Piracy Propagation of Information Goods:
Demand and Supply-side Dynamics in P2P Networks

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Abstract

The extent of piracy for a specific digital good can be characterized by the laws of demand and supply. The music industry and its trade organization, the RIAA, have recognized this and taken aggressive steps to diminish demand as well as supply. In this study, we develop a model that describes the dynamics of demand and supply-side of piracy propagation. Using the model, we investigate the effect of anti-piracy measures on piracy. We quantify the impact of anti-piracy measures that are administered prior to the official release of the song. We analyze two anti-piracy measures: the reduction of file supply and the reduction of file demand. The impact of a reduction of file supply is modest. Our results show that the removal of 1% of file supply about 6 weeks prior to the official song release date leads to a reduction of about 0.7% of cumulative file demand and 0.6% of cumulative file supply at the release date. However, the impact of a demand reduction is significantly greater. A decrease of 1% of file demand about six weeks prior to the release will cut about 1.57% of cumulative file demand and 0.6% of cumulative file supply at the release date. From a policy point of view, our results suggest that taking anti-piracy measure early on is important. In addition, and potentially more controversial, our results provide support to the notion that in order to reduce piracy, punishing end-users is more effective than efforts to control the supply of unauthorized music files.

I. Introduction

Sales of music albums have shown a remarkable decline over the last decade. For the year 2009, data from Nielsen SoundScan indicates that album sales have fallen 52% since 2000. Despite this significant slump, sales of CDs remain an important source of revenue. While online sales of music have grown significantly, 80% of all album revenues are still based on CD sales (NYT, 2010). The potential reasons for the drop in album sales are manifold; the disassociation of the physical medium (the CD) from information goods, the availability of all album components as individual songs (unbundling), changes in taste, piracy, and especially peer-to-peer (P2P) piracy have been discussed (Waldfogel 2009; Brynjolfsson and Saunders 2009; Mortimer and Sorensen 2005). The exact impact of P2P piracy on album sales remains a point of academic discussion (Oberholzer-Gee and Strumpf 2007; Smith and Telang 2008; Bhattacharjee et al. 2007), however, research estimates that P2P piracy accounts between 0 and 30 percent of lost album sales (Oberholzer-Gee and Strumpf 2010, Billboard.biz, 2009).

In order to combat widespread digital piracy on file sharing networks, the music industry has directed its efforts to reduce piracy through technology and legal means. Specific attention has been placed on reducing pre-release piracy, the act of illegally copying files before they are publicly released. According to Mitch Bainwol, the chairman and CEO of the Recording Industry Association of America:

> Pre-release piracy is a particularly damaging and onerous form of theft…. It robs artists of the chance to sell their music before it even hits the streets or becomes legally available online, and the ripple effects are felt far and wide throughout the entire music community (RIAA 2006).

This emphasis on fighting pre-release piracy is not surprising. First, a significant part of pre-release piracy may indicate true demand for a song that is met without any positive revenue implications for the labels. In addition, demand for a music album is typically highest right after the release date and diminishes at an exponential rate in the following months. Hence, most revenue is generated in the first few weeks after the release of an album or a song. Therefore, pre-release piracy is considered damaging the most lucrative revenue period.

Measures to reduce piracy levels can be categorized as focusing on demand- or supply-reduction. Supply-reduction efforts aim to lower the level of files available on P2P networks. Specific legal activities have focused on the deterioration of the supply side. For example, in September 2009, criminal copyright infringement charges were brought against high-level members of a pre-release music piracy group call

Rabid Neurosis. In addition, the music industry has employed technologies such as interdiction[1] that aim to reduce the effective supply level of specific songs. Demand-reduction efforts squarely aim at the end-user. Perhaps the most prominent example of a demand-reducing strategy is the litigation campaign that the RIAA undertook from 2003 to 2008 and that has surged again in 2010. According to Administrative Office of the Courts, more than 18,000 people were targeted from 2003-2008, while in 2010 alone more than 14,000 P2P users were targeted. Technologies such as the introduction of decoys and swarming[2] are also used to make it more costly to P2P users to search for and download music files.

However, to our knowledge, there is little understanding on the impact of demand-reduction and supply-reduction on the overall piracy level. Similarly, few people truly understand how supply and demand are formed in P2P networks and how they interact. This work was motivated by discussions with industry participants that raised questions about how the demand-side of file sharing networks (P2P users requesting music files) affects the supply-side (P2P users providing music files) in file sharing networks and vice versa. A better analysis of the demand- and supply-side interactions has been deemed critical to grasp the effectiveness of efforts to reduce piracy early in the product life cycle. Our research question can therefore be described as follows:

- How can we describe demand- and supply-side dynamics of individual songs in a P2P network?
- What is the potential impact of demand- and supply-reduction efforts on file-sharing activities in P2P networks?

Our model of piracy propagation is motivated by the unique properties of file sharing behavior and the topological system characteristics of P2P networks. A request for a song by a P2P user represents the unit of analysis for the demand side. The level of requests may change based on potential demand, based on its growth pattern, but also based upon the availability of the song. The individual file (a song) that is provided by a P2P user is the basic unit of analysis for the supply-side. The supply of a song can increase with the arrival of new supply sources that join the network, but it may also decrease depending on the retention rate. In addition, a file request that is fulfilled can potentially lead to an increase in supply. These characteristics lead to a system of differential equations. We present a unique closed-form solution

[1] Interdiction in the P2P file sharing context represents technologies in which a party deliberately and repeatedly connects to file suppliers in order to hinder the fulfillment of other requests.

[2] Swarming refers to a technique in which a part of the song is corrupted and distributed. The recipient receives a polluted file that often is unplayable.

of the model and analyze dynamic impact of initial conditions on piracy level over time. We compare and validate propositions obtained from the analytical model with numerical simulation and empirical analysis.

To the best of our knowledge, this work is the first to formally model and estimate the interaction of the demand- and supply-side of piracy. Despite the employment of diverse anti-piracy strategies such as file pollution technologies, there is little understanding about the propagation process of piracy itself. This work also provides insight into the effectiveness of anti-piracy measures. Content producing firms as well as public legislatures have invested time and resources to stem piracy on the demand- and supply-side. In this study, we discuss the impact of an early reduction in demand and supply on subsequent piracy levels. From a music label point of view, the rampant pre-release piracy in P2P networks has proven to be very disruptive to the marketing and sales process of albums. Many well-known artists had to release their albums earlier to minimize potential damages from pre-release piracy. Apart from damages to the crucial sales period, pre-release piracy also short-changed considerably expensive and elaborate marketing and promotion plans. [3] A thoughtful analysis of pre-release piracy and the effectiveness of anti-piracy measures can enable companies to incorporate these valuable insights into marketing and sales strategies.

The structure of the rest of the paper is organized as follows. In Section II we review related literature. In Section III, we develop an analytical model to describe the dynamics of piracy propagation. We derive analytic properties of the propagation processes described as a differential equations system. In Section IV, we empirically illustrate different types of piracy curves based on functional data analysis. We provide estimation results using actual piracy data and demonstrate how demand-side process influences supply-side piracy and vice versa by using numerical methods. Scenario analyses were performed to provide anti-piracy policy implications. We compare the impacts of various demand or supply shocks about a month prior to the release on the cumulative level of piracy by the release date. In Section V, we summarize results and discuss our findings. Finally, Section VI concludes this study and presents future work.

II. Related Literature

[3] Examples of artists that changed their release date and marketing and sales strategy include Eminem (‘Encore’), KoRn (‘Take A Look in the Mirror’), Brittney Spears (‘Blackout’), Jay-Z (‘The Black Album’, ‘The Blueprint 3’), G-Unit (‘Beg for Mercy’), YYY’s (‘It’s Blitz’).

Studies on file-sharing behavior in P2P networks examine participation incentives and network externalities (Asvanund et al 2004; Gu et al. 2008; Krishnan 2004). Asvanund et al. (2004) assume that an individual's utility from using a P2P network consists of positive utility from downloading contents and negative utility from congestion effects. They find that users contribute additional value to the network at a decreasing rate and impose costs on the network at an increasing rate. Gu et al. (2008) develop a two-sided network model based on two groups, sharers and downloaders, assuming heterogeneity of preference. They find positive externalities between contribution and consumption; more contribution leads to more consumption and vice versa. Krishnan et al. (2004) analyze the motivation of file-sharers that peers may share their content based on entirely on self-interest; it is to draw traffic away from other peers in the network to the sharing peer, thereby increasing the chance that the sharing peer will be able to get her desired content from other peers.

Recent studies examine the implications of piracy in diverse perspectives: the relationship between piracy and digital distribution, influence of piracy on sales of music albums and individual songs, effects of sampling on purchase decision, and the impacts of piracy on the popularity of artists. For instance, Danaher et al. (2010) study whether the emergence of digital distribution channels cannibalizes sales of physical product and deteriorate piracy participation. They found that while provision of digital products have no impacts on the demand for corresponding physical products, decrease in provision of digital contents is causally related to the increase in piracy about 11%.

The effect of online piracy upon sales is yet inconclusive. Oberholzer-Gee and Strumpf (2007) find that the impact of file-sharing activities on the legal sales of music albums is statistically indistinguishable from zero. Zentner (2006) estimates the impact of music downloads at file-sharing networks on the probability of purchasing music. He finds that peer-to-peer network usage reduces the probability of buying music by 30%, which indicates that the sales without downloads would have been around 7.8% higher. Gopal et al. (2006) studied whether students who sample music are more likely to purchase the CD albums. They found that lowering the cost of sampling music propel more consumers to purchase music online as the total cost of evaluation and acquisition decreases. Bhattacharjee et al. (2007) examined the impact of piracy on survival of music albums on the charts. They found that file-sharing has a negative impact on low-ranked albums while it doesn't hurt the survival of top-ranked albums. More recently, Leung (2009) estimated the effects of file sharing on sales of songs at iTune. He found that when students pirate 10% more music, they intend to buy 0.7% fewer iTunes songs and 0.4% fewer CDs based on a conjoint survey data.

Direct studies of content propagation process in P2P networks are limited. A few studies examine file dispersion speed and performance of file sharing mechanisms on large-scale P2P systems (Qiu and Srikant 2004; Kumar et al. 2006, 2007). Qiu and Srikant (2004) develop a fluid model to evaluate the scalability, performance and efficiency of BitTorrent like P2P systems. Kumar et al. (2006) examine pollution proliferation in P2P systems. They illustrate a variety of user behaviors such as propensity for popular versions, abandonment after repeated failure to obtain a good version, freeloading, etc. and obtained closed-form solutions for the nonlinear differential equations. Hosanagar et al. (2009) propose mixing models for distribution of legitimate content in P2P networks. Their model incorporates redistribution incentives for nodes by providing optimal payments to users who distribute content.

While piracy has been studied in many aspects, propagation process of piracy in file-sharing networks has not been well understood nor formally analyzed. To better understand the unique features of piracy, we define piracy propagation as the dissemination process which spreads copies of song files to a larger number of P2P users. The dissemination process is driven by participating P2P users, who may request a file or who may choose to fulfill a request for a file. The aggregate number of requests and the aggregate number of suppliers for a file represent the demand and the supply side respectively. The demand and supply sides do not develop independently; the exact shape is determined by the interaction of demand and supply.

The dissemination process of durable products has been pervasively developed in the diffusion literature. Early work on diffusion is focused upon a general demand generation process that is driven by two population segments. Bass (1969) assumed two types of adopters, innovators and imitators. Givon et al. (1995) develop a software piracy diffusion model based on buyers and pirates. While many studies analyze the effects of population segments, studies found that the initial phases of production introduction process hardly fits the Bass model (Goldenberg et al. 2009; Mahajan et al. 2000). To explain the early stages of the production introduction process, Goldenberg et al. (2009) develop an individual-level approach that shows how changes in sales patterns can self-emerge as a stochastic process. They treat the fluctuations observed in volatile daily sales data as information rather than noise and generate short-term prediction at the early stages of the penetration process.

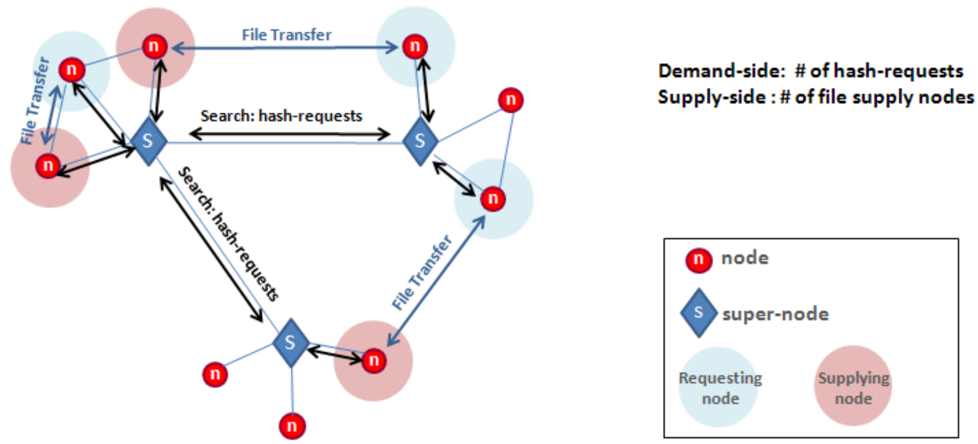
Some studies extended diffusion models by taking supply-side constraints into account while most diffusion assume that demand is automatically fulfilled by supply (Kumar and Swaminathan 2003; Ho et al. 2002; Jain et al. 1991). Ho et al. (2002) generalize the Bass model by allowing for a supply constraint where potential customers who join the waiting queue potentially reverse their adoption decision. They examine how companies should manage demand and supply to optimize their profits. Jain et al. (1991) and Kumar and Swaminathan (2003) also develop supply-constrained diffusion models that include limits on product capacity or distribution systems. While these models capture the dynamics of supply restrictions and its impact on demand, they mainly focus on the centralized distribution system. In this paper, we study early period of piracy propagation driven by individuals' requests, participation behavior and the characteristics of topological system in a decentralized network.

III. A Model of Piracy Propagation Process in P2P Networks

3-1. Characteristics of P2P network architecture

We develop an analytical model that formulates the interplay between demand and supply side of piracy in P2P networks. Peer-to-peer systems rely on the voluntary participation of users and reveal a significant amount of heterogeneity in the bandwidth, availability, latency and sharing based on the architecture (Saroiu et al. 2006). For instance, "hybrid" network structures consist of two types of nodes, i.e., nodes that represent end-users and super-nodes that maintain a database of shared items. Figure 1 shows the "hybrid" architecture that is employed in many P2P networks. A user who searches for a song initiates a search at a node by requesting a unique hash code that is associated with the song. At first, the search is performed across the neighboring nodes that are connected to the same super-node. These super-nodes are typically selected from peer nodes that have a fast processing unit, high bandwidth access to the Internet, and capable of supporting 200-300 simultaneous connections (Blackard, 2005). In a hybrid network, super-nodes do not transfer the file between peers; it aids connections and file transfers. If this search is not successful, then the super-node sends out hash-requests to neighboring super-nodes. These hash-requests represent the overall demand side of P2P networks. The nodes that supply the song file to the requesting node represent the supply side of P2P networks.

[Figure 1: Demand and Supply-side of Piracy in Hybrid Architecture of P2P Networks]



Demand-side of piracy in a P2P network can be defined as the quantity of file transfer between the nodes occurred as a result of a search process. While hash-requests among super-nodes will be followed by file transfer in hybrid architecture, file transfer occurs directly between nodes in the completely decentralized flat network architecture. Supply-side of piracy can be represented by the number of available supplying nodes that have the song file. This conceptualization of piracy demand and supply can be easily applied to any type of P2P network architecture. In this study, we employ a hybrid network represented in Figure 1 to describe our model.

The goal of this study is to examine the demand and supply-side dynamics of P2P traffic. A request for song file that has been fulfilled will then lead to a potentially greater supply. In turn, a larger supply of a song can indicate more future requests. In order to investigate such dynamics, we examine two processes of propagation; a demand and a supply process. All demand and supply-side of measures are counted as *flow* variables, i.e., demand is measured as the daily number of download based on (hash) requests, and supply is measured as the number of available supply nodes for a song. As we shall demonstrate, there is an intricate relationship between the proliferation of the downloading requests and the available supplying nodes.

The Demand- Side Propagation Process

The demand-side propagation rate depends on three processes: current requests process, supplying process by available nodes, and new requests process by potential demanders. Each process affects propagation in a different rate. The hash searching process introduces new peers with different resources, therefore altering the propagation rate. Peers are brought into P2P networks by requesting that a song file be added their accompanying network resources, e.g., resting CPU and the bandwidth into P2P network. This influx of additional resources influences the response rate of file-transfers followed by hash requests. When hash-requesting peers retain a superior quality of network resources compared to the average condition of the whole network, the increase in hash requests number speeds the file transfer due to faster response from the new peers. In turn, a lower quality of added resources can slow the propagation rate by creating congestion effect. Note that such influence on response rates from downloading peers occurs persists when peers remain in the network for downloads without being converted to supplying nodes. The coefficient p denotes this growth rate of piracy demand due to the quantity of current download requests.

The second source is the availability of supplying nodes. One particular aspect of piracy traffic is that downloading requests can be fulfilled only if when files are available. Increase in the number of supplying nodes, which represents the file availability, influences the response rate of hash requests. If the available files are spoofed or corrupted, or available only at limited bandwidth, then the downloading request could break down. Each additional supply node added to the system not only provides files but also introduces extra resources in the system, such as processing or connection speed, local network configuration and operating system. These network resources are known to be available to all peers at the application layer (Karagiannis et al. 2004). More supply nodes with extra resources positively influence file download speed. The parameter α_1 stands for these impacts of the supply-side file availability on the song demand rate.

Lastly, new requests also influence the current rate of the demand propagation process. For instance, a popular song might indicate a higher level of potential daily demand. If current download level is low with respect to its maximum potential daily traffic, the residual demand request get slow response in file transferring process due to the low level of network resources. In turn, when the download level is high, a small amount of residual demand requests will get fast response from hash-requests due to the existing peers staying in the network. The coefficient α_2 characterizes this effect of daily demand potential on the download request rate. The left side of Figure 2 illustrates these three main features that govern the demand-side downloading request rate.

The Supply- Side Propagation Process

The supply-side propagation rate is determined by three processes; a retention process of current nodes to be remained in the network, a conversion process of downloaders to be suppliers who intentionally or unintentionally share their files, and an influx process of new suppliers. A supply node might outflow after downloads by removing files to a personal folder; such outflow activity of current peer results in the decrease of supply rate for a song. The retention rate of supply nodes that remain in the supply pool extracting this outflow is denoted as γ .

A characteristic of P2P networks is that the multiplication and transmission of copies is for all practical matters instantaneous. In addition, digital files are a non-rivalry good; the consumption of digital files by one does not reduce the availability to others, including the original supplier. Hence, in P2P networks, satisfying a download request is equivalent to the production of a flawless copy that is transmitted instantaneously without reducing the supply of the file. More accurately, not only is supply not reduced by fulfilling demand, supply actually increases due to the multiplication of copies. This is true even in the case where a user immediately removes the copy from the shared folder. During the short period of time when a copy is produced, individual packets are made available to fulfill requests from other downloaders. Thus, an increase in the number of downloading requests for a song positively affects the supply rate. This fraction of demand converted to supply is denoted as coefficient β_1 .

Lastly, the quantity of new supply nodes affects the supply rate. For instance, some songs might be easily available to many nodes due to the early exposure and extensive propagation through the network; this affects an increase in the quantity of potential supply influences on the supply rate positively by the infusion rate of new suppliers. This new influx rate to the process as supply nodes is denoted as coefficient β_2 . The supply-side process of piracy propagation is depicted in the right side of Figure 2. Table 1 summarizes the determinants of demand and supply-side piracy propagation processes described in Figure 2.

[Figure 2: Song-Level Propagation Processes of P2P Piracy: Demand- side vs. Supply- side]

[Table 1: Determinants of Piracy Propagation Processes: Demand- side vs. Supply- side]

Demand Process	Growth rate (p)	Current file request rate
	Availability rate (α_1)	Availability rate of current supply nodes

	New request rate (α_2)	Arrival rate of new requests
	Download Requests (D)	Number of downloads for a song at time t
	Supply Nodes (S)	Number of supply nodes for a song at time t
	New Demand ($\overline{D} - D$)	Maximum potential demand-Number of downloads for a song at time t
Supply Process	Retention rate (γ)	Retention rate of supply nodes after outflow
	Conversion rate (β_1)	Conversion rate of file sharing from downloads
	New infusion rate (β_2)	Infusion rate of new supply nodes
	Supply Nodes (S)	Number of supply nodes for a song at time t
	Download Requests (D)	Number of downloads for a song at time t
	New Supply ($\overline{S} - S$)	Maximum potential supply-Number of supply nodes for a song at time t

3-2. A Model of Propagation Processes

We seek the relationship between the propagation rate of demand and supply with respect to the influential sources, e.g., current level of download requests, supply nodes, potential demand and supply level. Let $f(t)$ and $g(t)$ be distribution functions describing the piracy demand and supply for a song during the pre-release period. The rate of change in download requests, denoted as $f'(t)$, can be configured as follows. The number of download-requests at time t, $D(t)$ increases when a peer, who does not have copy of a song, requests a download, which happens at the quantity of $\alpha_2[\overline{D} - D(t)]$. The demand growth rate considering the current level of demand is represented by the coefficient p. The influence of the current demand level on the new request rate can be simply represented by $pD(t)$. The response rate to download requests from available supply nodes increases the number of downloads at the amount of $\alpha_1 \frac{\overline{D}}{\overline{S}} S(t)$. Similarly, the number of supply nodes for a song, $S(t)$, increases when new supply nodes appear, which happens at the quantity of $\beta_2[\overline{S} - S(t)]$. In addition, $S(t)$ increases when a user who downloaded the song, subsequently becomes a supply node for the song. This occurs at the quantity of $\beta_1 \frac{\overline{S}}{\overline{D}} D(t)$. The fraction of supply nodes that remain active (i.e., supply nodes that do not log out or make the song unavailable for download) are denoted as $\gamma S(t)$. While the other two terms describe the dynamics between the supply-level and the rate, the term $\beta_1 \frac{\overline{S}}{\overline{D}} D(t)$ measures the demand-side impact on the supply-side. This leads to the differential equation system (1)-(2). The following differential equation system has a unique analytical solution.

According to the demand propagation process described above, we define $f'(t) = pf(t) + \alpha_1 g(t) + \alpha_2(\max(f_t) - f(t))$, a linear combination of both demand- and supply-side distribution functions with the effect of three identified base sources introduced above. The rate of change in supply level for a song, $g'(t)$, can be also defined as $g'(t) = \beta_1 f(t) + \gamma g(t) + \beta_2(\max(f_t) - f(t))$. If the daily maximum number of download requests during the pre-release period is given by $\overline{D}$ and each user downloads one unit, then the demand rate of a song file can be reduced to $D(t) = \overline{D} \cdot f(t)$. Similarly, if the daily maximum supply for a song during the pre-release period is given by $\overline{S}$, then the supply rate of a song at time t can be reduced to $S(t) = \overline{S} \cdot g(t)$. This is the demand- and supply-side propagation processes of file on time domain. We can convert the expressions on the file adoption domain. Since $f(t) = D(t)/\overline{D}$ and $g(t) = S(t)/\overline{S}$, the fluctuation of file demand and supply level at each time t, our variables of interest, $dD(t)/dt$ and $dS(t)/dt$ can be described as follows:

$$\frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{\overline{S}} S(t) + \alpha_2[\overline{D} - D(t)] \qquad (1)$$

$$\frac{dS(t)}{dt} = \gamma S + \beta_1 \frac{\overline{S}}{\overline{D}} D(t) + \beta_2[\overline{S} - S(t)] \qquad (2)$$

[Table 2: Piracy Propagation Processes: Demand- side vs. Supply- side]

$f(t)$: Distribution of daily download requests level (Distribution of piracy demand)
$g(t)$: Distribution of daily supply nodes level (Distribution of piracy supply)

$\overline{D}$: Maximum quantity of potential daily demand
$\overline{S}$: Maximum quantity of potential daily supply

The system of differential equations (1)-(2) has the following unique solution presented in equations (1)'-(2)'. From the distribution of piracy demand can be described as the process denoted by $f(t)$ and the piracy supply process as $g(t)$, we have recovered the distribution function of demand- and supply- level of pre-release piracy, $D(t)$ and $S(t)$. A detailed proof can be found in Appendix I.

$$D(t) = \overline{D} \cdot f(t) = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{12} e^{\lambda_2 t} + c_3 v_{11} \frac{1}{\lambda_1}\left(e^{\lambda_1 (t-t_0)} - 1\right) + c_4 v_{12} \frac{1}{\lambda_2}\left(e^{\lambda_2 (t-t_0)} - 1\right) \qquad (1)'$$

$$S(t) = \overline{S} \cdot g(t) = c_1 v_{21} e^{\lambda_1 t} + c_2 v_{22} e^{\lambda_2 t} + c_3 v_{21} \frac{1}{\lambda_1}\left(e^{\lambda_1 (t-t_0)} - 1\right) + c_4 v_{22} \frac{1}{\lambda_2}\left(e^{\lambda_2 (t-t_0)} - 1\right) \qquad (2)'$$

where $\Delta = (\gamma - \beta_2 - p + \alpha_2)^2 + 4\alpha_1\beta_1$, $\lambda_1 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 - \sqrt{\Delta})$, $\lambda_2 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 + \sqrt{\Delta})$

$c_1 = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}}\left(v_{22} D(0) - v_{12} S(0)\right)$, $c_2 = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}}\left(- v_{21} D(0) + v_{11} S(0)\right)$

$c_3 = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}}\left(v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}\right)$, $c_4 = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}}\left(- v_{21}\alpha_2 \overline{D} + v_{11}\beta_2 \overline{S}\right)$

$v_1 = k_1 \left[1 \ , \ \frac{1}{2\alpha_1} \frac{\overline{D}}{\overline{S}}\left(\gamma - \beta_2 - p + \alpha_2 - \sqrt{\Delta}\right) \right]^T$, $v_2 = k_2 \left[1 \ , \ \frac{1}{2\alpha_1} \frac{\overline{D}}{\overline{S}}\left(\gamma - \beta_2 - p + \alpha_2 + \sqrt{\Delta}\right) \right]^T$

While each parameters in the model (1)-(2) stand for unique rates that affect demand and supply rate change, we are particularly interested in parameters α_1 and β_1 that characterize the cross-impacts. For instance, α_1 measures the supply-side impact on demand rate. The availability of file critically alters response rate to a hash request, thereby, governs demand rate. In turn, β_1 represents the demand-side effect on supply rate. Current downloaders can be simultaneously converted to supplying nodes of songs and increase the supply rate.

The following proposition I and II describes the characteristics analytical properties of the system. We first study the impact of initial conditions on the subsequent level of demand and supply. We then analyze the dynamics between demand and supply when these cross-impact parameters, α_1, β_1 are not zero.

Proposition I (Positive Impact of Initial Condition): The solution to the system of differential equation (1)-(2) satisfies following properties when $\alpha_1 > 0$, and $\beta_1 > 0$.

a. Both of the demand and supply level for any point of time increase in response to an increase in the initial demand level : $\frac{\partial D(t)}{\partial D(0)} > 0$ and $\frac{\partial S(t)}{\partial D(0)} > 0$.

b. Both of the demand and supply level for any point of time increase in response to an increase in the initial supply level : $\frac{\partial D(t)}{\partial S(0)} > 0$ and $\frac{\partial S(t)}{\partial S(0)} > 0$.

The following equations summarize the impact of initial condition on the demand and supply level over time. The partial derivatives of $D(t)$ and $S(t)$ with respect to their initial condition, $D(0)$ and $S(0)$ can be directly obtained from the equation. When $\alpha_1 > 0$, and $\beta_1 > 0$, we can show that $v_{21} < 0, v_{22} > 0$ from the unique solution. Since $\overline{D}, \overline{S} > 0$ and $\lambda_1 < \lambda_2$, we have the Proposition I.

$$\frac{\partial D(t)}{\partial D(0)} = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(v_{11} v_{22} \exp(\lambda_1 t) - v_{12} v_{21} \exp(\lambda_2 t) \right), \quad \frac{\partial D(t)}{\partial S(0)} = \frac{\alpha_1 v_{11} v_{12}}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(\exp(\lambda_2 t) - \exp(\lambda_1 t) \right)$$

$$\frac{\partial S(t)}{\partial D(0)} = \frac{\alpha_1 v_{21} v_{22}}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(\exp(\lambda_1 t) - \exp(\lambda_2 t) \right), \quad \frac{\partial S(t)}{\partial S(0)} = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(v_{11} v_{22} \exp(\lambda_2 t) - v_{12} v_{21} \exp(\lambda_1 t) \right)$$

Proposition II (Increasing Cross Effect of Initial Condition): The solution to the system of differential equation (1)-(2) satisfies following properties when $\alpha_1 > 0$, and $\beta_1 > 0$.

a. The impact of initial supply shock on demand level increases over time, i.e., $\frac{\partial D(t)}{\partial S(0)}$ is increasing in time t, i.e., $\frac{\partial^2 D(t)}{\partial t \, \partial S(0)} > 0$.

b. The impact of initial demand shock on supply level increases over time, i.e., $\frac{\partial S(t)}{\partial D(0)}$ is increasing in time t, i.e., $\frac{\partial^2 S(t)}{\partial t \partial D(0)} > 0$.

Proposition II satisfy the same conditions of Proposition I-1, i.e., $v_{21} < 0, v_{22} > 0$, $\overline{D}, \overline{S} > 0$, $\lambda_1 < \lambda_2$ and can be derived from the first derivative of the partial derivative functions as follows:

$$\frac{\partial^2 D(t)}{\partial t \partial S(0)} = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(\lambda_2 v_{11} v_{12} \exp(\lambda_2 t) - \lambda_1 v_{11} v_{12} \exp(\lambda_1 t) \right)$$

$$\frac{\partial^2 S(t)}{\partial t \partial D(0)} = \frac{\alpha_1}{\sqrt{\Delta}} \frac{\overline{D}}{\overline{S}} \left(\lambda_1 v_{21} v_{22} \exp(\lambda_1 t) - \lambda_2 v_{21} v_{22} \exp(\lambda_2 t) \right)$$

Proposition I suggests that the initial shock of each demand and supply level positively influences on both levels over time. Not only the initial shock affects its own level, its cross-effect positively persists over time. Proposition II implies that the cross-effects of each shock, i.e., demand shock effect on supply level and supply shock effect on demand level, actually increases over time. Notice that the impact of shocks on its own level does not always increase over time. This result emphasizes the importance of cross-interplay between demand and supply in the propagation process.

IV. Empirical Analysis

4-1. Data

Our piracy data is obtained from the Ares P2P network for the time period of April 2007 to September 2007. We collected data for the songs that were part of all newly released albums that appeared in the Billboards' Top 200s list during the period. The data comes from a leading P2P anti-piracy and marketing solutions provider. The company actively monitors all major P2P networks and collects data on downloading and sharing activities and provides services to all major record labels and movie studios. Ares was chosen primarily for three reasons: (i) the popularity of the P2P network for music download, (ii) the breadth of coverage of the P2P network, and (iii) the ability to monitor downloading and sharing activities. The raw data for the P2P network is about 60GB per month; this includes data for downloading as well as for sharing activities.[4] We selected songs in newly-released albums except movie soundtracks and re-entered albums from May to July 15, 2007. Movie soundtracks or songs in re-entered albums are well known for the atypical sales patterns. We ended up with total 651 newly released songs that have minimum four days of downloading and sharing activities in the P2P network starting from about two months before the release date.

The demand-side of data measures the traffic between super-nodes in Figure 1. This measure can be representative for the overall demand. To better account for the traffic inside a local neighborhood of a super-node that is not counted in the data, we consider the fact that a typical super-node maintains around 200-300 nodes in a neighborhood (Blackard, 2005). In the Appendix II, we show that the results of our model (1)-(2) remain identical even if the demand level scales up more than 200 due to the traffic inside the local neighborhood that we cannot capture in the data. Based on this result, we measure the total demand-side traffic by scaling up the traffic among super-nodes 200 times to represent the traffic in the whole network.

Table 3 summarizes the descriptive statistics of piracy data before release. It shows that on average 552 files were requested for a song each day during pre-release period. Around 59 files were available for an average song per day during the pre-release period. The maximum number of file requests for a song in the data is about 7,025 units. For a popular song, about 3,060 nodes were available to supply the existing song files the most.

[Table 3: Demand- and Supply-side of Piracy: Descriptive Statistics on Songs]

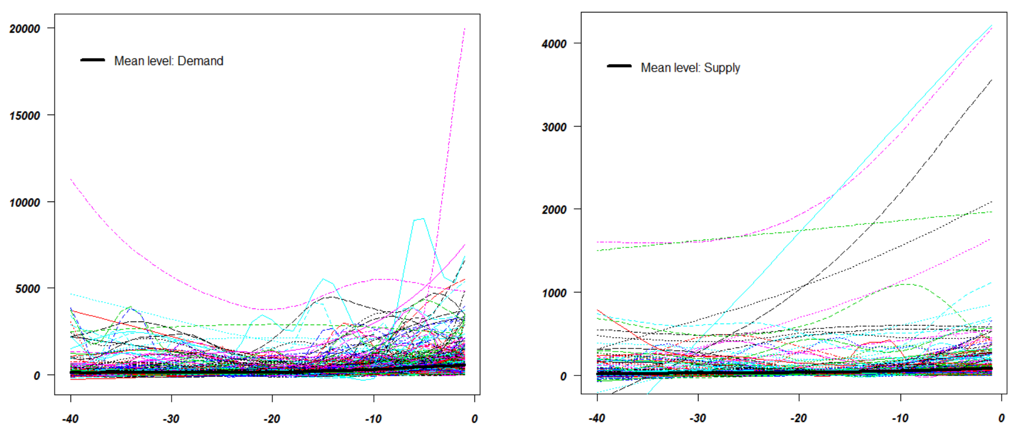
	Demand (All 651 Songs)	**Supply** (All 651 Songs)
Mean	551.8	58.8
Median	360.0	13.9
Minimum	200.0	1.0
Maximum	7025.0	3059.2
St. Dev.	614.3	225.9

Deriving Smooth Paths of File-sharing Behavior

[4] The data contains lists of the name of file, title of album, artist name, genre of music, unique hash of the file, and user IP address. According to industry experts, the data represents the most comprehensive collection of this kind.

In this paper, we use a penalized smoothing method that often employed as the first step of functional data analysis to recover a continuous type function which represents file sharing behavior (Ramsay and Silverman 2005). We recover the underlying continuous functional objects of the demand- and supply-side piracy for all songs during the pre-release period using discrete daily observations[5]. Figure 3 describes the smooth functional objects of the demand- and supply-side of piracy for all songs. A thick black line in Figure 3 illustrates the mean level function of songs. While both demand and supply level increases over time towards the release date, the specific shape and curvature differs and convey different information[6]. Our unit of analysis is mean level function of each demand and supply side of piracy.

[Figure 3: Mean Level Functions of Demand- and Supply- side Piracy]



4-2. Endogeneity and Estimation

We estimate the parameter values of the differential equation systems (3)-(4) using Generalized Method of Moments (GMM) (Hansen 1980). This system is identical to the system (1)-(2) with the error terms.

$$\frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{\overline{S}} S(t) + \alpha_2[\overline{D} - D(t)] + \varepsilon_D(t) \qquad (3)$$

[5] The details of penalized smoothing methods can be found in Foutz and Jank (2009) Appendix A.

[6] Derivative curves of each level curves drastically illustrates the difference in shape between demand and supply side level curves. One example is shown in Figure 4.

$$\frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\bar{S}}{\bar{D}} D(t) + \beta_2 [\bar{S} - S(t)] + \varepsilon_S(t) \qquad (4)$$

The typical source of endogeneity in demand equation (3) is unobservable exogenous shocks that simultaneously affect both current demand level and demand velocity. In addition, there could be some factors that influence both current supply level and demand velocity. In the supply equation (4), there could be also shocks that potentially affect both current supply level and supply velocity. Thus, using ordinary least squares estimation, we will likely overestimate the direct effect of demand and supply level on the demand rate. We follow Villas-Boas and Winer (1999) and use lagged demand and supply level variables, $D_{t-2}, D_{t-3}, S_{t-2}, S_{t-3}$ as instruments. The lagged level variables of demand and supply may be correlated over time and affect the current demand and supply level, D_t, S_t but not necessarily to the current growth rate, the dependent variable.

Table 4 presents parameter estimates using GMM estimation. The first column shows the parameter values estimated based on the variables D_t, S_t without using IVs. These parameter values are identical to the Ordinary Least Squares (OLS). The second column presents the estimated parameters using IVs. Demand equation (3) employed three IVs for both demand and supply level that are $Z_1 = [D_{t-2}, D_{t-3}, S_{t-3}]$. Also, variables $Z_2 = [D_{t-3}, S_{t-2}, S_{t-3}]$ were introduced to estimate parameters in the supply equation (4). We estimate the following moment conditions: $E(Z_{1t}\varepsilon_{Dt}) = 0$ and $E(Z_{2t}\varepsilon_{St}) = 0$.

The parameter values slightly differ between the two methods. To avoid model overidentification, we have restricted the number of lags to two. We perform some numerical analysis and policy analysis in the following sections using the estimation results based on IV estimation.

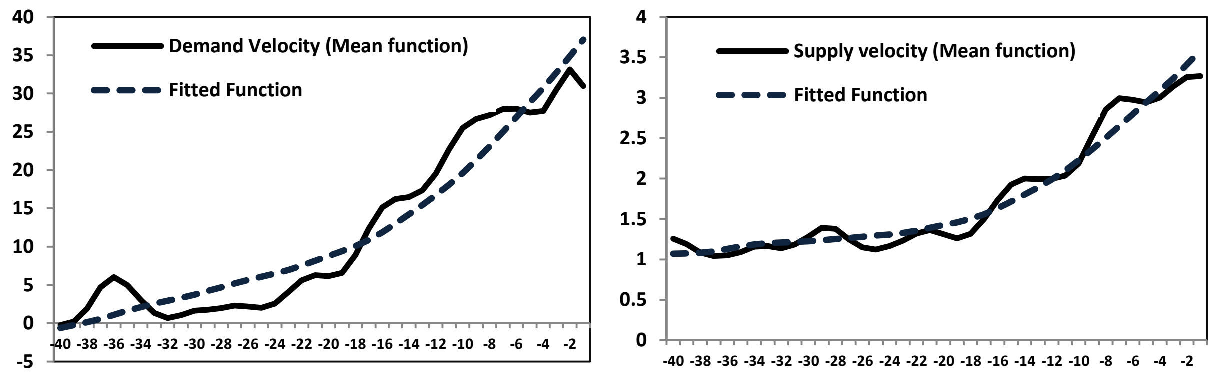
[Table 4: GMM Parameters of the System: Mean level]

	Exact Identification	IV estimation
Demand-side Coefficients		
Growth rate (p)	0.0211***	0.0259***
Availability rate (α_1)	0.0425***	0.0392**
New request rate (α_2)	-0.0154***	-0.0161***
Supply-side Coefficients		

Retention rate (γ)	0.0085***	0.0147**
Conversion rate (β_1)	0.0356***	0.0294***
Infusion rate (β_2)	0.0055***	0.0043***

Figure 4 illustrates the first-derivative of demand and supply-side piracy level in Figure 3. We call this the velocity function or piracy rate. This velocity function depicts the volatility involved in the daily rate change of file sharing activities. Both demand and supply velocity curves describe unique evolution pattern over time.

[Figure 4: Mean Velocity Functions of Demand- and Supply- side Piracy]



The parameter p represents the effect of current download request level on its own growth rate of demand. The parameter shows that on average, the current download request level has a positive influence on the growth rate of demand. For example, a daily download request level of 100 will increase the average growth rate of demand by 2.1 for the overall sample. The parameter α_1 explains the effect of supply on the demand rate. If a song can be supplied on any day by 100 nodes, then this will increase the average growth rate of demand by about 4.2 for the overall sample. This result indicates that the rate of download requests increases with respect to the song supply level. Larger levels of supply will increase the download requests for that song. The parameter α_2 measures the effect of potential download request level on the growth of demand. For a potential download request level of 100, the average growth rate of demand decreases by about 1.5 for the overall sample.

On the supply-side, the parameter γ denotes the effect of current supply on its own growth rate. The parameter shows that on average, the supply level has a positive impact on the growth rate of supply. For example, a supply level of 100 nodes will increase the growth rate of supply by 0.8 for the overall sample. This measures the retention rate, the effect of current supplying level on its change rate due to the outflux of free-riders. The parameter β_1 denotes the effect of download request level on the growth rate of supply. The results confirm that a download request leads to an increase of supply for a song. A download request level of 100 would lead to an average increase in supply node growth rate by 3.5 for the overall sample. Lastly, the parameter β_2 accounts for the effect of potential supply for a song on the growth rate of supply. For a potential supply of 100 nodes, the average growth rate is about 0.5 for the whole sample.

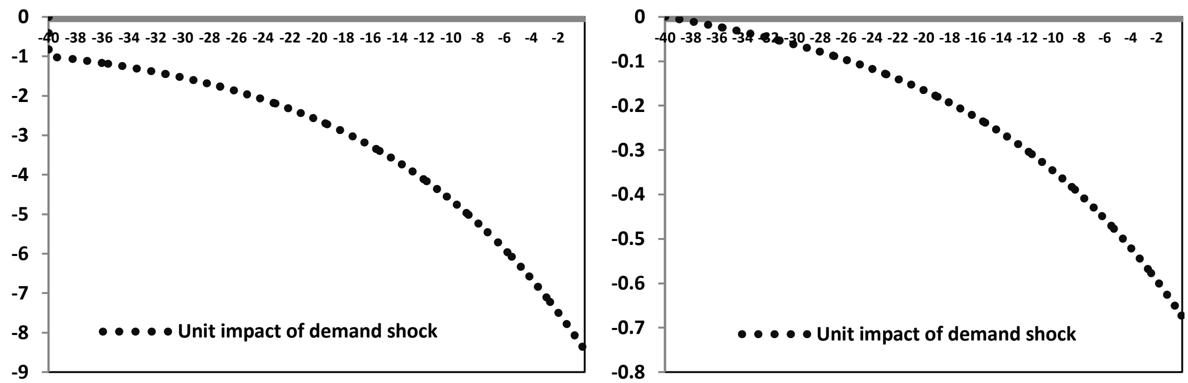
4-4. Model Validation: Numerical Results on Analytical Properties

We validate the model with a numerical simulation using the estimated parameters in Table 4. We utilize two types of numerical methods throughout the study. One is grounded on the analytical results; we assign the estimated parameter value on the functional form derived from the analytic solution (1)'-(2)'. Another is a numerical approximation on the equations (1)-(2); we recover the level curves from the first derivative curves obtained from assigning estimated value of parameters in the equation. We approximate the incremental level change ($\Delta D(t)$, $\Delta S(t)$) as the value of first derivative at time t multiplied by Δt, $((dD(t)/dt)*\Delta t)$ on a fine grid of time t.

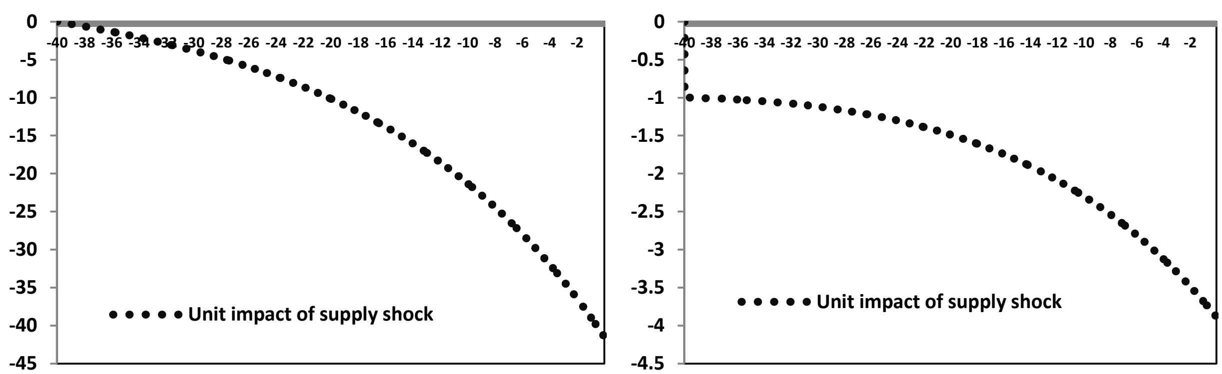
The first approach, calculating the exact amount of functional form derived from analytical solution can be utilized only when we have a closed-form of the expression, e.g., the impact of initial condition on the levels shown in Proposition I. In turn, the second approach, a numerical approximation can be applied to a broad range of analysis when there is no explicit form to analyze, e.g., impact of percentage shock in the middle of the period. We found that both numerical methods give us almost identical result. We employ both numerical methods in policy scenario analyses where we cannot derive and measure explicit form of the impacts of shocks.

Figure 5 and Figure 6 illustrate the result of Proposition I-1 and I-2 using the numerical approximation methods. The impact of initial reduction positively lasts through the time period. This result validates our Proposition I-1. In addition, the effect of initial one unit cut of demand on supply level is a decreasing function over time as shown in Figure 7-1. In turn, the effect of supply initial condition on demand level is a decreasing function over time. This result confirms the Proposition I-2.

[Figure 5: Impact of Demand Initial Condition on Demand- and Supply- side Piracy Level]



[Figure 6: Impact of Supply Initial Condition on Demand- and Supply- side Piracy Level]



4-5. Policy Scenario Analysis: Mean Level

We perform policy scenario analysis by introducing the effect of 1 % decrease shock in the file demand and supply level at different time points. Note that the impacts of a shock vary on the timing of policy interruption. Analyses on the timing of a policy shock provide an important criterion to evaluate the effectiveness of piracy policy. We illustrate the subsequent impact of a shock in a certain time point on the demand and supply level during the pre-release period. Here, we address a unit shock at the earliest time point about a month prior and mid-period shock about two weeks prior to the release date. Analytical results provide us the effect of incremental change only on initial condition of level. Without this numerical study, we cannot estimate the effects of various sizes and the timing of shocks. The equation (7) represents the subsequent impact of demand change on demand level during the period after the shock. To measure the total impact of an incremental shock on about 40 days prior on the release date, we will sum up the subsequent effect on demand at each time period and divide by the cumulative level on the release date.

$$\frac{\int_{t=-40}^{0} dD(t)/dt}{\int_{t=-40}^{0} D(t)} \qquad (7)$$

Impacts of Initial Period and Mid-Period Shock: Mean level

[Figure 7: 10% Decrease Demand Shock Effect from t= -40 to t=0 on Demand and Supply Level]

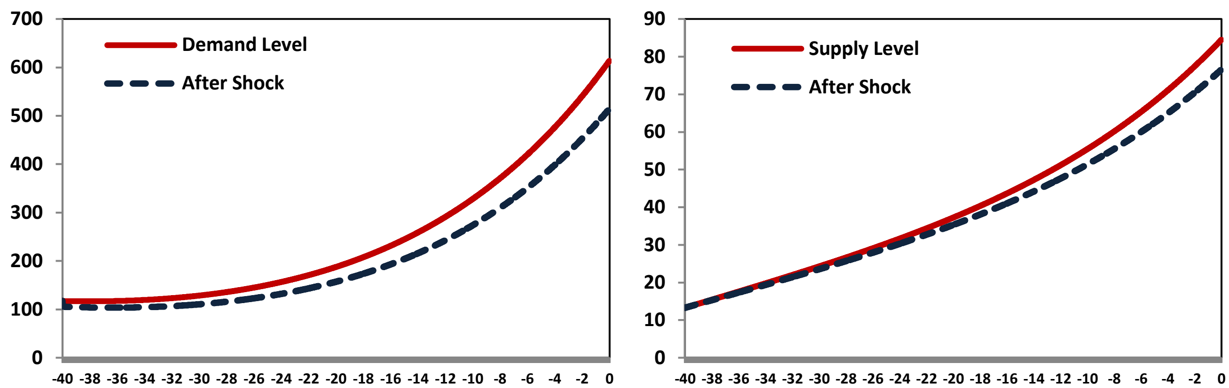


Figure 7 demonstrates the subsequent impact of 10% demand decrease on the initial level of demand and supply over time. The effect of initial period shock sustains over the time period. In addition, the sizes of incremental effect on demand and supply level are captured in Figure 8. While the effects of a shock gradually increase over time on demand-level, its impacts on supply-level increase more sharply over time. The effect of 10% decrease in demand shock ranges from 10% to 17% while its impact on supply level ranges from 0 to 9% on the flow level. The total effect of 10% decrease in demand level 40 days prior on the cumulative demand level at the release date is about -15.74%. The total impact of 10% decrease in demand level on the cumulative supply level is about -6.31%.

[Figure 8: Incremental Demand and Supply Level Change after 10% Decrease Demand Shock Effect]

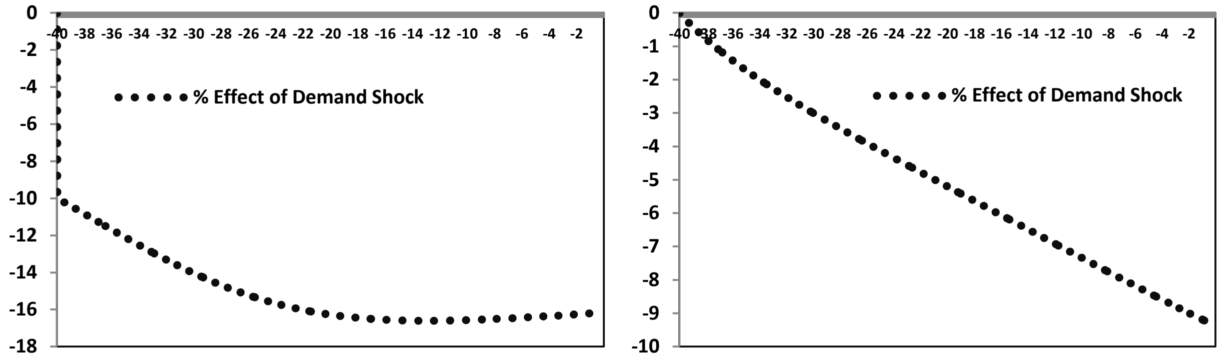
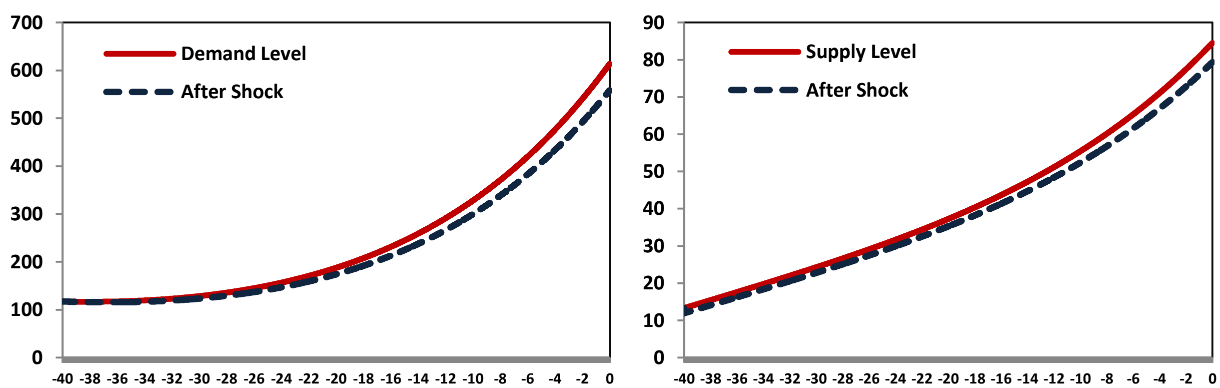
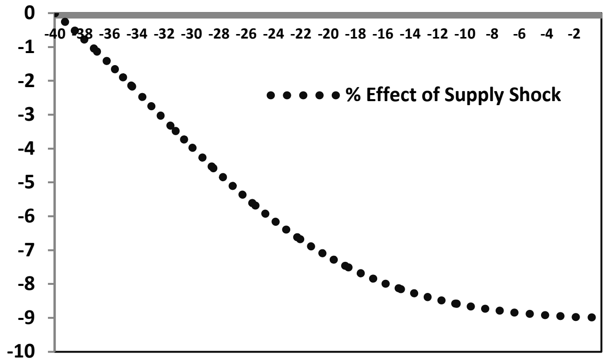


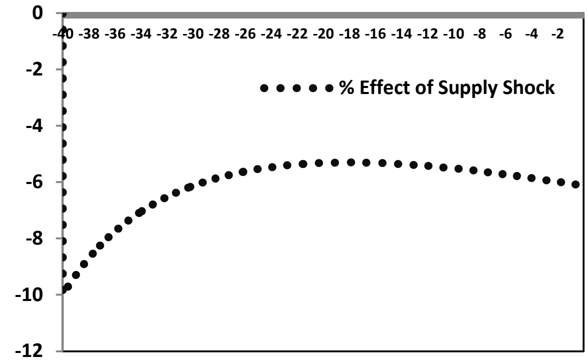
Figure 9 exhibits the subsequent effect of 10% supply reduction that occurred about 40 days prior to the release date on demand and supply level. The effect of supply shock on subsequent periods remains considerably on demand level during the period. Figure 10 shows the impact of subsequent impact of 10% supply shock on the flow-level of demand and supply over time. The subsequent impact of 10% decrease in supply level decreases the demand level up to 9% on its flow level. The aggregate effect of 10% decrease in supply level on initial period on the cumulative demand level at the release date is about -7.36%. The effect of initial supply reduction on the supply level dies out over time. The total effect of 10% decrease in supply level on the cumulative supply at the release date is -5.83%.

[Figure 9: 10% Decrease Supply Shock Effect from t= -27 to t=0 on Demand and Supply Level]



[Figure 10: Incremental Demand and Supply Level Change after 10% Decrease Supply Shock Effect]





Notice that introducing only one-side reduction shock of piracy policy influence both sides of evolution in piracy level as shown in Figure 9 and Figure 10. This result shows the importance of cross-influential parameters, α_1 and β_1 in piracy propagation.

Mid Period Shock Effect: Mean level

The dynamics between demand- and supply rate can be different in the initial period and in the mid period during the pre-release period. We measure the effect of policy shock in different timing to evaluate the effectiveness of policy implementation. Table 5 summarizes the effect of 10% demand shock on the cumulative demand and supply level when the reduction shock is introduced around three week prior to the release. We found that the effect of 10% demand shock on initial period is larger on both the demand and supply level than its impact of mid-period shock. In turn, the impact of 10% supply shock introduced on the mid-period is greater than the impact of initial period shock on the supply-level. The result implies that is critical to reduce demand-side piracy as early as possible. In turn, employing supplying-reduction policy can be more effective in the middle period than the initial period; the effect of 1% supply reduction in the middle period is around 7.5% on the cumulative supply level while the impact of initial reduction is 5.8%.

[Table 5: The Impact of 10% Shock on Cumulative Demand and Supply Level at the Release date]

Type of Shock	Types	Level Effect	Initial Period	Mid Period
Demand Shock (-10%)	Mean Level	Demand	-15.742%	-8.661%
		Supply	-6.307%	-2.259%
Supply Shock (-10%)	Mean Level	Demand	-7.357%	-4.476%
		Supply	-5.829%	-7.547%

V. Results and Discussions

This paper examines the propagation dynamics of pre-release piracy in P2P networks. Our study deals with three important issues in management: identifying pre-release penetration patterns for digital products, deriving the demand and supply relationship of piracy and providing informative measures to evaluate anti-piracy policies in early stages of digital products life. We demonstrate that the underlying patterns of piracy propagation differ among songs with respect to their popularity and also exhibit different dynamics between demand and supply-side propagation.

More precisely, the current approach enables us to identify patterns in how pre-release piracy propagates over time and how well differences in the characteristics of songs account for the variances in the mean pattern. Generalized patterns of songs estimated in this study also offers an explicit expression for the speed measures of demand and supply-side piracy propagation. We can examine how propagation speed differs among songs with different sets of antecedents such as the nature of products, level of diverse anti-piracy efforts, and marketing mix variables. Empirical evidence on the effect of product characteristics and anti-piracy measures on propagation speed can help managers and policy makers to use generalized empirical patterns to make forecasts when they have only very limited sets of data. Our functional approach utilizes the minimum data points that are available during pre-release period and generates a generalized form of distribution for demand and supply-side piracy to provide a framework that enhances market forecasts and policy decisions.

We found that the demand impact on demand level is more significant; decrease of a 1% file demand a month prior to the release will cut about 1.57% of file demand afterwards before release date. We found that it is more critical to curb piracy at the early stage to reduce both demand- and supply-side of piracy. In turn, employing supplying-reduction policy can be more effective in the middle period than the initial period; the effect of 1% supply reduction in the middle period is around 0.7% on the cumulative supply level while the impact of initial reduction is 0.6%. These results underline the importance of the timing of the anti-piracy regulatory activities. Especially, out results draw attention to the importance of demand control efforts to curb prerelease piracy propagation at the early period. While the economic theory posits several factors that might affect the demand of durable consumer products including initial adoptions, it is not clear what economic variables and environmental factors derive the demand and supply of digital information goods. Extending the current model to account for the unique characteristics of digital products will also provide meaningful insights into the emergence of piracy propagation trends.

VI. Conclusions and Implications

Online piracy of digital information goods has become an increasingly complex phenomenon in recent years. New information technologies have increased the variety of distribution channels available to consumers. At the same time, technologies have also raised the possible tools of redistribution. The general question addressed here is what different aspects of current file sharing phenomenon have evolved, compared to the long-existing copying and piracy phenomenon, and whether there exist any unique laws which address evolving online piracy. Specifically, we focus on the propagation features of piracy in P2P networks. Given the ease with which many files can be shared in online networks, not all of them can be traced. Starting with the first request for a specific song, the first occurrence of an unauthorized copy, the subsequent growth path of piracy propagation process can be traced in P2P networks. A firmer understanding of the dynamics of piracy propagation in P2P networks generates many implications to the theories of information goods and strategies to the content industry. Although companies cannot directly observe the piracy propagation process, they need not ignore it when planning for launch of new products and making predictions about consumers' preference for information goods.

There are several major findings of this paper. First, piracy propagation processes can be well characterized by current growth rate, availability rate, conversion rate, retention rate and new infusion rate. Our model represents known characteristics of file-sharing behavior and topological properties of P2P systems. We take into account the supply-side impact of availability in the demand-side process, which has been spotlighted in current anti-piracy efforts. Not only is the demand-process influenced by the supply-side, but the supply-side also critically affects the demand-process. Overall, we demonstrate that the underlying patterns of piracy propagation differ among songs with respect to their popularity and also exhibit different dynamics in piracy propagation.

Second, our results suggest that taking anti-piracy measure early on is important. From the policy scenario analysis, we found that the impact of demand shock to reduce piracy demand is at least 15 times larger than the impact of supply decrease shock. Especially, it is most efficient to focus anti-piracy efforts to reduce demand-side of piracy for popular titles in early periods. In addition, and potentially more controversial, our results provide support to the notion that in order to reduce piracy, punishing end-users is more effective than efforts to control the supply of unauthorized music files.

A major theoretical contribution of this study is to configure an underlying mechanism that specifically applies to the propagation of piracy. The conventional law of supply and demand implies that they will be adjusted based on the movement of price. Demand may not necessarily be satisfied by supply if the price remains rigid. If an excess demand exists, a conventional market system can reach equilibrium by price movement. While price does not exist in file sharing systems, there is a unique mechanism which resolves an excess demand. In P2P networks, a fulfillment of file demand by a supplier will be multiplied by the instantaneous sharing and copying of anonymous others. At the same time, multiplied numbers of suppliers bring about the fulfillment of additional demand. Our model suggests a unique law of supply and demand where demand for files increases file supply, and higher availability also facilitates fulfillment of file requests. As a result, demand produces supply and supply also drives demand in P2P piracy. An implication of this particular law is that an existing excess demand will be resolved in file-sharing systems by producing more supply responding to the demand generation.

This new law of supply and demand is not only constrained to file sharing activities in P2P systems. In fact, this reinforcing law of supply and demand can be applied to any form of online networks that provides a free connection to multiple users, and allows sharing of digital goods. Sharing a link of a products' website that is of interest on one's Facebook page, for example, is an instance of sharing a digital good. The link will be instantly viewed and shared by many users, while interested viewers will also request the link in other places at the same time. In this process, supplying a free link satisfies potential demand for the link and instantaneously propagates the supply of link. Thereby, sharing a products' link on a social networking site, or posting on a personal website generates potential demand and supply. In fact, propagation process of many digital goods can be studied in this framework.

An interesting direction for future study is to understand how new technology transforms existing economic assumptions and alters the conventional law of supply and demand. For instance, the wide spread use of Cloud Computing services will alter the conventional way of consuming, storing and sharing digital content. Individuals may not aim to download and store the song files on hardware devices; rather users flexibly migrate over multiple devices and manage the content products by virtually connecting to storage utilities without downloading them to hardware. Thereby, consumption of information goods can be less subject to the types of hardware platform, compatibility, and quality features. This decoupling of content from the physical platform influences the propagation of piracy. Many more technology will be introduced as P2P system or Cloud Computing will continue to in the future. New digital platforms will enable new behavior and inhibit others. This phenomenon may alter the fundamental economic assumptions and principles of markets, hierarchies, and networks in ways that we have not yet determined nor understood. These changes brought about by new digital technologies provide many new rich opportunities for research around digital piracy, information products, and digital distribution.

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VIII. Appendix

Part I. Proof of the solution of the differential equation system in (1)'-(2)'

The system of differential equation (1)-(2) has the following solution:

$$D(t) = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{12} e^{\lambda_2 t} + c_3 v_{11} \frac{1}{\lambda_1}\left(e^{\lambda_1 (t-t_0)} - 1\right) + c_4 v_{12} \frac{1}{\lambda_2}\left(e^{\lambda_2 (t-t_0)} - 1\right)$$

$$S(t) = c_1 v_{21} e^{\lambda_1 t} + c_2 v_{22} e^{\lambda_2 t} + c_3 v_{21} \frac{1}{\lambda_1}\left(e^{\lambda_1 (t-t_0)} - 1\right) + c_4 v_{22} \frac{1}{\lambda_2}\left(e^{\lambda_2 (t-t_0)} - 1\right)$$

where $\Delta = (\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1$,

$$c_1 = \frac{\hat{\alpha}_1}{\sqrt{\Delta}}\left(v_{22} D(0) - v_{12} S(0)\right), \quad c_2 = \frac{\hat{\alpha}_1}{\sqrt{\Delta}}\left(-v_{21} D(0) + v_{11} S(0)\right)$$

$$c_3 = \frac{\hat{\alpha}_1}{\sqrt{\Delta}}\left(v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}\right), \quad c_4 = \frac{\hat{\alpha}_1}{\sqrt{\Delta}}\left(-v_{21}\alpha_2 \overline{D} + v_{11}\beta_2 \overline{S}\right)$$

$$\lambda_1 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 - \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}) \qquad (Av_1 = \lambda_1 v_1)$$

$$\lambda_2 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 + \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}) \qquad (Av_2 = \lambda_1 v_2)$$

$$v_1 = k_1\left[1 \ , \ \frac{1}{2\hat{\alpha}_1}\left(\gamma - \beta_2 - p + \alpha_2 - \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}\right)\right]^T$$

$$v_2 = k_2\left[1 \ , \ \frac{1}{2\hat{\alpha}_1}\left(\gamma - \beta_2 - p + \alpha_2 + \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}\right)\right]^T$$

The solution of the differential equation system can be obtained as follows. The eigen-values and eigen-vectors denoted as λ_S and v_S are presented at the end of this part.

Let $y(t) = (D(t), S(t))^T$.

$$\frac{dy(t)}{dt} = \begin{pmatrix} p - \alpha_2 & \hat{\alpha}_1 \\ \hat{\beta}_1 & \gamma - \beta_2 \end{pmatrix} y(t) + \begin{pmatrix} \alpha_2 \overline{D} \\ \beta_2 \overline{S} \end{pmatrix}, \quad A = \begin{pmatrix} p - \alpha_2 & \hat{\alpha}_1 \\ \hat{\beta}_1 & \gamma - \beta_2 \end{pmatrix} \ , \ f(t) = \begin{pmatrix} \alpha_2 \overline{D} \\ \beta_2 \overline{S} \end{pmatrix}$$

Let the matrix $Q = (v_1 \ v_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ and $Q^{-1} = \dfrac{1}{v_{11}v_{22} - v_{12}v_{21}} \begin{pmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{pmatrix}$.

$$y(t)' = \begin{pmatrix} p - \alpha_2 & \hat{\alpha}_1 \\ \hat{\beta}_1 & \gamma - \beta_2 \end{pmatrix} y(t) + \begin{pmatrix} \alpha_2 \overline{D} \\ \beta_2 \overline{S} \end{pmatrix} = A\, y(t) + F \ = QDQ^{-1} y(t) + F$$

$Q^{-1} y' = DQ^{-1} y(t) + Q^{-1}F$ Then

$$Q^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + Q^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Let $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\tilde{F} = Q^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ Then $Q^{-1} y' = DQ^{-1} y(t) + Q^{-1}F$

Then $Z' = DZ + \tilde{F}$ where $Z = Q^{-1} y$ and $\tilde{F} = Q^{-1}F$ $\tilde{F} = Q^{-1}F$

$$Z = e^{Dt} \cdot Z_0 + \int_0^t e^{D(t-s)} \tilde{F}(s)ds$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} \cdot Z_1(0) \\ e^{\lambda_2 t} \cdot Z_2(0) \end{pmatrix} + \begin{pmatrix} \int_0^t e^{\lambda_1(t-s)} \tilde{f}_1(s)\, ds \\ \int_0^t e^{\lambda_1(t-s)} \tilde{f}_2(s)\, ds \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q\begin{pmatrix} e^{\lambda_1 t} \cdot Z_1(0) \\ e^{\lambda_2 t} \cdot Z_2(0) \end{pmatrix} + Q\begin{pmatrix} \int_0^t e^{\lambda_1(t-s)} \tilde{f}_1(s)\, ds \\ \int_0^t e^{\lambda_1(t-s)} \tilde{f}_2(s)\, ds \end{pmatrix}$$

$$= \frac{Q}{v_{11}v_{22} - v_{12}v_{21}} \begin{pmatrix} e^{\lambda_1 t} \cdot (v_{22} y_1(0) - v_{12} y_2(0)) \\ e^{\lambda_2 t} \cdot (- v_{21} y_1(0) + v_{11} y_2(0)) \end{pmatrix} + \frac{Q}{v_{11}v_{22} - v_{12}v_{21}} \begin{pmatrix} \int_0^t e^{\lambda_1 (t-s)} (v_{22} f_1(s) - v_{12} f_2(s)) ds \\ \int_0^t e^{\lambda_1 (t-s)} (- v_{21} f_1(s) + v_{11} f_2(s)) ds \end{pmatrix}$$

Since $\int_0^t e^{\lambda_1 (t-s)} (v_{22} f_1(s) - v_{12} f_2(s)) ds = \int_0^t e^{\lambda_1 (t-s)} (v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}) ds = (v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}) \int_0^t e^{\lambda_1 (t-s)} ds$

$$= (v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}) \left(-\frac{1}{\lambda_1} \right) [e^{\lambda_1 (t-s)}]_{t_0}^{t} = \left(\frac{1}{\lambda_1} \right) (v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}) (e^{\lambda_1 (t-t_0)} - 1)$$

$$\int_0^t e^{\lambda_2 (t-s)} (- v_{21} f_1(s) + v_{11} f_2(s)) ds = \left(\frac{1}{\lambda_2} \right) (- v_{21}\alpha_2 \overline{D} + v_{11}\beta_2 \overline{S}) (e^{\lambda_2 (t-s)} - 1)$$

We have following solution:

$$y_1(t) = c_1 v_{11} e^{\lambda_1 t} + c_2 v_{12} e^{\lambda_2 t} + c_3 v_{11} \frac{1}{\lambda_1} (e^{\lambda_1 (t-t_0)} - 1) + c_4 v_{12} \frac{1}{\lambda_2} (e^{\lambda_2 (t-t_0)} - 1)$$

$$y_2(t) = c_1 v_{21} e^{\lambda_1 t} + c_2 v_{22} e^{\lambda_2 t} + c_3 v_{21} \frac{1}{\lambda_1} (e^{\lambda_1 (t-t_0)} - 1) + c_4 v_{22} \frac{1}{\lambda_2} (e^{\lambda_2 (t-t_0)} - 1)$$

Where

$$c_2 = \frac{1}{2\sqrt{\Delta}} (v_{2s} y_1(0) - v_{12} y_2(0)) , \quad c_2 = \frac{1}{2\sqrt{\Delta}} (- v_{21} y_1(0) + v_{11} y_2(0))$$

$$c_3 = \frac{1}{2\sqrt{\Delta}} (v_{22}\alpha_2 \overline{D} - v_{12}\beta_2 \overline{S}) , \quad c_4 = \frac{1}{2\sqrt{\Delta}} (- v_{21}\alpha_2 \overline{D} + v_{11}\beta_2 \overline{S})$$

The eigen-values and eigen-vectors can be obtained as follows.

$$|A - \lambda E| = \begin{vmatrix} p - \alpha_2 & \hat{\alpha}_1 \\ \hat{\beta}_1 & \gamma - \beta_2 \end{vmatrix} = \lambda^2 - (\gamma - \beta_2 + p - \alpha_2)\lambda + (p - \alpha_2)(\gamma - \beta_2) - \hat{\alpha}_1 \hat{\beta}_1$$

$$\Delta = (\gamma - \beta_2 + p - \alpha_2)^2 - 4(p\gamma - p\beta_2 - \alpha_2\gamma + \alpha_2\beta_2 - \hat{\alpha}_1 \hat{\beta}_1)$$

$$= \alpha_2^2 + \beta_2^2 + p^2 + \gamma^2 - 2p\gamma + 2p\beta_2 + 2\alpha_2\gamma - 2\alpha_2\beta_2 - 2\beta_2\gamma - 2p\alpha_2 + 4\hat{\alpha}_1 \hat{\beta}_1$$

$$\Delta = (\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1 \hat{\beta}_1 \geq 0$$

Eigen-values are as follows:

$$\lambda_1 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 - \sqrt{\Delta}) \qquad \lambda_2 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 + \sqrt{\Delta}) \qquad (\lambda_1 < \lambda_2)$$

$$\lambda_1 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 - \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}) \qquad (Av_1 = \lambda_1 v_1)$$

$$\lambda_2 = \frac{1}{2}(\gamma - \beta_2 + p - \alpha_2 + \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}) \qquad (Av_2 = \lambda_1 v_2)$$

Eigenvectors are as follows:

$$v_1 = k_1\left[1 \quad , \quad \frac{1}{2\alpha_1}\left(\gamma - \beta_2 - p + \alpha_2 - \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}\right)\right]^T$$

$$v_2 = k_2\left[1 \quad , \quad \frac{1}{2\alpha_1}\left(\gamma - \beta_2 - p + \alpha_2 + \sqrt{(\gamma - \beta_2 - p + \alpha_2)^2 + 4\hat{\alpha}_1\hat{\beta}_1}\right)\right]^T$$

Part II-1. Proof of potential demand scalability

Let $D(t),\ S(t)$ denote the solution to the system of equations (1)-(2). Given the parameters $(p, \alpha_1, \alpha_2, \gamma, \beta_1, \beta_2, \overline{D}, \overline{S})$ and constant $c > 0$, $\tilde{D}(t) = cD(t),\ S(t)$ solve the following system of equations (1)'-(2)'.

$$(1) \quad \frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{\overline{S}} S(t) + \alpha_2[\overline{D} - D(t)]$$

$$(2) \quad \frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\overline{S}}{\overline{D}} D(t) + \beta_2[\overline{S} - S(t)]$$

$$(1)' \quad \frac{d\tilde{D}(t)}{dt} = p\tilde{D}(t) + \alpha_1 \frac{c\overline{D}}{\overline{S}} S(t) + \alpha_2[c\overline{D} - \tilde{D}(t)]$$

$$(2)' \quad \frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\overline{S}}{c\,\overline{D}} \tilde{D}(t) + \beta_2[\overline{S} - S(t)]$$

Proof) First, we show that $\tilde{D}(t) = cD(t),\ S(t)$ satisfies the equation (1)'.

Multiplying both sides of the equation (1) by the constant $c > 0$, we obtain :

$$c\frac{dD(t)}{dt} = pcD(t) + \alpha_1 \frac{c\overline{D}}{\overline{S}} S(t) + \alpha_2[c\overline{D} - cD(t)]$$

which is identical to (1)' :

$$\frac{d[cD(t)]}{dt} = p[cD(t)] + \alpha_1 \frac{c\overline{D}}{\overline{S}} S(t) + \alpha_2 [c\overline{D} - [cD(t)]]$$

because $\tilde{D}(t) \equiv cD(t)$.

Therefore, $\tilde{D}(t) = cD(t),\ S(t)$ satisfies the equation (1)'.

Next, we prove that $\tilde{D}(t) = cD(t),\ S(t)$ satisfies the equation (2)'.

Note that the equation (2) implies the following :

$$\frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\overline{S}}{c\overline{D}} cD(t) + \beta_2 [\overline{S} - S(t)]$$

Using $\tilde{D}(t) \equiv cD(t)$, the above equation becomes as follows.

$$\frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\overline{S}}{c\overline{D}} \tilde{D}(t) + \beta_2 [\overline{S} - S(t)]$$

which is identical to the equation (2)'. Therefore, we have shown that $\tilde{D}(t) = cD(t),\ S(t)$ satisfies the equation (2)'.

Having shown that $\tilde{D}(t) = cD(t),\ S(t)$ satisfies the two equations (1)' and (2)', we conclude that $\tilde{D}(t) = cD(t),\ S(t)$ solves the system of equations (1)' and (2)'.

Part II-2. Proof of potential supply scalability

Let $D(t),\ S(t)$ denote the solution to the system of equations (1)-(2). Given the parameters $(p, \alpha_1, \alpha_2, \gamma, \beta_1, \beta_2, \overline{D}, \overline{S})$ and constant $c > 0$, $D(t), \tilde{S}(t) = cS(t)$ solve the following system of equations (1)'-(2)'.

$$(1) \quad \frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{\overline{S}} S(t) + \alpha_2 [\overline{D} - D(t)]$$

$$(2) \quad \frac{dS(t)}{dt} = \gamma S(t) + \beta_1 \frac{\overline{S}}{\overline{D}} D(t) + \beta_2 [\overline{S} - S(t)]$$

$$(1)' \quad \frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{c\overline{S}} \tilde{S}(t) + \alpha_2[\overline{D} - D(t)]$$

$$(2)' \quad \frac{d\tilde{S}(t)}{dt} = \gamma\tilde{S}(t) + \beta_1 \frac{c\overline{S}}{\overline{D}} D(t) + \beta_2[c\overline{S} - \tilde{S}(t)]$$

Proof) First, we prove that $D(t),\ \tilde{S}(t) = cS(t)$ satisfies the equation (1)'.

The equation (1) implies the following:

$$\frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{c\overline{S}} cS(t) + \alpha_2[\overline{D} - D(t)]$$

Using $\tilde{S}(t) \equiv cS(t)$, the above equation becomes as follows.

$$\frac{dD(t)}{dt} = pD(t) + \alpha_1 \frac{\overline{D}}{c\overline{S}} \tilde{S}(t) + \alpha_2[\overline{D} - D(t)]$$

which is identical to the equation (1)'. Therefore, we have shown that $D(t),\ \tilde{S}(t) = cS(t)$ satisfies the equation (1)'.

Next, we show that $D(t),\ \tilde{S}(t) = cS(t)$ satisfies the equation (2)'.

Multiplying both sides of the equation (2) by the constant $c > 0$, we obtain :

$$c\frac{dS(t)}{dt} = \gamma cS(t) + \beta_1 \frac{c\overline{S}}{\overline{D}} D(t) + \beta_2[c\overline{S} - cS(t)]$$

which is identical to (2)' :

$$\frac{d[cS(t)]}{dt} = \gamma[cS(t)] + \beta_1 \frac{c\overline{S}}{\overline{D}} D(t) + \beta_2[c\overline{S} - [cS(t)]]$$

because $\tilde{S}(t) \equiv cS(t)$.

Therefore, $D(t),\ \tilde{S}(t) = cS(t)$ satisfies the equation (2)'.

Having shown that $D(t),\ \tilde{S}(t) = cS(t)$ satisfies the two equations (1)' and (2)', we conclude that $D(t),\ \tilde{S}(t) = cS(t)$ solves the system of equations (1)' and (2)'.