State-Dependent Intellectual Property Rights Policy*

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Abstract

Should a company with a large technological lead receive the same intellectual property rights (IPR) protection as a company with a more limited lead? We develop a general equilibrium framework to investigate this question. The economy consists of many industries and firms engage in cumulative (step-by-step) innovation. IPR policy regulates whether followers in an industry can copy the technology of the leader and also how much they have to pay if they build upon past innovations. With full patent protection, followers can only catch up to the leader in their industry by making the same innovation(s) themselves, whereas without full patent protection they may be able to copy the leading-edge technology. IPR policy creates two opposing effects. The first is a disincentive effect; relaxing full patent protection diminishes R&D incentives. The second is a composition effect; less than full patent protection makes it more likely that followers catch up to the leaders, bringing them "neck-and-neck", whereby each party undertakes greater R&D in order to surge ahead.

We prove the existence of a steady-state equilibrium and characterize some of its properties. We then quantitatively investigate the form of growth-maximizing ("optimal") IPR policy under three different variants of the model. In the first, there is quick catch-up, because followers can reach the leading-edge technology with a single innovation. In this case, most industries have a relatively small gap between leaders and followers. In this quick catch-up regime, uniform IPR policy leads to a very small increase in the equilibrium rate of growth of the economy because of the disincentive effects. In contrast, state-dependent IPR, which makes the extent of patent protection a function of the technological lead in the industry, can lead to significant growth gains. Growth-maximizing state-dependent IPR policy provides greater protection to firms that have a greater technological lead. This is because the prospect of high and secure profits for a leader that achieves a large technological lead creates a trickle-down effect and encourages R&D also by firms with smaller technological leads. The second environment has slow catch-up and followers need to advance one step at a time. In this case, most industries have a large gap between leaders and followers. Growth-maximizing IPR policy now involves more significant departures from full patent protection, but the trickle-down effect is still operational and implies stronger patent protection for firms that are technologically more advanced relative to their rivals. State-dependent IPR again generates significantly faster growth than optimal uniform IPR. Finally, we analyze a version of the model in which followers can build upon past innovations by paying a patent fee and investigate the implications of IPR policy modeled as the amount of patent fees followers have to make. Once again we find that trickle-down of incentives is important and growth-maximizing IPR policy affords greater protection to firms that are further ahead of their rivals.

Keywords: competition, economic growth, industry structure, intellectual property rights, patents.

JEL classification: O31, O34, O41, L16.

1 Introduction

Should a company with a large technological lead receive the same intellectual property rights (IPR) protection as a company with a more limited technological lead? This question is central to many discussions of patent policy (for example, to the debate on whether Microsoft or iPod should make their code available to competitors that are producing complementary products), and requires a framework for the analysis of *state-dependent* patent/IPR protection policy. By state-dependent IPR policy, we mean a policy that makes the extent of patent or intellectual property rights protection conditional on the technology gap between different firms in the industry. Existing work has investigated the optimal length and breadth of patents assuming a *uniform* patent/IPR policy. In this paper, we make a first attempt to develop a framework for state-dependent IPR policy, focusing on the length of patent protection.

Our basic framework builds on and extends the step-by-step innovation models of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001), where a number (typically two) of firms engage in price competition within an industry and undertake R&D in order to improve the quality of their product. The quality (technology) gap between the firms determines the price markups and profits. The purpose of R&D by the follower is to catch up and surpass the leader (as in standard Schumpeterian models of innovation, e.g., Reinganum, 1981, 1985, Aghion and Howitt, 1992, Grossman and Helpman, 1991), while the purpose of R&D by the leader is to escape the competition of the follower and increase its markup and profits. As in racing-type models in general (e.g., Harris and Vickers, 1987, Budd, Harris and Vickers, 1993), a large gap between the leader and the follower discourages R&D by both, and overall R&D and technological progress are greatest when the leader and the follower are "neck-and-neck".[1] Consequently, one may expect that full patent protection may be suboptimal in a world of step-by-step competition; by stochastically or deterministically allowing the follower to use the innovations of the technological leader, the likelihood of neck-and-neck competition and thus the amount of R&D may be raised. Based on this intuition, one may further conjecture that state-dependent IPR policy will also be useful and should afford less protection to firms that are technologically more advanced relative to their competitors.

We construct a general equilibrium model with step-by-step innovation to investigate these questions. We consider three variants of this framework. In the first, the *quick catch-up regime*, followers can catch up to the leading-edge technology with one innovation and then

[1] Aghion, Bloom, Blundell, Griffith and Howitt (2005) provide empirical evidence that there is greater R&D in British industries where there is a smaller technological gap between firms.

can become the leader with an additional innovation. In the second, the *slow catch-up regime*, followers need to advance step-by-step. In the third, the *equilibrium with patent fees*, we allow followers to build upon previous innovations by paying pre-determined patent (license) fees to the innovators of the previous leading-edge technologies, and if they choose to do so, they can surpass the previous frontier technology with a single innovation (without first catching up to the frontier). Each of these variants captures different aspects of an industry consisting of a leader and a follower engaged in product market and R&D competition.[2] In all three regimes, relaxation of full patent protection corresponds to allowing followers to use the technologies of the leaders (without any costs or at low costs).

We prove the existence of a stationary (steady-state) equilibrium and characterize a number of features of the equilibrium analytically under all three scenarios. For example, we prove that with uniform IPR policies, R&D investments decline when the gap between the leader and the follower increase, and the highest R&D takes place when the firms in the the industry are "neck-and-neck".

We then turn to a quantitative investigation of growth-maximizing ("optimal") IPR policy, by providing a number of simulations of the equilibrium for plausible parameter values and for different forms of IPR policies.[3] The equilibrium structure of industries is very different between these regimes. For example, in the quick catch-up regime, most industries have a small gap between the leader and the follower because followers can quickly innovate to the frontier technology and have strong incentives to do so. In contrast, in the slow catch-up regime, most industries have a large gap between the leader and the follower. This is both because followers can only advance slowly and because a large gap between leader and follower now discourages follower R&D more significantly (as it becomes very difficult for them to catch up and surpass the leader). Given these differences in industry structure, the level and structure of optimal IPR also differ between these two regimes.

In the quick catch-up regime, uniform IPR policy (patent protection independent of the technology gap between the leader and the follower) has limited beneficial effects, because it creates a large *disincentive effect* on R&D. Consequently, in most cases optimal uniform IPR policy is indistinguishable from full patent protection or involves a very small devia-

[2]See O'Donoghue, Scotchmer and Thisse (1998) for a discussion of how patent life may come to an end because of related innovations.

[3]Throughout, we simplify terminology by using the terms "optimal" and "growth-maximizing" interchangeably. It is well known that welfare-maximizing and growth-maximizing policies need not coincide, and in fact, in models with quality competition, the equilibrium may involve excessive innovation and growth (e.g., Aghion and Howitt, 1992). Nevertheless, when the discount rate is small, welfare and growth-maximizing policies will be close, which justifies our use of the term "optimal" in this context.

tion from this benchmark. In contrast, optimal state-dependent IPR policy can significantly increase innovation and growth in the economy. In our baseline parameterization, optimal state-dependent IPR policy increases the growth rate of the economy by about 0.1% per annum relative to full patent protection—while uniform IPR essentially has no effect on growth. State-dependent IPR policy achieves this both by exploiting the beneficial *composition effects* (by increasing the likelihood of neck-and-neck competition between the leader and the follower) and by creating largely beneficial incentive effects (reducing R&D by followers, but increasing R&D by leaders). The key to the success of state-dependent IPR policy is *trickle-down* of R&D incentives; when a particular state for the leader (say being n^* steps ahead of the follower) is very profitable, this not only increases the incentives to perform R&D for leaders that are $n^* - 1$ steps ahead, but for *all* leaders with a lead of size $n \leq n^* - 1$. Notably, the trickle-down effect can be powerful enough to *increase*—rather than reduce—the R&D investments by technological leaders relative to the benchmark with full patent protection. Thus, offering a high degree of protection to firms that are sufficiently advanced relative to their competitors is a potent way of boosting R&D incentives for a range of leading firms. Optimal state-dependent IPR is quite different from full patent protection, however, since it simultaneously exploits the composition effects by reducing patent protection afforded to firms with a small technological lead and increases the likelihood of neck-and-neck competition. Consequently, the trickle-down effect implies that, contrary to the conjecture above, the optimal state-dependent IPR policy provides *greater* protection to firms that have a larger technological lead.

The structure of optimal IPR policy is somewhat different in the slow catch-up regime, but the major economic insights from the quick catch-up regime continue to apply. When followers can only advance slowly, there will be large technology gaps between leaders and followers in most industries, especially because followers may become discouraged and stop performing R&D in this regime (see also Harris and Vickers, 1987). In this case, there is room for larger composition effects and optimal IPR may involve a greater relaxation of full patent protection, even for firms that are technologically further ahead than others. In particular, we show that with full patent protection, the equilibrium growth rate will often be equal to zero, whereas relaxation of full patent protection can increase the growth rate to positive levels. The optimal state-dependent IPR policy in this case does not involve full patent protection for firms that are technologically advanced relative to their rivals. Nevertheless, the trickle-down effect is still important and implies greater protection for more technologically advanced firms than those that are only one step ahead. Moreover, the growth rate under state-dependent IPR policy is now considerably larger than—about three times—the growth rate under uniform IPR.

The structure of optimal IPR policy is also qualitatively similar in the equilibrium with patent fees. The trickle-down of R&D incentives continues to be important and implies that the growth-maximizing IPR policy should provide greater protection to firms that are more advanced relative to their rivals. In this case, this type of state-dependent IPR can be implemented by setting higher patent fees for technologies with greater leads relative to the next best alternatives. State-dependent IPR again increases the equilibrium growth rate considerably more than uniform IPR.

Overall, our analysis suggests that optimal state-dependent IPR policy can significantly increase the growth rate of an economy relative to uniform IPR policy, thus the benefits of considering state-dependent policies could be substantial. Our analysis also shows that the structure of optimal IPR may depend on the equilibrium industry structure (which reflects the underlying technology of catch-up in the industry). A more detailed analysis of the relationship between industry structure and the optimal form of IPR policy is an area for future research.

Our paper is a contribution both to the literature on intellectual property rights and to endogenous growth. The industrial organization and growth literatures emphasize that ex-post monopoly rents and thus patents are central to generate the ex-ante investments in R&D and technological progress, even though monopoly power also creates distortions (e.g., Arrow, 1962, Reinganum, 1981, Tirole, 1988, Romer, 1990, Grossman and Helpman, 1991, Aghion and Howitt, 1992, Scotchmer, 1999, Gallini and Scotchmer, 2002, O'Donoghue and Zweimuller, 2004).[4]

Much of the literature discusses the trade-off between these two forces to determine the optimal length and breadth of patents. For example, Klemperer (1990) and Gilbert and Shapiro (1990) show that optimal patents should have a long duration in order to provide inducement to R&D, but a narrow breadth so as to limit monopoly distortions. A number of other papers, for example, Gallini (1992) and Gallini and Scotchmer (2002), reach opposite conclusions. Another branch of the literature, including the seminal paper by Scotchmer (1999) and the recent interesting papers by Llobet, Hopenhayn and Mitchell (2001) and Hopenhayn and Mitchell (2001), adopt a mechanism design approach to the determination of the optimal patent and intellectual property rights protection system. For example, Scotchmer (1999) derives the patent renewal system as an optimal mechanism in an environment where the cost and value of different projects are unobserved and the main problem is to decide which projects should

[4]Boldrin and Levine (2001, 2004) or Quah (2003) argue that patent systems are not necessary for innovation.

go ahead. To the best of our knowledge, no other paper in the literature has considered state-dependent patent or IPR policy, which is the focus of the current paper. As a first attempt, we only look at state-dependent optimal length of IPR protection (though similar ideas can be applied to an investigation of the gains from making the breadth of patent awards state-dependent).

Our paper also extends the results of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001) on endogenous growth with step-by step innovation. Although our model builds on these papers, it also differs in a number of respects. First and crucially, our economy includes uniform or state-dependent IPR policy as one of its parameters, and also considers the possibility of patent fees. Second, we consider both the quick catch-up and the slow catch-up regimes. Third, in our economy there is a general equilibrium interaction between production and R&D, since they both compete for scarce labor. Finally, we provide a number of analytical results for the general model (with or without IPR policy), while Aghion, Harris, Howitt and Vickers (2001) focus on the case where innovations are either "drastic" (so that the leader never undertakes R&D) or very small. They also do not prove the existence of a steady state or characterize the properties of the equilibrium in this class of economies.

The rest of the paper is organized as follows. Section 2 presents the basic environment focusing on the quick catch-up regime. Section 3 characterizes the form of the equilibrium in the case with uniform IPR policy (where the extent of patent protection does not depend on the technology gap between leaders and followers). Section 4 extends these results to the case in which IPR policy is state dependent. Section 5 presents the results for the slow catch-up regime. Section 6 provides an analysis of the equilibrium when followers can build on the leading-edge technology by paying a patent fee. Section 7, 8 and 9 quantitatively evaluate the form of the optimal state-dependent IPR policy and the contribution of such a policy to economic growth in the three different scenarios theoretically explored in the previous sections. Section 10 concludes, while the Appendix contains the proofs of all the results stated in the text.

2 Model

We now describe the basic environment. The characterization of equilibrium with uniform and state-dependent IPR policy is presented in the next three sections.

2.1 Preferences and Technology

Consider the following continuous time economy with a unique final good. The economy is populated with a continuum of 1 individuals, each with 1 unit of labor endowment, which they supply inelastically. Their preferences at time t are given by

$$\mathbb{E}_t \int_t^\infty \exp\left(-\rho\left(s-t\right)\right) \log C\left(s\right) ds, \tag{1}$$

where $\mathbb{E}_t$ denotes expectations at time t, $\rho > 0$ is the discount rate and $C\left(t\right)$ is consumption at date t. The logarithmic preferences in (1) facilitates the analysis by leading to a simple relationship between the interest rate, growth rate and the discount rate as given by (2) below.

Let $Y\left(t\right)$ be the total production of the final good at time t. We assume that the economy is closed and the final good is used only for consumption (i.e., there is no investment), so that $C\left(t\right) = Y\left(t\right)$. The standard Euler equation from (1) then implies that

$$g\left(t\right) \equiv \frac{\dot{C}\left(t\right)}{C\left(t\right)} = \frac{\dot{Y}\left(t\right)}{Y\left(t\right)} = r\left(t\right) - \rho, \tag{2}$$

where this equation defines $g\left(t\right)$ as the growth rate of consumption and thus output, and $r\left(t\right)$ is the interest rate at date t.

The final good Y is produced using a continuum 1 of intermediate goods according to the Cobb-Douglas production function

$$\ln Y\left(t\right) = \int_0^1 \ln y\left(j,t\right) dj, \tag{3}$$

where $y\left(j,t\right)$ is the output of jth intermediate at time t. Throughout, we take the price of the final good as the numeraire and denote the price of intermediate j at time t by $p\left(j,t\right)$. We also assume that there is free entry into the final good production sector. These assumptions together with the Cobb-Douglas production function (3) imply that each final good producer will have the following demand for intermediates

$$y\left(j,t\right) = \frac{Y\left(t\right)}{p\left(j,t\right)}, \quad \forall j \in \left[0,1\right]. \tag{4}$$

Each intermediate $j \in [0,1]$ comes in two different varieties, each produced by one of two infinitely-lived firms.[5] We assume that these two varieties are perfect substitutes and these

[5] Alternatively, we can assume that these two varieties are imperfect substitutes, for example, so that the output of intermediate j is given by

$$y\left(j,t\right) = \left[\varphi y_1\left(j,t\right)^{\frac{\sigma-1}{\sigma}} + \left(1-\varphi\right) y_2\left(j,t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

with $\sigma > 1$. This generalization has no effect on any of our qualitative results, but increases notation. We therefore prefer to focus on the case where the two varieties are perfect substitutes.

firms are assumed to compete a la Bertrand. Firm $i = 1, 2$ in industry j has the following technology

$$y_i(j,t) = q_i(j,t)\, l_i(j,t) \tag{5}$$

where $l_i(j,t)$ is the employment level of the firm and $q_i(j,t)$ is its level of technology at time t. The only difference between two firms is their technology, which will be determined endogenously.

The production function for intermediate goods, (5), implies that the marginal cost of producing intermediate j for firm i and industry j at time t is

$$MC_i(j,t) = \frac{w(t)}{q_i(j,t)} \tag{6}$$

where $w(t)$ is the wage rate in the economy at time t.

When this causes no confusion, we denote the *technological leader* in each industry by i and the follower by $-i$, so that we have:

$$q_i(j,t) \geq q_{-i}(j,t).$$

Bertrand competition between the two firms implies that all intermediates will be supplied by the leader at the "limit" price:[6]

$$p_i(j,t) = \frac{w(t)}{q_{-i}(j,t)}. \tag{7}$$

Equation (4) then implies the following demand for intermediates:

$$y(j,t) = \frac{q_{-i}(j,t)}{w(t)} Y(t). \tag{8}$$

2.2 Technology and R&D: Quick Catch-Up Regime

R&D by the leader or the follower stochastically leads to innovation. We assume that when the leader innovates, its technology improves by a factor $\lambda > 1$. When the follower innovates, there are three alternative scenarios. In the first, the *quick catch-up regime*, when followers successfully undertake an innovation, they jump to the leading-edge technology. In the second, the *slow catch-up regime*, they only advance by one step per innovation. In the third, *equilibrium with patent fees*, followers can choose to undertake R&D building on the leading-edge

[6] If the leader were to charge a higher price, then the market would be captured by the follower earning positive profits. A lower price can always be increased while making sure that all final good producers still prefer the intermediate supplied by the leader i rather than that by the follower $-i$ even if the latter were supplied at marginal cost. Since the monopoly price with the unit elastic demand curve is infinite, the leader always gains by increasing its price, making the price given in (7) the unique equilibrium price.

technology and if they are successful, they surpass the current leader. The cost of doing so is that they have to pay a pre-determined patent (license) fee to the original innovator of the technology (which is the previous leader in the industry). The fact that in the quick and slow catch-up regimes, followers first have to innovate up to the frontier can also be justified by appealing to the importance of tacit knowledge in innovation (e.g., Evenson and Westphal, 1995); without undertaking the R&D for a product and manufacturing that product, a firm may be unable to acquire the tacit knowledge necessary to improve upon it. In this and the next section, we focus on the quick catch-up regime and assume that one innovation by the follower is enough for catching up with the leader, but the follower cannot surpass the leader without first catching up. We discuss the slow catch-up regime in Section 5 and the equilibrium with patent fees in Section 6.

More specifically, suppose that leader i in industry j at time t has a technology level of

$$q_i(j,t) = \lambda^{n_{ij}(t)}, \tag{9}$$

and that the follower $-i$'s technology at time t is

$$q_{-i}(j,t) = \lambda^{n_{-ij}(t)}, \tag{10}$$

where $n_{ij}(t)$, $n_{-ij}(t) \in \mathbb{Z}_+$ and $n_{ij}(t) \geq n_{-ij}(t)$. We refer to $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ as the *technology gap* in industry j. If the leader undertakes an innovation within a time interval of Δt, then its technology increases to $q_i(j, t+\Delta t) = \lambda^{n_{ijt}+1}$ and technology gap rises to $n_j(t+\Delta t) = n_j(t) + 1$ (the probability of two or more innovations within the interval Δt will be $o(\Delta t)$, and thus for $\Delta t \to 0$, it will vanish, where $o(\Delta t)$ represents terms that satisfy $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t$).

Conversely, in the quick catch-up regime, when the follower undertakes an innovation within the interval Δt, then it catches up with the leader and its technology improves to $q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}}$ (and $n_{jt+\Delta t} = 0$). This assumption about the technology of catching up implies that followers always have the option to jump to the leading-edge technology. For this reason, as we will see in greater detail below, the quick catch-up regime will lead to a very different distribution of the technology gaps across industries than the slow catch-up regime.

Innovations follow a Poisson process, and firms can affect the rate of innovations by investing in R&D. Each firm (in every industry) has access to the following R&D technology:

$$x_i(j,t) = F(h_i(j,t)) \tag{11}$$

where $x_i(j,t)$ is the flow rate of innovation at time t and $h_i(j,t)$ is the number of workers hired by firm i in industry j to work in the R&D process at t. This specification implies that within a time interval of Δt, the probability of innovation for this firm is $x_i(j,t) \cdot \Delta t + o(\Delta t)$.

We assume that F is twice continuously differentiable and satisfies $F'(\cdot) > 0$, $F''(\cdot) < 0$, $F'(0) < \infty$ and that there exists $\bar{h} \in (0,\infty)$ such that $F'(h) = 0$ for all $h \geq \bar{h}$. The assumption that $F'(0) < \infty$ implies that there is no Inada condition when $h_i(j,t) = 0$. The last assumption ensures that there is an upper bound on the flow rate of innovation (which will be useful in some of the proofs). Recalling that the wage rate for labor is $w(t)$, the cost for R&D is therefore $w(t) G(x_i(j,t))$ where

$$G(x_i(j,t)) \equiv F^{-1}(x_i(j,t)), \tag{12}$$

and the assumptions on F immediately imply that G is twice continuously differentiable and satisfies $G'(\cdot) > 0$, $G''(\cdot) > 0$, $G'(0) > 0$ and $\lim_{x \to \bar{x}} G'(x) = \infty$, where $\bar{x} \equiv F(\bar{h})$ is the maximal flow rate of innovation (with $\bar{h}$ defined above).

2.3 Profits

We next write the instantaneous profits for the leader (clearly, followers always make zero profits, since they make no sales). The instantaneous profit function for leader i in industry j at time t is

$$\begin{aligned}
\Pi_i(j,t) &= [p_i(j,t) - MC_i(j,t)] \cdot y_i(j,t) \\
&= \left(\frac{w(t)}{q_{-i}(j,t)} - \frac{w(t)}{q_i(j,t)} \right) \frac{Y(t)}{p_i(j,t)} \\
&= \left(1 - \lambda^{-n_j(t)}\right) Y(t)
\end{aligned} \tag{13}$$

where $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ is the technology gap in industry j at time t. The first line simply uses the definition of profits as price minus marginal cost times quantity sold. The second line uses the fact that the equilibrium limit price of firm i is $p_i(j,t) = w(t)/q_{-i}(j,t)$ as given by (7), and the final equality uses the definitions of $q_i(j,t)$ and $q_{-i}(j,t)$ from (9) and (10). The expression in (13) also implies that there will be zero profits in an industry that is neck-and-neck, i.e., in industries with $n_j(t) = 0$.

The noteworthy implication of (13), which follows from the Cobb-Douglas aggregate production function in (3), is that profits only depend on the technology gap of the industry and aggregate output. This will simplify the analysis below by making the technology gap in each industry the only industry-specific payoff-relevant state variable.

The objective function of each firm at time t can then be written as

$$\max_{\{x_i(j,s)\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} \exp\left(-R\left(s-t\right)\right)\left[\Pi_i\left(j,s\right)-w\left(s\right)G\left(x_i\left(j,s\right)\right)\right]ds, \tag{14}$$

where $R(t) \equiv \int_0^t r(s)\,ds$ is the discount factor applying between time 0 and time t, and the sequence of wages $\{w(s)\}_{s=t}^{\infty}$ is taken as given by all firms.

2.4 IPR Policy

In the quick and the slow catch-up regimes, patent or intellectual property rights (IPR) policy is modeled as the flow probability with which the government allows a follower to copy the leader (without undertaking a successful innovation). In the equilibrium with patent fees, IPR policy will correspond to the level of patent fees. Recall that the leader and the follower are producing two different varieties. Successful R&D can then be interpreted as the follower progressing through the quality ladders of its own variety, while copying the leader would correspond to implementing advances from the leader's variety. Full patent protection corresponds to the case in which the government never allows a follower to copy the leader. Throughout, we assume that there is no technological linkage among industries, so there is no point in copying the innovation from another industry.

As argued in the Introduction, for our purposes it is essential to allow the degree of IPR enforcement to be a function of the technological gap between the leader and the follower. For this reason, we represent the government IPR policy by an infinite sequence $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$, such that $\eta_n \in \mathbb{R}_+$ for each $n \in \mathbb{N}$ is the flow rate at which a follower that is n-step behind can copy the leader's technology.[7] Throughout, we assume that the IPR policy is time-invariant, and we suppose that there exists $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$ for all $n \geq \bar{n}$. Uniform IPR corresponds to $\boldsymbol{\eta}^{uni} = (\eta, \eta, ...)$, while full patent protection corresponds to $\boldsymbol{\eta} = (0, ..., 0)$. With full patent protection the only way the follower can catch up with the leader is by undertaking a series of step-by-step innovations.[8]

Consequently, the law of motion of the technology gap in industry j in the quick catch-up

[7] Alternative modeling assumptions on IPR policy, such as a fixed patent length of $T > 0$ from the time of innovation, are not tractable, since they introduce a form of delayed-differential equations.

[8] In Section 6, we will assume that the follower can copy the technology of the leader, but has to pay some fee ζ_n. For now note that for any $\zeta_n > 0$, the follower will not copy the technology to undertake production, because Bertrand pricing will lead to zero ex post profits and thus to a loss equal to ζ_n for the follower. However, the follower may decide to undertake R&D building on the technology of the leader, so that after an innovation it directly advances one step ahead of the previous leader, but pays the patent fee ζ_n. This will be discussed and analyzed in Section 6.

regime can be written as

$$n_j(t+\Delta t) = \begin{cases} n_j(t)+1 & \text{with probability} \quad x_i(j,t)\cdot\Delta t + o(\Delta t) \\ 0 & \text{with probability} \quad \left(x_{-i}(j,t)+\eta_{n_j(t)}\right)\cdot\Delta t + o(\Delta t) \\ n_j(t) & \text{with probability} \quad 1-\left(x_i(j,t)+x_{-i}(j,t)+\eta_{n_j(t)}\right)\cdot\Delta t - o(\Delta t)) \end{cases} \tag{15}$$

with $o(\Delta t)$ representing second-order terms, in particular, the probabilities of more than one innovations within an interval of length Δt; $x_i(j,t)$ and $x_{-i}(j,t)$ are the flow rates of innovation by the leader and the follower, respectively; and $\eta_{n_j(t)}$ is the flow rate at which the follower is allowed to copy the technology of a leader that is $n_j(t)$ steps ahead.

2.5 Equilibrium

Let $\{\mu_n(t)\}_{n=0}^{\infty}$ denote the distribution of industries over different step sizes, with $\sum_{n=1}^{\infty}\mu_n(t) = 1$. For example, $\mu_0(t)$ denotes the fraction of industries that are neck-and-neck at time t. We define an allocation as follows:

Definition 1 ***(Allocation)*** *Let $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$ be the IPR policy sequence. Then an allocation is a sequence of decisions for a leader that is $n = 0, 1, ..., \infty$ step ahead, $\{\xi_n(t)\}_{t=0}^{\infty}$ where $\xi_n(t) \equiv \{x_n(t), p_i(j,t), y_i(j,t)\}$, a sequence of R&D decisions for a follower that is $n = 0, 1, ..., \infty$ step behind, $\{x_{-n}(t)\}_{t=0}^{\infty}$, a sequence of wage rates $\{w(t)\}_{t=0}^{\infty}$, and a sequence of industry distributions over step sizes $\{\{\mu_n(t)\}_{n=0}^{\infty}\}_{t=0}^{\infty}$.*

We focus on the Markov Perfect Equilibrium (MPE) where strategies are only a function of the payoff-relevant state variables, which is a natural equilibrium concept in this context. In particular, we will require equilibrium R&D strategies to be of the form:

$$x : \mathbb{Z} \times \mathbb{R}_+^2 \to \mathbb{R}_+,$$

thus depend on whether the firm is a follower or a leader in an industry with an n step technology gap ($\mathbb{Z}$), and the aggregate level of output and the wage ($\mathbb{R}_+^2$). Thus we have the following definition of equilibrium (below we will define a steady-state equilibrium in terms of a solution to maximization problems):

Definition 2 ***(Equilibrium)*** *Given $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$, a Markov Perfect Equilibrium is given by a sequence $\left\langle \{\xi_n^*(t)\}_{n=0}^{\infty}, \{x_n^*(t)\}_{n=-\infty}^{-1}, w^*(t), Y^*(t) \right\rangle_{t=0}^{\infty}$ such that (i) $\{p_i^*(j,t)\}_{t=0}^{\infty}$ and*

$\{y_i^*(j,t)\}_{t=0}^{\infty}$ *implied by* $\{\xi_n^*(t)\}_{n=0}^{\infty}$ *satisfy (7) and (8); (ii) R&D policies* $\left\langle \{x_n^*(t)\}_{n=-\infty}^{\infty} \right\rangle_{t=0}^{\infty}$ *are best responses themselves, i.e.,* $\left\langle \{x_n^*(t)\}_{n=-\infty}^{\infty} \right\rangle_{t=0}^{\infty}$ *maximizes the expected profits of firms (14) taking aggregate output* $\{Y^*(t)\}_{t=0}^{\infty}$*, wages* $\{w^*(t)\}_{t=0}^{\infty}$*, government policy* $\{\eta_n\}_{n=1}^{\infty}$ *and the R&D policies of other firms* $\left\langle \{x_n^*(t)\}_{n=-\infty}^{\infty} \right\rangle_{t=0}^{\infty}$ *as given; (iii) aggregate output* $\{Y^*(t)\}_{t=0}^{\infty}$ *is given by (3); and (iv) the labor market clears at all times given the wage sequence* $\{w^*(t)\}_{t=0}^{\infty}$.

To make more progress, we now represent the profit-maximization problem of the leaders and the followers in the form of dynamic programs.

2.6 The Labor Market

Since all decisions just depend on the technology gap of the industry and only the technological leader produces, labor demand in industry j with technology gap $n_j(t) = n$ can be expressed as

$$l_n(t) = \begin{cases} \frac{\lambda^{-n} Y(t)}{w(t)} & \text{if } n \in \mathbb{N} \\ \\ \frac{Y(t)}{w(t)} & \text{if } n = 0 \end{cases} . \tag{16}$$

In addition, there is demand for labor coming for R&D from both followers and leaders in all industries. Again, since this only depends on technology gap, from (11) and the definition of the G function, we can express industry demands for R&D labor as

$$h_n(t) = \begin{cases} G(x_n(t)) + G(x_{-n}(t)) & \text{if } n \in \mathbb{N} \\ \\ 2G(x_0(t)) & \text{if } n = 0 \end{cases} , \tag{17}$$

where $n < 0$ refers to the demand of a follower in an industry with a technology gap of n. This expression takes into account that in an industry with neck-and-neck competition, i.e., $n = 0$, there will be twice the demand for R&D coming from the two firms.

The labor market clearing condition can then be expressed as:

$$1 \geq \mu_0(t) \left[\frac{1}{\omega(t)} + 2G(x_0(t)) \right] + \sum_{n=1}^{\infty} \mu_n(t) \left[\frac{1}{\omega(t)\lambda^n} + G(x_n(t)) + G(x_{-n}(t)) \right], \tag{18}$$

and $\omega(t) \geq 0$, with complementary slackness, where

$$\omega(t) \equiv \frac{w(t)}{Y(t)} \tag{19}$$

is the labor share at time t. The labor market clearing condition, (18), uses the fact that total supply is equal to 1, and the demand cannot exceed this amount. If demand falls short of 1, then the wage rate, $w(t)$, and thus the labor share $\omega(t)$ has to be equal to zero (though this will never be the case in equilibrium). The right-hand side of (18) consists of the demand for production (the terms with ω in the denominator), the demand for R&D workers from the neck-and-neck industries ($2G(x_0(t))$) and the demand for R&D workers coming from leaders and followers in other industries ($G(x_n(t)) + G(x_{-n}(t))$).

Moreover, defining the index of aggregate quality in this economy by the aggregate of the qualities in the different industries, i.e.,

$$\ln Q(t) \equiv \int_0^1 \ln q(j,t)\, dj, \tag{20}$$

the equilibrium wage can be written as:[9]

$$w(t) = Q(t)\, \lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}. \tag{21}$$

2.7 Steady State and the Value Functions

Let us now focus on steady-state (Markov Perfect) equilibria, where $\omega(t)$ defined in (19), the distribution of industries $\{\mu_n(t)\}_{n=0}^{\infty}$ and of the growth rate of the economy, g, are constant over time. We will establish the existence of such an equilibrium and characterize a number of its properties. If the economy is in steady state at time $t = 0$, then by definition, we have $Y(t) = Y_0 e^{g^* t}$ and $w(t) = w_0 e^{g^* t}$, where g^* is the steady-state growth rate. The second equation also implies that $\omega(t) = \omega^*$ for all $t \geq 0$.

Let us next write the maximization problem of a leader that is $n > 0$ step ahead recursively. Given an optimal policy $\hat{x}$ for a firm, the net present discounted value of a leader that is n steps ahead at time t takes the form:

$$V_n(t) = \mathbb{E}_t \int_t^{\infty} \exp(-r(s-t)) \left[\hat{\Pi}(s) - w(s)\, G(\hat{x}(s))\right] ds$$

where $\hat{x}(t)$ is the optimal innovation rate and $\hat{\Pi}(t)$ is the resulting profit at time t (all of which are stochastic variables depending on the evolution of the technology gap within the industry). Taking the R&D policy on other firms, $x_{-n}(t)$ as given, this can be written as

$$\begin{aligned} V_n(t) &= \max_{\hat{x}_n(t)} \Bigg\{ [\Pi_n(t) - w(t)\, G(\hat{x}_n(t))] \cdot \Delta t + o(\Delta t) \\ &\quad + \exp(-r(t+\Delta t)\cdot \Delta t) \left[\begin{array}{c} (\hat{x}_n(t) \cdot \Delta t + o(\Delta t))\, V_{n+1}(t+\Delta t) \\ + (\eta_n \cdot \Delta t + x_{-n}(t) \cdot \Delta t + o(\Delta t))\, V_0(t+\Delta t) \\ + (1 - x_n(t) \cdot \Delta t - \eta_n \cdot \Delta t - x_{-n}(t)\cdot \Delta t - o(\Delta t))\, V_n(t+\Delta t) \end{array} \right] \Bigg\}, \end{aligned} \tag{22}$$

[9] Note that $\ln Y(t) = \int_0^1 \ln q(j,t)\, l(j,t)\, dj = \int_0^1 \left[\ln q_i(j,t) + \ln \frac{Y(t)}{w(t)} \lambda^{-n_j}\right] dj$, where the second equality uses (16). Thus we have $\ln Y(t) = \int_0^1 [\ln q_i(j,t) + \ln Y(t) - \ln w(t) - n_j \ln \lambda]\, dj$. Rearranging and canceling terms, and writing $\exp \int n_j \ln \lambda dj = \lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}$, we obtain (21).

where $\Pi_n(t)$ is the profits of the leader that is n steps ahead and $V_{n+1}(t)$ and $V_0(t)$ are defined as net present discounted values for a leader that is $n+1$ steps ahead and a firm in an industry that is neck-and-neck (i.e., $n=0$). This expression also uses the fact that in a short time interval Δt, the probability of innovation by the leader is $\hat{x}_n(t)\cdot\Delta t + o(\Delta t)$, and similarly the probability of innovation by the follower is $x_{-n}(t)\cdot\Delta t + o(\Delta t)$, where $o(\Delta t)$ again denotes second-order terms. Now subtract $V_n(t)$ from both sides, divide everything by Δt and take the limit as $\Delta t \to 0$, which yields:

$$\begin{aligned} r(t)V_n(t) - \dot{V}_n(t) &= \max_{\hat{x}_n(t)}\{[\Pi_n(t) - w(t)G(\hat{x}_n(t))] + \hat{x}_n(t)[V_{n+1}(t) - V_n(t)] \\ &\quad + (x_{-n}(t) + \eta_n)[V_0(t) - V_n(t)]\}. \end{aligned} \tag{23}$$

In steady state, the net present value of a firm that is n steps ahead, $V_n(t)$, will also grow at a constant rate g^* for all $n \in \mathbb{Z}_+$. Let us then define the normalized values as

$$v_n(t) \equiv \frac{V_n(t)}{Y_n(t)} \tag{24}$$

for all $n \in \mathbb{Z}$, which will be independent of time in steady state. Using (24) and the fact that $r(t) = g(t) + \rho$, the steady-state the value function (23) takes the form

$$\rho v_n = 1 - \lambda^{-n} - \omega G(\hat{x}_n) + \hat{x}_n[v_{n+1} - v_n] + (x_{-n} + \eta_n)[v_0 - v_n] \text{ for } n \in \mathbb{N},$$

where we suppress time-dependence since all variables take their steady-state values, and $\hat{x}_n$ refers to the R&D level that is optimal for a leader that is n steps ahead (and x_{-n} is taken as given). Consequently, the recursive form of the optimization problem of each leader can be written as:

$$\rho v_n = \max_{x_n}\left\{\left(1 - \lambda^{-n}\right) - \omega^* G(x_n) + x_n[v_{n+1} - v_n] + \left(x_{-n}^* + \eta_n\right)[v_0 - v_n]\right\}, \tag{25}$$

where x_{-n}^* is the equilibrium value of R&D by a follower that is n steps behind, and ω^* is the steady-state labor share (while x_n is now explicitly chosen to maximize v_n).

Similarly the value for neck-and-neck firms is

$$\rho v_0 = \max_{x_0}\{-\omega^* G(x_0) + x_0[v_1 - v_0] + x_0^*[v_{-1} - v_0]\}, \tag{26}$$

while the values for followers are

$$\rho v_{-n} = \max_{x_{-n}}\{-\omega^* G(x_{-n}) + (x_{-n} + \eta_n)[v_0 - v_{-n}] + x_n^*[v_{-n-1} - v_{-n}]\} \text{ for } n \in \mathbb{N}. \tag{27}$$

For neck-and-neck firms and followers, there are no instantaneous profits, which is reflected in (26) and (27). In the former case this is because neck-and-neck firms sell at marginal cost, and in the latter case, because followers have no sales. These normalized value functions emphasize that, because of growth, the effective discount rate of firms is $r(t) - g(t) = \rho$ rather than $r(t)$.

It is also straightforward to see that the maximization problems in (25)-(26) immediately yield:

$$\begin{aligned} x_n^* &= \max\left\{G'^{-1}\left(\frac{[v_{n+1} - v_n]}{\omega^*}\right), 0\right\} && (28) \\ x_{-n}^* &= \max\left\{G'^{-1}\left(\frac{[v_0 - v_{-n}]}{\omega^*}\right), 0\right\} && (29) \\ x_0^* &= \max\left\{G'^{-1}\left(\frac{[v_1 - v_0]}{\omega^*}\right), 0\right\}, && (30) \end{aligned}$$

where $G'^{-1}(\cdot)$ is the inverse of the derivative of the G function, and since G is twice continuously differentiable and strictly concave, G'^{-1} is continuously differentiable and strictly increasing. These equations therefore imply that innovation rates, the x_n^*'s, will increase whenever the increment in value to moving to the next step is greater and when the cost of R&D, as measured by the normalized wage rate, ω^*, is less. Note also that since $G'(0) > 0$, these R&D levels can be equal to zero, which is taken care of by the max operator.

The response of innovation rates, $\{x_n^*\}$, to the increments in values, $v_{n+1} - v_n$, is the key economic force in this model. For example, a policy that reduces the patent protection of leaders that are $n + 1$ steps ahead will make being $n + 1$ steps ahead less profitable, thus reduce $v_{n+1} - v_n$ and x_n^*—which corresponds to the *disincentive effect*. On the other hand, it is also typically the case that $\{v_{n+1} - v_n\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that x_0^* is higher than x_n^* for $n > 0$, so that there is a higher likelihood of productivity growth in neck-and-neck industries (see Propositions 4 and 5 below). Weaker patent protection will shift more industries into neck-and-neck state, and thus create a *composition* effect, increasing the innovation rate in the economy.[10] The optimal level and structure of IPR policy in this economy will be determined by the interplay of these opposing forces.

Given the R&D levels $\{x_n^*\}_{n=-\infty}^{\infty}$, the steady-state distribution of industries across states

[10]There is another benefit of weaker patent protection, which is that the R&D undertaken by followers is purely "duplicative" in the sense that, despite the fact that it uses resources, it does not contribute to the productive capacity of the economy (though a successful innovation enables further innovations contributing to productivity and also affects equilibrium prices as shown in equation (7) above). This additional benefit creates a level effect (increasing output by reallocating workers from R&D to production), but has no effect on the growth rate.

$\{\mu_n^*\}_{n=1}^{\infty}$ has to satisfy the following accounting identities:

$$\left(x_{n+1}^* + x_{-n-1}^* + \eta_{n+1}\right)\mu_{n+1}^* = x_n^* \mu_n^* \text{ for } n \in \mathbb{N}, \tag{31}$$

$$\left(x_1^* + x_{-1}^* + \eta_1\right)\mu_1^* = 2x_0^* \mu_0^*, \tag{32}$$

$$2x_0^* \mu_0^* = \sum_{n=1}^{\infty} \left(x_{-n}^* + \eta_n\right)\mu_n^*. \tag{33}$$

The first expression equates exit from state $n+1$ (which takes the form of the leader going one more step ahead or the follower catching up) to entry into the state (which takes the form of a leader from the state n making one more innovation). The second equation, (32), does the same for state 1, taking into account that entry into this state comes from innovation by either of the two firms that are competing neck-and-neck. Finally, equation (33) equates exit from state 0 with entry into this state, which comes from innovation by a follower in any industry with $n \geq 1$.

The labor market clearing condition in steady state can then be written as

$$1 \leq \mu_0^* \left[\frac{1}{\omega^*} + 2G\left(x_0^*\right)\right] + \sum_{n=1}^{\infty} \mu_n^* \left[\frac{1}{\omega^* \lambda^n} + G\left(x_n^*\right) + G\left(x_{-n}^*\right)\right] \text{ and } \omega^* \geq 0, \tag{34}$$

with complementary slackness.

The next proposition gives the characterization of the steady-state growth rate of the economy. As with all the other results in the paper, the proof of this proposition is provided in the Appendix.

Proposition 1 *Let the steady-state distribution of industries be* $\{\mu_n^*\}_{n=0}^{\infty}$ *and of R&D levels be* $\{x_n^*\}_{n=0}^{\infty}$*, then the steady-state growth rate is*

$$g^* = \ln \lambda \left[2\mu_0^* x_0^* + \sum_{n=1}^{\infty} \mu_n^* x_n^*\right]. \tag{35}$$

This proposition clarifies that the steady-state growth rate of the economy is determined by two factors:

1. R&D decisions of industries at different levels of technology gap.

2. The distribution of industries across different technology gaps.

IPR policy affects these two margins in different directions as illustrated by the discussion above.

We now define a steady-state equilibrium in a more convenient form, which will be used to establish existence and derive some of the properties of the equilibrium.

Definition 3 ***(Steady-State Equilibrium)*** *Given an IPR policy $\boldsymbol{\eta} = \{\eta_n\}_{n=0}^{\infty}$, a steady-state equilibrium is a tuple $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ such that the distribution of industries $\{\mu_n^*\}_{n=0}^{\infty}$ satisfy (31), (32) and (33), the values $\{v_n\}_{n=-\infty}^{\infty}$ satisfy (25), (26) and (27), the R&D decisions $\{x_n^*\}_{n=-\infty}^{\infty}$ are given by (28), (29) and (30), the steady-state labor share ω^* satisfies (34) and the steady-state growth rate g^* is given by (35).*

We next provide a characterization of the steady-state equilibrium, starting first with the case in which there is uniform IPR policy.

3 Uniform IPR Policy

Let us first assume that the only available policy option is uniform IPR policy (as assumed in existing models), whereby $\eta_n = \eta < \infty$ for all $n \in \mathbb{N}$, which we denote by $\boldsymbol{\eta}^{uni}$. In this case, it is straightforward to observe from (27) that the problem is identical for all $n < 0$, so it is clear that $v_{-n} = v_{-1}$ for all $n < 0$. Consequently, (27) can be replaced with

$$\rho v_{-1} = \max_{x_{-1}} \left\{ -\omega^* G\left(x_{-1}\right) + \left(x_{-1}^* + \eta\right)\left[v_0 - v_{-1}\right] \right\}, \tag{36}$$

implying optimal R&D decisions for all followers of the form

$$x_{-1}^* = \max\left\{ G'^{-1}\left(\frac{\left[v_0 - v_{-1}\right]}{\omega^*}\right), 0 \right\}. \tag{37}$$

Let us denote the sequence of the value funtions with uniform IPR as $\{v_n\}_{n=-1}^{\infty}$. We start with two results that characterize the form of a steady-state equilibrium in this economy.

Proposition 2 *Consider the case of uniform IPR policy $\boldsymbol{\eta}^{uni}$ and suppose that $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ is a steady-state equilibrium. Then, the values $\{v_n\}_{n=-1}^{\infty}$ form a bounded and strictly increasing sequence converging to some positive value v_∞.*

Proposition 2 establishes that the equilibrium can be represented by an increasing sequence of values. The next result shows that we can restrict attention to a finite sequence of values:

Proposition 3 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$ and a corresponding steady-state equilibrium $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$. Then, there exists $n^* \in \mathbb{N}$ such that $x_n^* = 0$ for all $n \geq n^*$.*

Moreover, we can show that highest R&D investments are when firms are neck-and-neck, that is, we have both $x_0^* > x_{-1}^*$ and $x_0^* > x_n^*$ for all $n \in (0, n^*)$. The following proposition proves the first of these and the subsequent one shows that $\{x_n^*\}_{n=0}^{\infty}$ is a decreasing sequence.

Proposition 4 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$. Then in any steady-state equilibrium, we have $x_0^* > x_{-1}^*$.*

Proposition 5 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$. Then in any steady-state equilibrium with uniform IPR, we have $x_{n+1}^* \leq x_n^*$ for all $n \in \mathbb{N}$ and moreover, $x_{n+1}^* < x_n^*$ if $x_n^* > 0$.*

Next, we prove the existence of a steady-state equilibrium $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ with $\omega^* < 1$. A steady state with with $\omega^* < 1$ is economically more interesting, since when $\omega^* = 1$, there are no profits, thus it must be the case that $\mu_0^* = 1$. Equation (35) together with $\mu_0^* = 1$ and $x_n^* = 0$ for all $n \in \mathbb{Z}_+$ then implies that there is no economic growth, i.e., $g^* = 0$.

Establishing the existence of a steady-state equilibrium in this economy is made complicated by the fact that, the equilibrium allocation cannot be represented as a solution to a maximization problem. Instead, as emphasized by Definition 3, each firm maximizes its value taking the R&D decisions of other firms; thus an equilibrium corresponds to a set of R&D decisions that are best responses to each other and a labor share (wage rate) ω^* that clears the labor market. We will establish the existence of a steady-state equilibrium in a number of steps.

First, note that each x_n belongs to a compact interval $[0, \bar{x}]$, where $\bar{x}$ is the maximal flow rate of innovation defined above. Now fix a labor share $\tilde{\omega} \in [0,1]$ and a sequence $\{\tilde{x}_n\}_{n=-1}^{\infty}$ of (Markovian) steady-state strategies for all other firms in the economy, and consider the dynamic optimization problem of a single firm. Our next result characterizes this problem and shows that given $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)$, the value function and the optimal policies of an individual firm are uniquely determined.

Proposition 6 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$, and suppose that the labor share and the R&D policies of all other firms are given by $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)$. Then the dynamic optimization problem of an individual firm leads to a unique value function $\mathbf{v}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+$ and unique optimal R&D policy $\mathbf{X}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \to [0, \bar{x}]$. Moreover, let $\mathbf{X}[\mathbf{z}] = \{x_n(\mathbf{z})\}_{n=-1}^{\infty}$, then $x_n(\mathbf{z})$ is continuous in $\mathbf{z}$ for $n \in \{-1\} \cup \mathbb{Z}_+$.*

Now let us start with an arbitrary $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right) \in \mathbf{Z} \equiv [0,1] \times [0, \bar{x}]^{\infty}$. From Proposition 6, this $\mathbf{z}$ is uniquely mapped into an optimal R&D policy $\mathbf{X}[\mathbf{z}] = \mathbf{x}$, where $\mathbf{x} \equiv \{x_n\}_{n=-1}^{\infty}$. From $\mathbf{x}$, calculate $\boldsymbol{\mu} = \{\mu_n\}_{n=0}^{\infty}$ using equations (31), (32) and (33). Then we can rewrite the labor market clearing condition (34) as

$$\begin{aligned}\omega &= \min\left\{\mu_0\left[1+2G(x_0)\tilde{\omega}\right]+\sum_{n=1}^{\infty}\mu_n\left[\frac{1}{\lambda^n}+G(x_n)\tilde{\omega}\right]\right.\\&\qquad\left.+G(x_{-1})\tilde{\omega}\sum_{n=1}^{\infty}\mu_n;1\right\},\\&\equiv \varphi(\tilde{\omega}).\end{aligned}\tag{38}$$

We now consider the mapping $\Phi \equiv (\varphi, \mathbf{X})$, such that Φ: $\mathbf{Z} \to \mathbf{Z}$. That Φ maps $\mathbf{Z}$ into itself follows from the fact that $\{\tilde{x}_n\}_{n=-1}^{\infty} \in [0, \bar{x}]^{\infty}$ and $\{x_n\}_{n=-1}^{\infty} \in [0, \bar{x}]^{\infty}$, and also that $\tilde{\omega} \in [0, 1]$, and the range of (38) is clearly $[0, 1]$ (since the right-hand side is nonnegative and bounded above by 1). Moreover, from Proposition 6, x_n is continuous in $\mathbf{z}$ for all n, and thus so is (38). Using this construction, we can establish:

Proposition 7 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$ and suppose that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho\right) > 0$. Then a steady-state equilibrium $\left\langle\{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-1}^{\infty}, \{x_n^*\}_{n=-1}^{\infty}, \omega^*, g^*\right\rangle$ with $\omega^* < 1$ and $g^* > 0$ exists.*

Remark 1 If the assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/r\right) > 0$ is relaxed, then there may exist a trivial equilibrium in which $\mu_0^* = 1$, and $x_0^* > 0$ for all n, i.e., an equilibrium in which there is no innovation and thus no growth. In such an equilibrium, in every industry two firms with equal costs compete a la Bertrand and charge price equal to marginal cost, so that there are no profits and the labor share of output is equal to 1.

An immediate consequence of Proposition 3, combined with (31) is that $\mu_n = 0$ for all $n \geq n^*$ (since there is no innovation in industries with technology gap greater than n^*). Thus the law of motion of an industry can be represented by a finite Markov chain. Moreover, because all industries jump to the neck-and-neck situation after innovation by a follower, this Markov chain is irreducible (and is also aperiodic), thus converges to a unique steady-state distribution of industries. This is stated and proved in the next proposition.

Proposition 8 *Consider a uniform IPR policy $\boldsymbol{\eta}^{uni}$ and steady-state equilibrium values of $\{x_n^*\}_{n=-\infty}^{\infty}$. Then, there exists a unique steady-state distribution of industries $\{\mu_n^*\}_{n=0}^{\infty}$.*

4 State-Dependent IPR Policy

We now extend the results from the previous section to the environment with state-dependent IPR policy. The main results from the previous section generalize, but the argument is slightly modified.

It is no longer necessarily the case that the sequence of values $\{v_n\}_{n=-\infty}^{\infty}$ is increasing, since IPR policies could be very sharply increasing, making a particular state very unattractive for the leader.[11] For this reason, we do not have the equivalent of Proposition 2. Nevertheless, it can be established that only a finite number of states will have positive weight in the steady-state distribution:

Proposition 9 *Consider the state-dependent IPR policy* $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$ *and suppose that* $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ *is a steady-state equilibrium. Then there exists a state* $n^* \in \mathbb{N}$ *such that* $\mu_n^* = 0$ *for all* $n \geq n^*$.

Because of the reasons highlighted in footnote 11, Propositions 4 and 5 also do not necessarily hold with state-dependent IPR. Nevertheless, the proofs of these propositions make it clear that as long as $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$ is not too far from uniform IPR, the conclusions in these propositions will continue to hold. In fact, our numerical results with optimal state-dependent IPR always verify the conclusions of Propositions 4 and 5.

Our next result is a generalization of Proposition 6, which shows that each individual firm's maximization problem is well-behaved with state-dependent IPR.

Proposition 10 *Consider the state-dependent IPR policy* $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$ *and suppose that the labor share and the R&D policies of all other firms are given by* $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}\right)$. *Then the dynamic optimization problem of an individual firm leads to a unique value function* $v[\mathbf{z}] : \mathbb{Z} \rightarrow \mathbb{R}_+$ *and a unique optimal R&D policy* $\mathbf{X}[\mathbf{z}] : \mathbb{Z} \rightarrow [0, \bar{x}]$. *Moreover, let* $\mathbf{X}[\mathbf{z}] = \{x_n(\mathbf{z})\}_{n=-\infty}^{\infty}$, *then* $x_n(\mathbf{z})$ *is continuous in* $\mathbf{z}$ *for* $n \in \mathbb{Z}$.

Finally, the next result generalizes Proposition 7 and establishes the existence of a steady-state equilibrium with positive growth.

Proposition 11 *Consider the state-dependent IPR policy* $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$ *and suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho\right) > 0$. *Then a steady-state equilibrium* $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ *with* $\omega^* < 1$ *and* $g^* > 0$ *exists.*

Although the analysis so far has established the existence of a steady-state equilibrium and characterized some of its properties, it is not possible to determine the optimal state-dependent IPR policy analytically. For this reason, in Section 7, we undertake a quantitative

[11] For example, we could have $\eta_n = 0$ and $\eta_{n+1} \rightarrow \infty$, which would imply that $v_{n+1} - v_n$ is negative.

investigation of the form and structure of optimal state-dependent IPR policy using plausible parameter values. Before doing so, we discuss equilibrium in the slow catch-up regime.

5 Equilibrium in the Slow Catch-Up Regime

The analysis so far focused on the quick catch-up regime and assumed that any follower reaches the leading-edge technology with a single innovation. Although catching up to the leader with one innovation may be a good approximation for certain industries and technologies, in other instances followers may need multiple innovations to catch up with the technology frontier. Therefore, an analysis of the slow catch-up regime and its differences from the quick catch-up regime is important to understand what types of IPR policies would be optimal for industries with different technology structures.

Since our purpose is to vary the technology (thus structure) of an industry, while keeping the set of policies fixed, we continue to assume that relaxation of IPR policies still allows the follower to catch up with the leading-edge technology. The rest of the environment is the same as before, in particular, equations (7) and (8) give the equilibrium prices and intermediate demands. The main difference is in the law of motion of the technology gap, which now changes from (15) to:

$$n_j(t+\Delta t) = \begin{cases} n_j(t)+1 & \text{with probability} \quad x_i(j,t)\cdot\Delta t + o(\Delta t) \\ n_j(t)-1 & \text{with probability} \quad (x_{-i}(j,t)+\alpha)\cdot\Delta t + o(\Delta t) \\ 0 & \text{with probability} \quad \eta_{n_j(t)}\cdot\Delta t + o(\Delta t) \\ n_j(t) & \text{with probability} \quad 1-\left(x_i(j,t)+x_{-i}(j,t)+\alpha+\eta_{n_j(t)}\right)\cdot\Delta t - o(\Delta t)) \end{cases} \tag{39}$$

This expression emphasizes that when the follower makes a successful innovation, it advances by one step, but relaxation of full patent protection allows the follower to copy the leading-edge technology. In addition, equation (39) introduces an additional parameter $\alpha \geq 0$, which is the flow rate at which the follower advances one step even without an innovation. This may correspond to discoveries of already existing technologies that take place without R&D effort. The reason for introducing this parameter is that, as we will see in Section 8, when $\alpha = 0$, the steady-state industry distribution without IPR will be very concentrated at a single (or a few) technology gap(s). Allowing for $\alpha > 0$ enables us to vary the shape of the steady-state industry distribution and investigate whether our results hold for industry distributions. In any case, all of the results stated in this section hold for $\alpha = 0$, and our quantitative investigation of optimal IPR in the slow catch-up regime in Section 8 will start with $\alpha = 0$.

Given (39), the normalized value functions change in an intuitive way. In particular, for a leader that is $n \in \mathbb{N}$ steps ahead, we have

$$\rho v_n = \max_{x_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(x_n\right) + x_n \left[v_{n+1} - v_n\right] + \left(x^*_{-n} + \alpha\right) \left[v_{n-1} - v_n\right] + \eta_n \left[v_0 - v_n\right] \right\}, \tag{40}$$

where x^*_{-n} is again the equilibrium value of R&D by a follower that is n steps behind, ω^* is the steady-state labor share, and the key difference is that now of innovation by a follower (at the rate x^*_{-n}) enables the follower to advance by one step, thus the value change for the leader in this case is $v_{n-1} - v_n$. The value function for neck-and-neck firms is unchanged, and is still given by (26). Finally, the value for a follower that is $n \in \mathbb{N}$ steps behind is now given by

$$\rho v_{-n} = \max_{x_{-n}} \left\{ -\omega^* G\left(x_{-n}\right) + \left(x_{-n} + \alpha\right) \left[v_{-n+1} - v_{-n}\right] + \eta_n \left[v_0 - v_{-n}\right] + x^*_n \left[v_{-n-1} - v_{-n}\right] \right\} \text{ for } n \in \mathbb{N}, \tag{41}$$

which, like (40), takes into account that an innovation by the follower leads to a one step improvement in technology.

Given these value functions, the equilibrium innovation rates are now

$$\begin{aligned} x^*_n &= \max\left\{ G'^{-1}\left(\frac{\left[v_{n+1} - v_n\right]}{\omega^*}\right), 0 \right\} && (42) \\ x^*_{-n} &= \max\left\{ G'^{-1}\left(\frac{\left[v_{-n+1} - v_{-n}\right]}{\omega^*}\right), 0 \right\} && (43) \\ x^*_0 &= \max\left\{ G'^{-1}\left(\frac{\left[v_1 - v_0\right]}{\omega^*}\right), 0 \right\}, && (44) \end{aligned}$$

where $G'^{-1}(\cdot)$ is again the inverse of the derivative of the G function. These equations show that followers now will have weaker incentives for research, because research does not immediately enable them to jump to the frontier. This, naturally, changes the entire sequence of value functions and affects the innovation rates of leaders and neck-and-neck firms. In particular, surging ahead is now more valuable (since followers have a harder time catching up with the leader once they fall behind). Consequently, in the slow catch-up regime we would expect x^*_0 to be even greater than innovation rates at $n \neq 0$ relative to the quick catch-up regime.

Finally, the distribution of industries across different levels of technology gaps now satisfies:

$$\left(x^*_{n+1} + x^*_{-n-1} + \alpha + \eta_{n+1}\right) \mu^*_{n+1} = x^*_n \mu^*_n + \left(x^*_{-n-2} + \alpha\right) \mu^*_{n+2} \text{ for } n \in \mathbb{N}, \tag{45}$$

$$\left(x^*_1 + x^*_{-1} + \alpha + \eta_1\right) \mu^*_1 = 2 x^*_0 \mu^*_0, \tag{46}$$

$$2x_0^*\mu_0^* = \left(x_{-1}^* + \alpha\right)\mu_1^* + \sum_{n=1}^{\infty}\eta_n\mu_n^*. \tag{47}$$

The key difference between these equations and those in the quick catch-up regime is in equation (45), which states that an industry can enter the state $n+1$ both from state n (with the leader undertaking an innovation) and from state $n+2$ (with the follower undertaking an innovation). Correspondingly, the entry rate into state 0 is also modified as reflected in equation (47). In addition, equations (45), (46) and (47) also feature the parameter α, which could be positive.

The labor market clearing condition in steady state (34) and the growth rate equation (35) still apply as before. Consequently, we can define a steady-state equilibrium as follows:

Definition 4 ***(Steady-State Equilibrium)*** *Given an IPR policy $\boldsymbol{\eta} = \{\eta_n\}_{n=0}^{\infty}$, a steady-state equilibrium in the slow catch-up regime is a tuple $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ such that the distribution of industries $\{\mu_n^*\}_{n=0}^{\infty}$ satisfy (45), (46) and (47), the values $\{v_n\}_{n=-\infty}^{\infty}$ satisfy (40), (26) and (41), the R&D decisions $\{x_n^*\}_{n=-\infty}^{\infty}$ are given by (42), (43) and (44), the steady-state labor share ω^* satisfies (34) and the steady-state growth rate g^* is given by (35).*

A similar analysis can be applied to characterize the steady-state equilibrium both with uniform and state-dependent IPR policy. Here, for brevity, we focus on the case with state-dependent IPR policy. In particular, the proofs of the following results are essentially identical to those of Propositions 9-11 and are omitted.

Proposition 12 *Consider the slow catch-up regime and the state-dependent IPR policy $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$, and suppose that $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ is a steady-state equilibrium. Then there exists a state $n^* \in \mathbb{N}$ such that $\mu_n^* = 0$ for all $n \geq n^*$.*

Proposition 13 *Consider the slow catch-up regime and the state-dependent IPR policy $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$, and suppose that the labor share and the R&D policies of all other firms are given by $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}\right)$. Then the dynamic optimization problem of an individual firm leads to a unique value function $v[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$ and a unique optimal R&D policy $\mathbf{X}[\mathbf{z}] : \mathbb{Z} \to [0, \bar{x}]$. Moreover, let $\mathbf{X}[\mathbf{z}] = \{x_n(\mathbf{z})\}_{n=-\infty}^{\infty}$, then $x_n(\mathbf{z})$ is continuous in $\mathbf{z}$ for $n \in \mathbb{Z}$.*

Proposition 14 *Consider the slow catch-up regime and the state-dependent IPR policy $\boldsymbol{\eta} = (\eta_1, ..., \eta_\infty)$. Then a steady-state equilibrium $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \omega^*, g^* \right\rangle$ exists.*

Although the existence and characterization results apply as before, there is a major difference between the quick catch-up and the slow catch-up regimes; followers have weaker incentives to invest in R&D in the slow catch-up regime than in the quick catch-up regime. In particular, equation (43) gives the optimal R&D of a follower. Once a follower is n steps behind, it needs to undertake at least $n+1$ innovations in order to be able to surpass the leader. In this process, it will incur the costs of R&D and generate no profits. This implies that once followers fall behind a certain number of steps, they will stop investing in R&D (since $G'(0) > 0$). Consequently, in the slow catch-up regime the steady-state distribution of industries $\{\mu_n^*\}_{n=0}^{\infty}$ will involve considerably more weights in the tails.

To illustrate this, suppose that there is full patent protection, i.e., $\boldsymbol{\eta} = (0, 0, ...)$. Let $k = -n$ and consider the sequence $\{v_k\}_{k=0}^{\infty}$. This is clearly a decreasing sequence in a compact set (by the same argument as in the proof of Proposition 2). Consequently, it is also a convergent and thus Cauchy sequence. Therefore, there exists $\bar{n}^*$ such that $[v_{-\bar{n}^*+1} - v_{\bar{n}^*}] < G'(0)$. Since $\omega^* \leq 1$, this imply that there exists $n^* \leq \bar{n}^*$ such that $[v_{-n^*+1} - v_{n^*}] / \omega^* < G'(0)$, and thus $x_{-n^*} = 0$. Now let us look at the R&D incentives of a leader that is n^* steps ahead. From (25), we have that $v_{n^*+1} - v_{n^*} \geq \lambda^{-n^*-1} (\lambda - 1) / \rho$. If we have that $\lambda^{-n^*-1} (\lambda - 1) / \rho > \omega^* G'(0)$, then a leader that is n^* steps ahead will continue to undertake R&D in order to increase the markup that it can charge. By the same argument as in Proposition 2, there will then exist some n^{**}, such that $x_{n^{**}} = 0$ (and $x_n > 0$ for $0 < n < n^{**} - 1$). Moreover, since $n^{**} \geq n^*$, we have that followers undertake no innovation, i.e., $x_{-n^{**}} = 0$. Inspection of (45), (46) and (47) then shows that in this case $\mu_n = 0$ for all $n \neq n^{**}$—that is, all industries will have a technology gap equal to n^{**}. However, since $x_{n^{**}} = x_{-n^{**}} = 0$ and $\mu_{n^{**}} = 1$, from equation (35) we have that $g^* = 0$, that is, there will be no economic growth in this equilibrium. Consequently, in the slow catch-up regime, full patent protection can easily lead to stagnation.

6 Equilibrium with Patent Fees

In the environments analyzed so far, patent protection prevented followers from using innovations previously undertaken by technological leaders. This formulation was theoretically attractive and made the model similar to previous work by Aghion, Harris, Howitt and Vickers (2001). In practice, however, followers may be able to use innovations that are patented by leaders as long as they pay some pre-determined or negotiated price. In fact, placing previous innovations in the public domain is often viewed as one of the important advantages of a patent system (e.g., Scotchmer, 2005). In such a patent system, followers could undertake R&D to reach the next technology (rather than try to catch up with the frontier first). Whether they can do so or not depends not only on the patent system, but also on whether tacit knowledge with existing technologies is necessary to improve over existing technologies (e.g., Evanson and Westphal, 1995). Viewed from this perspective, the models analyzed so far implicitly assumed that such tacit knowledge was necessary and a follower could only exceed the leading-edge technology by first reaching (and perhaps producing at) the frontier. We now relax this assumption and investigate the robustness of our conclusions to allowing followers to pay patent (license) fees in order to build on previous innovations by technological leaders.

More specifically, we modify the previous setup by setting $\boldsymbol{\eta} = (0, ..0, ...)$, but allowing followers to build on and use existing innovations at some patent fee that depends on the size of the technology gap in the sector. In other words, if a follower that is n steps behind uses the technology of the leader and builds upon it to create the next most advanced technology, it has to pay a patent fee of $\zeta_n \geq 0$ to the leader. The IPR policy is then represented by a sequence $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, ..., \zeta_\infty)$.

The follower always has the option to try to undertake the same innovation in its own variety by expending R&D effort, in which case it does not have to pay this patent fee. We denote this decision by a follower that is n steps behind by $a_{-n} \in \{0, 1\}$, where $a_{-n} = 1$ implies that the follower is building on the innovation of the leader and is undertaking R&D to surpass the frontier, while $h_{-n} = 0$ implies that, as in the quick catch-up regime, a follower is undertaking R&D to reach the frontier technology. Note also that because of Bertrand competition, the follower would never use the leading-edge technology for production. Finally, it is clear that full patent protection in this setup corresponds to $\zeta_n \to \infty$ for all n, so that it becomes prohibitively costly for the followers to use the technologies developed by the leaders.

This discussion implies that given the IPR vector $\boldsymbol{\zeta}$, the value functions now change from (25)-(27) to the following. First, for the leader that is n steps ahead, we have

$$\begin{aligned} \rho v_n &= \max_{x_n} \{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(x_n\right) + x_n \left[v_{n+1} - v_n\right] \\ &\quad + \left(1 - a^*_{-n}\right) x^*_{-n} \left[v_0 - v_n\right] + a^*_{-n} x^*_{-n} \left[v_{-1} - v_n + \zeta_n\right] \}, \end{aligned} \tag{48}$$

where x^*_{-n} is the equilibrium value of R&D by a follower that is n steps behind, $a^*_{-n} \in \{0, 1\}$ is its decision of whether to build upon the leading-edge innovation, and ω^* is the steady-state labor share. Intuitively, this equation is very similar to (25)—in fact when $a^*_{-n} = 0$, it is identical to (25). When $a^*_{-n} = 1$, the follower does not undertake R&D to catch up to the leader, but to surpass the leader. If this happens, because the follower is building up on the frontier technology developed by the leader, it makes patent fee payment equal to ζ_n.

On the other hand, the value function for a follower is given by:

$$\begin{aligned} \rho v_{-n} &= \max_{x_{-n}, a_{-n}} \{ -\omega^* G(x_{-n}) + (1 - a_{-n}) x_{-n} [v_0 - v_{-n}] + \\ & \quad a_{-n} x_{-n} [v_1 - v_{-n} - \zeta_n] + x_n^* [v_{-n-1} - v_{-n}] \} \quad \text{for } n \in \mathbb{N}. \end{aligned} \tag{49}$$

This equation makes it clear that the follower can choose either $a_{-n} = 0$, in which case the problem is equivalent to the one it faced in the quick catch-up regime, in particular, in the optimization problem given in (27). Alternatively, it can choose $a_{-n} = 1$ and build upon the innovation by the leader. In this case, a successful innovation takes the follower one step ahead of the leader, but after such a successful innovation, it has to pay a patent fee equal to ζ_n to the innovator of the previous leading-edge technology (i.e., the previous leader in the industry).

Finally, the value functions for neck-and-neck firms are the same as before, in particular, we have:

$$\rho v_0 = \max_{x_0} \{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x_0^* [v_{-1} - v_0] \}. \tag{50}$$

The steady-state distribution Ove industries $\{\mu_n^*\}_{n=1}^{\infty}$ now satisfies:

$$\left(x_{n+1}^* + x_{-n-1}^* \right) \mu_{n+1}^* = x_n^* \mu_n^* \text{ for } n \in \mathbb{N}, \tag{51}$$

$$\left(x_1^* + x_{-1}^* \right) \mu_1^* = 2 x_0^* \mu_0^* + \sum_{n=1}^{\infty} a_{-n}^* x_{-n}^* \mu_n^*, \tag{52}$$

$$2 x_0^* \mu_0^* = \sum_{n=1}^{\infty} \left(1 - a_{-n}^* \right) x_{-n}^* \mu_n^*. \tag{53}$$

These equations take into account that depending on their choices, followers can now jump either to the frontier technology, or by paying the patent fee, one step ahead of the frontier technology.

It is a straightforward to check that the labor market clearing condition is still given by (34).

The growth rate in this economy is potentially different because followers can also advance the technology frontier. The growth rate is derived in the next proposition. The proof of this proposition is similar to that of Proposition 1 and is thus omitted.

Proposition 15 *Consider the equilibrium with patent fees. Let the steady-state distribution of industries be $\{\mu_n^*\}_{n=0}^{\infty}$ and of R&D levels be $\{x_n^*\}_{n=0}^{\infty}$, then the steady-state growth rate is*

$$g^* = \ln \lambda \left[2 \mu_0^* x_0^* + \sum_{n=1}^{\infty} \mu_n^* \left(x_n^* + a_{-n}^* x_{-n}^* \right) \right]. \tag{54}$$

In light of this result, a steady-state equilibrium can now be defined as:

Definition 5 ***(Steady-State Equilibrium)*** *Consider the equilibrium with patent fees and an IPR policy* $\boldsymbol{\zeta} = \{\zeta_n\}_{n=1}^{\infty}$*, a steady-state equilibrium is a tuple* $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \{a_n^*\}_{n=-\infty}^{-1}, \omega^*, g^* \right\rangle$ *such that the distribution of industries* $\{\mu_n^*\}_{n=0}^{\infty}$ *satisfy (51), (52) and (53), the values* $\{v_n\}_{n=-\infty}^{\infty}$ *satisfy (48), (49) and (50), the R&D decisions* $\{x_n^*\}_{n=-\infty}^{\infty}$ *and* $\{a_n\}_{n=-\infty}^{-1}$ *maximize the value functions, the steady-state labor share* ω^* *satisfies (34) and the steady-state growth rate* g^* *is given by (54).*

It is straightforward to derive the basic results from the quick catch-up regime in this case. The following propositions parallel previous results and are stated without proof.

Proposition 16 *Consider the equilibrium with patent fees and the state-dependent IPR policy* $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_\infty)$*, and suppose that* $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \{a_n^*\}_{n=-\infty}^{-1}, \omega^*, g^* \right\rangle$ *is a steady-state equilibrium. Then there exists a state* $n^* \in \mathbb{N}$ *such that* $\mu_n^* = 0$ *for all* $n \geq n^*$*.*

Proposition 17 *Consider the equilibrium with patent fees and the state-dependent IPR policy* $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_\infty)$ *and suppose that the labor share and the R&D policies of all other firms are given by* $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}, \{\tilde{a}_n\}_{n=-\infty}^{-1}\right)$*. Then the dynamic optimization problem of an individual firm leads to a unique value function* $v[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$ *and a unique optimal R&D policy* $\mathbf{X}[\mathbf{z}] : \mathbb{Z} \to [0, \bar{x}]$*. Moreover, let* $\mathbf{X}[\mathbf{z}] = \{x_n(\mathbf{z})\}_{n=-\infty}^{\infty}$*, then* $x_n(\mathbf{z})$ *is continuous in* $\mathbf{z}$ *for* $n \in \mathbb{Z}$*.*

Proposition 18 *Consider the equilibrium with patent fees and the state-dependent IPR policy* $\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_\infty)$*. Then a steady-state equilibrium* $\left\langle \{\mu_n^*\}_{n=0}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}, \{x_n^*\}_{n=-\infty}^{\infty}, \{a_n^*\}_{n=-\infty}^{-1}, \omega^*, g^* \right\rangle$ *exists.*

In addition, with uniform IPR, where $\zeta_n = \zeta$ for all n, we can also prove the equivalent of Proposition 5, showing that R&D investments are decreasing in the technology gap.

Nevertheless, as in the previous sections the growth-maximizing IPR policy cannot be characterized explicitly, and thus we next turn to a quantitative analysis of the optimal (state-dependent) IPR policy in the quick catch-up regime, slow catch-up regime and in the equilibrium with patent fees.

7 Optimal IPR Policy in the Quick Catch-Up Regime

In this section, we investigate how uniform and optimal state-dependent IPR policies affect the equilibrium growth and R&D rates using simulations of the steady-state equilibrium characterized above for plausible parameter values. Our purpose is not to provide a detailed calibration of the model economy, but to highlight the broad quantitative characteristics of the model and its implications for optimal IPR policy. As we will see, the structure of optimal IPR policy and the innovation gains from such policy are relatively invariant to the range of parameter values we consider, though they do depend on whether we are in the quick catch-up or the slow catch-up regime. We start with the quick catch-up regime.

7.1 Methodology

We take the annual discount rate as 5%, i.e., $\rho_{year} = 0.05$ and without loss of generality, we normalize labor supply to 1. The theoretical analysis considered a general production function for R&D given by (11). The empirical literature typically assumes a Cobb-Douglas production function. For example, Kortum (1993) considers a function of the form

$$\text{Innovation}\,(t) = B_0 \exp(\kappa t)\,(\text{R\&D inputs})^{\gamma}\,, \tag{55}$$

where B_0 is a constant and $\exp(\kappa t)$ is a trend term, which may depend on general technological trends, a drift technological opportunities, or changes in general equilibrium prices (such as wages of researchers etc.). The advantage of this form is not only its simplicity, but also the fact that most empirical work estimates a single elasticity for the response of innovation rates to R&D inputs. Consequently, they essentially only give information about the parameter γ in terms of equation (55). For example, Kortum (1993) reports that estimates of γ vary between 0.1 and 0.6 (see also Pakes and Griliches, 1980, or Hall, Hausman and Griliches, 1988). For these reasons, throughout, we adopt a R&D production function similar to (55):

$$x = Ba^{\gamma} \tag{56}$$

where $B, \gamma > 0$. In terms of our previous notation, equation (56) implies that $G(x) = [x/B]^{\frac{1}{\gamma}}\, w$, where w is the wage rate in the economy (thus in terms of the above function, it is captured by the $\exp(\kappa t)$ term).[12] The form in equation (56) does not satisfy the bound-

[12] More specifically, (56) can be alternatively written as

$$\text{Innovation}\,(t) = Bw\,(t)^{-\gamma}\,(\text{R\&D expenditure})^{\gamma}\,,$$

thus would be equivalent to (55) as long as the growth of $w(t)$ can be approximated by constant rate.

ary conditions we imposed in the previous two sections, and can be easily modified to do so without affecting any of the results, since in all numerical exercises only a finite number of states are reached.[13] Following the estimates reported in Kortum (1993), we start with a benchmark value of $\gamma = 0.35$, and then report sensitivity checks for $\gamma = 0.1$ and $\gamma = 0.6$. The other parameter in (56), B, is chosen so as to ensure an annual growth rate of approximately 1.2%, i.e., $g \simeq 0.012$. This is about half of the annual growth rate of US output, which is a reasonable estimate of the growth resulting from innovations. This growth rate together with $\rho_{year} = 0.05$ also pins down the annual interest rate as $r_{year} = 0.062$ from equation (2).

We choose the value of λ using a reasoning similar to Stokey (1995). Equation (35) implies that if about 15% of the industries had one major innovation during one year, then a growth rate of about 0.12% would require $\lambda = 1.1$ (from the equation $g \simeq \ln \lambda \cdot x$). We take $\lambda = 1.1$ as the benchmark value, and check the robustness of the results to $\lambda = 1.05$ and $\lambda = 1.2$. This completes all of the parameters of the model except the IPR policy.

We start with full patent protection $\boldsymbol{\eta} = (0, 0, ...)$. We then consider the optimal uniform IPR policy $\boldsymbol{\eta}^{uni}$ and the optimal state-dependent IPR policy. With state-dependent IPR policy, since it is computationally impossible to calculate the optimal value of each η_n, we limit our investigation to a particular form of state-dependent IPR policy, whereby the same η applies to all industries that have a technology gap of $n = 5$ or more. In other words, the IPR policy matrix takes the form:

$$\begin{array}{|l|ccccc|}\hline \text{IPR policy} \to & \textit{none} & \eta_1 & \eta_2 & \eta_3 & \eta_4 \quad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{\eta_5} \\ \text{Technology gap: } n \to & 0 & 1 & 2 & 3 & 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9 \;\; 10 \;\; 11 \;\; . \;\; . \;\; \infty \\ \hline \end{array}.$$

We checked and verified that allowing for further flexibility (e.g., allowing η_5 and η_6 to differ) has little effect on our results.

The numerical methodology we pursue relies on (multiple) uses on the forward-shooting algorithm (e.g., Judd 1999, Chapter 10). We first take an IPR policy $\boldsymbol{\eta}$ as given. We then make an initial guess for the equilibrium labor share ω, then conditional on ω, we generate a sequence of values $\{v_n\}_{n=-\infty}^{\infty}$ (or $\{v_n\}_{n=-1}^{\infty}$ depending on whether there is state-dependent IPR policy or not), derive the optimal R&D policies, $\{x_n\}_{n=-\infty}^{\infty}$ and the steady-state distribution of industries, $\{\mu_n\}_{n=0}^{\infty}$. We then check for the labor market clearing condition (18), and iterate.

[13]For example, we could add a small linear term to the production function for R&D, (56), and also make it flat after some level $\bar{h}$. For example, the following generalization of (56),

$$x = \min\left\{Bh^{\gamma} + \varepsilon h; B\bar{h}^{\gamma} + \varepsilon \bar{h}\right\}$$

for ε small and $\bar{h}$ large, makes no difference to our simulation results.

The difficulty arises in generating the sequence of values. We perform this by using the fact that v_∞ and $v_{-\infty}$ are given by the equations

$$v_\infty = \frac{1 + \left(G'^{-1}\left(\frac{[v_0 - v_{-\infty}]}{\omega}\right) + \eta_\infty\right) v_0}{\rho + \left(G'^{-1}\left(\frac{[v_0 - v_{-\infty}]}{\omega}\right) + \eta_\infty\right)}, \tag{57}$$

and

$$v_\infty = \frac{-\omega G\left(G'^{-1}\left(\frac{v_0 - v_{-\infty}}{\omega}\right)\right) + \left(G'^{-1}\left(\frac{v_0 - v_{-\infty}}{\omega}\right) + \eta_\infty\right) v_0}{\rho + \left(G'^{-1}\left(\frac{[v_0 - v_{-\infty}]}{\omega}\right) + \eta_\infty\right)}. \tag{58}$$

We make an initial guess for $v_{-\infty}$, which implies a value for v_0 from (58). Then given the value for v_0, we compute v_∞ from (57). Given these guesses, we can compute the entire sequence $\{v_n\}_{n=-\infty}^{\infty}$ from (25) by shooting forward; in particular, we make further guesses for v_1 and v_{-1} iterate forward to v_∞ and $v_{-\infty}$. This process is iterated until convergence with different initial guesses (i.e., until the implied values for v_∞ and $v_{-\infty}$ from the iterations match the initial values).[14] After convergence, we compute the growth rate g^*, and then check for market clearing. Depending on whether there is excess demand or supply of labor, ω is varied and the whole process is repeated until the entire steady-state equilibrium for a given IPR policy is computed. The process is then repeated for different IPR policies.

In the state-dependent IPR case, the optimal (growth-maximizing) IPR policy $\boldsymbol{\eta}$ is computed one element at a time, until we find the growth-maximizing value for that component, for example, η_1. We then move the next component, for example, η_2. Once the growth-maximizing value of η_2 is determined, we go back to optimize over η_1 again, et cetera.

We now present our results, first for the benchmark with full protection, then for the case of optimal uniform IPR policy and then for optimal state-dependent IPR policy.

7.2 Benchmark With Full Protection

We first compute the steady-state equilibrium for our benchmark transition with full patent protection (i.e., $\boldsymbol{\eta} = (0, 0, ...)$). The growth rate in the benchmark economy with full protection turns out to be $g^* = 0.0113$. Since we take $\rho = 0.05$, this implies an annual interest rate of $r = 0.061$. This benchmark economy also leads to an equilibrium labor share of $\omega = 0.885$. This is clearly higher than the labor share in the US data, but recall that here the only income that does not accrue to labor is profits. Thus, alternatively, this number implies a pure profit

[14] With uniform IPR, the second step is not necessary, since all followers have the same value function v_{-1}.

rate of 11.5% of GDP, which is not unreasonable. Finally, 4% of workers are employed in the research and development sector (see Table 1).

Table 1
Simulations of the Full Protection IPR Case for Different Parameter Values

λ	γ	x_{-1}^*	x_0^*	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	researcher ratio
1.05	0.35	0.0039	0.0070	0	0	0	0	0	0.0045	0.930	0.019
1.10	0.35	0.0053	0.0092	0	0	0	0	0	0.0113	0.886	0.040
1.20	0.35	0.0070	0.0119	0	0	0	0	0	0.0268	0.823	0.078
λ	γ	x_{-1}^*	x_0^*	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	researcher ratio
1.10	0.10	0.0165	0.0181	0	0	0	0	0	0.0269	0.895	0.014
1.10	0.35	0.0053	0.0092	0	0	0	0	0	0.0113	0.886	0.040
1.10	0.60	0.0011	0.0084	0	0	0	0	0	0.0053	0.740	0.055

Note: Growth rates refer to annual rates. See text for explanations.

Figure 1 depicts the values for this case with the solid line (the figure connects the values for different n's to show the shape more clearly). As expected, for $n \geq 0$, the increments, $v_{n+1} - v_n$, are decreasing, which translates into decreasing levels of innovation, $\{x_n\}$, as shown in Figure 2 (again with the solid line). It is also important that $v_0 - v_{-1} < v_1 - v_0$, which implies $x_{-1} < x_0$; consequently, as suggested above, neck-and-neck competition gives the highest incentives for innovation.[15] The steady-state distribution of industries over different values of technology gaps is shown in Figure 3 (again with the solid line). The highest fraction of industries are at $n = 1$. Moreover, since any innovation takes an industry to the neck-and-neck state, μ_0 is also relatively high (over 20% of industries are in this state). It can also be seen that there are practically no firms after $n = 12$, despite the fact that the R&D production function used here, (56), does satisfy the Inada conditions.

7.3 Benchmark With Optimal Uniform IPR Protection

We next look at the optimal uniform IPR policy starting from the benchmark steady-state equilibrium presented in the previous subsection. It turns out that with the benchmark parameters, the optimal uniform IPR protection is indistinguishable from full protection. This is essentially because with most of the firms concentrated in industries between $n = 0$ and $n = 2$, there are strong disincentive effects. In particular, the disincentive effects with uniform IPR always reduce the R&D effort by leaders. Since, as emphasized by equation (35), R&D by technological leaders is essential for the growth rate of the economy, these disincentive effects are costly, and, to a first approximation, offset the benefits created by composition effects.

Optimal uniform IPR is also indistinguishable from full patent protection when we vary the parameter values for λ and γ, except in the case where $\lambda = 1.3$ and $\gamma = 0.1$.

[15]See also Table 1. Note that throughout the x's and η's are in monthly rates. Only the growth rate g is reported in annual rate for convenience.

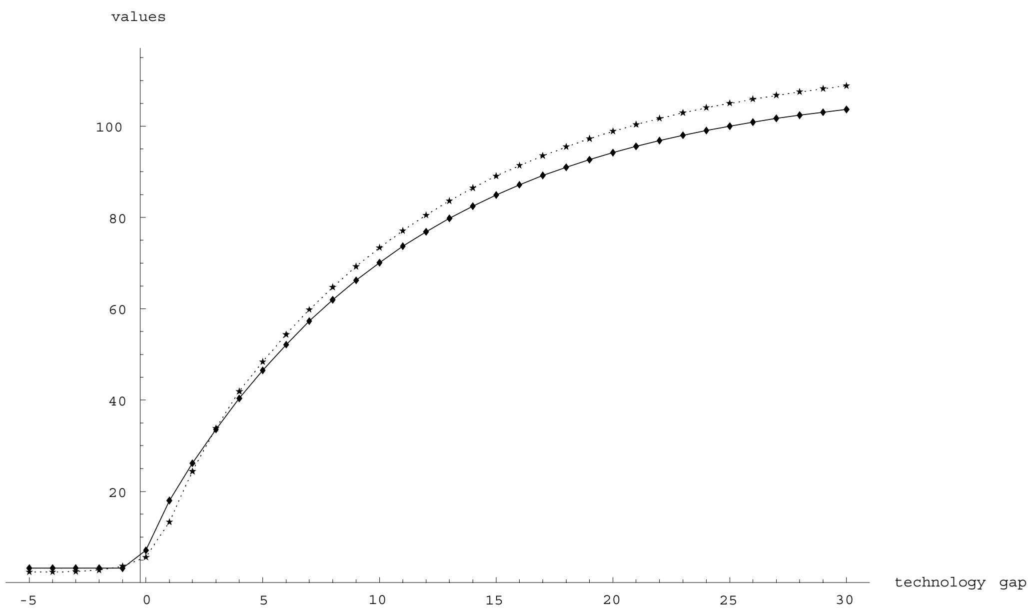

Figure 1: Value functions with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

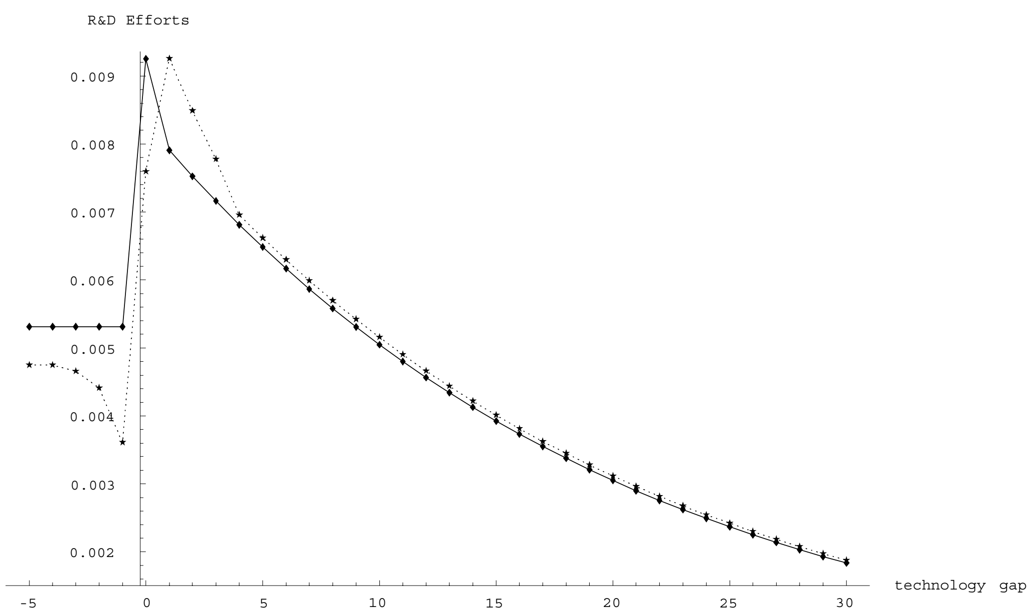

Figure 2: R&D levels with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

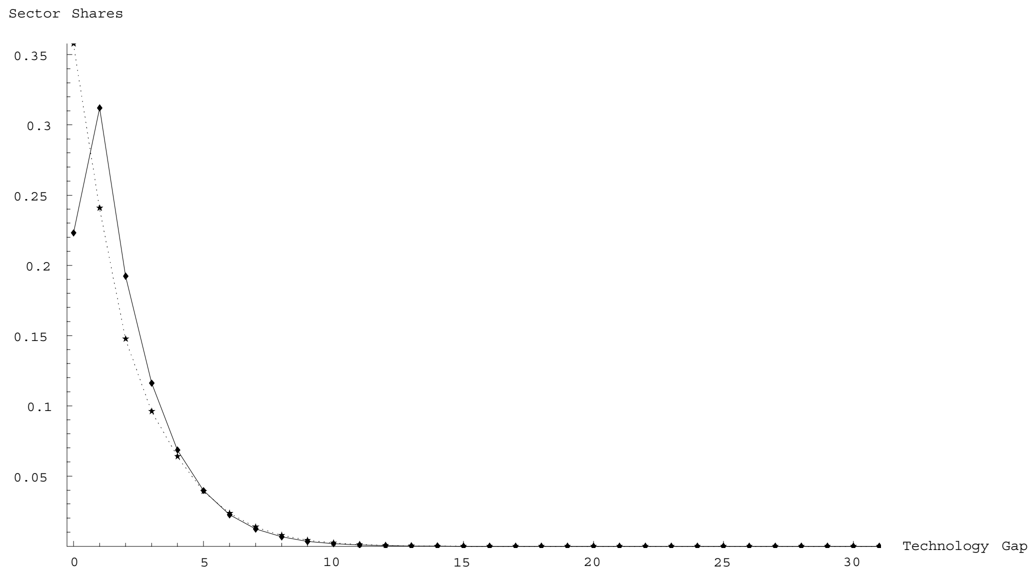

Figure 3: Distribution of industries with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

7.4 State-Dependent IPR Policy

We next calculate the optimal state-dependent IPR policy in the benchmark parameterization of our economy (see Table 2). In the benchmark case, we find that the optimal policy involves $\eta_1 = 0.0097$, $\eta_2 = 0.0022$, $\eta_3 = 0.0006$ and $\eta_4 = \eta_5 = 0$. This corresponds to very little patent protection for firms that are one step ahead of the followers. In particular, the flow rate at which a follower that is one step behind can copy the leader, $\eta_1 = 0.0097$, is about three times the rate of innovation of a firm that is one step behind, $x_{-1} \simeq 0.0036$. This implies that a firm that is one step behind is almost three times as likely to catch-up with the leader because of relaxation of patents rather than an innovation. In contrast, once a firm reaches the technology gap of four steps or more, it receives full patent protection, i.e., $\eta_4 = \eta_5 = 0$.

Table 2
Simulations of the Optimal State-Dependent IPR Case for Different Parameter Values

λ	γ	x^*_{-1}	x^*_0	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	researcher ratio
1.05	0.35	0.0024	0.0056	0.0094	0.0021	0.0005	0	0	0.0049	0.938	0.018
1.10	0.35	0.0036	0.0076	0.0097	0.0022	0.0006	0	0	0.0123	0.900	0.038
1.20	0.35	0.0052	0.0102	0.0090	0.0018	0.0007	0	0	0.0289	0.842	0.073
λ	γ	x^*_{-1}	x^*_0	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	researcher ratio
1.10	0.10	0.0142	0.0166	0.0520	0.0044	0.0008	0	0	0.0306	0.933	0.010
1.10	0.35	0.0036	0.0076	0.0097	0.0022	0.0006	0	0	0.0123	0.900	0.038
1.10	0.60	0.0004	0.0056	0.0053	0.0017	0.0008	0.0003	0	0.0060	0.694	0.063

Note: Growth rates refer to annual rates. See text for explanations.

The value functions, R&D efforts and the steady-state distribution of industries across technology gaps under the optimal state-dependent IPR policies are depicted in Figures 1-3 with the dotted lines. The value function looks very similar to that with full protection. Notably, however, while the R&D levels by followers are reduced (because they can copy the leader costlessly, and this increases v_{-1} and reduces $v_0 - v_{-1}$), R&D levels by technology leaders, especially at steps 1, 2 and 3, are increased. This is achieved thanks to the trickle-down effect induced by the optimal state-dependent IPR policy. By providing secure patent protection to firms that are four or more steps ahead of their rivals, optimal state-dependent IPR increases the R&D effort of leaders that are 1, 2 and 3 steps ahead as well—despite the fact that these firms now face less secure protection of their intellectual property. More specifically, high levels of η_1 and η_2 reduce v_1 and v_2, while high IPR protection for more advanced firms increases v_n for $n \geq 4$. Consequently, it becomes more beneficial for leaders that are one or two steps ahead to undertake R&D to reach the higher level of IPR protection. This pattern of increased R&D for leaders under state-dependent IPR contrasts with uniform IPR, which always reduces R&D by all leaders.

State-dependent IPR does not only increase the growth rate of the economy by boosting R&D by leaders, but also by creating the beneficial composition effects emphasized above. This is illustrated in Figure 3, which shows that under the optimal state-dependent IPR policy, the steady-state distribution of industries is shifted to the left, with the highest fraction of industries (over 35%) being neck-and-neck. This shows that state-dependent IPR policy successfully exploits the composition effect by shifting more of the industries to the neck-and-neck status where R&D effort is higher. Uniform IPR policy also induces such a shift, but also creates significant disincentive costs, so R&D levels of all firms are reduced. In contrast, as Figure 2 shows, state-dependent IPR policy exploits the composition effect without reducing R&D investments (in fact, while simultaneously increasing R&D by leaders). It can do so thanks to the "trickle-down" effects discussed above.

Because there are more neck-and-neck firms that compete for labor, the labor share under the optimal state-dependent IPR policy is higher, $\omega^* = 0.9$. Instead, the fraction of the workforce employed in research is slightly less than in the benchmark model with full patent protection. Despite the fact that there are fewer researchers, the growth rate of the economy is considerably higher, $g^* = 0.0123$ (by about 10% relative to the full patent protection benchmark). This is achieved because less of the researchers are working to duplicate existing innovations and more of them are employed in neck-and-neck industries, which again reflects the composition effect.

Therefore, the benchmark case leads to the following conclusions: (1) while uniform IPR is unable to increase the growth rate of the economy, state-dependent IPR can generate a significant increase in innovation and economic growth; (2) optimal state-dependent IPR provides more protection to firms that are technologically more advanced relative to their rivals.

7.5 Robustness

The general pattern that emerges from the benchmark is relatively robust. Tables 1 and 2 show the equilibrium patterns under full protection and optimal state-dependent IPR when we change λ and γ starting from the benchmark (and not change any of the other parameters). Since we do not change any of the other parameters (in particular, B), the growth rate of the economy differs significantly between different parameterizations. Consequently, what is relevant is not the absolute change in the growth rate but the change relative to the full patent protection benchmark.

Comparison of Tables 1 and 2 shows that in all cases, optimal state-dependent IPR leads to approximately a 10% increase in the growth rate of the economy relative to the growth rate under full patent protection (while uniform IPR typically has no effect). Moreover, in all of the cases, optimal state-dependent IPR involves providing more protection to technologically more advanced firms. In particular, optimal state-dependent IPR policy always involves $\eta_4 = \eta_5 = 0$, except when $\lambda = 1.1$ and $\gamma = 0.6$, in which case it involves $\eta_4 = 0.003$ and $\eta_5 = 0$.

As an additional robustness check, we have also experimented with different values of the discount factor ρ. It may be conjectured that high discount factors might weaken the trickle-down effect, which results from the forward-looking behavior of firms. We verified that the form of the optimal IPR policy is not much affected by the extent of discounting; for discount factors as high as $\rho = 0.5$, optimal IPR policy involves technologically more advanced firms

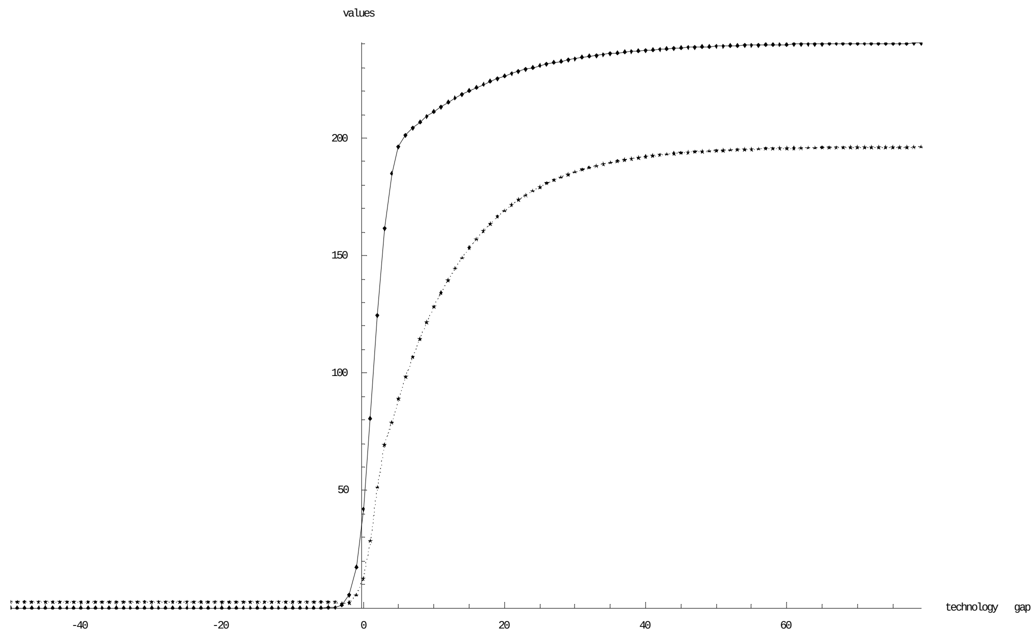

Figure 4: Value functions in the slow catch-up regime with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

receiving greater patent protection than those that a a few steps ahead of their rivals. We do not report these results to save space.

8 Optimal IPR Policy in the Slow Catch-Up Regime

8.1 Slow Catch-Up With $\alpha = 0$

We next turn to an analysis of optimal IPR policy in the slow catch-up regime. We start with the case in which $\alpha = 0$, thus there are no innovations without R&D effort and followers can catch up with leaders only by undertaking a series of successful innovations.

The benchmark parameterization again involves $\rho = 0.05$, $\lambda = 1.1$ and $\gamma = 0.35$. As in the example discussed in Section 5, it turns out that there is no growth in the economy with full patent protection; in the stationary distribution, all industries are at a level of the technology gap where neither the leader nor the follower undertakes any R&D. This is illustrated in the first row of Table 3 and in Figures 4-6, which show the value function, the R&D levels and the stationary industry distribution under full patent protection in the slow catch-up regime (again solid lines referring to the full patent protection case).

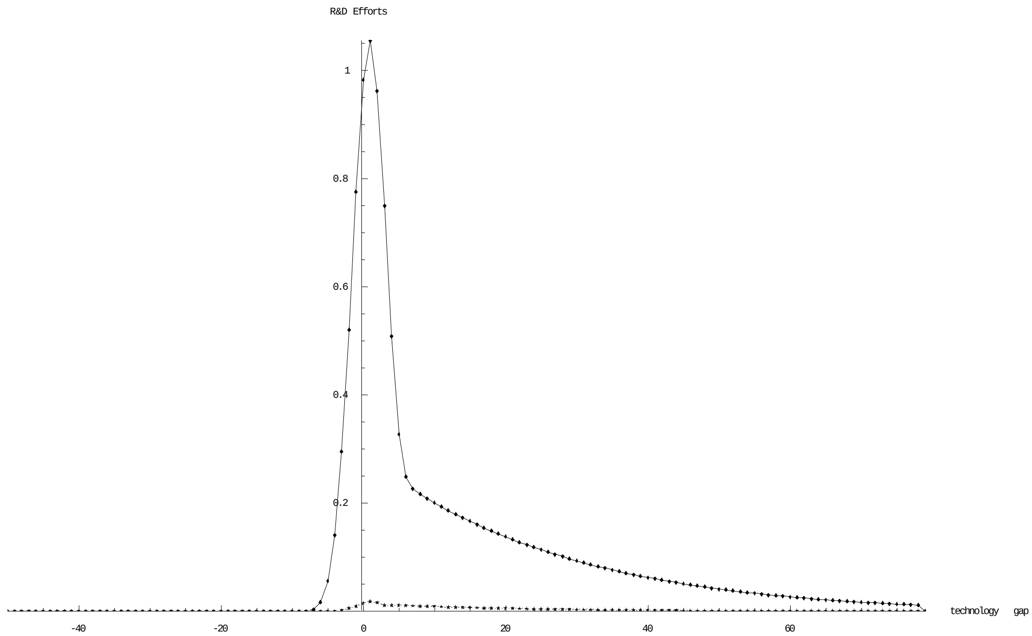

Figure 5: R&D levels in the slow catch-up regime with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

Even though the R&D investments are relatively high as shown in Figure 5, Table 3 indicates that these high R&D levels have no effect on the long-run growth rate of the economy, which is equal to zero. Figure 6 shows the reason for this; in the stationary distribution, all industries are at $n = 80$, where there is no R&D either by followers or leaders. Since there is no growth, the ratio of the workforce working in research is equal to zero. Moreover, because there is a very large technology gap in all sectors, markups are also very large and thus the labor share is extremely small 0.05% (as opposed to around 90% in the quick catch-up regime before).

The second row of Table 3 shows the equilibrium under optimal uniform IPR. The pattern in Figure 6 suggests that there is room for beneficial composition effects, and by exploiting these, uniform IPR can increase the annual growth rate of the economy to approximately 0.5%. Despite the substantial composition effects, uniform IPR policy has limited impact on economic growth because it again creates significant disincentive effects.

The situation is rather different with optimal state-dependent IPR. The benchmark with optimal state-dependent IPR is shown in Figures 4-6 and in row 4 of Table 3. Figure 6 shows that the industry distribution under the optimal IPR policy is very different from the distri-

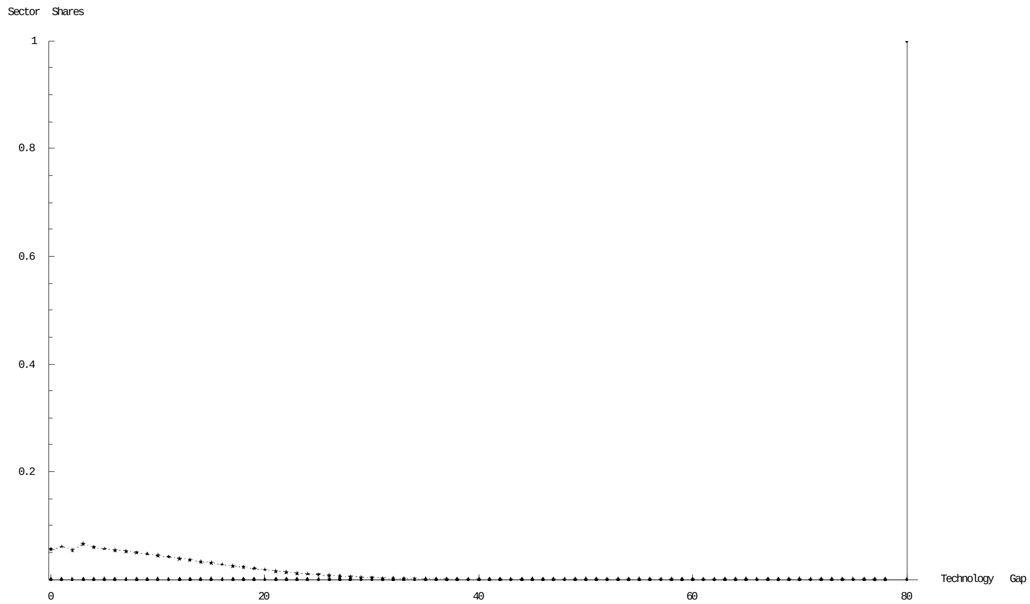

Figure 6: Industry distribution in the slow catch-up regime with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for the case of full patent protection and the dotted line is for optimal state-dependent IPR policy.

bution under full protection. It can be seen from Table 3 that now η_4 and η_5 are relatively high (they are both equal to 0.001). Consequently, as shown in Figure 5, there is a very substantial disincentive effect, and R&D levels are much lower than under full patent protection. However, a direct comparison of the two curves in Figure 5 would be misleading; as Figure 6 shows, under full patent protection, all firms are in the part of the distribution where there is no R&D, while under optimal IPR, firms have been shifted to parts of the distribution where there is positive R&D. As a result, the growth rate increases from zero to 1.2% a year. This is associated with an increase in the researcher ratio to 8% of the workforce and an increase in the labor share to 52%.

Table 3

Simulation Results for Slow Catch-Up Regime

λ	γ	x^*_{-1}	x^*_0	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	res. ratio
1.10	0.35	0.7764	0.9829	0.0000	0.0000	0.0000	0.0000	0.0000	0.000000	0.0005	0.0000
1.10	0.35	0.0051	0.0087	0.0012	0.0012	0.0012	0.0012	0.0012	0.004971	0.7806	0.0289
λ	γ	x^*_{-1}	x^*_0	η_1	η_2	η_3	η_4	η_5	g^*	ω^*	res. ratio
1.05	0.35	0.0050	0.00881	0.0064	0	0	0.0006	0.0010	0.004945	0.7325	0.0301
1.10	0.35	0.0097	0.01515	0.0054	0	0	0.0010	0.0010	0.012386	0.5233	0.0812
1.20	0.35	0.0140	0.02093	0.0050	0	0	0.0010	0.0016	0.027924	0.4521	0.1630

Note: Growth rates refer to annual rates. See text for explanations.

It is noteworthy, however, that despite the significant relaxation of patent protection for firms that are technologically advanced (four or five steps ahead of their followers), the trickle-down effect is still operational. In fact, the trickle-down effect is the reason why the R&D distribution shifts significantly down in Figure 5. The trickle-down effect also implies that it is more economical to exploit the composition effect by reducing the patent protection of firms that are only one step ahead of their rivals, and η_1 is substantially higher than η_4 and η_5 (0.0054 versus 0.0010). Note that in the stationary industry distribution under optimal IPR, a large fraction of industries feature a technology gap of one between the leader and the follower, thus the value of η_1 has a major impact on the equilibrium growth rate. Overall, as noted above, optimal state-dependent IPR increases the growth rate of the economy to just over 1.2% a year. This is almost 3 times the growth rate of the economy under uniform IPR (while, recall that, the growth rate with full patent protection was equal to 0).

Rows 3 and 5 of Table 3 show that the general pattern illustrated by the benchmark calibration is robust. For different values of λ, the economy with full patent protection continues to generate zero growth (while uniform IPR still generates only small improvements). In contrast, state-dependent IPR policy increases the growth rate of the economy substantially. Furthermore, it again achieves this by providing relatively greater security to firms that are 5 steps or more ahead and thus exploiting the trickle-down effect.

8.2 Slow Catch-Up With $\alpha > 0$ (in progress)

We next investigate how the above conclusions are affected when we allow for $\alpha > 0$, so that followers can advance by one step even without R&D effort.

TO BE COMPLETED

9 Optimal IPR With Patent Fees (in progress)

We now briefly investigate the optimal structure of patent fees when followers can build on the innovations of leaders as in the model with equilibrium patent fees analyzed in Section 6. The results in this section are still preliminary. Nevertheless, our findings so far show that, similar to our analysis of quick and slow catch-up regimes, the trickle-down of incentives continues to be a powerful force, leading to greater protection for relatively advanced firms.

We choose the same baseline parameters as above, in particular, $\lambda = 1.1$ and $\gamma = 0.35$. Note also that the case with full patent protection, i.e., $\zeta_n \to \infty$ for all n, is clearly identical to the full patent protection case under the quick catch-up regime. We therefore show the behavior of the economy under optimal uniform and state-dependent IPR policies. In the first case, a single value of ζ is chosen to maximize the growth rate, whereas in the latter, the sequence $\{\zeta_n\}_{n=1}^{\infty}$ is chosen in order to maximize the growth rate (again with the computational constraint that $\zeta_n = \zeta_5$ for all $n \geq 5$). In both cases, we report results from simulations that assume $a_{-n} = 1$ for all n.[16]

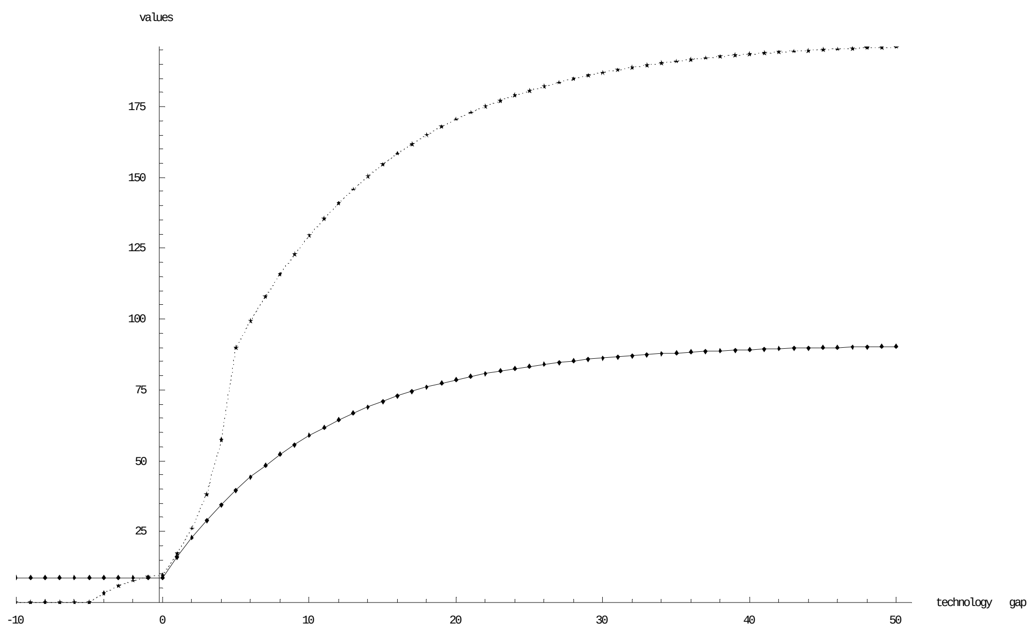

Figure 7: Value functions with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for optimal uniform IPR and the dotted line is for optimal state-dependent IPR policy.

[16]In principle, this is not optimal, and solving for the full equilibrium with $h_n = 0$ for some n appears to increase the growth rate. This will be investigated in greater detail in our future work.

Figures 7-9 show the value function, the R&D levels and the stationary industry distribution under the growth-maximizing level of uniform IPR and state-dependent IPR policies. We find that the growth-maximizing level on uniform IPR is to set $\zeta = 13$, which is equal to the value of $v_1 - v_0$. This implies that ζ takes the maximum level consistent with $a_{-n} = 1$ for all n (i.e., $v_1 - v_0 = 13$ in this case; if ζ were greater than 13, followers would prefer to develop their own technologies to reach the frontier). Consequently, uniform IPR, as in previous cases, is quite close to full patent protection. The reason is again the disincentive effects that uniform IPR creates, which depressed the R&D investments by technology leaders.

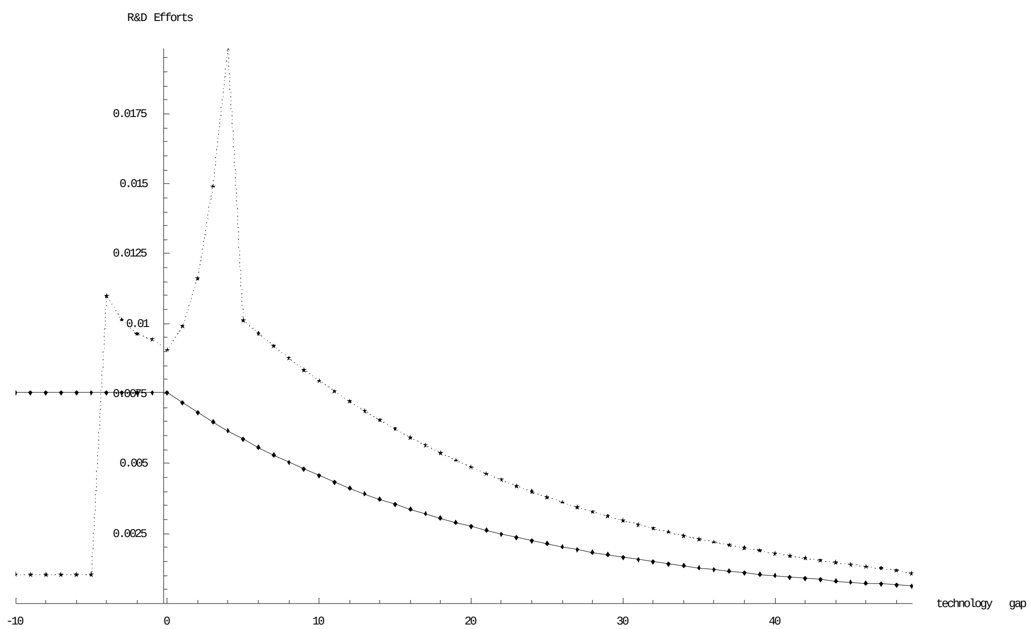

Figure 8: R&D levels with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for optimal uniform IPR and the dotted line is for optimal state-dependent IPR policy.

In contrast, optimal state-dependent IPR provides little protection to firms that are one or two steps ahead, but gives substantial protection to firms that are technologically more advanced. In particular, the optimal state-dependent IPR involves $\zeta_1 = 0$, which implies no patent protection for firms that are one step ahead. However, it also imposes $\zeta_2 = 2$, $\zeta_3 = 5$, $\zeta_4 = 7$ and $\zeta_5 = 17$, so that firms that are five or more steps ahead receive substantial protection. This again reflects the role of trickle-down of R&D incentives emphasized previously. By exploiting the trickle-down of incentives, state-dependent IPR can increase the growth rate of the economy by about 30% (0.3 percentage points) relative to the optimal uniform IPR

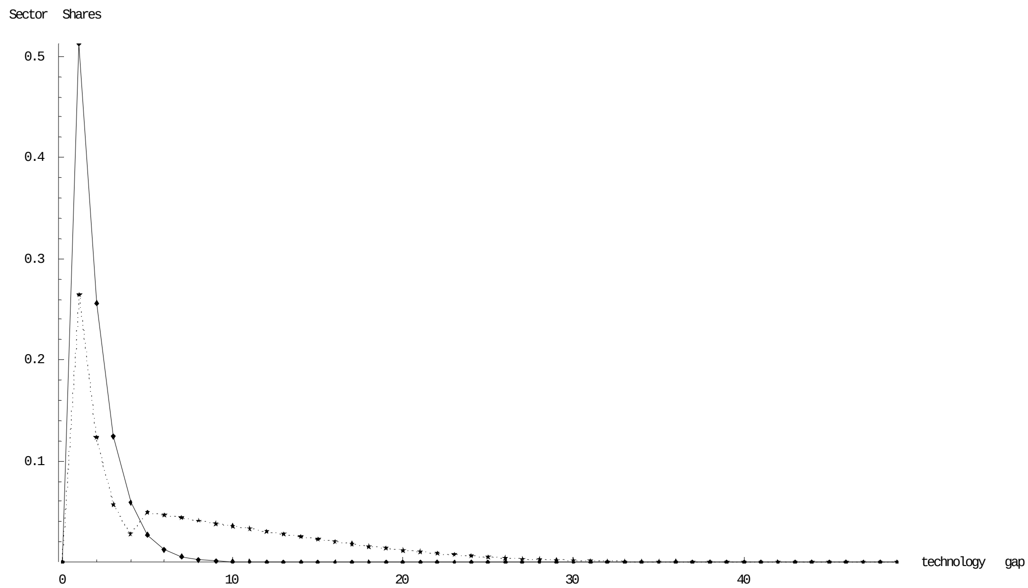

Figure 9: Distribution of industries with $\lambda = 1.1$ and $\gamma = 0.35$. The solid line is for optimal uniform IPR and the dotted line is for optimal state-dependent IPR policy.

policy. This can be seen in Figure 8, where R&D investments are substantially higher under state-dependent IPR than under uniform IPR.

Overall, this preliminary analysis suggests that the trickle-down of incentives is also a powerful force in the equilibrium with patent fees and implies that growth-maximizing IPR policy should provide greater patent protection to firms with greater technological leads over their rivals.

10 Conclusions

In this paper, we developed a general equilibrium framework to investigate the optimal form of intellectual property rights (IPR) policy and whether the same patent protection should be given to companies that are further ahead of their competitors as those that are technologically "neck-and-neck" with other firms. The latter question, i.e., whether optimal IPR policy should be "state dependent", is important not only theoretically, but also for many of the policy debates about the implementation of existing patent laws in the United States and in Europe.

In our model economy, firms engage in cumulative (step-by-step) innovation. Leaders can innovate in order to widen the technological gap between themselves and the followers, which enables them to charge higher markups. On the other hand, followers innovate to catch up with leaders. We considered three different scenarios about the technology of catch-up: (i) quick catch-up regime, in which followers can jump to the leading-edge technology with a single innovation; (ii) slow catch-up regime, in which followers can only advance one step at a time; (iii) equilibrium with patent fees, where followers can undertake R&D building on the leading-edge technology to directly surpass it, but must pay a patent fee when they use this knowledge in their products.

In the model economy, IPR policy regulates whether followers in an industry can copy the technology of the leader (and in the equilibrium with patent fees, how much they have to pay when they use this technology). In all cases, full patent protection implies that followers can only catch up to the leader in their industry by making the same innovation(s) themselves, whereas without full patent protection they may be able to copy the leading-edge technology without the innovation or build upon it to shift the technology frontier forward.

We characterized the form of the state-state (stationary) equilibrium and proved its existence in all three cases. In all three scenarios, R&D investments are determined by the same trade-offs. Consequently, in all cases relaxing full patent protection creates two opposing forces. The first is a *disincentive effect*, which diminishes R&D effort, because being successful in R&D is less valuable without secure patent protection. Opposing this is a *composition effect*, which increases R&D by raising the fraction of industries where firms are technologically "neck-and-neck" (i.e., technologically closer together). In an attempt to become the leader in their industry, neck-and-neck firms undertake greater R&D than leaders and followers in industries with a large technology gap; as a result, the composition effect tends to increase R&D and economic growth. The overall impact of the level and the structure of patent protection on economic growth depends on the interplay of these two forces, which in turn depends on the steady-state distribution of industries across different levels of technology gaps.

We investigated the form of "optimal" (growth-maximizing) IPR policy quantitatively, i.e., by simulating the equilibrium of the economy for plausible parameter values. We found that the level and form of optimal IPR policy differs between the quick and slow catch-up regimes.

In the quick catch-up regime, most industries feature a small technological gap between leaders and followers. As a result, there are limited composition effects. Therefore, optimal uniform IPR policy, which creates significant disincentive effects, does not generate additional growth relative to the full patent protection benchmark. In contrast, state-dependent IPR policy can significantly increase the growth rate of the economy, typically by about 10% to relative to the full patent protection benchmark growth rate. The reason is that state-dependent IPR can increase the fraction of industries where firms are neck-and-neck, without diminishing the overall R&D incentives. On the contrary, optimal state-dependent IPR *increases* the R&D incentives of leaders. It achieves this, somewhat surprisingly, by providing greater patent protection to firms that have a larger technological lead. The intuition for this result is that the prospect of high and secure profits for a leader that achieves a large technological lead creates a powerful *trickle-down* effect and encourages R&D also by firms with smaller technological leads.

The results are somewhat different when we look at the slow catch-up regime. In this case, most industries feature a large technological gap between leaders and followers, and there is more to be gained by exploiting the composition effects. Consequently, optimal IPR policy involves less strong patent protection, even for firms that are technologically advanced relative to their rivals, and can increase the growth rate of the economy more significantly relative to full patent protection. Nevertheless, the trickle-down effect continues to play an important role in the slow catch-up regime and implies that optimal IPR policy should feature greater patent protection for firms that are technologically more advanced relative to their rivals than those that only have a small lead. In the slow catch-up regime, as in the quick catch-up regime, state-dependent IPR can exploit the composition effects without creating severe disincentive effects, and thus generates a much more significant increase in economic growth than uniform IPR policy can.

Finally, in the equilibrium with patent fees followers advance much more rapidly relative to leaders, but the trickle-down of R&D incentives continues to be a powerful force and implies that optimal IPR policy should provide greater protection to firms that are technologically more advanced compared to their rivals. Quantitative gains from state-dependent policy are again substantial.

The analysis in this paper suggests that a move towards optimal state-dependent policies may significantly increase innovation and economic growth both under the quick and the slow catch-up regimes. The results also show that the form of optimal IPR policy may depend on the industry structure (and the technology of catch-up within the industry). Nevertheless, these conclusions are based on a quantitative evaluation of a rather simple model, and both the quick catch-up and the slow catch-up regimes are extreme. It would be interesting to investigate the robustness of these results to different models of industry dynamics and study whether the relationship between the form of optimal IPR policy and industry structure suggested by our analysis also applies when variation in industry structure has other sources (for example, differences in the extent of fixed costs). The most important area for future work is a detailed empirical investigation of the form of optimal IPR policy, using both better estimates of the effects of IPR policy on innovation rates and also structural models where the effect of different policies on equilibrium can be evaluated.

11 Appendix: Proofs

Proof of Proposition 1. From (19) and (21), we can write

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n^*(t)}}{\omega(t)}.$$

Since $\omega(t) = \omega^*$ and $\{\mu_n^*\}_{n=0}^{\infty}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Therefore, $g = \lim_{\Delta t \to 0} (\ln Q(t+\Delta t) - \ln Q(t))/\Delta t$. Since a fraction μ_n^* of industries have technology gap of $n \geq 1$ and face a rate of innovation of $x_n^* \cdot \Delta t + o(\Delta t)$ during an interval of length Δt (and the industries with technology gap of $n = 0$ face an innovation rate of $2x_0^* \cdot \Delta t + o(\Delta t)$) and each innovation increase productivity by a factor λ, we have

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu_0^* x_0^* \Delta t + \sum_{n=1}^{\infty} \mu_n^* x_n^* \Delta t + o(\Delta t) \right].$$

Subtracting $\ln Q(t)$, dividing by Δt and taking the limit $\Delta t \to 0$ gives (35). ■

Proof of Proposition 2. Let $\{x_n\}_{n=-1}^{\infty}$ be the R&D policy sequence of the firm and $\{v_n\}_{n=-1}^{\infty}$ be its value functions, taking decisions by other firms and equilibrium values, $\{x_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g^* as given. By choosing $x_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$. Moreover, since $\pi_n \leq 1$ for all $n \geq -1$, $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$

We show that it is increasing by proof by contradiction.

Suppose, first, $v_{-1} \geq v_0$. Then (37) implies $x_{-1} = 0$, and substituting this back into (36), we obtain

$$\frac{\rho + \eta}{\eta} v_{-1} = v_0,$$

which implies $v_{-1} < v_0$, contradicting the hypothesis $v_{-1} \geq v_0$ and establishing that $v_{-1} < v_0$ as desired.

Next, again to obtain a contradiction, suppose that $v_0 \geq v_1$, then (30) implies $x_0 = 0$, and (26) becomes

$$\rho v_0 = x_0^* (v_{-1} - v_0),$$

which is impossible in view of the fact that $v_{-1} < v_0$ (established in the previous step) and that $v_n \geq 0$ for all $n \geq -1$.

Finally, again to obtain a contradiction, suppose that $v_n \geq v_{n+1}$. Now (28) implies $x_n = 0$, and (25) becomes

$$\rho v_n = \left(1 - \lambda^{-n}\right) + \left(x_{-1}^* + \eta\right)[v_0 - v_n] \tag{59}$$

Also from (25), the value for state $n+1$ satisfies

$$\rho v_{n+1} \geq \left(1 - \lambda^{-n-1}\right) + \left(x_{-1}^* + \eta\right)[v_0 - v_{n+1}]. \tag{60}$$

Combining these two, we obtain

$$\left(1 - \lambda^{-n}\right) + \left(x_{-1}^* + \eta\right)[v_0 - v_n] > 1 - \lambda^{-n-1} + \left(x_{-1}^* + \eta\right)[v_0 - v_{n+1}].$$

Since $\lambda^{-n-1} < \lambda^{-n}$, this implies $v_n < v_{n+1}$, contradicting the hypothesis that $v_n \geq v_{n+1}$, and establishing the desired result, $v_n < v_{n+1}$. Consequently, $\{v_n\}_{n=-1}^{\infty}$ is strictly increasing.

Since an increasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_∞, which must be strictly positive, since $\{v_n\}_{n=-1}^{\infty}$ is strictly increasing starting from 0. This completes the proof. ■

Proof of Proposition 3. The first-order condition of the maximization of the value function (25) implies:

$$G'(x_n) \geq \frac{v_{n+1} - v_n}{\omega^*} \text{ and } x_n \geq 0,$$

with complementary slackness. $G'(0)$ is strictly positive by assumption. If $(v_{n+1} - v_n)/\omega^* < G'(0)$, then $x_n = 0$. Proposition 2 implies that $\{v_n\}_{n=-1}^{\infty}$ is a convergent and thus a Cauchy sequence, which implies that there exists $\exists n^* \in \mathbb{N}$ such that $v_{n+1} - v_n < \omega^* G'(0)$ for all $n \geq n^*$. ■

Proof of Proposition 4. In equilibrium

$$\rho v_0 = -\omega^* G(x_0) + x_0^*[v_{-1} + v_1 - 2v_0], \tag{61}$$

Since $v_0 > 0$ from Proposition 2, it must be that

$$\begin{aligned} v_{-1} + v_1 - 2v_0 &> 0 \\ v_1 - v_0 &> v_0 - v_{-1.} \end{aligned}$$

This inequality combined with (30) and (37) implies the result. ■

Proof of Proposition 5. From equation (28),

$$z_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv z_n \tag{62}$$

is a sufficient to establish that $x^*_{n+1} \leq x^*_n$.

Let us write:

$$\bar{\rho} v_n = \max_{x_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(x_n\right) + x_n \left[v_{n+1} - v_n\right] + \left(x^*_{-1} + \eta\right) v_0 \right\}, \tag{63}$$

where $\bar{\rho} \equiv \rho + x^*_{-1} + \eta$. By definition of x^*_{n+1}, x^*_n and x^*_{n-1} being maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (63) implies:

$$\begin{aligned}
\bar{\rho} v_{n+1} &= 1 - \lambda^{-n-1} - \omega^* G\left(x^*_{n+1}\right) + x^*_{n+1}\left[v_{n+2} - v_{n+1}\right] + \left(x^*_{-1} + \eta\right) v_0, \\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(x^*_{n+1}\right) + x^*_{n+1}\left[v_{n+1} - v_n\right] + \left(x^*_{-1} + \eta\right) v_0, \\
\bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(x^*_{n-1}\right) + x^*_{n-1}\left[v_{n+1} - v_n\right] + \left(x^*_{-1} + \eta\right) v_0, \\
\bar{\rho} v_{n-1} &= 1 - \lambda^{-n+1} - \omega^* G\left(x^*_{n-1}\right) + x^*_{n-1}\left[v_n - v_{n-1}\right] + \left(x^*_{-1} + \eta\right) v_0.
\end{aligned}$$

Now taking differences with $\bar{\rho} v_n$ and using the definitions of z_n's, we obtain

$$\begin{aligned}
\bar{\rho} z_{n+1} &\leq \lambda^{-n}\left(1 - \lambda^{-1}\right) + x^*_{n+1}\left(z_{n+2} - z_{n+1}\right) \\
\bar{\rho} z_n &\geq \lambda^{-n+1}\left(1 - \lambda^{-1}\right) + x^*_{n-1}\left(z_{n+1} - z_n\right).
\end{aligned}$$

Therefore,

$$\left(\bar{\rho} + x^*_{n-1}\right)\left(z_{n+1} - z_n\right) \leq -k_n + x^*_{n+1}\left(z_{n+2} - z_{n+1}\right), \tag{64}$$

where

$$k_n \equiv (\lambda - 1)^2 \lambda^{-n-1} > 0.$$

Now to obtain a contradiction, suppose that $z_{n+1} - z_n \geq 0$. From (64), this implies $z_{n+2} - z_{n+1} > 0$ since k_n is strictly positive. Repeating this argument successively, we have that if $z_{n'+1} - z_{n'} \geq 0$, then $z_{n+1} - z_n > 0$ for all $n \geq n'$. However, we know from Proposition 2 that $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $z_n \downarrow 0$, which contradicts the hypothesis that $z_{n+1} - z_n \geq 0$ for all $n \geq n'$, and establishes that $x^*_{n+1} \leq x^*_n$. To see that the inequality is strict when $x^*_n > 0$ it suffices to note that we have already established (62), i.e., $z_{n+1} - z_n > 0$, thus if equation (28) has a positive solution, then we necessarily have $x^*_{n+1} < x^*_n$. ■

Proof of Proposition 6. Fix $\mathbf{z} = \left(\left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)\right)$, and consider the optimization problem of a representative firm, written recursively as:

$$\begin{aligned}
\rho v_n &= \max_{x_n \in [0, \bar{x}]} \left\{ \left(1 - \lambda^{-n}\right) - \tilde{\omega} G\left(x_n\right) + x_n\left[v_{n+1} - v_n\right] + \left(\tilde{x}_{-1} + \eta\right)\left[v_0 - v_n\right] \right\} \text{ for } n \in \mathbb{N} \\
\rho v_0 &= \max_{x_0 \in [0, \bar{x}]} \left\{ -\tilde{\omega} G\left(x_0\right) + x_0\left[v_1 - v_0\right] + \tilde{x}_0\left[v_{-1} - v_0\right] \right\} \\
\rho v_{-1} &= \max_{x_{-1} \in [0, \bar{x}]} \left\{ -\tilde{\omega} G\left(x_0\right) + \left(x_{-1} + \eta\right)\left[v_0 - v_{-1}\right] \right\}.
\end{aligned}$$

We now transform this dynamic optimization problem into a form that can be represented as a contraction mapping using the method of "uniformization" (see, for example, Ross, 1996, Chapter 5). Let $p_{n,n'}(\mathbf{x} \mid \tilde{\mathbf{x}})$ be the probability that the next state will be n' starting with state n when the firm in question chooses $\mathbf{x} = \{x_n\}_{n=-1}^{\infty}$ and the R&D policy of other firms is given by $\tilde{\mathbf{x}} = \{\tilde{x}_n\}_{n=-1}^{\infty}$. Using the fact that $x_{-n} = x_{-1}$ for all $n \in \mathbb{N}$, these transition probabilities are:

	$p_{-1,0}(\mathbf{x} \mid \tilde{\mathbf{x}}) = 1$
$p_{0,-1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{\tilde{x}_0}{x_0+\tilde{x}_0}$	$p_{0,1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{x_0}{x_0+\tilde{x}_0}$
$p_{n,0}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{\tilde{x}_{-1}+\eta}{x_n+\tilde{x}_{-1}+\eta}$	$p_{n,n+1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{x_n}{x_n+\tilde{x}_{-1}+\eta}$

Uniformization involves adding fictitious transitions from a state into itself, which do not change the program, but allow us to represent the optimization problem as a contraction. For this purpose, define the transition rates ψ_n as $\psi_n(\mathbf{x} \mid \tilde{\mathbf{x}}) = x_n + x_{-1} + \eta$ for $n \in \{-1, 1, 2, ...\}$ and $\psi_n(\mathbf{x} \mid \tilde{\mathbf{x}}) = 2x_n$ for $n = 0$. These transition rates are clearly finite, in particular, $\psi_n(\cdot \mid \tilde{\mathbf{x}}) \leq \psi \equiv 2\bar{x} + \eta$, for all n.

Now following equation (5.8.3) in Ross (1996), we can use these transition rates and define the new transition probabilities (including the fictitious transitions from a state to itself) as:

$$\tilde{p}_{n,n'}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \begin{cases} \frac{\psi_n(\mathbf{x}|\tilde{\mathbf{x}})}{\psi} p_{n,n'}(\mathbf{x} \mid \tilde{\mathbf{x}}) & \text{if } n \neq n' \\ \frac{\psi_n(\mathbf{x}|\tilde{\mathbf{x}})}{\psi} p_{n,n}(\mathbf{x} \mid \tilde{\mathbf{x}}) - \frac{\psi_n(\mathbf{x}|\tilde{\mathbf{x}})}{\psi} & \text{if } n = n' \end{cases}.$$

This yields equivalent transition probabilities

	$\tilde{p}_{-1,-1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{2\bar{x}-(x_{-1}+\tilde{x}_n)}{2\bar{x}+\eta}$	$\tilde{p}_{-1,0}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{x_{-1}+\eta}{2\bar{x}+\eta}$
$\tilde{p}_{0,-1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,0}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{2\bar{x}+\eta-(x_0+\tilde{x}_0)}{2\bar{x}+\eta}$	$\tilde{p}_{0,1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{x_0}{2\bar{x}+\eta}$
$\tilde{p}_{n,0}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{2\bar{x}-(x_{-1}+x_n)}{2\bar{x}+\eta}$	$\tilde{p}_{n,n+1}(\mathbf{x} \mid \tilde{\mathbf{x}}) = \frac{x_n}{2\bar{x}+\eta}$

and also defines an effective discount factor β given by

$$\beta \equiv \frac{\psi}{\rho + \psi} = \frac{2\bar{x} + \eta}{\rho + 2\bar{x} + \eta}.$$

Also define the per period return function (profit net of R&D expenditures) as

$$\Pi_n(x_n) = \begin{cases} \frac{1-\lambda^{-n}-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{if } n \geq 1 \\ \frac{-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{otherwise} \end{cases}. \tag{65}$$

Using these transformations, the dynamic optimization problem can be written as:

$$\begin{aligned} v_n &= \max_{x_n} \left\{ \Pi_n(x_n) + \beta \sum_{n'} \tilde{p}_{n,n'}(x_n \mid \tilde{\mathbf{x}}) v_{n'} \right\}, \ n \in \mathbb{Z}. \\ &= T\mathbf{v}, \end{aligned} \tag{66}$$

where $\mathbf{v} = \{v_n\}_{n=-1}^{\infty}$ and the second line defines the operator T, mapping from the space of functions $\mathbf{V} = \{\mathbf{v} : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+\}$ into itself. T is clearly a contraction, thus, for given $\mathbf{z} = \left(\left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)\right)$, possesses a unique fixed point $\mathbf{v}^* = \{v_n^*\}_{n=-1}^{\infty}$ (e.g., Stokey, Lucas and Prescott, 1989).

Moreover, $x_n \in [0, \bar{x}]$ and the right-hand side of (66) is strictly concave, so the optimal policy $\mathbf{x}^* = \{x_n^*\}_{n=-1}^{\infty}$ is uniquely defined (in fact, given by (28), (29) and (30) above). Then by Berge's maximum theorem (e.g., Stokey, Lucas and Prescott, 1989, Theorem 3.6, p. 62), each x_n^* is an upper hemi-continuous correspondence of each component of $\mathbf{z} = \left(\left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)\right)$. Since x_n^* is uniquely defined, upper hemi-continuity implies continuity, so that x_n^* is a continuous function of each component of $\mathbf{z} = \left(\left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)\right)$. This also implies that for any sequence $\mathbf{z}^k \to \mathbf{z}$, we have $x_n^*\left[\mathbf{z}^k\right] \to x_n^*\left[\mathbf{z}^k\right]$, thus x_n^* is continuous in $\mathbf{z} = \left(\left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}\right)\right)$ in the product topology, completing the proof. ■

Proof of Proposition 7. We first show that the mapping Φ: $\mathbf{Z} \to \mathbf{Z}$ constructed above has a fixed point, and then establish that when $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/\rho\right) > 0$ this fixed point corresponds to a steady state with $\omega^* < 1$ and $g^* > 0$.

First, it has already been established that Φ maps $\mathbf{Z}$ into itself. We first show that $\mathbf{Z}$ is compact in the product topology. This follows from the fact that $\mathbf{Z}$ can be written as the Cartesian product of compact subsets, $\mathbf{Z} = [0,1] \times \prod_{n=-1}^{\infty} [0, \bar{x}]$. Then by Tychonoff's theorem (e.g., Aliprantis and Border, 1999, Theorem 2.57, p. 52; Kelley, 1955, p. 143), $\mathbf{Z}$ is compact in the product topology. Moreover, clearly $\mathbf{Z}$ is nonempty and convex, since for any $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ and $\lambda \in [0,1]$, we have $\lambda\mathbf{z} + (1-\lambda)\mathbf{z}' \in \mathbf{Z}$. Finally, since $\mathbf{Z}$ is a product of intervals on the real line, it is a subset of a locally convex Hausdorff space (see Aliprantis and Border, 1999, Lemma 5.54, p. 192). Next let A be an arbitrary set; suppose that $\prod_{\alpha \in A} Y_\alpha$ is endowed with the product topology, and let $P_\alpha : \prod_{\alpha \in A} Y_\alpha \to Y_\alpha$ to be the projection mapping. Then a function $f : X \to \prod_{\alpha \in A} Y_\alpha$ is continuous if and only if $P_\alpha \circ f$ is continues for each $\alpha \in A$ (see, Kelley, 1995, p. 91). The function φ is a continuous function from a subset of $\mathbb{R}_+$ into itself and therefore is continuous in the product topology by the previous observation. More importantly, from Proposition 6, $x_n(\mathbf{z})$ is continuous in $\mathbf{z}$ for $n \in \{-1\} \cup \mathbb{Z}_+$ for $\mathbf{X}[\mathbf{z}] = \{x_n(\mathbf{z})\}_{n=-1}^{\infty}$, which implies that $\mathbf{X}[\mathbf{z}]$ is continuous in $\mathbf{z}$ in the product topology. Consequently, the function $\Phi \equiv (\varphi, \mathbf{X}) : \mathbf{Z} \to \mathbf{Z}$ is continuous in the product topology. The Brouwer-Schauder-Tychonoff fixed point theorem implies that if the function Φ maps a convex, compact and nonempty subset of a locally convex Hausdorff space into itself and is continuous, then it possesses a fixed point $\mathbf{z}^* = \Phi(\mathbf{z}^*)$ (see Aliprantis and Border, 1999, Theorem 16.50 and Corollary 16.52, pp. 549-550). This establishes the existence of a fixed point $\mathbf{z}^*$ of Φ.

Second, we show that a fixed point $\mathbf{z}^*$ corresponds to a steady-state equilibrium. Clearly, if $x_n\left(\left(\omega^*, \{x_n^*\}_{n=-1}^{\infty}\right)\right) = x_n^*$ for $n \in \{-1\} \cup \mathbb{Z}_+$, then given a labor share of ω^*, $\{x_n^*\}_{n=-1}^{\infty}$ is an R&D policy sequence that is best response to itself, as required by steady-state equilibrium (Definition 3). Thus we only have to show that $\varphi\left(\left(\omega^*, \{x_n^*\}_{n=-1}^{\infty}\right)\right) = \omega^*$ is a steady-state equilibrium labor share. First, we show that $\omega^* = 1$ is not possible. Suppose, to obtain a contradiction, that $\omega^* = 1$. Then, as noted above, we must have $\mu_0^* = 1$. From (31), (32) and (33), this implies $x_n^* = 0$ for $n \in \{-1\} \cup \mathbb{Z}_+$. Now consider the value of an innovation. Since each industry is small, ω remains at 1, and since $x_n^* = 0$, there are no further innovations, so the present discounted value of profits for the innovating firm is: $\hat{v}_1 = \left(1 - \lambda^{-1}\right)/r$. The assumption that $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/r\right) > 0$ then makes sure that (38) cannot be satisfied at $\omega^* = 1$, implying that $\omega^* < 1$. When $\omega^* < 1$, the labor market clearing condition (34) is satisfied at ω^* as an equality, so ω^* is an equilibrium given $\{x_n^*\}_{n=-1}^{\infty}$, and thus $\mathbf{z}^* = \left(\omega^*, \{x_n^*\}_{n=-1}^{\infty}\right)$ is a steady-state equilibrium as desired. Finally, with the same argument as above, $G'^{-1}\left(\left(1 - \lambda^{-1}\right)/r\right) > 0$ implies that $x_0^* = 0$ cannot be optimal, thus $x_0^* > 0$. This together with the observation that $\mu_0^* < 1$ (which follows from $\omega^* < 1$) and equation (35) implies that $g^* > 0$, completing the proof. ■

Proof of Proposition 8. Since $x_n^* = 0$ for all $n > n^*$, the states $n \geq n^* - 1$ are transient and can be ignored. More specifically, denoting the probability of being in state $\tilde{n}$ starting in state n after τ periods by $P^\tau(n, \tilde{n})$, we have that $\lim_{\tau \to \infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n. Thus we can focus on the finite Markov chain over the states $n = 1, ..., n^* - 1$, and $\{\mu_n^*\}_{n=0}^{n^*-1}$ is the limiting (invariant) distribution of this Markov chain. Since $\{x_n^*\}_{n=-1}^{n^*}$ is uniquely defined, so is $\{\mu_n^*\}_{n=0}^{n^*-1}$. Moreover, the underlying Markov chain is irreducible (since $x_n > 0$ for $n = -1, 1, ..., n^* - 1$, so that all states communicate with $n = 0$). Therefore, by Theorem 11.2, p. 62 of Stokey, Lucas and Prescott, 1989, there exists a unique stationary distribution $\{\mu_n^*\}_{n=0}^{\infty}$. ■

Proof of Proposition 9. There are two cases to consider. First, suppose that $\{v_n\}_{n \in \mathbb{Z}_+}$ is strictly increasing. Then it follows from the proof of Proposition 3 that there exists a state $n^* \in \mathbb{N}$ such that $x_n^* = 0$ for all $n \geq n^*$, and as in the proof of Proposition 8, states $n \geq n^*$ are transient (i.e., $\lim_{\tau \to \infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n), so $\mu_n^* = 0$ for all $n \geq n^*$.

Second, in contrast to the first case, suppose that there exists some $n^{**} \in \mathbb{Z}_+$ such that $v_{n^{**}} \geq v_{n^{**}+1}$. Then, let $n^* = \min\{n \in \mathbb{N}: v_n \geq v_{n+1}\}$. Since $v_{n^{**}} \geq v_{n^{**}+1}$ by hypothesis, such an n^* is well defined. Then, optimal R&D decision (28) immediately implies that $x_n^* = 0$, so that all states with $n \geq n^*$ are transient and again $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n, completing the proof. ■

Proof of Proposition 10. The proof follows closely that of Proposition 6. In particular, again using uniformization, the maximization problem of an individual firm can be written as a contraction mapping, which uniquely determines the value function $v[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$. Optimal R&D decisions are again uniquely determined (e.g., from (28), (29) and (30), or from the fact that each $v[\mathbf{z}]$ is strictly concave in $\{x_n\}_{n=-\infty}^{\infty}$). Berge's maximum theorem then implies that $x_n(\mathbf{z})$ is continuous in each element of $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}\right)$ for $n \in \mathbb{Z}$. The same argument as in the proof of Proposition 6 then establishes that $\mathbf{X}[\mathbf{z}]$ is continuous in $\mathbf{z}$ in the product topology. ■

Proof of Proposition 11. The proof follows that of Proposition 7 closely. Fix $\mathbf{z} = \left(\tilde{\omega}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}\right)$, and define $\mathbf{Z} = [0,1] \times \prod_{n=-\infty}^{\infty} [0, \bar{x}]$. Again by Tychonoff's theorem, $\mathbf{Z}$ is compact in the product topology. Then consider the mapping $\Phi\colon \mathbf{Z} \to \mathbf{Z}$ constructed as $\Phi \equiv (\varphi, \mathbf{X})$, where φ is given by (38) and $\mathbf{X}$ is defined in Proposition 10. Clearly Φ maps $\mathbf{Z}$ into itself. Moreover, as in the proof of Proposition 7, $\mathbf{Z}$ is nonempty, convex, and a subset of a locally convex Hausdorff space, and Φ is continuous in the product topology. Then the Brouwer-Schauder-Tychonoff fixed point theorem again applies and implies that Φ has a fixed point $\mathbf{z}^* = \Phi(\mathbf{z}^*)$. The argument that the fixed point $\mathbf{z}^*$ corresponds to a steady-state equilibrium is identical to that in Proposition 7. ■

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