it to the following problem in each period

$$(12) \quad V^{R,U} = \max_{p(z_1),\pi(z)} \sum_s h(s)R(p(z_1),\pi(z),s)$$

subject to

$$(13) \quad p(z_1) = \sum_s f(z_2|z_1)\pi(z)$$

As we show in Appendix A, under (2) the Ramsey policy that solves (12) is given by

$$(14) \quad \pi^{R,U}(z) = -\frac{\bar{A}(z_2)}{1+\kappa}$$

$\bar{A}(z_2) \equiv \sum_{\nu_2} g^2(\nu_2)A(\nu_2, z_2)$ is the expectation of the productivity shock over idiosyncratic shocks v_2 conditional on the aggregate shock z_2. The equilibrium price is $p(z_1) = 0$. Substituting these policies back into the objective function (5) gives that

$$(15) \quad V^{R,U} = -\frac{1}{2}\left[E\theta^2 + EA^2 - \frac{E\bar{A}(z_2)^2}{1+\kappa}\right]$$

where $E\bar{A}(z_2)^2 = \sum_{z_2} f^2(z_2)\left[\sum_{\tilde{\nu}_2} g^2(\tilde{\nu}_2)A(\tilde{\nu}_2, z_2)\right]^2$. The intuition for why the policy (14) is optimal for the union is virtually identical to the intuition for why (7) is optimal under flexible exchange rates once we integrate out the idiosyncratic shocks in the first order conditions.

Our first proposition compares the value of the Ramsey problem under flexible exchange rates and the union and illustrates the standard Friedman–Mundell logic. A key term in this proposition is $\sum_{z_2} f^2(z_2) var(A(s_2)|z_2)$ which measures the expected variance of the productivity shock conditional on the aggregate shock z_2. The term $var(A(s_2)|z_2)$ represents the residual uncertainty that each country faces conditional on the relevant aggregate shock.

Proposition 1. The ex-ante expected utility under the Ramsey policy is higher in a flexible exchange rates regime than in a monetary union as long as productivity shocks have an idiosyncratic component in that

$$V^R - V^{R,U} = \frac{1}{2(1+\kappa)} \sum_{z_2} f^2(z_2) var(A(s_2)|z_2) > 0 \quad (16)$$

whenever $\sum_{z_2} f^2(z_2) var(A(s_2)|z_2) > 0$ so that productivity shocks have an idiosyncratic component.

The details are provided in Appendix A. Here the inability to target monetary policy to country-specific ex post idiosyncratic shocks entails a cost of joining a union. These costs are lower the less different are the shocks that countries experience as measured by the expected variance of the productivity shock conditional on the aggregate shock z_2. In particular, if there are no such idiosyncratic shocks then $var(A(s_2)|z_2) = 0$ and there are no losses in being part of a union. This conforms with the Mundellian criteria for an optimal currency area: under commitment, the larger the common component of shocks the lower the cost from losing monetary independence.

B. Policy Without Commitment

We turn now to a similar comparison when monetary authorities lack commitment. Our main result is that the standard Friedman–Mundell logic can be overturned when the monetary authority lacks commitment. In particular, when countries markup shocks are sufficiently different, as measured by their ex-ante conditional variances, countries can gain from giving up their monetary independence. The key idea is that in face of such shocks, giving up the ability to target policy to country-specific shocks can raise credibility and hence raise welfare.

Consider now the case in which the monetary authority cannot commit to its policy. We model this lack of commitment by supposing that policy is set in a Markovian fashion.

Consider first the flexible exchange rate regime. To characterize the Markov equilibrium, we begin with the best response of the monetary authority to an arbitrary price set by private agents, p, and an arbitrary shock vector s. The monetary authority's best response solves

$$(17) \quad U^{BR}(p,s) = \max_{\pi} R(p,\pi,s)$$

In Appendix A we show that the resulting best response is given by

$$\pi^{BR}(p,s) = \frac{\theta(s_1) - A(s_2) + p}{1+\kappa}$$

Imposing that in equilibrium the decision of private agents satisfies $p(s_1) = E\left(\pi^{BR}(p,s)|s_1\right)$ we obtain that

$$(18) \quad \pi^{M}(s) = \frac{\theta(s_1)}{\kappa} - \frac{A(s_2)}{1+\kappa}$$

and $p(s_1) = \theta(s_1)/\kappa$. Comparing (18) to (7) we see that the Markov policy under flexible exchange rates is simply the Ramsey policy under flexible exchange rates shifted up by the inflationary bias of $\theta(s_1)/\kappa$ that depends on the size of the markup and the relative weight on inflation. This bias is higher the larger is the markup and the lower is the weight on inflation.

We can substitute the optimal policy for the monetary authority and for private agents into the objective function and take expectations over the state s to obtain that the welfare in Markov equilibrium under flexible exchange rates is

$$(19) \quad V^{M} = -\frac{1}{2}\left[\frac{1+\kappa}{\kappa}E\theta^2 + \frac{\kappa}{1+\kappa}EA^2\right]$$

Comparing (19) to (8) we see that in a union the utility without commitment is lower than the utility with commitment by

$$(20) \quad -\frac{1}{2}\left[\frac{E\theta^2}{\kappa}\right] = -\frac{1}{2}\left[\frac{\bar{\theta}^2 + var(\theta)}{\kappa}\right]$$

where $\bar{\theta} = \sum_{s_1} h^1(s_1)\theta(s_1)$. Here the loss in utility from lack of commitment has two sources: first the average inflation is higher without commitment by $\bar{\theta}/\kappa$ and second, the volatility of inflation is higher as the monetary authority adjusts the inflation rate as the temptation to offset the markup varies.

Consider now the problem for the union. In any equilibrium the price set by private agents only depends on the aggregate shock at the beginning of the period, so it is without

loss of generality to consider the monetary authority's best responses to prices $p(z_1)$ that do not depend on ν_1. So, given the pre-set price $p = p(z_1)$ and aggregate state $z = (z_1, z_2)$, the best response for the union monetary authority solves

$$U^{BR,U}(p, z) = \max_{\pi} \sum_{\nu} g(\nu) R(p, \pi, (z, \nu))$$

In Appendix A we show that the best response is given by

$$\pi^{BR,U}(p, z) = \frac{\bar{\theta}(z_1) - \bar{A}(z_2) + p}{1 + \kappa}$$

Imposing that, in equilibrium, the decision of private agents satisfies $p(z_1) = E\left(\pi^{BR}(p(z_1), z)|z_1\right)$, we obtain

$$\pi^{M,U}(z) = \frac{\bar{\theta}(z_1)}{\kappa} - \frac{\bar{A}(z_2)}{1 + \kappa} \tag{21}$$

and $p(z_1) = \bar{\theta}(z_1)/\kappa$. Comparing (21) to (14) we see that also here the Markov policy in a union is simply the Ramsey policy in the union shifted up by the inflationary bias term $\bar{\theta}(z_1)/\kappa$. Here the bias from the Markov only depends on the union-wide average markup.

We can substitute the optimal policy for the monetary authority and for private agents into the objective function and take expectations over the state s to obtain that the welfare in a Markov equilibrium in the union is

$$V^{M,U} = -\frac{1}{2}\left[E\theta^2 + \frac{1}{\kappa} E\bar{\theta}(z_1)^2 + EA^2 - \frac{E\bar{A}(z_2)^2}{1 + \kappa}\right] \tag{22}$$

Comparing (22) to (15) we see that the utility without commitment is lower than the utility with commitment by

$$-\frac{1}{2}\left[\frac{E\bar{\theta}(z_1)^2}{\kappa}\right] = -\frac{1}{2}\left[\frac{var(\bar{\theta}(z_1)) + \bar{\theta}^2}{\kappa}\right] \tag{23}$$

where $\bar{\theta} = \sum_{s_1} h^1(s_1)\theta(s_1)$. Here the loss in utility from lack of commitment again comes from the average inflation being higher without commitment by $\bar{\theta}/\kappa$ and from the volatility of inflation being higher as the monetary authority adjusts the inflation rate to the aggregate markup shock.

Next we turn to our main issue. In joining a union a country gives up its monetary independence, that is, the ability to adjust its monetary policy to offset country-specific

shocks. When countries have full commitment, as they do in the standard Friedman–Mundell analysis, this loss of independence necessarily involves a cost. Here we ask does the logic apply when countries do not have such commitment. We find that it does not: without commitment, countries may gain by giving up their monetary independence. The reason is that by doing so it increases their credibility.

We formalize this claim by comparing the ex-ante value of the Markov equilibrium under the two regimes.

Proposition 2. If the expected variance of the markup shock conditional on the aggregate shock z_1 is sufficiently high relative to the expected variance of the productivity shock conditional on the aggregate shock z_2 then the ex-ante expected utility under the Markov policy is higher in a monetary union than in a flexible exchange rates regime in that

$$V^{M,U} - V^{M} = \frac{1}{2}\frac{\sum_{z_1} f^1(z_1) var\left(\theta(s_1)|z_1\right)}{\kappa} - \frac{1}{2}\frac{\sum_{z_2} f^2(z_2) var(A(s_2)|z_2)}{1+\kappa} > 0 \tag{24}$$

The proof of Proposition 2 is provided in Appendix A.

To understand the economics behind Proposition 2, consider the extreme case in which $A(s_2) = 0$ for all s_2 and in which the markup shock only depends on idiosyncratic shocks so that $\theta(s_1) = \theta(\nu_1)$. Under flexible exchange rates each monetary authority is tempted to respond to the country-specific markup θ in order to minimize the deviation from the natural level of unemployment. However, since private agents choose their price after this shock is realized and anticipate the monetary authority's actions, the monetary authority's desire to stabilize unemployment is always frustrated in equilibrium. Hence the monetary authority's reaction to the country specific θ shocks just adds undesirable variance to inflation. More precisely substituting $p(\nu_1) = \pi(\nu_1)$ into the period objective function (2), it follows that the indirect utility function associated with any policies is

$$V(\pi, \nu_1) = -\frac{1}{2}\left[\theta(\nu_1)^2 + \kappa\pi^2\right] \tag{25}$$

It is evident from (25) that the monetary authority cannot affect the first term. The incentive to accommodate θ only results in excessive volatility of π that is detrimental given the second term in (25). Here with $A(s_2) = 0$, inflation under flexible exchange rates equals

$$\pi^M(s) = \frac{\theta(s_1)}{\kappa} \tag{26}$$

and hence the unwanted volatility in inflation arises both from the idiosyncratic shock v_1 and the aggregate shock z_1.

In a monetary union the monetary authority is only tempted to respond to the component of the markup that is common across countries. Here also the monetary authority's desire to stabilize unemployment is frustrated in equilibrium as the private agents anticipate the monetary authority's action. In a monetary union, however, the monetary authority only reacts to the aggregate component of the markup shock and the policy in the union is

$$\pi^{M,U}(z) = \frac{\bar{\theta}(z_1)}{\kappa} \tag{27}$$

Clearly, the undesirable volatility of inflation in the union is lower than it is under flexible exchange rates. This decrease in volatility is the source of the credibility gains in the union.

So far we have described the source of the credibility gains in the monetary union arising from the inability to react to idiosyncratic markup shocks. Of course, the union also has standard Friedman–Mundell losses from its inability to react to idiosyncratic productivity shocks. To illustrate these losses consider an alternative extreme in which there is no idiosyncratic component to markup shocks so that $\theta(s_1)$ can be written as $\theta(z_1)$ but there is an idiosyncratic component to productivity shocks. Then the first term in (24) is zero and giving up monetary independence involves a cost. Indeed, in this case

$$V^{M,U} - V^M = V^{R,U} - V^R$$

so that the cost of giving up monetary independence has the same Friedman–Mundell form as it did with commitment. To understand this result better consider the policies in this case. Under flexible exchange rates the Markov policy is given by

$$\pi^M(s) = \frac{\theta(z_1)}{\kappa} - \frac{A(s_2)}{1+\kappa} \tag{28}$$

while in a union the Markov policy is given by

$$\pi^{M,U}(z) = \frac{\theta(z_1)}{\kappa} - \frac{\bar{A}(z_2)}{1+\kappa}. \tag{29}$$

Both policies respond in the same way to markup shocks and differ only in how they respond to productivity shocks. In a union the policies respond only to the aggregate component of productivity shocks while under flexible exchange rates they can also respond to idiosyncratic

productivity shocks. Hence, here there are no credibility gains from joining a union but there are the standard Friedman–Mundell costs, so a regime of flexible exchange rates dominates that of a monetary union.

These two extreme cases make clear that overall, joining a monetary union brings credibility gains and Friedman–Mundell costs. The key implication of Proposition 2 is that joining a union is desirable if the credibility gains are sufficiently large relative to the Friedman–Mundell costs

2. A General Equilibrium Model

The economy consists of a continuum of countries, each of which produces traded and nontraded goods and uses currency to purchase goods. The traded goods sector in each country is perfectly competitive and the production function in this sector is not subject to shocks. The nontraded goods consists of a continuum of firms each of which produces a differentiated product. The production function for each of the nontraded goods producers is subject to cboth aggregate and country-specific shocks to *productivity* and to shocks to the elasticity of substitution between the varieties of nontraded goods, referred to as *markup* shocks. The traded goods prices are flexible and they are bought with cash while the nontraded goods prices are sticky and they are bought with credit.

We begin by describing the equilibrium for exogenous sequences of policies. We then turn to the classic comparison of a flexible exchange rate system and a currency union in an environment in which each monetary authority is fully committed to its policies. In the flexible exchange rate system the nominal price of traded goods can differ across countries while in a currency union this price must be equated across countries. We show that with commitment, the lack of monetary independence makes the currency union less desirable than a system of flexible exchange rates. We turn to making the same comparison when monetary authorities have no such commitment and instead set policy in a Markovian fashion. The inability to react to the idiosyncratic component of markup shocks leads the union to have credibility gains, while the inability to react to the idiosyncratic component of productivity shocks leads to Friedman–Mundell costs. Our main result is that if the variability of the idiosyncratic markup shocks is sufficiently high relative to that of the idiosyncratic productivity shocks

then a monetary union is preferable to a regime of flexible exchange rates.

A. Environment

In each period t, an i.i.d. aggregate shock $z_t = (z_{1t}, z_{2t}) \in Z$ is drawn and each of a continuum of countries draws a vector of idiosyncratic shocks $v_t = (v_{1t}, v_{2t}) \in V$ which are i.i.d. both over time and across countries. The probability of aggregate shocks is $f(z_{1t}, z_{2t}) = f^1(z_{1t})f^2(z_{2t})$ and the probability of the idiosyncratic shocks is $g(v_{1t}, v_{2t}) = g^1(v_{1t})g^2(v_{2t})$. Here Z and V are finite sets. We let $s_t = (s_{1t}, s_{2t})$ with $s_{it} = (z_{it}, v_{it})$ and let $h(s_t) = h^1(s_{1t})h^2(s_{2t})$ with $h^i(s_{it}) = f^i(z_{it})g^i(s_{it})$. These aggregate and idiosyncratic shocks are to the nontraded goods sector and affect the elasticity of substitution between goods in this sector denoted $\theta(s_{1t})$ and referred to as markup shock and the productivity in this sector denoted $A(s_{2t})$. We let s^t and $h_t(s^t)$ denote the history and probability of these shocks and use similar notation for any components of these shocks.

The timing of events with a period is the following: first the markup shocks are realized, then the sticky price firms make their decisions, then the productivity shocks are realized, then the monetary authority chooses its policy, then households and flexible price firms make their decisions.

Production of Traded and Nontraded Goods

Consider first the production of traded and nontraded goods. The production function for a traded goods in a given country is simply $Y_{Tt}(s^t) = L_{Tt}(s^t)$ where $s^t = (s_0, \ldots, s_t)$ is the history of aggregate and idiosyncratic shocks in that country, $Y_{Tt}(s^t)$ is the output of traded goods and $L_{Tt}(s^t)$ are the inputs of labor in the traded goods sector. The problem of traded goods firms is then to solve

$$\max_{L_{Tt}(s^t)} P_{Tt}(s^t)L_{Tt}(s^t) - W_t(s^t)L_{Tt}(s^t)$$

so that in equilibrium

$$P_{Tt}(s^t) = W_t(s^t). \tag{30}$$

The non-traded good in any given country is produced by a competitive final consumption firm using $j \in [0,1]$ intermediates according to

$$(31) \quad Y_{Nt}(s^t) = \left[\int Y_N(j,s^t)^{\frac{\varepsilon(s_{1t})-1}{\varepsilon(s_{1t})}} dj\right]^{\frac{\varepsilon(s_{1t})}{\varepsilon(s_{1t})-1}}$$

where $\varepsilon(s_{1t}) = \theta(s_{1t})/(\theta(s_{1t})-1)$. This firm maximizes

$$P_N(s^{t-1},s_{1t})Y_{Nt}(s^t) - \int P_N(j,s^{t-1},s_{1t})y_{Nt}(j,s^t)dj$$

where the notation makes clear that, consistent with our timing assumption, the prices of nontraded goods cannot vary with s_{2t}. The demand for an intermediate of type j is thus given by

$$(32) \quad Y_{Nt}(j,s^t) = \left(\frac{P_{Nt}(s^{t-1},s_{1t})}{P_{Nt}(j,s^{t-1},s_{1t})}\right)^{\varepsilon(s_{1t})} Y_{Nt}(s^t)$$

The intermediate goods are produced by monopolistic competitive firms using a linear technology:

$$(33) \quad Y_{Nt}(j,s^t) = A(s_{2t})L_N(j,s^t)$$

The problem of an intermediate good firm of type j is to choose $P = P_t(j,s^{t-1},s_{1t})$ to solve

$$(34) \quad \max_P \sum_{s_t} Q_t(s^t)\left[P - W_t(s^t)\right]\left(\frac{P_{Nt}(s^t)}{P}\right)^{\varepsilon(s_{1t})} C_{Nt}(s^t)$$

subject to (33) where $Q_t(s^t)$ is the nominal stochastic discount factor. The solution to this problem gives that for all intermediate goods producers j,

$$(35) \quad P_N(j,s^{t-1},s_{1t}) = \theta(s_{1t})\frac{\sum_{s_{2t}} Q_t(s^t)C_{Nt}(s^t)\frac{W_t(s^t)}{A(s_t)}}{\sum_{s_{2t}} Q_t(s^t)C_{Nt}(s^t)}.$$

where $\theta(s_{1t})$ is the markup in period t. Since this price does not depend on j we note that $P_{Nt}(j,s^{t-1},s_{1t}) = P_{Nt}(s^{t-1},s_{1t})$. This result implies that the labor hired by each intermediate goods firm is the same so that $L_N(j,s^t)$ can be written $L_{Nt}(s^t)$ and the final output of nontraded goods is simply

$$(36) \quad Y_{Nt}(s^t) = A(s_{2t})L_{Nt}(s^t).$$

Consumers and the Government

The consumers in any given country have preferences given by

$$(37) \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t h_t(s^t) U\left(C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t)\right)$$

where $C_{Tt}(s^t)$ is the consumption of traded goods, $C_{Nt}(s^t)$ is the consumption of the (final) nontraded good, and $L_t(s^t)$ is (total) labor supply. For our main results we will assume that the utility function takes the form

$$(38) \quad U(C_T, C_N, L) = \alpha \log C_T + (1-\alpha) \log C_N - bL$$

Consumers are subject to a cash-in-advance constraint that requires them to buy traded goods at t using domestic money brought in from period $t-1$, namely $M_{t-1}(s^{t-1})$, so that

$$(39) \quad P_{Tt}(s^t) C_{Tt}(s^t) \leq M_{t-1}(s^{t-1}).$$

The budget constraint of the consumer is given by

$$\begin{aligned} (40) \qquad & P_{Tt}(s^t) C_{Tt}(s^t) + P_N(s^{t-1}, s_{1t}) C_{Nt}(s^t) + M_t(s^t) + B_t(s^t) \\ \leq \; & W_t(s^t) L_t(s^t) + M_{t-1}(s^{t-1}) + (1 + r_t(s^t)) B_{t-1}(s^{t-1}) + T_t(s^t) + \Pi_t(s^t). \end{aligned}$$

where $T_t(s^t)$ are nominal transfers, $\Pi_t(s^t)$ are the profits from the nontraded goods firms, $r_t(s^t)$ is the nominal interest rate in the domestic currency, and $B_t(s^t)$ are nominal bonds.

We assume that these bonds are exchanged only among agents in any given countries, but not across countries, so that an equilibrium condition is that

$$(41) \quad B_t(s^t) = 0$$

Thus we assume that countries as a whole do not intertemporally borrow or lend from each other. As we will discuss, under (38) and our structure of shocks the constraint (41) will not bind in the relevant Ramsey problems.

The first order conditions for the consumer are summarized by

$$(42) \quad \frac{U_{Nt}(s^t)}{P_{Nt}(s^{t-1}, s_{1t})} = -\frac{U_{Lt}(s^t)}{W_t(s^t)}$$

$$(43) \quad \frac{U_{Tt}(s^t)}{P_{Tt}(s^t)} = -\frac{U_{Lt}(s^t)}{W_t(s^t)} + \xi_t(s^t) \geq -\frac{U_{Lt}(s^t)}{W_t(s^t)}$$

$$(44) \quad \frac{U_{Nt}(s^t)}{P_{Nt}(s^{t-1})} = \beta \sum_{s^{t+1}} h(s^{t+1}|s^t) \frac{U_T(s^{t+1})}{P_T(s^{t+1})}$$

$$(45) \quad \frac{1}{1+r_t(s^t)} = \beta \sum_{s^{t+1}} h_{t+1}(s^{t+1}|s^t) \frac{U_N(s^{t+1})}{P_{Nt}(s^t, s_{1t+1})} \frac{P_{Nt}(s^{t-1}, s_{1t})}{U_{Nt}(s^t)}.$$

where $\xi_t(s^t) \geq 0$ is the (normalized) multiplier on the cash-in-advance constraint. Notice also that the nominal stochastic discount factor for the country is

$$(46) \quad Q_{t+1}(s^{t+1}) = \beta h_{t+1}(s^{t+1}|s^t) \frac{U_N(s^{t+1})}{P_{Nt}(s^t, s_{1t+1})} \frac{P_{Nt}(s^{t-1}, s_{1t})}{U_{Nt}(s^t)}$$

where $Q_t(s^t)$ is the price of a state-contingent claim to local currency units at s^t in units of local currency at s^t. This is the relevant price that firms use to discount profits in (34).[1] The monetary authority's budget constraint is simply that newly created money is transferred to consumers in a lump-sum fashion

$$(47) \quad T_t(s^t) = M_t(s^t) - M_{t-1}(s^{t-1}).$$

In this economy policies can be be described as a sequence of interest rates, money supplies, and transfers that satisfy (45) and (47). In terms of what follows, we can either let the monetary authority a nominal interest rate policy and letting nominal transfers and money growth being endogenously determined or we can let the monetary authority choose money growth rates and letting interest rates and transfers be endogenously determined.

Equilibrium for given policies

We focus on symmetric equilibria, in which any two countries with the same history of idiosyncratic shocks s^t have the same allocations, policies, and prices. Given this symmetry, consider first the definition of an equilibrium with flexible exchange rates. Given initial conditions $\{M_{-1}, B_{-1}\}$ and a policy $\{r_t(s^t), M_t(s^t), T_t(s^t)\}$ an *equilibrium with flexible exchange rates* is a set of allocations $\{C_{Tt}(s^t), C_{Nt}(s^t), L_{Tt}(s^t), L_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ and prices $\{W_t(s^t), P_{Tt}(s^t), P_{Nt}(s^{t-1}, s_{1t}), Q_t(s^t), r_t(s^t)\}$ such that: i) at these prices, the decisions of

[1] To check this claim add to the left side of the consumer's budget constraint period t purchases of nominal contigent claims $\sum_{s^{t+1}} Q_{t+1}(s^{t+1}) D_{t+1}(s^{t+1})$ and to the right side the payments for period $t-1$ purchases $D_t(s^t)$ and note that the resulting first order condition gives the formula for $Q_{t+1}(s^{t+1})$.

households are optimal, ii) at these prices, the decisions of firms are optimal, iii) the labor market clears in each country

$$L_{Nt}(s^t) + L_{Tt}(s^t) = L_t(s^t), \tag{48}$$

iv) the traded and nontraded goods markets clear

$$C_{Tt}(s^t) = Y_{Tt}(s^t), C_{Nt}(s^t) = Y_{Nt}(s^t), \tag{49}$$

v) the monetary authority's budget constraint holds

$$T_t(s^t) = M_t(s^t) - M_{t-1}(s^{t-1}), \tag{50}$$

and the interest rate $r_t(s^t)$ satisfies (45).

We model a monetary union as the restriction that the nominal price of traded goods is the same for all countries, so that at time t, if one country has a history $s^t = (z^t, v^t)$ and another has history $\tilde{s}^t = (z^t, \tilde{v}^t)$ then $P_{Tt}(s^t) = P_{Tt}(\tilde{s}^t)$. Hence, P_{Tt} depends on the history of aggregate shocks but not on the history of any country's idiosyncratic shocks. An *equilibrium with fixed exchange rates* is defined analogously to an equilibrium with flexible exchange rates with the added restriction that for any s^t and $\tilde{s}^t$,

$$P_{Tt}(s^t) = P_{Tt}(\tilde{s}^t) \tag{51}$$

for all $s^t = (z^t, v^t)$ and $\tilde{s}^t = (z^t, \tilde{v}^t)$. Note that here we model the union as restricting the implicit nominal exchange rate between countries to be equal, say to 1, but otherwise we let the rest of monetary policy differ across countries.

In a monetary union the set of allocations that can be implemented as a competitive equilibrium is more restricted than under flexible exchange rates. In particular, when the cash-in-advance constraint is slack, as it will turn out to be under Ramsey allocations, combining (42), (43), (30) and imposing that the price of traded goods can only depend on aggregate shocks gives

$$-\frac{U_L(s^t)}{U_N(s^t)} P_N(s^{t-1}, s_{1t}) = P_T(z^t) \tag{52}$$

Clearly, (52) imposes restrictions on how allocations are related across countries that are not present under flexible exchange rates. (Note that the one-period cashless economy of

Farhi and Werning (2013) has this same condition.) To be precise consider two countries A and B at time t given an aggregate history z^t. Consider two histories for country A that differ in the period t idiosyncratic shock component v_{A2t} so that $s_A^t = (z^t, v_A^{t-1}, v_{A1t}, v_{A2t})$ and $\tilde{s}_A^t = (z^t, v_A^{t-1}, v_{A1t}, \tilde{v}_{A2t})$ and two analogous histories of shocks for country B, say $s_B^t = (z^t, v_B^{t-1}, v_{B1t}, v_{B2t})$ and $\tilde{s}_B^t = (z^t, v_B^{t-1}, v_{B1t}, \tilde{v}_{B2t})$. Then (52) immediately implies that

$$\frac{U_L(s_A^t)/U_N(s_A^t)}{U_L(\tilde{s}_A^t)/U_N(\tilde{s}_A^t)} = \frac{U_L(s_B^t)/U_N(s_B^t)}{U_L(\tilde{s}_B^t)/U_N(\tilde{s}_B^t)} \tag{53}$$

We begin with a preliminary result that will be useful in setting up the Ramsey problem under flexible exchange rates.

Lemma 1. Allocations $\{C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ and prices $\{P_{Tt}(s^t), P_{Nt}(s^{t-1}, s_{1t})\}$ given initial conditions $\{M_{-1}, B_{-1}\}$ are part of a competitive equilibrium under flexible exchange rates iff the following conditions hold: i) the consumer's first order conditions and cash-in-advance constraint are satisfied after replacing $W_t(s^t) = P_{Tt}(s^t)$, i.e. (42), (43), (44), and (39) are satisfied and if (39) holds a strict inequality then (43) holds as an equality; ii) a version of the sticky price first order condition holds with $P_{Tt}(s^t)$ replacing $W_t(s^t)$ in (35); iii) a version of market clearing holds

$$L_t(s^t) = C_{Tt}\left(s^t\right) + \frac{C_{Nt}(s^t)}{A(s_{2t})} \tag{54}$$

Proof. First notice that these conditions are necessary for a competitive equilibrium. In fact, (42)–(44) and (39) are necessary first order conditions for the households problem using (30) to substitute for $W_t(s^t)$. The modified version of (35) follows from (35) and (30). Finally (54) is implied by the market clearing conditions for the consumption goods and labor.

Conversely, suppose that (42), (43), (44), (39), (35), and (54) are satisfied. Letting $W_t(s^t) = P_{Tt}(s^t)$ and defining $Q(s^t)$ and $r(s^t)$ from (46) and (45) from (42)–(44) and (39) it follows that $\{C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ is optimal for the household problem. Optimality for the sticky price firms is implied by the modified version of (35). Finally the market clearing conditions for consumption goods and labor are implied by (54) and for money holdings by the fact that $M(s^t)$ is optimal for the the household problem. $Q.E.D.$

The next lemma will be useful in characterizing the Ramsey problem.

Lemma 2. The allocations in both a flexible exchange rate equilibrium and a fixed exchange rate equilibrium satisfy the following constraints

$$C_{Tt}(s^t) = L_{Tt}(s^t) \tag{55}$$

$$C_{Nt}(s^t) = A_{Nt}(s^t)L_{Nt}(s^t) \tag{56}$$

$$\sum_{s_{2t}} h(s^t|s^{t-1}, s_{1t})C_{Nt}(s^t)\left[U_{Nt}(s^t) + \theta(s_{1t})\frac{U_{Lt}(s^t)}{A(s_t)}\right] = 0 \tag{57}$$

$$L_{Tt}(s^t) + L_{Nt}(s^t) = L_t(s^t) \tag{58}$$

$$U_{Tt}(s^t) \geq -U_{Lt}(s^t) \tag{59}$$

Proof. Consider either a flexible exchange rate or a fixed exchange rate equilibrium. Constraints (55), (56), and (58) clearly hold since they are the market clearing conditions. Equation (57) follows from substituting the consumer first order conditions into the price setting equation for nontraded goods firms. Specifically, substituting for $W_t(s^t)$ and $Q_t(s^t)$ from (42) and (46) gives (57). (59) follows from substituting $P_T(s^t) = W(s^t)$ in (43). *Q.E.D.*

We refer to (57) as the *labor market distortion* constraint.

3. Optimal Policy with Commitment

We turn now to analyzing optimal policy under flexible exchange rates and in a monetary union. We will show that the lack of monetary independence in a monetary union imposes a loss on member countries. The intuition for this result is based on the standard Friedman-Mundell logic: under fixed exchange rates countries are less able to target monetary policy to their country specific shocks. Of course, since we have abstracted from the standard Mundellian gains to trade that accompanies a monetary union this result is consistent with Mundell's optimal currency criterion. For any given gains from trade of a currency union (here zero) countries should join the union only if the idiosyncratic component of their shocks is small enough.

We start by defining the *Ramsey problem* for a country under *flexible exchange rates*. The problem is to choose allocations $\{C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ and prices $\{P_{Tt}(s^t), P_{Nt}(s^{t-1}, s_{1t})$ given initial conditions $\{M_{-1}, B_{-1}\}$ to maximize date 0 utility

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t h_t(s^t)U\left(C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t)\right) \tag{60}$$

subject to (42), (43), (44), (39), (35) replacing $W_t(s^t)$ with $P_t(s^t)$ and (54).

In a monetary union allocations must also satisfy (51) for all $s^t = (z^t, v^t)$ and $\tilde{s}^t = (z^t, \tilde{v}^t)$. The *Ramsey problem* in a *monetary union* can thus be written as choosing allocations $\{C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ and prices $\{P_{Tt}(s^t), P_{Nt}(s^{t-1}, s_{1t})\}$ to maximize date 0 utility (60) subject to (42), (43), (44), (39), (35) replacing $W_t(s^t)$ with $P_t(s^t)$, (54), and the additional constraint (51) for all $s^t = (z^t, v^t)$ and $\tilde{s}^t = (z^t, \tilde{v}^t)$.

The fact that the Ramsey problem under flexible exchange rates is a more relaxed version of the Ramsey problem in a monetary union immediately implies the following result:

Proposition 3. The Ramsey problem under flexible exchange rates leads to weakly higher welfare than the Ramsey problem in a monetary union.

The ex-ante value of the Ramsey problem under flexible exchange rates is an upper bound for the value that can be attained by the Ramsey problem in a monetary union. Next we show that under conditions the additional constraint in the Ramsey problem for the monetary union necessarily binds at some point so that the Ramsey problem under flexible exchange rates leads to *strictly* higher welfare than the Ramsey problem in a monetary union. We show that this is the case under (38).

To do so, we begin by considering a *relaxed Ramsey problem* under flexible exchange rates, written in primal form. That problem is to choose allocations to maximize date 0 utility

$$(61) \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t h_t(s^t) U\left(C_{Tt}(s^t), C_{Nt}(s^t), L_t(s^t)\right)$$

subject to the constraints (55)–(59).

As we show in Lemma 2, (55)–(59) are necessary conditions for allocations to be part of a competitive equilibrium. In this sense (61) is a relaxed version of the Ramsey problem (60). The next lemma shows that under our preference specification, (38), the relaxed Ramsey problem and the Ramsey problem attain the same value.

Lemma 3. Under (38), the solution to the relaxed Ramsey problem (61) can be implemented as a competitive equilibrium with flexible exchange rates.

The proof for this Lemma is provided in Appendix B.

The next lemma contains a key characteristic of the solution to this problem that will

allow us to rank welfare under fixed and flexible exchange rates. That characteristic is that in any such Ramsey allocation ratio of the marginal rate of substitution of labor to nontraded consumption across states equals the marginal rate of transformation across these states.

Lemma 4. Under (38) the Ramsey allocations under flexible exchange rates satisfy

$$\frac{U_L(s^t)}{U_N(s^t)} / \frac{U_L(\tilde{s}^t)}{U_N(\tilde{s}^t)} = \frac{A(s_t)}{A(\tilde{s}_t)} \tag{62}$$

where $\tilde{s}^t = (s^{t-1}, s_{1t}, z_{2t}, \tilde{v}_{2t})$, so that the resulting shocks differ in the idiosyncratic component of productivity shocks at time t, $A(z_{2t}, v_{2t})$ and $A(z_{2t}, \tilde{v}_{2t})$.

Proof. Consider a relaxed version of the Ramsey problem in which we drop constraint (59). Since the solution to this relaxed problem will satisfy this dropped constraint then the solution to the relaxed problem is a solution to the original problem.

Now dividing the first order condition for $C_N(s^t)$ by that for $L(s^t)$ and using additive separability gives

$$\frac{U_N(s^t) + \xi(s^{t-1}, s_{1t}) \left[\left[U_N(s^t) + \frac{1}{\theta(s_{1t})} \frac{U_L(s^t)}{A(s_t)} \right] + C_N(s^t) U_{NN}(s^t) \right]}{-U_L(s^t) + \xi(s^{t-1}, s_{1t}) \left[C_N(s^t) \frac{1}{\theta(s_{1t})} \frac{U_{LL}(s^t)}{A(s_t)} \right]} = \frac{1}{A(s_t)}$$

where $\xi(s^{t-1}, s_{1t})$ is the normalized multiplier on the labor market distortion constraint (57). Then using (38) and manipulating this equation we can reduce it to

$$-\frac{U_{Nt}(s^t)}{U_{Lt}(s^t)} = \frac{1}{A(s_{2t})} \left[1 + \xi_t(s^{t-1}, s_{1t}) \frac{1}{\theta_t(s_{1t})} \right]. \tag{63}$$

Our restriction follows from the feature that the term in square brackets on the right side of this equation does not vary with the s_{2t}. More formally, defining $\tilde{s}^t = (s^{t-1}, s_{1t}, \tilde{s}_{2t})$ so that the shock histories s^t and $\tilde{s}^t$ differ only in the period t productivity shocks they produce, we can divide (63) evaluated at s^t with that for $\tilde{s}^t$ to give (62). *Q.E.D.*

We then have

Proposition 4. Under (38) the Ramsey problem under flexible exchange rates leads to strictly higher welfare than the Ramsey problem under fixed exchange rates as long as countries are subject to idiosyncratic productivity shocks.

Proof. Since the Ramsey problem under flexible exchange rates a relaxed version of the Ramsey problem under fixed exchange rates in which the key restriction (53) has been

dropped, we need only show that the solution to the Ramsey problem under flexible exchange rates violates this restriction. Since under flexible exchange rates

$$(64) \quad \frac{U_L(s_i^t)}{U_N(s_i^t)} / \frac{U_L(\tilde{s}_i^t)}{U_N(\tilde{s}_i^t)} = \frac{A(z_{2t}, v_{i2t})}{A(z_{2t}, \tilde{v}_{i2t})}$$

holds for any two countries $i = A, B$, the flexible exchange rate solution will be inconsistent with the fixed exchange rate restriction (53) unless

$$(65) \quad \frac{A(z_{2t}, v_{A2t})}{A(z_{2t}, \tilde{v}_{A2t})} = \frac{A(z_{2t}, v_{B2t})}{A(z_{2t}, \tilde{v}_{B2t})}$$

for all possible idiosyncratic productivity shocks $v_{A2t}, \tilde{v}_{A2t}, v_{B2t}$,and $\tilde{v}_{B2t}$ Letting $\tilde{v}_{A2t} = \tilde{v}_{B2t}$ it is clear that (65) holds if and only if

$$(66) \quad A(z_{2t}, v_{2t}) = A(z_{2t}, v'_{2t})$$

for all v_{2t} and v'_{2t}, which implies that productivity shocks in all countries do not vary with idiosyncratic shocks. *Q.E.D.*

Proposition 4 exemplifies the standard Friedman-Mundell intuition: the inability to target monetary policy to country specific shocks under fixed exchange rates implies a cost of adopting a common currency.

There is one subtlety of interpretation of the Ramsey problem under the two regimes. Under flexible exchange rates the objective function represents the utility of a single country. Under fixed exchange rates the objective function represents the equally weighted integral of the continuum of ex-ante objective functions of the continuum of countries in the union.

In the proof of Proposition 4 we have used the general logic behind the Friedman–Mundell intuition: the union simply adds constraints to the Ramsey problem and hence must lower welfare. We supplement this general intuition by working out the allocations and prices in closed form for our preference specification (38) in the case of no aggregate shocks Specifically, as we show in Appendix B, in this case under flexible exchange rates the consumption of nontraded goods is given by

$$(67) \quad C_N^R(v) = \frac{(1-\alpha)}{b} \frac{A(v_2)}{\theta(v_1)},$$

the consumption of traded goods is given by $C_T = \alpha/b$, and labor is given by

$$(68) \quad L^R(v_1) = \frac{1}{b} \left[\alpha + \frac{(1-\alpha)}{\theta(v_1)} \right]$$

In a monetary union in this case the consumption of nontraded goods is given by

$$(69) \quad C_N^{R,U}(v_1) = \frac{1-\alpha}{b}\frac{1}{\theta(v_1)}\frac{1}{\sum_{v_2} g^2(v_2)/A(v_2)},$$

the consumption of traded goods is $C_T = \alpha/b$, and labor is given by

$$(70) \quad L^{R,U}(v_1) = \frac{1}{b}\left[\alpha + \frac{(1-\alpha)}{\theta(v_1)}\frac{1/A(v_2)}{\sum_{\tilde{v}_2} g^2(\tilde{v}_2)/A(\tilde{v}_2)}\right]$$

Since the consumption of traded goods is equal across regimes and the expected value of labor supply is equal across regimes then difference in utility in the regimes is that due to the differences in the consumption of nontraded goods. That is,

$$EU^R - EU^{R,U} = \alpha \sum_{v_1,v_2} g^1(v_1)g^2(v_2)\left[\log C_N^R(v) - \log C_N^{R,U}(v)\right]$$

which equals

$$(71) \quad \alpha\left[\log\left(\sum_{v_2} g^2(v_2)\frac{1}{A(v_2)}\right) - \sum g^2(v_2)\log\frac{1}{A(v_2)}\right] > 0$$

Clearly, (71) is strictly positive since the log function is a concave function.

To get a better understanding of the forces that lead to lower utility in the union we consider the labor wedge in nontraded goods defined implicitly as the $\tau_N(s)$ such that

$$(72) \quad -\frac{U_L(s)}{U_N(s)} = (1-\tau_N(s))A(s_2)$$

That is, $\tau_N(s)$ measures the distortion between the marginal rate of substitution between labor and consumption of nontraded goods and the marginal rate of transformation between these same goods. Here with no aggregate shocks and assuming (38) the nontraded labor wedge satisfies $1-\tau_N(v) = \frac{b}{1-\alpha}\frac{C_N(v)}{A(v_2)}$. With flexible exchange rates this labor wedge

$$(73) \quad 1-\tau_N^R(v) = \frac{1}{\theta(\nu_1)}$$

so, in particular, this wedge does not move with the country-specific productivity shock. In the monetary union instead,

$$(74) \quad 1-\tau_N^R(v) = \frac{1}{\theta(v_1)}\frac{1/A(v_2)}{\sum_{\tilde{v}_2} g^2(\tilde{v}_2)/A(\tilde{v}_2)},$$

this labor wedge varies with the country specific productivity shock. Notice that the mean of the labor wedge in the union coincides with the mean of the labor wedge under flexible

exchange rates. The key to the welfare losses in the union is that the higher volatility of the labor wedge in the union leads to welfare losses.

Note for later that the corresponding labor wedge in the tradable goods sector $\tau_T(v)$ is defined by

$$(75) \quad -\frac{U_L(s)}{U_T(s)} = (1 - \tau_T(s)).$$

With (38) and no aggregate shocks this formula reduces to

$$(76) \quad \frac{b}{\alpha C_T(v)} = (1 - \tau_T(v)).$$

Since $C_T(v) = \alpha/b$ in both regimes the traded goods labor wedge is identically equal to zero under both regimes.

4. Optimal Policy without Commitment

Consider now the same physical environment except that the monetary authorities cannot commit. We model this lack of commitment as having these authorities as choose policies in a Markovian fashion.

The timing is the same as before: the monetary authority sets its policies in each period t after all the shocks have been realized for the period and immediately before production and consumption take place. There are three relevant stages. The first stage—the *sticky price* stage—occurs at the beginning of the period after the markup shocks associated with (z_1, v_1) have been realized. At this stage the sticky price firms make their decisions. At the next stage—the *policy* stage—monetary policy is set after the productivity shocks associated with (z_2, v_2) have been realized. Then at the *household* stage, the household and the flexible price firms make their decisions.

We begin by describing the state variables for the sticky and flexible price firms, the households, and the monetary authority. We normalize all nominal variables by the beginning of period aggregate stock of money $\bar{M}_{-1}$. Note that for an arbitrary measure Λ over entering nominal money stocks over countries we can define the aggregate nominal money stock as

$$\bar{M}_{-1} = \int M_{-1} d\Lambda(M_{-1})$$

The *sticky price firm state* in (x_F, S_F) where $x_F = (m, v_1)$ and $S_F = (z_1, \lambda_F)$ and λ_F is a measure over x_F. Denote the sticky price firm's normalized decision rule as $\bar{p}_N(x_F, S_F)$.

The *monetary authority state* in a country consists of a country-specific component $x_G = (m, p_N, v)$ and an aggregate component $S_G = (z, \lambda_G)$ where λ_G is a measure over x_G. The corresponding union-wide monetary authority state is simply S_G. Denote the monetary authority's policy decision for money as $\bar{\mu}(x_G, S_G)$ and use similar notation for transfers and nominal interest rates.

Finally, the *household state* has a household-specific component, a country-specific component, and a aggregate component. The household-specific component is the normalized level of that household's money $m_H = M_H/\bar{M}_{-1}$. The country-specific component $x_H = (m, p_N, v, \mu)$ consists of the normalized money balances for the country as a whole $m = M/\bar{M}_{-1}$, the normalized price of nontraded goods $p_N = P_N/\bar{M}_{-1}$, the idiosyncratic shocks v, and the country-specific growth rate of money μ. The aggregate component $S_H = (z, \lambda_H)$ consists of the aggregate shock z and a measure λ_H over the country-specific components (m, p_N, v, μ) for all countries. Thus, the household state is (m_H, x_H, S_H). Denote the household decision rule for the consumption of the traded good C_T as $C_T(m_H, x_H, S_H)$ and use similar notation for other household choices. The *flexible price firm state* is (x_H, S_H). Denote the rule for normalized traded goods prices as $\bar{p}_T(x_H, S_H)$. Note for later use that the marginal measure of λ_H over x_G is λ_G and the marginal measure of either λ_H or λ_G over x_F is λ_F. We will use these properties repeatedly below.

With this notation in hand we can set up the consumer's problem as follows.

$$\tilde{V}(m_H, x_H, S_H) = \max_{C_T, C_N, L, m'_H} U(C_T, C_N, L) + \beta \sum_s h(s') V(m'_H, x'_H, S'_H) \tag{77}$$

subject to the cash-in-advance constraint

$$\bar{p}_T(x_H, S_H) C_T \leq m_H$$

and the budget constraint

$$\bar{p}_T(x_H, S_H) C_T + p_N C_N + \gamma m'_H \leq m_H + \bar{w}(x_H, S_H) L + [\mu(x_G, S_G) - 1] m + \pi(x_H, S_H)$$

where m, m_H, p_N and μ are in the state and the aggregate money growth rate γ defined as $\bar{M}/\bar{M}_{-1}$

$$\gamma = \int [\mu(x_G, S_G) m] \, d\lambda_G$$

where we are using the feature that λ_G is the marginal of λ_H. This problem is also subject to the law of motion for aggregate states. These laws of motions are determined by applying the relevant decision rules to the current state in the obvious way. For example, consider the new normalized money holdings for a country:

$$(78) \quad m' = \frac{M}{\bar{M}} = \mu \frac{M_{-1}}{\bar{M}_{-1}} \frac{\bar{M}_{-1}}{\bar{M}} = \frac{\bar{\mu}(x_G, S_G)m}{\gamma(S_H)}$$

Likewise the new normalized price of nontraded goods

$$(79) \quad p'_N = \bar{p}_N(x'_F, S'_F)$$

where $x'_F = (m', v'_1)$ and $S'_F = (z'_1, \lambda'_F)$. Since S_F and S_G are the marginal distributions of S_H, the evolution of S_H implies the evolution of S_F and S_G.

In order to set up the problem confronting a monetary authority's, it is convenient to begin by defining a continuation competitive equilibrium under both flexible and fixed exchange rates for some aggregate state $S_G = (z, \lambda_G)$ from an arbitrary choice of money growth today $\mu(x_G)$ for some given monetary authority policy $\bar{\mu}(\cdot, \cdot)$ from tomorrow on.

For an arbitrary choice of money growth today $\mu(x_G, S_G)$ and some given monetary authority policy $\bar{\mu}(\cdot, \cdot)$ from tomorrow on, a *continuation competitive equilibrium under flexible exchange rates* consists of sticky price decision rules $\bar{p}_N(x_F, S_F)$, households decision rules $C_N(m_H, x_H, S_H)$, $C_T(m_H, x_H, S_H)$, $L(m_H, x_H, S_H)$, $m'_H(m_H, x_H, S_H)$, and value function $V(m_H, x_H, S_H)$, price rules $\bar{w}(x_H, S_H)$ and $\bar{p}_T(x_H, S_H)$, profit rules $\pi(x_H, S_H)$, such that i) in the current period and all future periods the flexible price firm and the household decision rules are optimal in that the flexible price firms' price rule satisfies

$$(80) \quad \bar{p}_T(x_H, S_H) = \bar{w}(x_H, S_H)$$

and the household decision rules are optimal for problem (77) and the value function $V$and the profit rule satisfies

$$(81) \quad \pi(x_H, S_H) = \left(p_N - \frac{\bar{w}(x_H, S_H)}{A(s_2)} \right) C_N(m, x_H, S_H)$$

ii) the sticky price firms' price rule satisfies

$$(82) \quad \bar{p}_N(x_F, S_F) = \theta(s_1) \frac{\sum_{s_2} h^2(s_2) U_N(m, x_H, S_H)) C_N(m, x_H, S_H) \bar{w}(x_H, S_H)/A(s_2)}{\sum_{s_2} h^2(s_2) U_N(m, x_H, S_H)) C_N(m, x_H, S_H)}$$

where (x_H, S_H) are induced from (x_F, S_F) from $\bar{\mu}$, iii) the market clearing conditions hold, $C_N(m, x_H, S_H) = A(s_2)L_N(x_H, S_H)$, $C_T(m, x_H, S_H) = L_T(x_H, S_H)$, $L(m, x_H, S_H) = L_N(x_H, S_H) + L_T(x_H, S_H)$, as well as money market clearing in the current period

$$(83) \quad m'_H(m, x_H, S_H) = \frac{\mu(x_G, S_G)m}{\int \mu(x_G, S_G)m d\lambda_G}$$

where x_H is induced from x_G by $\mu(x_G, S_G)$ and money market clearing in all future periods

$$(84) \quad m'_H(m, x_H, S_H) = \frac{\bar{\mu}(x_G, S_G)m}{\int \bar{\mu}(x_G, S_G)m d\lambda_G}$$

where x_H is induced from x_G by $\bar{\mu}(x_G, S_G)$

For an arbitrary choice of money growth today for all countries $\mu(x_G, S_G)$ and some given monetary authority policy $\bar{\mu}(\cdot, \cdot)$ from tomorrow on, a *continuation competitive equilibrium under fixed exchange rates* is a continuation competitive equilibrium under flexible exchange rates that satisfies the following additional restriction

$$(85) \quad \bar{p}_T(x_H, S_H) = \bar{p}_T(S_H) \text{ for all } x_H, S_H$$

We can now use the notion of a continuation competitive equilibrium to define a Markov equilibrium. A *Markov equilibrium with flexible exchange rates* is a continuation competitive equilibrium such that the policy chosen today by the monetary authority coincides with the rule chosen by future monetary authorities in that

$$\mu(x_G, S_G) = \bar{\mu}(x_G, S_G)$$

and, for all S_G the policy maximizes

$$(86) \quad \int \left[U(C_T, C_N, L) + \beta \sum_s h(s') V(m'_H, x'_F, S'_F) \right] d\lambda_G$$

where $C_T = C_T(m, x_H, S_H)$, $C_N = C_N(m, x_H, S_H)$, $L = L(m, x_H, S_H)$, and $m'_H = m'_H(m, x_H, S_H)$ and the monetary authority takes into consideration that its policy influences the future history of households according to $x_H = (x_G, \bar{\mu}(x_G, S_G))$.

(Remark: don't need to impose that sticky prices are optimal in the first period, this is implied by (82) as μ today $= \bar{\mu}$).

A *Markov equilibrium with fixed exchange rates* is a Markov equilibrium with flexible exchange rates with the additional restriction (85).

A. Characterizing Markov Equilibrium

We begin with a simple lemma that characterizes the conditions that define a continuation competitive equilibrium.

Lemma 5. Given a monetary authority policy $\bar{\mu}(\cdot,\cdot)$ from tomorrow on, households decision rules and price functions for traded and nontraded goods can be part of a continuation competitive equilibrium under flexible exchange rates iff there exists a function $\mu(\cdot, S_G)$ such that the following conditions hold. First, the consumer's first order conditions and cash-in-advance constraint are satisfied

$$\frac{U_N(m, x_H, S_H)}{p_N} = -\frac{U_L(m, x_H, S_H)}{p_T(x_H, S_H)} \tag{87}$$

$$\frac{U_T(m, x_H, S_H)}{p_T(x_H, S_H)} \geq -\frac{U_L(m, x_H, S_H)}{p_T(x_H, S_H)} \tag{88}$$

$$p_T(x_H, S_H) C_T(m, x_H, S_H) \leq m \tag{89}$$

where if (89) is a strict inequality then (88) holds as an equality, and

$$\gamma \frac{-U_L(m, x_H, S_H)}{p_T(x_H, S_H)} = \beta \sum_{s'} h(s') \frac{U_T(m'_H, x'_H, S'_H)}{p_T(x'_H, S'_H)} \tag{90}$$

holds both in the current period in which policy is set by $\mu(x_G, S_G)$ and m'_H, x'_H, and S'_H are induced from $\mu(x_G, S_G)$ and in similar fashion for a future period in which policy is set by $\bar{\mu}(x_G, S_G)$. Second, a version of the sticky price first order condition holds with $\bar{p}_T(x_H, S_H)$ replacing $\bar{w}(x_T, S_H)$ in (82). Finally, the following market clearing condition holds

$$L(m, x_H, S_H) = C_T(m, x_H, S_H) + \frac{C_N(m, x_H, S_H)}{A(s_2)} \tag{91}$$

Proof. First notice that these conditions are necessary for a continuation competitive equilibrium. In fact, (87)-(90) are the necessary first order conditions for the households problem (77) using (80) to substitute for the wage. Condition (91) is implied by the market clearing conditions for the consumption goods and labor and finally the modified version of (82) follows from (82) and (80).

Conversely, suppose that conditions (87)-(91) are satisfied. Conditions (87)-(90) imply that C_N, C_T, L, m'_H are optimal for the household problem given the policy rule μ and $\bar{\mu}$ defining $\bar{w}(x_H, S_H) = \bar{p}_T(x_H, S_H)$ so that condition i) in the definition of a continuation

competitive equilibrium is met. Condition ii) in the definition of a continuation competitive equilibrium is met by letting $\bar{w}(x_H, S_H) = \bar{p}_T(x_H, S_H)$ and substituting it into the modified version of (82). Finally, the market clearing conditions for consumption goods and labor are implied by (91) and for money holdings by the fact that we impose $m'_H = \mu(x_G, S_G)m/\gamma$ in (90). *Q.E.D.*

We turn now to rewriting the problem faced by the monetary authority by substituting out the decision rules and instead using the first order conditions and market clearing conditions that characterize them. We will use this rewritten problem to characterize the policy of the monetary authority.

Combining Lemma 5 and the definition of a Markov equilibrium immediately gives the following result: A continuation competitive equilibrium with flexible exchange rates is a Markov equilibrium if and only if i) $\mu(x_G, S_G) = \bar{\mu}(x_G, S_G)$ for all (x_G, S_G), and ii) for all S_G the policy rule $\bar{\mu}(\cdot, S_G)$ solves

$$W(S_G) = \max_{p_T, C_T, C_N, L, \mu} \int \left[U\left(C_T(x_G), C_N(x_G), L(x_G)\right)\right] d\lambda_G + \beta \sum_s h(s') W\left(S'_G\right) \tag{92}$$

subject to

$$\frac{U_N(x_G)}{p_N} = -\frac{U_L(x_G)}{p_T(x_G)} \tag{93}$$

$$\frac{U_T(x_G)}{p_T(x_G)} \geq -\frac{U_L(x_G)}{p_T(x_G)} \tag{94}$$

$$p_T(x_G) C_T(x_G) \leq m \tag{95}$$

where if (89) is a strict inequality then (88) holds as an equality, and

$$\gamma \frac{-U_L(x_G)}{p_T(x_G)} = \beta \sum_{s'} h(s') \frac{U_T(m'_H, x'_H, S'_H)}{p_T(x'_H, S'_H)} \tag{96}$$

$$L(x_G) = C_T\left(x_G\right) + \frac{C_N(x_G)}{A(s_2)} \tag{97}$$

where $\gamma = \int \left[\mu(x_G, S_G)m\right] d\lambda_G$, $m'_H = m' = \mu(x_G)m/\gamma$ and the continuation histories are induced by μ, $\bar{p}_N$, and $\bar{\mu}$, in that $p'_N = \bar{p}_N\left(\nu_1, m', S'_F\right)$ and $x'_H = \left(\nu, m', p'_N, \bar{\mu}\left(x'_G, S'_G\right)\right)$ where $m' = \mu(x_G)m/\gamma$ and iii) W is the fixed point of (92).

Likewise, a continuation competitive equilibrium with fixed exchange rates is a Markov equilibrium if and only if the above conditions i) - iii) hold where the constraints on (92) also include

(98) $p_T(x_G) = p_T$ for all x_G.

We turn now to simplifying the constraints in the Markov problem using our functional form. Specifically, under our preference specification (38) the constraints (93)–(96) can be simplified to

(99) $$C_N(x_G) = \frac{1-\alpha}{b}\frac{p_T(x_G)}{p_N}$$

(100) $$C_T(x_G) = \min\left\{\frac{m}{p_T(x_G)}, \frac{\alpha}{b}\right\}$$

(101) $$\gamma\frac{b}{p_T(x_G)} = \beta\sum_{s'} h(s')\frac{\alpha}{p_T(x'_H, S'_H)C_T(m'_H, x'_H, S'_H)}$$

where the continuation histories m'_H, x'_H, and S'_H are induced by the sticky price firm decision rules $\bar{p}_N$ and the monetary policy rule $\bar{\mu}$. Likewise, the sticky price firm's rule can be simplified to

(102) $$\bar{p}_N(x_F, S_F) = \theta(s_1)\sum_{s_2} h^2(s_2)\bar{p}_T(x_H, S_H)/A(s_2).$$

Thus, under (38) the policy in a Markov equilibrium under flexible exchange rates maximizes (92) subject to (97) and (99)–(101) and while the policy in a Markov equilibrium under fixed exchange rates maximizes (92) subject to (97), (98) and (99)–(101).

We turn now to showing that under our preference specification the analysis of the Markov equilibrium can be greatly simplified. Consider first the equilibrium with fixed exchange rates

Lemma 6. Under the preference specification (38) if the markup is strictly positive in all states in that $\theta(s_1) > 1$ for all s_1 then in any Markov equilibrium in a monetary union, given any initial distribution of money at the beginning of the period then end of period money holdings are concentrated on a single point.

There are two ideas underlying this lemma. The first is that if two agents have differing money holdings, say $m_1 < m_2$,.and the cash-in-advance constraint binds in at least

one state in the next period then these agents experience different consumption levels of traded goods in at least one state, and hence differing levels of expected marginal utility of traded goods consumption. But, for each of these agents the first order conditions imply that the marginal disutility of labor must be equated to the expected marginal utility of traded goods consumption. Since under (38) the marginal disutility of labor of each agent is equal so must be the expected marginal utility of traded goods consumption, which is a contradiction. The second idea is that in equilibrium if the markup is positive in some state the monetary authority will continue to increase the price of traded goods until the benefit of surprise inflation is balanced against some cost from this inflation. Since the cost from surprise inflation only occurs when the cash-in-advance constraint is binding, then we know that the cash-in-advance constraint must bind whenever the markup is positive. Combining these two ideas gives Lemma 6.

In light of Lemma 6, if we choose the date 0 initial nominal money holdings of all countries to be equal then we know these money holdings will continue to be equal over time. Lemma 6, which is proven in the Appendix, greatly simplifies the characterization of the Markov equilibrium in a monetary union. In fact, in light of Lemma 6, we can rewrite problem (92) for (38) using (99)-(101) as

$$
\begin{aligned}
(103)\quad W(S_G) \;=\; & \max_{p_T} \int \left[\alpha \log\left(\min\left\{\frac{m}{p_T}, \frac{\alpha}{b}\right\}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{p_T}{p_N}\right)\right] d\lambda_G \\
& -b\int \left[\min\left\{\frac{m}{p_T}, \frac{\alpha}{b}\right\} + \frac{1-\alpha}{bA(s_2)}\frac{p_T}{p_N}\right] d\lambda_G + \beta \sum_{s} h(s') W\left(S'_G\right)
\end{aligned}
$$

where S'_G is such that $m = 1$ for all countries and the distribution over p_N next period is induced by $\bar{p}_N$ in (102) starting from a degenerate money holding distribution on $m = 1$. Notice that (103) is a simple static problem.

A partial characterization of the Markov equilibria under the two regimes is provided in Lemma A2 and A3 in Appendix. Here we consider the simple case in which productivity shocks are constant across countries so that

$$(104)\quad A(s_2) = 1 \text{ for all } s_2$$

and the markups lie in the following range

$$(105)\quad 1 < \theta(s_1) < \frac{1-\alpha}{1-2\alpha} \text{ for all } s_1.$$

Note that $\theta(s_1) > 1$ simply implies that there are monopoly distortions in each state and $\theta(s_1) < (1-\alpha)/(1-2\alpha)$ guarantees that the monopoly distortions are sufficiently small so that a Markov equilibrium exists. (At an intuitive level, if $\theta(s_1) > (1-\alpha)/(1-2\alpha)$ then the gains to the government of inflating in order to reduce the distortion ex post are sufficiently high that no matter what the level of p_N the government will always have an incentive to increase the inflation rate a bit, so that no fixed point exists.) Under (104), and (105) the formulasLemmas A2 and A3 greatly simplify and we can obtain simple closed form solutions for the equilibrium outcome under both regimes.

Consider first the flexible exchange rates regime. The Markov equilibrium outcome with flexible exchange rates is such that the ratio $p_T(s^t)/m(s^{t-1})$ denoted $q_T(s_t)$ only depends on s_t and it is given by $q_N(s_1) = \theta(s_1)q_T(s_1)$, the ratio $p_N(s^{t-1}, s_{1t})/m(s^{t-1})$ denoted $q_N(s_{1t})$ only depends on s_{1t} and is given by $q_T(s_1) = b/\left[(1-\alpha)/\theta(s_1) - (1-2\alpha)\right]$. Furthermore the cash-in-advance constraint always holds with equality along the equilibrium path and consumption of traded and nontraded goods are given by

$$(106)\quad C_T(s) = \frac{(1-\alpha)(1/\theta(s_1) - 1) + \alpha}{b}$$

and

$$(107)\quad C_N(s) = \frac{1}{\theta(s_1)}\frac{(1-\alpha)}{b}$$

and the formula for $L(s)$ follows from (91).

In a monetary union, further assuming that all agents start with the same holdings of money, we can explicitly solve for the equilibrium value for the price of the nontraded goods $p_N(s_1) = \theta(s_1)p_T(z_1)$ where $\bar{\eta}(z_1) = \sum_{\nu_1} g(\nu_1)/\theta(z_1, \nu_1)$, and for the price of the traded goods: $p_T(z_1) = b/[(1-\alpha)\,\bar{\eta}(z_1) - (1-2\alpha)]$. Furthermore, under the stated conditions, the cash-in-advance constraint always binds and the formulas for traded and nontraded goods consumption are given by

$$(108)\quad C_T(z_1) = \frac{(1-\alpha)\,(\bar{\eta}(z_1) - 1) + \alpha}{b}$$

and

$$(109)\quad C_N(s_1) = \frac{1}{\theta(s_1)}\frac{(1-\alpha)}{b}$$

and the formula for $L(s)$ follows from (91).

Comparing (106)–(107) and (108)–(109), notice that under the stated assumptions, the Markov equilibrium outcome under flexible exchange rates differs from the one in a monetary union only in terms of the consumption of the traded good and the labor needed to produce it. In particular, from (106) and (108) it follows that the expected consumption of traded goods is constant in both regimes but the traded goods consumption is more volatile under flexible exchange rates. Hence, because of concavity of preferences over traded consumption goods, the ex-ante welfare associated with the Markov equilibrium in a monetary union is higher than under flexible exchange rates. The next proposition formalizes this argument.

B. Comparing Utility Under Flexible and Fixed Exchange Rates

For the next proposition we will make four assumptions: (38), (104), (105), and at date zero all agents start with the same nominal money balances.

Proposition 5. Under our four assumptions the ex ante utility in the Markov equilibrium for a monetary union is strictly higher than the ex ante utility in the Markov equilibrium with flexible exchange rates.

Proof. Plugging the formulas for tradable and non-tradable consumption under the two regimes, (106)–(107) and (108)–(109), in the objective function and simplifying gives that the difference in value for a given initial aggregate state z_1 is given by

$$U^{MU}(z_1) - U^{MA}(z_1) = K(\bar{\eta}(z_1)) - \sum_{\nu_1} K(\eta(z_1,\nu_1))g^1(\nu_1)$$

where $\eta(s_1) \equiv 1/\theta(s_1)$ and $K(\eta;\alpha) \equiv \alpha\log\left((1-\alpha)(\eta-1)+\alpha\right)$. Note that the function $K(\eta;\alpha)$ is concave since

$$K^{''}(\eta;\alpha) = -\alpha\left(\frac{1}{(1-\alpha)(\eta-1)+\alpha}\right)^2(1-\alpha) < 0$$

The concavity of K implies that for all z_1

$$U^{MU}(z_1) - U^{MA}(z_1) = K(\bar{\eta}(z_1)) - \sum_{\nu_1} K(\eta(z_1,\nu_1))g^1(\nu_1) \geq 0$$

with strict inequality if there is variability in the idiosyncratic shock ν_1. *Q.E.D.*

Consider now how money growth and inflation compare in the two regimes. Under (104) the expression for the money growth rate reduces to $\mu^A(s) = \Delta(\eta(s_1))$ under flexible exchange rates, and to $\gamma(z) = \Delta(\bar{\eta}(z_1))$ in a monetary union where

$$\Delta(\eta) = \frac{\beta\alpha}{[(1-\alpha)\eta - (1-2\alpha)]}.$$

Since Δ is a convex function of $\eta = 1/\theta$ the expected value of money growth rate is higher under flexible exchange rates than in the union.

Consider next the inflation rates in the tradable and non-tradable sector. Under flexible exchange rates these inflation rates are given by

$$\pi_T^M(s,s') = \Delta(\eta(s_1)) \text{ and } \pi_N^M(s,s') = \frac{\theta(s_1')}{\theta(s_1)}\Delta(\eta(s_1))$$

and in the union they are given by

$$\pi_T^U(s,s') = \Delta(\bar{\eta}(z_1)) \text{ and } \pi_N^U(s,s') = \frac{\theta(s_1')}{\theta(s_1)}\Delta(\bar{\eta}(z_1))$$

The convexity of Δ implies that in a monetary union inflation is not only less volatile than under flexible exchange rates but also is lower on average. This lower and less volatile inflation rate is beneficial because it results in distortions in the consumption the tradable good that are on average lower and less volatile.

So far we have abstracted from productivity shocks. In general, our equilibrium model implies a tradeoff between markup shocks and productivity shocks similar to that present in the reduced form model. Recall, that in that model, when the idiosyncratic component of the markup shocks are sufficiently volatile relative to the idiosyncratic component of the productivity shocks then a monetary union is preferred to flexible exchange rates. Here we state a similar result for the general equilibrium model.

Corollary. Under the assumptions in Proposition 5, the ex ante utility in the Markov equilibrium for a monetary union is strictly higher than the ex ante utility in the Markov equilibrium with flexible exchange rates as long as the variability of productivity shocks is sufficiently small.

Note the corollary immediately follows from Proposition 5 and continuity of the equilibrium values in the parameters of the model. Thus, when the monetary authority cannot

commit to its policy, a group of ex-ante homogeneous countries can gain from joining a union when the variability of ex-ante idiosyncratic shock is large relative to the variability of ex-post idiosyncratic shock.

We illustrate this corollary in Figure 1. In this figure we plot the ex-ante value of the Markov equilibrium under the two regimes as we vary the relative volatility of the idiosyncratic component of the ex-post productivity shock in the non-tradable sector. We parameterize the model by considering a simple case with no aggregate shocks: $\theta(s_1) = \nu_1$ and $A(s_2) = \nu_2$ where $v_1 \in \{1.1, 1.2\}$ with $g^1(\nu_1) = 1/2$ and $\nu_2 \in \{1-\varepsilon, 1+\varepsilon\}$ with $g^2(\nu_2) = 1/2$ and $\varepsilon \geq 0$ is a parameter that we let vary. As shown in Proposition 5, when $\varepsilon = 0$ the ex-ante value of a Markov equilibrium for a country in a monetary union is higher than the what the same country can attain under flexible exchange rates. As ε increases and the variability of the idiosyncratic component of the ex-post productivity shocks increases the losses of monetary independence gets larger: the country cannot accommodate the idiosyncratic shocks in the tradable sector and cannot increase production of non-traded good when its productivity is high.

To better understand the mechanics of the model we explore the decision rules from a numerical example under (38). For expositional reasons, we consider an example without aggregate shocks. We consider a utility function of the form (38) with $\alpha = 1/2$ and $b = 1$. For the idiosyncratic shocks we let the markup shock be such that θ is uniform on $[1.1, 1.2]$ and let the productivity shock A be such that $1/A$ is uniform on $[.95, 1.05]$.

Figure 2 shows the price of the tradable good - normalized by the country nominal money balance - as a function of the country specific productivity shocks. In a Markov equilibrium with flexible exchange rates p_T not only varies with the country-specific productivity shock but it also moves with the markup shock. This is because after the realization of a high markup shock the monetary authority is more tempted to generate ex-post surprise inflation to reduce the monopoly distortion in the non-traded sector. In equilibrium this temptation is frustrated by the behavior of the sticky price firm and it results only in excessive inflation (level) and volatility.

This can be seen from Figure 3 that displays the behavior of the labor wedge in the tradable sector. Without commitment, the cash in advance constraint is binding, introducing

a wedge between the marginal rate of substitution between labor and consumption of traded goods and the marginal rate of transformation between these same goods. This wedge is increasing in both the country-specific productivity shock $A(\nu_1)$ and in particular in the markup shock.

In the Markov equilibrium for a monetary union instead, the price of the tradable good only reacts to union-wide variation in the markup shock. Thus, in our example with no aggregate risk, p_T is constant and its level is lower than the expected value of p_T under flexible exchange rates. This results in a distortion due to a positive multiplier on the cash-in-advance constraint that is less volatile (no volatility if there are no aggregate shocks) relative to the flexible exchange rates regime, as shown in Figure 5. (Note that the expected value of the wedge is equal, but the average money growth rate - and hence average inflation from the formulas in Lemma A2 and A3 - is lower in the monetary union than with flexible exchange rates). Because of concavity of preferences, this has a positive effect on welfare.

However, as we emphasized for the Ramsey outcome, a standard Friedman-Mundell effect is operating: the wedge in the nontradable sector is more volatile in the union relative to the flexible exchange rate regime (again same expected value, wedges are linear). This is illustrated in Figure 4. Because of concavity of preferences, the higher volatility in the labor wedge in the nontraded sector contribute to lower the utility for a monetary union relative to a flexible exchange rate regime.

Thus there is a trade off: in a Markov equilibrium a monetary union can attain a higher ex-ante welfare than a flexible exchange rate regime depending on relative importance on the country-specific component of the volatility of the ex-ante markup shock and ex-post productivity shock as we illustrated in Figure 1.

5. Conclusion

We have presented a new argument for why forming a monetary union among symmetric countries may be desirable.

References

Albanesi, S., Chari, V. V., & Christiano, L. J. (2003). Expectation traps and monetary policy. *The Review of Economic Studies*, 70(4), 715-741.

Alesina, A., & Barro, R. J. (2002). Currency unions. *The Quarterly Journal of Economics*, 117(2), 409-436.

Alesina, A., Barro, R. J., & Tenreyro, S. (2003). Optimal currency areas. In NBER Macroeconomics Annual 2002, Volume 17 (pp. 301-356). MIT Press.

Barro, R. J., & Gordon, D. B. (1983). Rules, discretion and reputation in a model of monetary policy. *Journal of monetary economics*, 12(1), 101-121.

Dixit, A. (2000). A repeated game model of monetary union. *The Economic Journal*, 110(466), 759-780.

DeGrauwe

Dellas, H., & Tavlas, G. S. (2009). An optimum-currency-area odyssey. *Journal of International Money and Finance*, 28(7), 1117-1137.

Farhi, E., & Werning, I. (2013). Fiscal Unions. mimeo MIT

Fuchs, W., & Lippi, F. (2006). Monetary union with voluntary participation. *The Review of Economic Studies*, 73(2), 437-457.

Friedman, Milton (1953). The case for flexible exchange rates in *Essays in Positive Economics*, University of Chicago Press.

Friedman, Milton (1971). Monetary policy in developing economies.

Gali, J., & Monacelli, T. (2005). Monetary policy and exchange rate volatility in a small open economy. *The Review of Economic Studies*, 72(3), 707-734.

Gali, J., & Monacelli, T. (2008). Optimal monetary and fiscal policy in a currency union. *Journal of International Economics*, 76(1), 116-132.

Giavazzi, F., & Pagano, M. (1988). The advantage of tying one's hands: EMS discipline and central bank credibility. *European Economic Review*, 32(5), 1055-1075.

Kydland, F. E., & Prescott, E. C. (1977). Rules rather than discretion: The inconsistency of optimal plans. *The Journal of Political Economy*, 473-491.

Meade, J. E. (1957). The balance-of-payments problems of a European free-trade area. *The Economic Journal*, 67(267), 379-396.

Mundell, R. A. (1961). A theory of optimum currency areas. *The American Economic Review*, 51(4), 657-665.

Nicolini, J. P. (1998). More on the time consistency of monetary policy. *Journal of Monetary Economics*, 41(2), 333-350.

Obstfeld, M., & Rogoff, K. (1995). Exchange Rate Dynamics Redux. *The Journal of Political Economy*, 103(3), 624-660.

Svensson, L. (1983). Money and asset prices in cash-in-advance economies. *The Journal of Political Economy,* 93, 914-944.

6. Appendix A: Derivations for the Reduced Form Model

A. Under Commitment

$$\max_{p(s_1),\pi(z)} -\frac{1}{2}\sum_s h(s)\left[(\theta(s_1)-A(s)+p(s_1)-\pi(s))^2+\kappa\pi(s)^2\right]$$

subject to

$$(110)\quad p(s_1)=\sum_s g(s|s_1)\pi(s)$$

Consider first the problem of an individual country under flexible exchange rates. Letting $\lambda(s_1)$ be the multiplier on the constraint (110), the foc for the problem are:

$$(111)\quad \pi(s):0=(\theta(s_1)-A(s)+p(s_1)-\pi(s))-\kappa\pi(s)-\lambda(s_1)$$

$$(112)\quad p(s_1):0=-\sum_s g(s|s_1)\left(\theta(s_1)-A(s)+p(s_1)-\pi(s)\right)+\lambda(s_1)$$

Using the constraint (110) in the (112) gives

$$(113)\quad \lambda(s_1)=\sum_s g(s|s_1)\left(\theta(s_1)-A(s)\right)=\theta(s_1)-E(A|s_1)$$

Solve for $\pi(s)$ from (111)

$$\theta(s_1)-A(s)+p(s_1)-(1+\kappa)\pi(s)=\lambda(s_1)$$

$$\pi(s)=\frac{1}{1+\kappa}\left[\theta(s_1)-A(s)+p(s_1)-\lambda(s_1)\right]$$

Now substitute for $\lambda(s_1)$ using (113)

$$(114)\quad \pi(s)=\frac{1}{1+\kappa}\left[\theta(s_1)-A(s)+p(s_1)-\theta(s_1)+E(A|s_1)\right].$$

this explains why it is not optimal to respond to θ shocks

$$\pi(s) = \frac{1}{1+\kappa}\left[E(A|s_1) - A(s) + E\pi\right]$$

to find $E\pi$, take expectation of this equation to get

$$E\pi = 0$$

so the Ramsey policy is

$$\pi(s) = \frac{1}{1+\kappa}\left[E(A|s_1) - A(s)\right] \tag{115}$$

and, of course,

$$p(s_1) = E(\pi(s)|s_1) = 0 \tag{116}$$

so under flexible exchange rates

$$\max_{p(s_1),\pi(z)} -\frac{1}{2}\sum_s h(s)\left(\theta(s_1) - A(s) - \frac{1}{1+\kappa}\left[E(A|s_1) - A(s)\right]\right)^2 + \kappa\left[\frac{1}{1+\kappa}\left[E(A|s_1) - A(s)\right]\right]^2$$

Consider now the problem for the union:

$$\max_{p(s_1),\pi(z)} -\frac{1}{2}\sum_{\nu_1} g^1(\nu_1)\sum_s h^2(s_2)\left[\left(\theta(s_1) - A(s_2) + p(z_1) - \pi(z)\right)^2 + \kappa\pi(z)^2\right]$$

subject to

$$p(z_1) = \sum_s f(z_2|z_1)\pi(z) \tag{117}$$

which requires that p only depends on z_1. The foc for the problem are

$$\pi(z) : 0 = \sum_{\nu_1} g^1(\nu_1)\sum_{v_2} g^2(\nu_2)\left[\theta(s_1) - A(z_2,\nu_2) + p(z_1) - \pi(z) - \kappa\pi(z)\right] - \lambda \tag{118}$$

$$\pi(z) : 0 = \sum_{\nu_1} g^1(\nu_1) f^2(z_2)\sum_{v_2} g^2(\nu_2)\left[\theta(s_1) - A(z_2,\nu_2) + p(z_1) - \pi(z) - \kappa\pi(z)\right] - \lambda f^2(z_2) \tag{119}$$

$$p(z_1) : 0 = -\sum_s h(s|z_1)\left(\theta(s_1) - A(s) + p(z_1) - \pi(z)\right) + \lambda \tag{120}$$

Using the constraint (117) in the (120) gives

$$\lambda = \sum_s h(s|z_1)\left(\theta(s_1) - A(s)\right) = \bar{\theta}(z_1) - \bar{A} \tag{121}$$

where $\bar{\theta}(z_1) \equiv \sum_{\nu_1} g^1(\nu_1)\theta(z_1,\nu_1)$. Letting $\bar{A}(z_2) \equiv \sum_{\nu_2} g^2(\nu_2)A(z_2,\nu_2)$, we can solve for $\pi(s)$ from (119)

$$\bar{\theta}(z_1) - \bar{A}(z_2) + p(z_1) - (1+\kappa)\pi(z) = \lambda$$

$$\pi(z) = \frac{1}{1+\kappa}\left[\bar{\theta}(z_1) - \bar{A}(z_2) + p(z_1) - \lambda\right]$$

Now substitute for λ using (121) we have

$$\pi(z) = \frac{1}{1+\kappa}\left[\theta(s_1) - A(s) + p(s_1) - \theta(s_1) + \bar{A}\right]$$

this explains why it is not optimal to respond to θ shocks

$$\pi(z) = \frac{1}{1+\kappa}\left[\bar{A} - \bar{A}(z_2) + E\pi\right]$$

to find $E\pi$, take expectation of this equation to get $E\pi = 0$. Therefore the Ramsey policy in a union is

$$\pi(z) = \frac{1}{1+\kappa}\left[\bar{A} - \bar{A}(z_2)\right] \tag{122}$$

$$p(z_1) = 0 \tag{123}$$

Proof of Proposition 1. The welfare associated to the Ramsey policy for a country in isolation is:

$$\begin{aligned} V^R &= -\frac{1}{2}\sum_s h(s)\left[\left(\theta(s_1) - A(s_2) - \frac{1}{1+\kappa}\left[\bar{A} - A(s_2)\right]\right)^2 + \kappa\left(\frac{1}{1+\kappa}\left[\bar{A} - A(s_2)\right]\right)^2\right] \\ &= -\frac{1}{2}\left[E\theta^2 - 2\bar{\theta}\bar{A} + \frac{1}{1+\kappa}\bar{A}^2 + \left(\frac{\kappa}{(1+\kappa)}\right)EA^2\right] \end{aligned}$$

which reduces to

$$V^R = -\frac{1}{2}\left[E\theta^2 + \left(\frac{\kappa}{(1+\kappa)}\right)EA^2\right]$$

when $\bar{A} = 0$. For the union the value associated to the Ramsey policy is:

$$\begin{aligned} V^{R,U} &= -\frac{1}{2}\sum_s h(s)\left[\left(\theta(s) - A(s) - \frac{1}{1+\kappa}\left[\bar{A} - \bar{A}(z_2)\right]\right)^2 + \kappa\left(\frac{1}{1+\kappa}\left[\bar{A} - \bar{A}(z_2)\right]\right)^2\right] \\ &= -\frac{1}{2}\left[E\theta^2 - 2\bar{\theta}\bar{A} + \left(\frac{1}{1+\kappa}\right)\left[\bar{A}^2 - E\bar{A}(z_2)A(s) + (1+\kappa)EA^2\right]\right] \end{aligned}$$

and when $\bar{A} = 0$ it reduces to

$$V^{R,U} - \frac{1}{2}\left[E\theta^2 - \left(\frac{1}{1+\kappa}\right)\left[E\bar{A}(z_2)A(s) - (1+\kappa)EA^2\right]\right]$$

Thus the difference in the ex-ante value is:

$$\begin{aligned} V^R - V^{R,U} &= -\frac{1}{2}\frac{1}{1+\kappa}\left[\left[\bar{A}^2 + \kappa EA^2\right] - \left[\bar{A}^2 - E\bar{A}(z_2)A(s) + (1+\kappa)EA^2\right]\right] \\ &= \frac{1}{2}\frac{\sum_{z_2} f^2(z_2)var(A(s_2)|z_2)}{1+\kappa} \end{aligned}$$

B. Without Commitment

In a flexible exchange rates regime, the best response to any price set by private agents and shocks s can be found solving

$$U^{BR}(p,s) = \max_{\pi} -\frac{1}{2}\left[\left(\theta(s_1) - A(s_2) + p - \pi\right)^2 + \kappa\pi^2\right]$$

The foc is:

$$\pi : 0 = \left(\theta(s_1) - A(s_2) + p - \pi\right) - \kappa\pi$$

Thus

$$\pi^{BR}(p,s) = \frac{\theta(s_1) - A(s_2) + p}{1+\kappa} \tag{124}$$

To find an equilibrium, we must impose the equilibrium condition $p(s_1) = E\left(\pi^{BR}(p,s)|s_1\right)$. Solving for $p(s_1)$ gives

$$p(s_1) = \sum_s h(s|s_1)\frac{\theta(s_1) - A(s_2) + p(s_1)}{1+\kappa} \tag{125}$$

and finally

$$p(s_1) = \frac{\theta(s_1)}{\kappa} - \frac{1}{\kappa}\sum_s h^2(s_2)A(s_2) \tag{126}$$

Substituting back into the best response (124) we can solve for the equilibrium policy:

$$\pi(s) = \frac{\theta(s_1) - \bar{A}}{\kappa} - \frac{1}{1+\kappa}\left[A(s_2) - \bar{A}\right] = \frac{\theta(s_1)}{\kappa} - \frac{A(s_2)}{1+\kappa} \tag{127}$$

where in the last step we used the fact that $\bar{A} = 0$.

Consider now the problem for the union. In any equilibrium it must be that the price set by private agents only depends on the aggregate shock at the beginning of the period, so it is without loss of generality to consider only best response to $p(z_1)$ that do not depend on ν_1. So, given the pre set price $p = p(z_1)$ and the aggregate state $z = (z_1, z_2)$, the union monetary authority solves:

$$U^{BR,U}(p,z) = \max_{\pi} -\frac{1}{2}\sum_{\nu} g(\nu)\left[(\theta(z_1,\nu_1) - A(z_2,\nu_2) + p - \pi)^2 + \kappa\pi^2\right]$$

The foc is

$$0 = \sum_{\nu} g(v)\left(\theta(z_1,\nu_1) - A(z_2,\nu_2) + p - \pi\right) - \kappa\pi$$

Solving for π we obtain the best response function:

$$\pi^{BR,U}(p,z) = \frac{\bar{\theta}(z_1) - \bar{A}(z_2) + p}{1+\kappa} \tag{128}$$

Using (128) into the equilibrium condition $p(z_1) = E\left(\pi^{BR}(p(z_1),z)|z_1\right)$ we can be solve for the equilibrium $p(z_1)$ to get

$$p(z_1) = \frac{\bar{\theta}(z_1) - \bar{A}}{\kappa} = \frac{\bar{\theta}(z_1)}{\kappa} \tag{129}$$

Substituting back into (128) we obtain

$$\pi(s) = \frac{\bar{\theta}(z_1) - \bar{A}}{\kappa} - \frac{1}{1+\kappa}\left[\bar{A}(z_2) - \bar{A}\right] = \frac{\bar{\theta}(z_1)}{\kappa} - \frac{\bar{A}(z_2)}{1+\kappa} \tag{130}$$

Proof of Proposition 2.

Consider first the Markov equilibrium for a country alone. Substituting the decision rules in the objective function we obtain:

$$V^M = -\frac{1}{2}\left[\left(\frac{1+\kappa}{\kappa}\right)E\theta^2 - 2\left(\frac{1+\kappa}{\kappa}\right)\bar{\theta}\bar{A} + \left(\frac{1}{1+\kappa}\right)^2\left[\kappa(1+\kappa)EA^2 + \left(\frac{3\kappa+2\kappa^2+1}{\kappa}\right)\bar{A}^2\right]\right]$$

When $\bar{A} = 0$ this reduces to

$$V^M = -\frac{1}{2}\left[\left(\frac{1+\kappa}{\kappa}\right)E\theta^2 + \left(\frac{\kappa}{1+\kappa}\right)EA^2\right]$$

For a monetary union union, substituting the decision rules in the objective function we obtain:

$$\begin{aligned} V^{M,U} &= -\frac{1}{2}\left[E\theta^2 - 2\frac{1+\kappa}{\kappa}\bar{\theta}\bar{A} + EA^2 - \frac{2}{1+\kappa}EA\bar{A}(z_2) + \left(\frac{1}{1+\kappa}\right)E\bar{A}(z_2)^2\right] \\ &\quad -\frac{1}{2}\left[\frac{1}{\kappa}E\bar{\theta}(z_1)^2 + \left(\frac{1}{1+\kappa}\frac{1+2\kappa}{\kappa}\right)\bar{A}^2\right] \end{aligned}$$

When $\bar{A} = 0$ this reduces to

$$V^{M,U} = -\frac{1}{2}\left[E\theta^2 + EA^2 - \frac{2}{1+\kappa}EA\bar{A}(z_2) + \left(\frac{1}{1+\kappa}\right)E\bar{A}(z_2)^2 + \frac{1}{\kappa}E\bar{\theta}(z_1)^2\right]$$

So, we can combine the value of the Markov equilibrium under the two regimes to obtain:

$$V^{M,U} - V^{M} = \frac{1}{2}\frac{\sum_{z_1} f^1(z_1) var\left(\theta(s_1)|z_1\right)}{\kappa} - \frac{1}{2}\frac{\sum_{z_2} f^2(z_2) var(A(s_2)|z_2)}{1+\kappa}$$

7. Appendix B: Proofs for the General Equilibrium Model

A. Derivation of the Ramsey Outcome and Proof of Lemma 3

Here we derive the Ramsey outcome under (38). Consider an even more relaxed version of the relaxed Ramsey problem (61) by dropping (59). Letting $\xi(s^{t-1}, s_{1t})$ be the multiplier associated with (57), dividing the first order condition for $C_N(s^t)$ by that for $L(s^t)$ gives us

$$(131)\quad \frac{1-\alpha}{bC_N(s^t)} = \frac{1}{A(s_{2t})}\left[1 + \xi(s^{t-1}, s_{1t})\frac{1}{\theta(s_{1t})}\right]$$

which can be solved for $C_N(s^t)$ to yield:

$$(132)\quad C_N(s^t) = \frac{A(s_{2t})(1-\alpha)}{b}\frac{1}{1+\xi(s^{t-1}, s_{1t})/\theta(s_{1t})}$$

Clearly the consumption of nontraded goods is given by

$$(133)\quad C_T(s^t) = \frac{\alpha}{b}$$

Then, substituting (132) into (57) for all s^{t-1}, s_1 and solving for $\xi(s^{t-1}, s_1)$ we get

$$(134)\quad \xi(s^{t-1}, s_{1t}) = \theta(s_{1t})\left[\theta(s_{1t}) - 1\right]$$

Thus consumption of nontraded is given by

$$(135)\quad C_N(s^t) = \frac{A(s_{2t})(1-\alpha)}{b}\frac{1}{1+\theta(s_{1t})-1} = \frac{A(s_{2t})}{\theta(s_{1t})}\frac{(1-\alpha)}{b}$$

and obviously

$$(136)\quad L(s^t) = C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}$$

We next show that this allocation can be implemented as a competitive equilibrium, proving Lemma 3 in the text.

Proof of Lemma 3. Consider implementing $\{C_T(s^t), C_N(s^t), L(s^t)\}$ given by (133)–(136) as a competitive equilibrium. We construct the prices so that the cash-in-advance constraint holds with equality holds at the highest level of productivity of the nontraded goods and is slack at all other shocks. (Of course, one could have the cash in advance slack at all shocks and this would shift the prices down for the same money supplies). For all t, s^t, recursively construct prices normalized by the beginning of the period money holdings, $p_T(s^t) = P_T(s^t)/M(s^{t-1})$ and $p_N(s^{t-1}, s_{1t}) = P_N(s^{t-1}, s_{1t})/M(s^{t-1})$ and money growth rate as:

$$(137)\quad p_N(s^{t-1}, s_{1t}) = \min_{s_2}\left\{\frac{b}{\alpha}\frac{\theta(s_{1t})}{A(s_{2t})}\right\} = \frac{b}{\alpha}\frac{\theta(s_{1t})}{\max A(s_{2t})}$$

$$(138)\quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})}p_N(s^{t-1}, s_{1t})$$

$$(139)\quad \frac{M(s^t)}{M(s^{t-1})} = \beta\sum_{s^{t+1}} h(s^{t+1}|s^t)\frac{\alpha}{p_T(s')C_T(s')}\Big/\frac{(1-\alpha)}{C_N(s^t)p_N(s^{t-1}, s_{1t})} = \beta$$

The allocations $\{C_T(s^t), C_N(s^t), L(s^t)\}$ and the process $\{P_T(s^t), P_N(s^{t-1}, s_{1t}), M(s^t), W(s^t)\}$ obtained from (137)–(139) where we let $W(s^t) = P_T(s^t)$ is a competitive equilibrium outcome. First notice that the sufficient conditions for households optimality are satisfied. $W(s^t) = P_T(s^t)$ and (133)gives (43); combining (138), (137), (135) and using $W(s^t) = P_T(s^t)$ gives (42); (139), (133), (135), (138), and (137) imply (44); finally notice that (39) is satisfied by substituting (138) and (133) in the cash-in-advance constraint. Nominal interest rates $\{r_t(s^t)\}$ and state-prices $\{Q_t(s^t)\}$ are given by (45) and (46). The constructed prices satisfy (35) because the allocations satisfy (57). Finally, market clearing follows from the feasibility of the allocations. $Q.E.D.$

We now turn to the Ramsey problem for a monetary union under (38). Consider the following relaxed problem:

$$(140)\quad \max\sum_t\sum_{s^t}\beta^t h(s^t)\left[\alpha\log\left(C_T(s^t)\right) + (1-\alpha)\log\left(C_N\left(s^t\right)\right) - b\left(C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}\right)\right]$$

subject to (57) and

$$(141)\quad C_N(s^t) = C_N(s^{t-1}, s_{1t}, z_{2t}) \text{ for all } \nu_{2t}$$

where the last constraint imposes that $C_N(s^t)$ cannot vary with v_{2t}.and follows from (53).

After substituting the last constraint in the objective function, the first order condition for $C_N(s^{t-1}, s_{1t}, z_{2t})$ can be written as

$$(142)\quad \frac{1-\alpha}{C_N(s^{t-1}, s_{1t}, z_{2t})} = \left(1 + \xi(s^{t-1}, s_{1t})\right) \theta(s_{1t}) \sum_{\nu_2} g^2(\nu_{2t}) \frac{b}{A(s_{2t})}$$

where $\xi(s^{t-1}, s_1)$ is the multiplier on (57). The first order condition for $C_T(s^t)$ simply gives

$$(143)\quad C_T(s^t) = \frac{\alpha}{b}$$

Defining $\bar{X}(z_2) = \sum_{\nu_2} g^2(\nu_2)/A(s_2)$, we can solve (142) for C_N obtaining

$$(144)\quad C_N(s^{t-1}, s_{1t}, z_{2t}) = \frac{1-\alpha}{\left(1 + \xi(s^{t-1}, s_{1t})\right) \theta(s_{1t}) b \bar{X}(z_{2t})}$$

and substituting back into the labor market distortion constraint, (57), we can solve for the multiplier, obtaining:

$$(145)\quad \left(1 + \xi(s^{t-1}, s_{1t})\right) = \sum_{s_2} h^2(s_{2t}) \frac{1/A(s_{2t})}{\bar{X}(z_{2t})}$$

Plugging back the expression for $\xi(s^{t-1}, s_{1t})$ into (144) gives:

$$(146)\quad C_N(s^{t-1}, s_{1t}, z_{2t}) = \frac{1}{\theta(s_{1t})} \frac{1-\alpha}{b} \frac{1}{\bar{X}(z_{2t}) \sum_{\tilde{s}_2} h^2(\tilde{s}_2) \frac{1/A(\tilde{s}_2)}{\bar{X}(\tilde{z}_2)}}$$

and obviously

$$(147)\quad L(s^t) = C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}$$

We now show that the allocations in (143), (146)–(147) can be implemented as a competitive equilibrium under a monetary union. In particular, we construct prices such that the cash-in-advance constraint holds with equality in all states. For all t, s^t, construct prices normalized by the beginning of the period money holdings, $p_T(s^t) = P_T(s^t)/M(s^{t-1})$ and $p_N(s^{t-1}, s_{1t}) = P_N(s^{t-1}, s_{1t})/M(s^{t-1})$ and money growth rate as follows:

$$(148)\quad p_N(s^{t-1}, s_{1t}) = \frac{b}{\alpha} \theta(s_1) \min_{z_2} \left\{\bar{X}(z_2)\right\} \sum_{s_2} h^2(s_2) \frac{1/A(s_2)}{\bar{X}(z_2)}$$

$$(149)\quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})} p_N(s^{t-1}, s_{1t}) = \frac{b}{\alpha} \frac{\min_{z_2} \left\{\bar{X}(z_2)\right\}}{\bar{X}(z_2)}$$

$$(150)\quad \frac{M(s^t)}{M(s^{t-1})} = \beta \frac{\bar{X}}{\bar{X}(z_2)}$$

The allocations $\{C_T(s^t), C_N(s^t), L(s^t)\}$ and the process $\{P_T(s^t), P_N(s^{t-1}, s_{1t}), M(s^t), W(s^t)\}$ obtained from (148)–(150) where we let $W(s^t) = P_T(s^t)$ is a competitive equilibrium outcome in a monetary union. First notice that the sufficient conditions for households optimality are satisfied. $W(s^t) = P_T(s^t)$ and (143)gives (43); combining (149), (148), (146) and using $W(s^t) = P_T(s^t)$ gives (42); (150), (143), (146), (149), and (148) imply (44); finally notice that (39) is satisfied by substituting (149) and (143) in the cash-in-advance constraint. Nominal interest rates $\{r_t(s^t)\}$ and state-prices $\{Q_t(s^t)\}$ are given by (45) and (46). The constructed prices satisfy (35) because the allocations satisfy (57). Finally market clearing follows from the feasibility of the allocations.

B. Proof of Lemma 6

We prove a preliminary lemma that immediately implies Lemma 6.

Lemma A1. i) If at the beginning of period t there is a non-degenerate money holding distribution then the date t cash-in-advance constraint $p_T(z)C_T(m, p_T(z)) \leq m$ has a zero multiplier for all m and all z and ii) If $\theta(s_1) > 1$ for all s_1 then in any continuation Markov equilibrium the multiplier on the cash-in-advance is binding for at least one level of aggregate shocks z and some normalized money holding m in the support of λ_m.

Proof of part i. Suppose that the money holding distribution is not degenerate. Let m_1 and m_2 be two arbitrary points in the support of λ_m (λ_m is the marginal of λ_F for money holdings) with $m_1 < m_2$. From (100) it follows that for all possible aggregate state S'_G tomorrow that the consumption of the agent with the lower money holdings is weakly lower than that of the agent with the higher money holdings and is strictly lower for any state in which the cash-in-advance constraint is strictly binding (in the sense that the multiplier on this constraint is positive). It is convenient to write this inequality as

$$(151)\quad \frac{1}{C_T(m_2, p'_{N2}, \mu'_2, S'_G)p_T(S'_G)} \leq \frac{1}{C_T(m_1, p'_{N1}, \mu'_1, p'_{N2}, s')p_T(S'_G)}.$$

The first order conditions require that an agent be indifferent between working an extra unit today, getting paid $p_T(x_G)$ in normalized terms and using that money to increase consumption of traded goods in all states tomorrow and not doing so. That is, the condition

$$(152)\quad \frac{b}{p_T(x_G)} = \beta \sum_{s'} h(s') \frac{\alpha}{\gamma p_T(x'_H, S'_H) C_T(m'_H, x'_H, S'_H)}$$

must hold. Now, if the cash in advance is binding at some state s', then (151) implies that the right hand side of (152) is strictly higher for the agent with the lower holdings m_1 units than it is for the agent with the higher holdings m_2 but the left hand side of (152) is equal for both agents, which is a contradiction. At an intuitive level, either the agent with the lower money holdings would want to hold more money or the agent with the higher money holdings would want to hold less money, but it cannot be that these agents want to hold differing amounts of money when there is a binding CIA constraint in the next period.

Proof of part ii. Suppose by way of contradiction that in a continuation of a Markov equilibrium for all x_G the multiplier on the CIA is zero for all m in the support of λ_m - say the support is $[\underline{m}, \bar{m}]$ - then it must be that

$$(153)\quad \underline{m} \geq \bar{p}_T(z, \lambda_G)\frac{\alpha}{b} \quad \text{or equivalently} \quad \bar{p}_T(z, \lambda_G) \leq \underline{m}\frac{b}{\alpha}$$

where $\bar{p}_T(z, \lambda_G)$ is the equilibrium price chosen by the monetary authority. That is, for all p_T

$$W\left(z, \lambda_F | \bar{p}_T(z, \lambda_G)\right) \geq W\left(z, \lambda_F | p_T\right)$$

where $W(z, \lambda_G | p_T)$ is the current payoff for the monetary authority if it chooses p_T. Using (100), we can write $C_T(m; p_T) = \min\{m/p_T, \alpha/b\}$ and

$$(154)\quad u^T(m; p_T) \equiv \alpha \log C_T(m; p_T) - bC_T(m; p_T)$$

where $u^T(m; p_T)$ is the component of period utility coming from the consumption of traded goods and the disutility of labor used to make those traded goods. Using (154) and (99), we can write the payoff for the monetary authority as

$$W(z, \lambda_G | p_T) = \int u^T(m; p_T) d\lambda_G(m, p_N, \nu) + \int \left[(1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{p_T}{p_N}\right) - (1-\alpha)\frac{1}{A(z,\nu)}\frac{p_T}{p_N}\right] d\lambda_G($$

$$+\beta \sum_{z_1'} h(z_1') \bar{W}\left(z_1, \lambda_m\right)$$

Here we have broken the current period utility into the sum of the component of traded goods $u^T(m; p_T)$ and the corresponding component for nontraded goods: the sum of the utility of the consumption of nontraded goods and the disutility of labor used to make those nontraded goods, namely the term in the second integral in (155). The foc with respect to p_T is

$$(156)\quad 0 = \int u_2^T\left(m; p_T\right) d\lambda_G(m, p_N, \nu) + (1-\alpha)\left[\frac{1}{p_T} - \int \frac{1}{A(z,\nu)} \frac{1}{\bar{p}_N\left(\nu_1, S_F\right)} d\lambda_G(m, p_N, \nu)\right]$$

where

$$(157)\quad u_2^T\left(m; p_T\right) = \left(\frac{\alpha}{C_T(m, p_T)} - b\right) \frac{\partial C_T(m, p_T)}{\partial p_T} = \begin{cases} -\left(\frac{\alpha}{m/p_T} - b\right) \frac{1}{p_T^2} & \text{if } m < \frac{\alpha}{b} p_T \\ \left(\frac{\alpha}{\alpha/b} - b\right) 0 = 0 \times 0 = 0 & \text{if } m \geq \frac{\alpha}{b} p_T \end{cases}$$

That $u_2^T\left(m; p_T\right) = 0$ for $m > \alpha p_T / b$ follows from a standard envelope argument: when the CIA constraint is slack, optimality by private agents already ensures that the marginal utility of traded goods consumption is equated to the marginal disutility of working. In contrast, when the CIA constraint is binding, the marginal utility of traded goods consumption is strictly higher than the marginal disutility of working. In such a case, a marginal increase in p_T decreases this component of utility.

Then, using (157) in (156) and evaluating this condition at the optimal solution $\bar{p}_T\left(z, \lambda_G\right) \leq \underline{m} b / \alpha$ (so that the bottom branch of (157) applies) we can rewrite (156) as

$$(158)\quad (1-\alpha)\left[\frac{1}{\bar{p}_T(z, \lambda_G)} - \int \frac{1}{A(z,\nu)} \frac{1}{\bar{p}_N\left(\nu_1, S_F\right)} d\lambda_G(m, p_N, \nu)\right] = 0$$

Now, from the sticky price first order condition (102), rewritten here

$$(159)\quad \bar{p}_N(\nu_1, S_F) = \theta(s_1) \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2)}$$

it follows that for all $s_1 = (z_1, \nu_1)$:

$$(160)\quad 1 = \theta(s_1) \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2) \bar{p}_N(\nu_1, S_F)} > \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2) \bar{p}_N(\nu_1, S_F)}$$

where the last inequality follows from the fact that $\theta(s_1) > 1$ for all s_1. But (158) implies that for all z

$$(161)\quad 1 = \int \frac{\bar{p}_T(z, \lambda_G)}{A(z,\nu) \bar{p}_N\left(\nu_1, z_1, \lambda_F\right)} d\lambda_G(m, p_N, \nu)$$

which is not consistent with (160) hence we have a contradiction. *Q.E.D.*

Combining parts i) and ii) of Lemma A1 immediately implies Lemma 6

Lemma 6. Under (38) if the markup is strictly positive in all states in that $\theta(s_1) > 1$ for all s_1 then in any Markov equilibrium with fixed exchange rates, given any initial distribution of money at the beginning of the period then the end of period money holdings are concentrated on a single point.

Proof: Suppose for contradiction that in a continuation Markov equilibrium the money holdings distribution, λ_m, is *not* degenerate. By part i) of Lemma A1, it must be that for all z and m in support of λ_m the multiplier on the cash-in-advance constraint is zero. This is a contradiction because by part ii) of Lemma A2 in any continuation Markov equilibrium the multiplier on the cash-in-advance is binding for at least one z and some m in the support of λ_m. *Q.E.D.*

C. Lemmas A2 and A3

We start with the characterization of the Markov equilibrium under flexible exchange rates.

Lemma A2. Under (38) the Markov equilibrium outcome with flexible exchange rates is such that the ratio $p_T(s^t)/m(s^{t-1})$ denoted $q_T(s_t)$ only depends on s_t and solves

$$(162)\; q_T(s_t) = \max\left\{\frac{q_N(s_{1t})A(s_{2t})}{2(1-\alpha)}\left[(1-2\alpha)+\sqrt{(1-2\alpha)^2+4(1-\alpha)\frac{1}{A(s_{2t})}\frac{b}{q_N(s_{1t})}}\right], \frac{b}{\alpha}\right\},$$

the ratio $p_N(s^{t-1}, s_{1t})/m(s^{t-1})$ denoted $q_N(s_{1t})$ only depends on s_{1t} and solves

$$(163)\quad q_N(s_{1t}) = \theta(s_{1t})\sum_{s_{2t}} h^2(s_{2t})\frac{q_T(s_t)}{A(s_{2t})},$$

furthermore, $C_T(s_t) = \min\left\{\frac{1}{q_T(s_t)}, \frac{\alpha}{b}\right\}$, and $C_N(s_t) = \frac{1-\alpha}{b}\frac{q_T(s_t)}{q_N(s_{1t})}$. Finally, the money growth rate is $\mu(s_t) = \frac{\beta\alpha}{b}q_T(s_t)$ and the inflation rate in sector $i = T, N$, defined as $\pi_i(z_{t-1}, z_t) = P_i(s_t)/P_i(s_{t-1})$, is $\pi_i(s_{t-1}, s_t) = \mu(s_{t-1})q_i(z_t)/q_i(z_{t-1})$.

Proof. Start by solving (92), which under (38), using (100) and (99) can be written as

$$\begin{aligned} W(S_G) \;=\; & \max_{p_T(x_G),\mu(x_G)} \int\left[\alpha\log\left(\min\left\{\frac{m}{p_T(x_G)}, \frac{\alpha}{b}\right\}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{p_T(x_G)}{p_N}\right)\right] d\lambda_G \\ & -b\int\left[\min\left\{\frac{m}{p_T(x_G)}, \frac{\alpha}{b}\right\} + \frac{1-\alpha}{bA(s_2)}\frac{p_T(x_G)}{p_N}\right] d\lambda_G + \beta\sum_s h(s')W\left(S'_G\right) \end{aligned}$$

subject to

$$\gamma \frac{b}{p_T(x_G)} = \beta \sum_{s'} h(s') \frac{\alpha}{p_T(x'_H, S'_H) C_T(m'_H, x'_H, S'_H)}$$

Now consider a change of variable: let

$$q_T(x_G) = p_T(x_G)/m \quad \text{and} \quad q_N = p_N/m \tag{164}$$

and define S_G^q to be a measure over q_N. S_G^q is the relevant state variable for the problem, which can be rewritten as

$$\begin{aligned} W(S_G^q) \quad = \quad & \max_{q_T(x_G^q), \mu(x_G^q)} \int \left[\alpha \log \left(\min \left\{ \frac{1}{q_T\left(x_G^q\right)}, \frac{\alpha}{b} \right\} \right) + (1-\alpha) \log \left(\frac{1-\alpha}{b} \frac{q_T(x_G^q)}{q_N} \right) \right] d\lambda_G^q \\ & -b \int \left[\min \left\{ \frac{1}{q_T(x_G^q)}, \frac{\alpha}{b} \right\} + \frac{1-\alpha}{bA(s_2)} \frac{q_T(x_G^q)}{q_N} \right] d\lambda_G^q + \beta \sum_s h(s') W\left(S_G^{q\prime}\right) \end{aligned}$$

subject to

$$\gamma \mu(x_G^q) \frac{b}{q_T(x_G^q)} = \beta \sum_{s'} h(s') \frac{\alpha}{q_T(x'_H, S'_H) C_T(m'_H, x'_H, S'_H)}$$

Notice that the optimal $q_T(x_G^q)$ can be found by solving pointwise for all x_G^q is support of λ_G^q the following static problem: for all x_G^q

$$\begin{aligned} & \max_{q_T(x_G^q)} \left[\alpha \log \left(\min \left\{ \frac{1}{q_T\left(x_G^q\right)}, \frac{\alpha}{b} \right\} \right) + (1-\alpha) \log \left(\frac{1-\alpha}{b} \frac{q_T(x_G^q)}{q_N} \right) \right] \\ & -b \left[\min \left\{ \frac{1}{q_T(x_G^q)}, \frac{\alpha}{b} \right\} + \frac{1-\alpha}{bA(s_2)} \frac{q_T(x_G^q)}{q_N} \right] \end{aligned}$$

or equivalently - dropping the dependence from x_G^q - and defining $x = q_T/q_N$ we can write

$$\begin{aligned} & \max_x \left[\alpha \log \left(\frac{1}{x q_N} \right) + (1-\alpha) \log(x) - b \frac{1}{q_N x} - (1-\alpha) \frac{1}{A(s_2)} x \right] \\ & = \max_x (1-2\alpha) \log(x) - \frac{b}{q_N x} - (1-\alpha) \frac{1}{A(s_2)} x + \text{constants} \end{aligned}$$

subject to

$$x \geq \frac{b}{q_N \alpha} \tag{165}$$

If (165) does not bind, the solution to the above problem satisfies:

$$0 = \frac{1-2\alpha}{x} - (1-\alpha) \frac{1}{A(s_2)} + \frac{b}{q_N} \frac{1}{x^2}$$

$$0 = (1-\alpha)\frac{1}{A(s_2)}x^2 - (1-2\alpha)\,x - \frac{b}{q_N}$$

Then the monetary authority best response is:

$$x(q_N, s) = \max\left\{ A(s_2)\frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4(1-\alpha)\frac{1}{A(s_2)}\frac{b}{q_N}}}{2(1-\alpha)}, \frac{b}{q_N\alpha} \right\} \tag{166}$$

or

$$q_T(q_N, s) = \max\left\{ q_N A(s_2)\frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4(1-\alpha)\frac{1}{A(s_2)}\frac{b}{q_N}}}{2(1-\alpha)}, \frac{b}{\alpha} \right\} \tag{167}$$

Now, from (102), in equilibrium it must be that the private best response to government $p_T^i(s)$:

$$q_N(s_1) = \theta(s_1)\sum_{s_2} h^2(s_2)\frac{q_T(q_N(s_1), s)}{A(s_2)} \tag{168}$$

We can combine (167) and (168) to get

$$1 = \theta(s_1)\sum_{s_2} h^2(s_2)\max\left\{ \frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + \frac{4}{A(s_2)}\frac{(1-\alpha)b}{q_N(s_1)}}}{2(1-\alpha)}, \frac{1}{A(s_2)q_N(s_1)}\frac{b}{\alpha} \right\} \tag{169}$$

or, if (165) never binds, simply

$$1 = \theta(s_1)\frac{(1-2\alpha) + \sum h^2(s_2)\sqrt{(1-2\alpha)^2 + \frac{4}{A(s_2)}\frac{(1-\alpha)b}{q_N(s_1)}}}{2(1-\alpha)} \tag{170}$$

which implicitly defines $q_N(s_1)$. Using $q_N(s_1)$ in (167) gives and expression for the equilibrium $q_T(s)$ and finally the other relevant equilibrium objects can be recovered using $q_N(s_1)$ and $q_T(s)$ in (100) and (99). *Q.E.D.*

It turns out that it is particularly simple to characterize the Markov equilibrium with fixed exchange rates when the cash-in-advance constraint always holds with equality. It follows from the proof of Lemma 6 that a sufficient condition for this to be true is that there productivity shocks in the nontraded goods sector have no aggregate component, that is the set Z_2 is a singleton.

Lemma A3. Assume the all agents begin with the same initial holdings of money initial distribution of money, (38) holds, the markup is strictly positive in all states in that $\theta(s_1) > 1$ for all s_1, and the cash-in-advance constraint holds with equality in all states. Then

the Markov equilibrium outcome in a monetary union is such that the prices of consumptions of nontraded and traded goods can be written as $p_N(s_{1t}), C_N(s_{1t}, z_{2t})$, $p_T(z_t)$, and $C_T(z_t)$ and solve

$$p_N(s_{1t}) = \theta(s_{1t}) \sum h^2(s_{2t}) \frac{p_T(z_t)}{A(s_{2t})} \tag{171}$$

where

$$p_T(z_t) = \frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4(1-\alpha)b \sum_{\tilde{s}_t} \left[\frac{1}{A(\tilde{s}_{2t}) p_N(\tilde{s}_{1t})}\right] h(\tilde{s}_t | z_t)}}{\sum_{\tilde{s}} \left[\frac{2(1-\alpha)}{A(\tilde{s}_2) p_N(\tilde{s}_1)}\right] h(\tilde{s}_t | z_t)} \tag{172}$$

furthermore $C_T(z_t) = 1/p_T(z_t)$ and $C_N(s_{1t}, z_{2t}) = \frac{1-\alpha}{b} \frac{p_T(z_t)}{p_N(s_{1t})}$. Finally, the aggregate money growth rate is $\gamma(z_t) = \frac{\beta\alpha}{b} p_T(z_t)$ and the inflation rate in sector $i = T, N$, defined as $\pi_i(z_{t-1}, z_t) = P_i(z_t)/P_i(z_{t-1})$, is $\pi_i(z_{t-1}, z_t) = \gamma(z_{t-1}) p_i(z_t)/p_i(z_{t-1})$.

Proof. First if the distribution λ_G puts all mass on $m = 1$ and the cash-in-advance constraint holds with equality so that $p_T C_T = 1$ then problem (103) can be written for all (z, λ_G) as

$$\max_{p_T} \int \left[-\alpha \log(p_T) + (1-\alpha) \log\left(\frac{1-\alpha}{b} \frac{p_T}{p_N}\right) - b\frac{1}{p_T} - \frac{1-\alpha}{A(s_2)} \frac{p_T}{p_N} \right] d\lambda_G \tag{173}$$

$$= \max_{p_T} (1-2\alpha) \log(p_T) - b\frac{1}{p_T} - \int \left[\frac{1-\alpha}{A(s_2)} \frac{p_T}{p_N}\right] d\lambda_G + \text{constants}$$

where in (173) the integral is effectively over p_N and s_2. The solution to the problem above satisfies:

$$0 = \frac{1-2\alpha}{p_T} + b\left(\frac{1}{p_T}\right)^2 - \int \frac{(1-\alpha)}{A(s_2)} \frac{1}{p_N} d\lambda_G$$

or equivalently

$$0 = p_T^2 \left[\int \frac{(1-\alpha)}{A(s_2)} \frac{1}{p_N} d\lambda_G \right] - (1-2\alpha) p_T - b$$

We can thus solve for the monetary authority best response to some given aggregate shock z and distribution λ_G is

$$\bar{p}_T(z, \lambda_G) = \frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4b \left[\int \frac{(1-\alpha)}{A(s_2)} \frac{1}{p_N} d\lambda_G\right]}}{2\left[\int \frac{(1-\alpha)}{A(s_2)} \frac{1}{p_N} d\lambda_G\right]} \tag{174}$$

In equilibrium we must impose that (102) is satisfied. Substituting (174) into (102) for all s_1 and using that $\bar{p}_N(x_F, S_F) = \bar{p}_N(s_1)$ reduces (174) to (171). For all z_1,equations (171) for all ν_1 give rise to a system of equation in $\bar{p}_N(z_1, \nu_1)$ that can be solved, yielding the price of the non-traded good on the equilibrium path . Given the solution for $\bar{p}_N(s_1)$, $\bar{p}_T(z)$ can be determined from (174) as in (172). Finally, $C_T(s)$ and $C_N(s)$ can be recovered using (171) and (172) in (99), (100) with a cash-in-advance constraint holding with equality. *Q.E.D.*

8. Figures

Figure 1.

Figure 2.

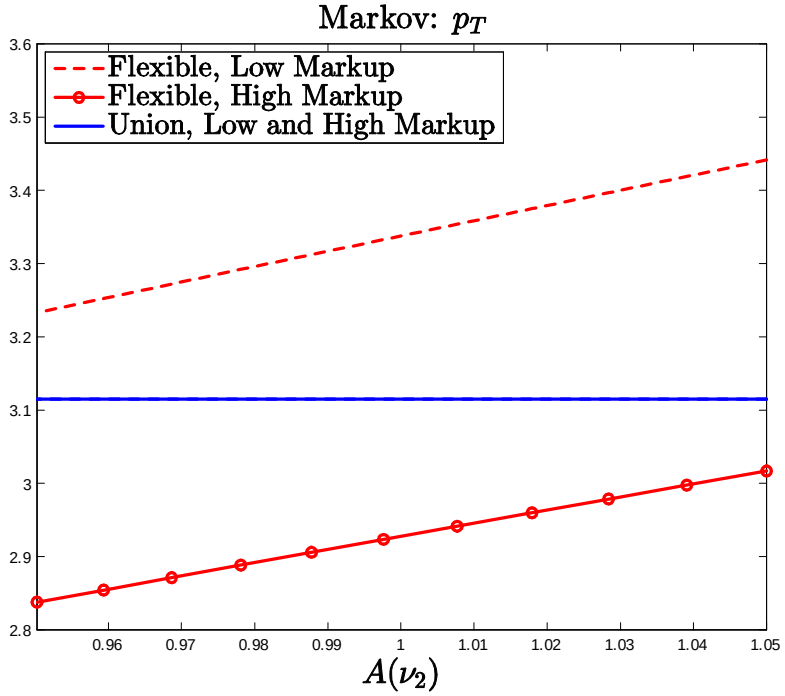

Figure 3.

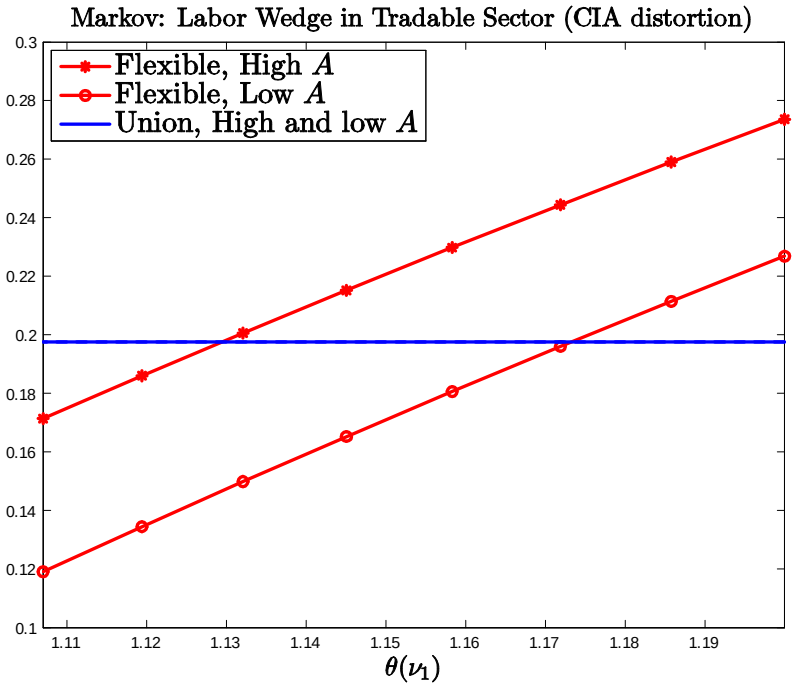
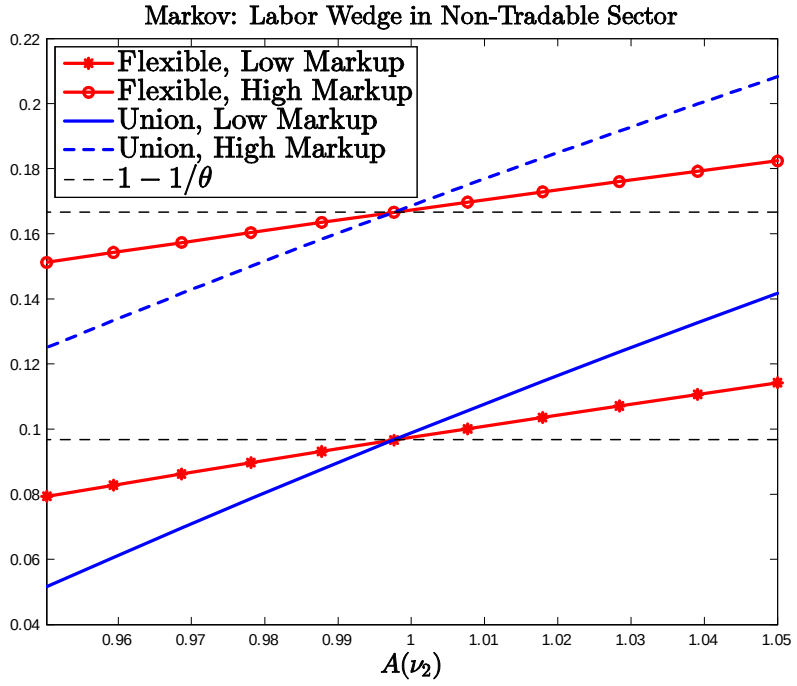
Figure 4.