Heterogeneous Beliefs and Bond Yields*

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Abstract

Motivated by the prevalence of heterogeneous beliefs held by market participants regarding future economic fundamentals, we provide an equilibrium model to analyze the effects of belief dispersion on bond yields. Our model shows that the disagreement among agents in regards to their expectations for future interest rates raises bond prices and reduces bond yields relative to the corresponding values in a homogeneous economy, wherein all agents share the wealth weighted average belief. Furthermore, the relative-wealth fluctuation caused by agents' speculative trading amplifies shocks to the economy and increases yield volatility. Taken together, these results highlight the importance of incorporating belief dispersion into economic analysis of bond yields.

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1 Introduction

Agents' expectations about the future course of short rates is a crucial determinant of bond yields. This idea underlies the expectations hypothesis, one of the classic theories of the yield curve dating at least back as Fisher (1896), Hicks (1939), and Lutz (1940). According to Lutz (1940, pp. 37), "An owner of funds will go into the long market if he thinks the return he can make there over the time for which he has funds available will be above the return he can make in the short market over the same time, and vice versa." Thus, the long rate should be determined by the agent's expectation of future short rates during the corresponding period. While expectations theory has attracted numerous academic studies,[1] most assume that all agents have identical expectations.[2]

The data easily reject the homogeneous expectations assumption. For example, Mankiw, Reis and Wolfer (2004) show that in December 2002, the interquartile range of inflation expectations for 2003 among professional economists in the Livingston Survey goes from 1.5% to 2.5%. The interquartile range among the general public in the Michigan Survey of Consumer Attitudes and Behavior goes from 0% to 5%. Welch (2000) surveys 226 academic financial economists' and finds that their forecasts of 30-year equity premium vary from 2% to 13%. While surveys may not be the most effective method for extracting agents' expectations, these surveys demonstrate significant belief dispersion among a broad range of respondents about the future values of macroeconomic variables.

Is belief dispersion among market participants important for understanding bond yields? An intuitive argument is that even when agents have heterogeneous beliefs, markets would aggregate these beliefs and bond prices would be equivalent to those in a homogeneous model, whereby all agents possess the average belief of the heterogeneous economy. If this argument holds true, it justifies the common practice in the literature of ignoring belief dispersion and simply focusing on agents' average belief. In this paper, we provide an equilibrium model to examine this argument.

Our model adopts the equilibrium framework of Cox, Ingersoll and Ross (1985a)

[1]See Campbell, Lo and MacKinlay (1997, chapter 10) for a review.
[2]For example, see Cox, Ingersoll and Ross (1981) and Campbell (1986).

with log-utility agents and a constant-return-to-scale risky investment technology. Since the risky technology represents an alternative investment channel to risk free bonds, the expected instantaneous return from investing in the risky technology determines the equilibrium short rate. Unlike Cox, Ingersoll and Ross (1985a), we assume that there is an unobservable variable that determines the future returns of the risky technology and, therefore, the future short rates as well. Furthermore, we allow agents to hold heterogeneous prior beliefs about the unobservable variable. This assumption is consistent with the observation that agents often disagree about their expectations of the future values of macroeconomic variables such as inflation and economic growth, although they base these expectations on a shared information set. Dispersion in agents' beliefs causes them to speculate with each other in capital markets. We study a competitive equilibrium, in which each agent optimizes consumption and investment decisions based on her beliefs. Market clearing conditions determine the equilibrium interest rates and bond prices.

Our model shows that the price of a bond is the wealth weighted average of bond prices in homogeneous economies, in each of which only one type of agent is present. Note that each agent's bond valuation in a homogeneous economy is a convex function of the agent's belief about future short rates. Thus, Jensen's inequality implies that the bond price in a heterogeneous economy is higher than the price in a homogeneous economy wherein the representative agent possesses the average belief of those agents in the heterogeneous economy. In other words, dispersion in agents' beliefs about future short rates increases bond prices and decreases bond yields. This effect is especially stronger for long-term bonds. Furthermore, belief dispersion increases yield volatility, because the relative-wealth fluctuation caused by agents' speculative trading tends to amplify shocks to the economy. We also provide numerical examples to show that the magnitudes of these belief dispersion effects could be substantial. Our model thus highlights the importance of incorporating belief dispersion into economic analysis of bond yields.

The belief dispersion effects in our model arise from the aggregation of beliefs in equilibrium across agents. This mechanism is distinct from effects generated by uncertainty, i.e., the lack of confidence in expectation. Although belief dispersion across agents and uncertainty in each agent's belief are sometimes correlated, these are two different concepts and can be measured separately.[3] Our model predicts that after controlling for uncertainty effects, dispersion in agents' beliefs would decrease bond yields and increase yield volatility. Our model also has potential implications for monetary policies. In particular, it highlights the importance for the monetary authority to manage the dispersion of market participants' expectations.

Our model adds to the theoretical literature on term structure of interest rates, e.g., Vasicek (1977), Cox, Ingersoll and Ross (1985b) and Duffie and Kan (1996), by establishing belief dispersion as an additional determinant of bond yields. Our study complements Williams (1977), Detemple and Murthy (1994) and Basak (2000), who analyze the effects of heterogeneous beliefs on stock returns and short rates, but not on bond yields. Our paper is also related to the literature that analyzes the joint effects of heterogeneous beliefs and short-sales constraints on stock overvaluation, e.g., Miller (1977), Harrison and Kreps (1978), and Scheinkman and Xiong (2003). We show that heterogeneous beliefs can increase bond prices even in the absence of short-sales constraints. Our mechanism is based on the aggregation of convex bond valuations across agents. Yan (2006) also uses a similar mechanism to show that noise trading in stock markets may not be cancelled out by aggregation.

2 The Model

Our model adopts the equilibrium framework of Cox, Ingersoll and Ross (1985a) with log-utility agents and a constant-return-to-scale risky investment technology. Unlike their model, ours assumes that agents cannot directly observe a random variable that drives the future expected return of the investment technology. Rather, one has to infer its value. There are N groups of agents with different beliefs regarding this variable. Because of the belief dispersion, agents speculate in capital markets. We study a competitive equilibrium, in which each agent optimizes consumption and investment

[3]See Zarnowitz and Lambros (1987) for a discussion and an empirical analysis of the difference between belief dispersion and uncertainty.

decisions based on her own beliefs. Market clearing conditions determine the equilibrium short rate and bond prices.

2.1 Investment technology

We consider a production economy with a constant-return-to-scale technology. The return of the technology follows a diffusion process:

$$\frac{dI_t}{I_t} = f_t dt + \sigma_I dZ_I(t) \tag{1}$$

where f_t is the expected instantaneous return, σ_I is a volatility parameter, and $Z_I(t)$ is a standard Brownian motion. The expected instantaneous return from the risky technology f_t follows another linear diffusion process:

$$df_t = -\lambda_f (f_t - l_t)dt + \sigma_f dZ_f(t), \tag{2}$$

where λ_f is a constant governing the mean reverting speed of f_t, l_t represents a moving long-run mean of the risky technology's expected return, σ_f is a volatility parameter, and $Z_f(t)$ is a standard Brownian motion independent of $Z_I(t)$. As we will show later, the expected instantaneous return of the technology f_t, after adjusting for risk, determines the equilibrium short rate.

The long-run mean l_t is unobservable, and it follows an *Ornstein-Uhlenbeck* process:

$$dl_t = -\lambda_l (l_t - \bar{l})dt + \sigma_l dZ_l(t), \tag{3}$$

where λ_l is a parameter governing the mean-reverting speed of l_t, $\bar{l}$ is the long-run mean of l_t, σ_l is a volatility parameter, and $Z_l(t)$ is a standard Brownian motion independent of $Z_I(t)$ and $Z_f(t)$. Since l_t is the level, to which f_t mean-reverts, it determines the future values of short rates.

Intuitively, we can interpret f_t as the current level of inflation, and l_t as the monetary authority's inflation target. According to this interpretation, the mean-reverting speed of f_t measures the monetary authority's effectiveness in controlling inflation, while the l_t process is determined by the evolution of the monetary policy. Since monetary authorities are often unwilling to clearly disclose their inflation targets, market participants have to infer these targets through the observed inflation.

We can also interpret f_t as the expected growth rate of the economy, and l_t as the "optimal" growth rate that the current technology level could have achieved in a frictionless economy. The existence of adjustment cost could prevent an immediate adjustment of the production capacity in the economy to reflect fluctuations in technology, causing f_t to trail behind l_t. According to this interpretation, the mean-reverting speed of f_t is determined by the adjustment cost, while the l_t process is determined by the evolution of technology. Since the optimal growth rate is not directly observable to market participants, it is reasonable to assume that they derive its value through the actual growth rate.

2.2 Agents' learning processes

Agents understand the structure of the economy, i.e., they know the processes and parameters in equations (1)-(3). They also observe the values of dI_t/I_t, f_t, although not l_t. There are N groups of agents with different prior beliefs about the value of l_t. We assume that agents in group i, $i \in \{1, 2, \cdots, N\}$, have the following Gaussian prior belief at time 0:

$$l_0 \sim N\left(\hat{l}_0^i, \gamma_0\right),$$

where $\hat{l}_0^i$ is the mean and γ_0 is the variance. That is we allow agents in different groups to hold different means in regards to l_0, but, for simplicity, we let their belief variance be identical.

As time goes on, agents use the observed values of f_t to update their beliefs about the current value of l_t. Because agents' prior beliefs and signals all have Gaussian distributions, their posterior beliefs must also have Gaussian distributions. We denote the group i's posterior belief at time t as

$$l_t | \{f_\tau\}_{\tau=0}^t \sim N\left(\hat{l}_t^i, \gamma_t\right).$$

According to Theorem 12.7 of Liptser and Shiryaev (1977), the mean of the posterior belief, which we later refer to as group-i agents' belief, is determined by

$$d\hat{l}_t^i = -\lambda_l(\hat{l}_t^i - \bar{l})dt + \frac{\lambda_f \gamma_t}{\sigma_f} d\hat{Z}_f^i \tag{4}$$

where

$$d\hat{Z}_f^i = \frac{1}{\sigma_f}\left[df_t + \lambda_f(f_t - \hat{l}_t^i)dt\right] \tag{5}$$

is surprise in the information flow, df_t. This surprise is a standard Brownian motion in group-i agents' probability measure. Since agents in different groups agree on the structure of the economy, they use the same learning rule in equation (4). They simply use different benchmarks defined by their current beliefs to determine surprise in the information flow, as in equation (5).

Theorem 12.7 of Liptser and Shiryaev (1977) also implies that agents in all groups share the same belief variance and that the variance changes deterministically over time:

$$\frac{d\gamma_t}{dt} = -\frac{\lambda_f^2}{\sigma_f^2}\gamma_t^2 - 2\lambda_l\gamma_t + \sigma_l^2. \tag{6}$$

As time goes to infinity, the belief variance converges to a stationary level $\bar{\gamma}$, which is the positive root to the following quadratic equation:

$$\frac{\lambda_f^2}{\sigma_f^2}\bar{\gamma}^2 + 2\lambda_l\bar{\gamma} - \sigma_l^2 = 0.$$

Given the difference in agents' beliefs, they have different views about the dynamics of f_t. For agents in group i,

$$df_t = -\lambda_f(f_t - \hat{l}_t^i)dt + \sigma_f d\hat{Z}_f^i(t). \tag{7}$$

In the probability measure of group-i agents, equations (1), (4), and (7) describe the evolution of the economy. There are two sources of shocks that directly affect the markets, dZ_I and $d\hat{Z}_f^i$.

For our discussion, we will use the probability measure of one of the groups– group K– as a benchmark. K could be any group in the population. We denote the difference between the beliefs of group-i and group-K agents by

$$g_t^i = \hat{l}_t^i - \hat{l}_t^K.$$

The following lemma describes the dynamics of the difference in beliefs with the proof given in Appendix A.1.

Lemma 1 g_t^i *changes deterministically according to*

$$dg_t^i = -\left(\lambda_l + \frac{\lambda_f^2}{\sigma_f^2}\gamma_t\right) g_t^i \; dt.$$

Lemma 1 suggests that, over time, the difference in agents' beliefs converges deterministically to zero. This feature comes from our setup, which states that agents only differ in their prior beliefs, but that they use the same information set and the same learning rule to update their beliefs. We view the heterogeneous-prior-belief assumption as the simplest way of generating some belief dispersion among agents. The basic mechanism in our model to demonstrate the way belief dispersion affects bond yields also applies to other setups, even those in which heterogeneous beliefs arise from alternative channels.[4]

2.3 Capital markets

The difference in agents' beliefs can cause trading among them. In particular, agents who are more optimistic about l_t would bet on interest rates going up against more pessimistic agents. As we discussed earlier, in each agent's probability measure, there are two sources of random shocks that directly affect the markets. Thus, markets are complete if agents can trade a risk free asset and two risky assets that span the two sources of random shocks. The existence of a bond market could make markets complete. Thus, we analyze agents' investment and consumption decisions, as well as their valuations of financial securities, in a complete-markets equilibrium.

[4]See Morris (1995) for a lucid discussion of the heterogeneous-prior-belief assumption in economic applications. Heterogeneous beliefs could also arise from several other channels. For example, Scheinkman and Xiong (2003) use overconfidence, a behavioral bias in agents' learning processes, as cause. Overconfidence causes agents to react differently to different sources of information, and thus generates stationary heterogeneous belief processes among agents– on one hand, agents' beliefs might diverge in response to a random shock; on the other hand, as in our model, the difference in their beliefs mean-reverts to zero over time. Kurz (2001) and the references therein argue that limited data make it difficult for rational agents to identify the correct model of the economy from alternative ones. As a result, model uncertainty could cause them to use different learning models and therefore to possess heterogeneous beliefs. Furthermore, Mankiw, Reis and Wolfer (2004) argue that stickiness in agents' information caused by the cost of collecting and processing information could also generate heterogeneous beliefs.

For simplicity, we introduce a zero-net-supply risk free asset and a zero-net-supply risky financial security, in addition to the risky technology. These securities facilitate agents' speculation need and complete the markets in our model.[5] At time t, the risk free asset offers a short rate r_t, which is determined endogenously in the equilibrium. The risky financial securities, which we call security f, offers the following return process:

$$dp_f/p_f = \mu_f(t)dt + df_t.$$

This security could be viewed as a synthetic position constructed from a dynamic trading strategy of bonds so that the resulting return process responds proportionally to the innovation in df_t. Since investors have different opinions about df_t, they disagree on the expected return of security f. As a result, some investors want to take long positions, while others want to take short positions. In equilibrium, $\mu_f(t)$ is determined so that the aggregate demand is zero.

By substituting equation (7) into the price process of security f, we obtain that, in the probability measure of group-i agents,

$$dp_f/p_f = \hat{\mu}_f^i(t)dt + \sigma_f d\hat{Z}_f^i(t),$$

where the expected return is given by

$$\hat{\mu}_f^i(t) = \mu_f(t) - \lambda_f(f_t - \hat{l}_t^i). \tag{8}$$

This equation shows that agents who hold higher beliefs about l_t perceive higher expected returns from security f.

We assume that all agents have logarithmic utility. Agents in group i maximize their lifetime utility from consumption by investing in all available securities under their beliefs:

$$\max_{\{c_t^i, \theta_I^i, \theta_f^i\}} E^i \int_0^\infty e^{-\beta t} u(c_t^i) dt,$$

[5]We also allow agents to short-sell the risky technology. This can be implemented by offering a derivative contract on the return of the technology. The market clearing conditions, however, require that agents in aggregate hold a long position in the risky technology.

where c^i is their consumption choice, θ_I^i and θ_f^i are the fractions of their wealth invested in the risky technology and security f, E^i is the expectation operator under their probability measure, β is their time-preference parameter, and

$$u(c_t^i) = \log(c_t^i)$$

is their utility function. Given group-i agents' investment and consumption strategies, their wealth process follows

$$\begin{aligned}\frac{dW_t^i}{W_t^i} &= \left(r_t - c_t^i/W_t^i\right) dt + \theta_I^i \left(dI_t/I_t - r_t dt\right) + \theta_f^i \left(dp_f/p_f - r_t dt\right) \\ &= \left[r_t - c_t^i/W_t^i + \theta_I^i \left(f_t - r_t\right) + \theta_f^i \left(\hat{\mu}_f^i - r_t\right)\right] dt \\ &\quad + \theta_I^i \sigma_I dZ_I(t) + \theta_f^i \sigma_f d\hat{Z}_f^i(t). \end{aligned} \tag{9}$$

We can solve these agents' consumption and investment problems using the standard dynamic programming approach following Merton (1971). The results for agents with logarithmic utility are well known. They always consume wealth at a constant rate equal to their time preference parameter:

$$c_t^i = \beta W_t^i.$$

They invest in risky assets according to their instantaneous risk-return tradeoff (the ratio between expected excess return and return variance):

$$\theta_I^i = \frac{f_t - r_t}{\sigma_I^2}, \quad \theta_f^i = \frac{\hat{\mu}_f^i - r_t}{\sigma_f^2}. \tag{10}$$

2.4 Equilibrium

We adopt a standard definition of competitive equilibrium. In the equilibrium, each agent chooses optimal consumption and investment decisions under her beliefs, and all markets clear. Market clearing conditions include 1) the aggregate investment to the risky technology is equal to the total wealth in the economy; 2) the aggregate investment to the risk free asset is zero; and 3) the aggregate investments to security f is also zero. We describe the equilibrium in the following theorem, and provide the proof in Appendix A.2.

Theorem 1 *The equilibrium short rate is*

$$r_t = f_t - \sigma_I^2. \tag{11}$$

The μ_f term of the return process of security f is

$$\mu_f(t) = r_t + \sum_{i=1}^{N} \omega_t^i \lambda_f (f_t - \hat{l}_t^i), \tag{12}$$

where ω_t^i is the wealth share of group-i agents in the economy:

$$\omega_t^i \equiv \frac{W_t^i}{W_t}, \qquad W_t \equiv \sum_{i=1}^{N} W_t^i.$$

The aggregate wealth in the economy fluctuates according to

$$\frac{dW_t}{W_t} = (f_t - \beta)\, dt + \sigma_I dZ_I(t). \tag{13}$$

In the equilibrium, group-i agents invest an amount of $\frac{\lambda_f}{\sigma_f^2}\left(\hat{l}_t^i - \sum_{j=1}^{N} \omega_t^j\ \hat{l}_t^j\right) W_t^i$ in security f.

Equation (11) in Theorem 1 shows that the equilibrium short rate is the expected instantaneous return of the risky technology adjusted for risk. This is because agents would demand a higher return from lending out capital when the expected return from the alternative option of investing in the risky technology is higher. Equation (11) implies that the short rate fluctuates according to the following process:

$$dr_t = -\lambda_f [r_t - (l_t - \sigma_I^2)]dt + \sigma_f dZ_f.$$

The short rate locally mean reverts to a time-varying long-term expected value, $l_t - \sigma_I^2$. This feature of local mean reversion in the short rate has also been pointed out by Fama (2006).

Equation(12) shows that the μ_f term of the return process of security f is determined by the short rate, r_t, minus the wealth weighted average of agents' beliefs about the factor f_t's drift rate, $\sum_{i=1}^{N} -\omega_t^i \lambda_f (f_t - \hat{l}_t^i)$. Equation (13) shows that the aggregate wealth in the economy grows at a rate determined by the return from the risky technology, $f_t dt + \sigma_I dZ_I(t)$, minus agents' consumption rate, βdt. This is because that the risky technology is the only storage technology in the economy.

Theorem 1 also shows that when group-i agents' belief $\hat{l}_t^i$ is higher than the wealth weighted average belief of all agents in the economy $\sum_{j=1}^{N} \omega_t^j \, \hat{l}_t^j$, they perceive a high expected return from security f. As a result, they take a long position. The different positions caused by agents' belief dispersion also affects their relative wealth. We define the wealth ratio between agents in group i and the benchmark group (group K) by

$$\eta_t^i \equiv \frac{W_t^i}{W_t^K}.$$

The following Lemma characterizes the dynamics of the wealth ratio, with the proof in Appendix A.3.

Proposition 1 *In the probability measure of group-K agents, the wealth ratio process η_t^i evolves according to*

$$\frac{d\eta_t^i}{\eta_t^i} = \frac{\lambda_f}{\sigma_f} g_t^i d\hat{Z}_f^K(t). \tag{14}$$

Proposition 1 shows that the wealth ratio between agents in groups i and K is more volatile when the difference in their beliefs g_t^i is higher. If $g_t^i = 0$, agents in the two groups invest in the same way, thus the wealth ratio stays constant. If $g_t^i > 0$, group-i agents are more optimistic. As a result, they put a bigger bet on the price of security f going up. If the shock to f_t turns out to be positive, i.e., $d\hat{Z}_f^K > 0$, the wealth ratio η_t^i would rise.

By directly solving equation (14), we further obtain that

$$\log\left(\eta_t^i\right) = \log\left(\eta_0^i\right) - \frac{1}{2}\int_0^t \frac{\lambda_f^2}{\sigma_f^2}\left(g_s^i\right)^2 ds + \int_0^t \frac{\lambda_f}{\sigma_f} g_s^i d\hat{Z}_f^K(s).$$

The logarithm of the wealth ratio has a Gaussian distribution. In the long run, it does not converge to any constant. Even in the limit, it has a Gaussian distribution, with a mean of

$$-\frac{1}{2}\int_0^\infty \frac{\lambda_f^2}{\sigma_f^2}\left(g_s^i\right)^2 ds$$

and a variance of

$$\int_0^\infty \frac{\lambda_f^2}{\sigma_f^2} \left(g_s^i\right)^2 ds.$$

Since the wealth ratio converges neither to 0, nor to 1, every group will survive in the long run.[6]

2.5 Stochastic discount factor

When agents are homogeneous, they share the same stochastic discount factor, which is determined by their marginal utility of consumption in future states. With a logarithmic preference, agents consume a fixed fraction of their wealth and the stochastic discount factor is inversely related to their aggregate wealth. More specifically, the stochastic discount factor, which we denote by M_t^H, is

$$M_t^H = e^{-\beta t} \frac{u'(c_t)}{u'(c_0)} = e^{-\beta t} \frac{c_0}{c_t} = e^{-\beta t} \frac{W_0}{W_t}. \tag{15}$$

When agents have heterogeneous beliefs about the probabilities of future states, they have different stochastic discount factors. However, in the absence of arbitrage, they have to share the same security valuations. For our discussion of security prices, we will use the stochastic discount factor of group-K agents.

Before we discuss the stochastic discount factor of group-K agents, we provide a lemma, with the proof in Appendix A.4, relating the fluctuation in agents' relative wealth to the difference in their probability measures of future states.

Lemma 2 *If X_T is a random variable to be realized at time $T > t$ and $E^i[X_T] < \infty$, then group-i agents' expectation of this variable at time t can be transformed into group-K agents' expectation through the wealth ratio process between the two groups:*

$$E_t^i \left[X_T\right] = E_t^K \left[\frac{\eta_T^i}{\eta_t^i} X_T\right].$$

Lemma 2 shows that the wealth ratio process between agents in groups i and K acts as the Randon-Nikodyn derivative of group-i agents' probability measure with respect

[6]For more general discussion of survival issues caused by heterogeneous beliefs, see recent studies of Kogan, et al (2005), Yan (2005), and Dumas, Kurshev and Uppal (2005).

to group-K agents' measure. The intuition is as follows. If group-i agents assign a higher probability to a future state than group-K agents, these agents would invest in such a way that the wealth ratio between them, W^i/W^K, is also higher in that state. Lemma 2 further implies that, as a consequence of logarithmic preference, the ratio of probabilities assigned by these groups to different states is perfectly correlated with their wealth ratio.

Group-K agents's stochastic discount factor is determined by their wealth dynamics, which is affected by their trading with agents in other groups. Intuitively, group-i agents have a larger impact on group-K agents' wealth if group-i agents have a larger wealth share, and/or disagree more with group-K agents. Hence, relative to the homogeneous economy case, group-K agents' stochastic discount factor is affected by other groups through two channels: each group's wealth share and its disagreement with group K. This intuition is made precise in the following Theorem, with the proof in Appendix A.5.

Theorem 2 *When agents have heterogeneous beliefs, group-K agents' stochastic discount factor is*

$$M_t = \left(\sum_{i=1}^{N} \omega_0^i \frac{\eta_t^i}{\eta_0^i} \right) e^{-\beta t} \frac{W_0}{W_t} = \left(\sum_{i=1}^{N} \omega_0^i \frac{\eta_t^i}{\eta_0^i} \right) M_t^H.$$

At time t, the price of a financial security, which provides a single payoff X_T at time T, is given by

$$P_t = E_t^K \left[\frac{M_T}{M_t} X_T \right] = \sum_{i=1}^{N} \omega_t^i P_t^i,$$

where

$$P_t^i = E_t^i \left[\frac{M_T^H}{M_t^H} X_T \right]$$

is the value of the security in a homogeneous economy, whereby only group-i agents are present.

Theorem 2 shows that when agents have heterogeneous beliefs, group-K agents' stochastic discount factor is adjusted by a factor of $\sum_{i=1}^{N} \omega_0^i \frac{\eta_t^i}{\eta_0^i}$, relative to M_t^H. In this adjustment, $\frac{\eta_t^i}{\eta_0^i}$ represents the probability ratio between agents in groups i and

K, while ω_0^i represents the initial wealth share of group i. Thus, the net adjustment in the stochastic discount factor is the wealth weighted average of the probability disagreements between agents in group K and other groups.

Theorem 2 further shows that the price of a financial security is the wealth weighted average of each group's valuation of the security in a corresponding homogeneous economy. This result allows us to decompose the security price in a heterogeneous economy into prices in homogeneous economies.

3 Bond Yields

In this section, we first derive a bond pricing formula in homogeneous economies, and then combine the formula with Theorem 2 to analyze the effects of heterogeneous beliefs on bond yields.

3.1 Bond pricing with homogeneous agents

We consider a homogeneous economy with only group-i agents. These agents have access to the risky technology. They can also trade the risk free short rate and security f. All agents share the same belief about the unobservable factor l_t. The mean and variance of their belief are given in equations (4) and (6). We derive bond prices in this economy by using the stochastic discount factor in equation (15). An analytical formula is given in the following proposition, with the proof in Appendix A.6.

Proposition 2 *In a homogeneous economy with only group-i agents, the price of a zero coupon bond at time t with a maturity τ is determined by*

$$B^H(f_t, \hat{l}_t^i, \tau, \gamma_t) = \exp\left[-a(\tau)f_t - b(\tau)\hat{l}_t^i - c(\tau, \gamma_t)\right]$$

where

$$a(\tau) = \frac{1}{\lambda_f}\left(1 - e^{-\lambda_f \tau}\right), \quad (16)$$

$$b(\tau) = \frac{1}{\lambda_l}\left(1 - e^{-\lambda_l \tau}\right) + \frac{1}{\lambda_f - \lambda_l}\left(e^{-\lambda_f \tau} - e^{-\lambda_l \tau}\right), \quad (17)$$

and $c(\tau, \gamma_t)$ satisfies the following partial differential equation:

$$\frac{\partial c(\tau, \gamma_t)}{\partial \tau} - \lambda_l \bar{l} b(\tau) + \frac{1}{2}\sigma_f^2 a^2(\tau) + \frac{\lambda_f^2 \gamma_t^2}{2\sigma_f^2} b^2(\tau) + \lambda_f \gamma_t a(\tau) b(\tau) + \sigma_I^2$$
$$+ \left(\frac{\lambda_f^2}{\sigma_f^2}\gamma_t^2 + 2\lambda_l \gamma_t - \sigma_l^2 \right) \frac{\partial c(\tau, \gamma_t)}{\partial \gamma_t} = 0.$$

If the agents' belief variance γ_t is equal to its stationary level $\bar{\gamma}$, we have an explicit solution for c :

$$c(\tau, \bar{\gamma}) = \int_0^{\tau} \left[\lambda_l \bar{l} b(s) - \frac{1}{2}\sigma_f^2 a^2(s) - \frac{\lambda_f^2 \bar{\gamma}^2}{2\sigma_f^2} b^2(s) - \lambda_f \bar{\gamma} a(s) b(s) - \sigma_I^2 \right] ds.$$

Proposition 2 implies that the yield of a τ-year bond in a homogeneous economy

$$Y^H(f_t, \hat{l}_t^i, \tau, \gamma_t) = -\frac{1}{\tau} \log\left(B^H\right) = \frac{a(\tau)}{\tau} f_t + \frac{b(\tau)}{\tau} \hat{l}_t^i + \frac{c(\tau, \gamma_t)}{\tau}$$

is a linear function of two fundamental factors: f_t and $\hat{l}_t^i$. This specific form belongs to the general affine structure proposed by Duffie and Kan (1996). Balduzzi, Das and Foresi (1998) have also analyzed a two-factor term structure model of interest rates, similar to the one above. However, they neither derive investors' learning processes, nor address the effects caused by heterogeneous beliefs.

The loadings of the bond yield on the two factors are plotted in Figure 1. The loading on f_t, $a(\tau)/\tau$, has a value of 1 when the bond maturity τ is zero and monotonically decreases to zero as the maturity increases, suggesting that short-term yields are more exposed to fluctuations in f_t. The intuition of this pattern works as follows. f_t is the expected instantaneous return from the risky technology, which is a close substitute of investing in short-term bonds. As a result, the fluctuation in f_t has a greater impact on short-term yields. As bond maturity increases, the impact of f_t becomes smaller; on the other hand, agents' expectation of future returns from the risky technology becomes more important.

Agents' belief about l_t determines their expectation of future returns from the risky technology, because l_t is the level, to which f_t mean-reverts. The loading of the bond yield on $\hat{l}_t^i$, $b(\tau)/\tau$, has a humped shape. As the bond maturity increases from 0 to an intermediate value, $b(\tau)/\tau$ increases from 0 to a positive value less than 1, suggesting that agents' expectation has a greater impact on longer term yields. As the bond maturity increases further, $b(\tau)/\tau$ drops. This is caused by the mean reversion of l_t, which makes any shock to l_t eventually die out. This force makes the yields of very long-term bonds have low exposure to agents' belief about l_t. If l_t has no mean-reversion ($\lambda_l = 0$), the factor loading $b(\tau)/\tau$ is a monotonically increasing function of bond maturity.

It is important to note that the bond price is a convex function of agents' belief:

$$B^H(f_t, \hat{l}_t^i, \tau, \gamma_t) \sim e^{-b(\tau)\hat{l}_t^i}.$$

This property is a natural outcome of the fact that bond prices are a convex function of bond yields which are determined by agents' expectation of future interest rates. As will become clear in the next subsection, this price convexity implies that belief dispersion is an important determinant for bond yields in a heterogenous economy.

3.2 Effects of belief dispersion

We can now combine Proposition 2 with Theorem 2 to analyze the effects of heterogeneous beliefs on bond yields. Consider a heterogeneous economy with N groups. At time t, their beliefs of l_t are $\hat{l}_t^1, \cdots, \hat{l}_t^N$, and their wealth shares in the economy are $\omega_t^1, \cdots, \omega_t^N$. Then, the price of a τ-year zero-coupon bond is

$$B_t = \sum_{i=1}^{N} \omega_t^i B^H(f_t, \hat{l}_t^i, \tau, \gamma_t), \tag{18}$$

where $B^H(f_t, \hat{l}_t^i, \tau, \gamma_t)$ is given in Proposition 2. The bond price in the heterogeneous economy is the wealth weighted average of each group's bond valuation in a homogeneous economy. The implied bond yield is

$$\begin{aligned} Y_t(\tau) &= -\frac{1}{\tau}\log(B_t) \\ &= \frac{a(\tau)}{\tau}f_t + \frac{c(\tau,\gamma_t)}{\tau} - \frac{1}{\tau}\log\left[\sum_{i=1}^{N}\omega_t^i e^{-b(\tau)\hat{l}_t^i}\right]. \end{aligned}$$

Note that Y_t is not a linear function of agents' beliefs $\hat{l}_t^1, \cdots, \hat{l}_t^N$, that is, the bond yield in the heterogeneous economy has a non-affine structure. Proposition 3 characterizes the impact of belief dispersion on bond yields, with the proof in Appendix A.7.

Proposition 3 *Bond prices in a heterogeneous economy are higher than those in a homogeneous economy whereby all agents hold an identical belief equal to the wealth weighted average belief in the heterogeneous economy.*

Proposition 3 shows that bond prices in a heterogeneous economy are not the same as those in a homogeneous economy where all agents hold the average belief of the heterogeneous economy. The failure of the homogeneous economy to capture bond prices in the heterogeneous economy lies in the aggregation of non-linear bond valuations across agents.[7] The intuition can be illustrated by the following example. Suppose there are two groups, groups 1 and 2, with an equal wealth at time t, and with a belief of l^1 and l^2, respectively. Figure 2 shows that the bond price would be B^i $(i = 1, 2)$ if the economy is populated by group-i agents only. Equation (18) implies that the bond price in this heterogeneous economy is B, the average of B^1 and B^2. If all the investors have the average belief $l^* = (l^1 + l^2)/2$, the bond price would be B^*. As noticed before, the bond pricing function $B^H(f_t, \hat{l}_t^i, \tau, \gamma_t)$ is convex with respect to $\hat{l}_t^i$. Hence, Jensen's inequality implies that averaging bond valuations across groups (B) is higher than B^*, the bond price that would prevail if all agents have the average belief. That is, belief dispersion increases bond prices and reduces bond yields.

It is also interesting to note that the impact of belief dispersion is small for short term bonds and the impact disappears for short rate, i.e. when the maturity approaches zero. The reason is as follows. As the bond maturity τ goes down to zero, $b(\tau)$ converges to zero, as can be seen in equation (17). This implies that the bond price convexity is small for short-term bonds and disappears when bond maturity approaches zero.

Belief dispersion not only affects the level of bond yields, but also increases their conditional volatility. We show this result in the following proposition, and provide

[7] Since markets are complete in the heterogeneous economy, one can construct a representative-agent to compute the equilibrium (Basak, 2000). However, the representative agent does not have the average belief, instead she possesses stochastic weights that incorporate each agent's belief.

the proof in Appendix A.8.

Proposition 4 *The conditional volatility of the bond yield $Y_t(\tau)$ in a heterogeneous economy is higher than the corresponding one in a homogeneous economy, whereby all agents hold an identical belief.*

Proposition 4 shows that when agents differ in their beliefs, bond yields become more volatile. This effect arises from an amplification mechanism generated by agents' relative-wealth fluctuation. Loosely speaking, bond yields are determined by agents' average belief about future short rates weighted by their wealth. Since agents who are more optimistic about future short rates bet on the rise of interest rates against those pessimistic agents (Theorem 1), any positive news about future short rates would cause wealth to flow from pessimistic agents to optimistic agents, making optimistic beliefs carry greater weights in bond yields. This relative-wealth fluctuation amplifies the effect of the initial news on bond yields.

3.3 An example

To further characterize the effects of belief dispersion, we derive bond yields and yield volatility in a specific case with a continuum of groups. Then, we illustrate the effects of heterogeneous beliefs for various model parameters. We assume that at time t, each group's belief can take a value in the range $[l^* - \Delta,\ l^* + \Delta]$ and that all groups have the same belief variance of γ_t. In addition, the wealth share across groups has a uniform distribution over the feasible range of beliefs. Note that this uniform wealth distribution can only hold for this instant, and would change to another distribution as wealth flows across groups with trading gains and losses. The following proposition gives analytical formulas for bond prices, bond yields and yield volatility in this case, with the proof in Appendix A.9.

Proposition 5 *The price of a τ-year zero-coupon bond is*

$$B_t = B^H(f_t, l^*, \tau, \gamma_t) K\left[b(\tau)\Delta\right],$$

where $K(x) = \frac{e^x - e^{-x}}{2x}$ *is an increasing function. The bond yield is*

$$Y_t = Y_t^H - \frac{1}{\tau}\log\left\{\frac{\exp[b(\tau)\Delta] - \exp[-b(\tau)\Delta]}{2b(\tau)\Delta}\right\}, \tag{19}$$

where $Y_t^H = Y^H(f_t, l^*, \tau, \gamma_t)$ *is the bond yield in a homogeneous economy whereby all agents hold the average belief* l^*. *The conditional volatility of the bond yield is*

$$\nu(\tau) = \frac{a(\tau)\sigma_f^2 + \lambda_f b(\tau)\gamma_t + \lambda_f \xi}{\tau\sigma_f}, \tag{20}$$

where

$$\xi = \frac{1}{b(\tau)} - \frac{e^{b(\tau)\Delta} + e^{-b(\tau)\Delta}}{e^{b(\tau)\Delta} - e^{-b(\tau)\Delta}}\Delta$$

increases with Δ.

To illustrate the magnitudes and some basic properties of the belief dispersion effects, we choose the following set of parameters. First, we set the mean-reversion parameters as

$$\lambda_f = 1, \;\; \lambda_l = 0.02.$$

These numbers imply that it takes $\ln(2)/\lambda_f = 0.69$ year for the difference between the short rate and its long-run expected value to converge by half, while it takes $\ln(2)/\lambda_l = 34.66$ years for the effect of a shock on the long-run expected value to die out by half. The length of these periods are consistent with the finding of Fama (2006) that the long-run expected value of short rate varies over time and is highly persistent. From equation (19), these two mean-reversion parameters, together with the magnitude of belief dispersion, determine the effect of belief dispersion on bond yields. We will vary these parameters around the chosen values to examine their impact.

The other parameters affect the benchmark bond yields in the homogeneous economy, but not the impact of belief dispersion on bond yields. There are three sources of random shocks in our model, shocks to the return of the risky technology, to its instantaneous expected return, and to the long-run mean of the expected return. We choose the following volatility parameters for these shocks:

$$\sigma_I = 0.03, \;\; \sigma_f = 0.03, \;\; \sigma_l = 0.01.$$

We choose γ_t as its stationary level $\bar{\gamma}$, and the remaining parameters as

$$f_t = 0.03,\ l^* = 0.06,\ \bar{l} = 0.08.$$

Figure 3 illustrates the yield curve and the conditional volatility curve for four different values of Δ, the half distance between the most optimistic and pessimistic beliefs. When there is no belief dispersion ($\Delta = 0$), the yield curve, which is $Y^H(f_t, \hat{l}^*, \tau, \gamma_t)$ of homogeneous economies, has a typical shape– it increases from 3% to 5.5% for maturities between 0 and 10 years, and then decreases slightly from 10 years to longer maturities. When the belief dispersion increases (Δ takes values of 2%, 4% and 6%), the yield curve, as $Y_t(\tau)$ given in equation (19), shifts lower. The reduction in bond yields is small for maturities shorter than 5 years. This is because that the price convexity of short-term bonds with respect to bond yields is small. The impact of belief dispersion becomes much more dramatic for longer maturities. When $\Delta = 6\%$, the reduction in bond yields could be larger than 80 basis points for maturities longer than 20 years.

When there is no belief dispersion ($\Delta = 0$), the conditional volatility curve, which we denote as $\nu^H(\tau)$, decreases from 3% to 1% as bond maturity increases from 0 to 30 years. The downward slope is generated by the mean reversion of the fundamental factors, f_t and l_t. Belief dispersion always raises the yield volatility, which is given as $\nu(\tau)$ in equation (20), and this effect is substantial. When $\Delta = 6\%$, the yield volatility increases from 3% to 4.5% as bond maturity goes from 0 to around 7 years, and then gradually decreases to 3.5% as bond maturity increases up to 30 years.

Figure 4 shows the impact of λ_l, the mean reversion parameter of the long run mean of the expected instantaneous return of the risky technology, on the belief dispersion effects on 20-year bond yield and its conditional volatility. For each of the three values of belief dispersion ($\Delta = 2\%, 4\%, 6\%$), the yield reduction and the volatility increase caused by belief dispersion decrease monotonically with λ_l. This is because that as λ_l becomes larger, l_t reverts faster to its long-run mean. As a result, the belief dispersion among agents is shorter lived and therefore has weaker effects on bond yields and yield volatility.

Figure 5 shows the impact of λ_f, the mean reversion parameter of the expected instantaneous return of the risky technology, on the belief dispersion effects on 20-year bond yield and its conditional volatility. Contrary to the impact of λ_l, the yield reduction and the volatility increase caused by belief dispersion increase monotonically with λ_f. This is because that as λ_f becomes larger, f_t reverts faster to l_t. As a result, agents' disagreement about l_t becomes more important for the market dynamics and therefore has stronger effects on bond yields and yield volatility.

3.4 Discussion

The existence of heterogeneous beliefs is evident in the data. Survey data provide a direct method for examining agents' beliefs. Mankiw, Reis and Wolfers (2004) give a thorough analysis of agents' disagreement about inflation expectations in several surveys. In Figure 6, we show time-series plots of belief dispersion from their paper. It directly reveals several patterns. There is substantial belief dispersion among both professionals and the general public, and the belief dispersion in each of the surveys varies dramatically over time. The interquartile range of inflation expectation among the general public, as shown in the Michigan Survey, fluctuates from as high as 10% in the early 1980s to around 4% in the early 2000s, while the interquartile range among professionals, as shown in the Livingston Survey and the Survey of Professional Forecasters, varies from above 2% to 0.5% in the same period. Although the dispersion among the general public is higher than that among professionals, their time-series patterns are similar. Kurz (2001) finds substantial and persistent forecast dispersion of future GDP growth in the Blue Chip Economic Indicators survey of major U.S. corporations and financial institutions. The 5% to 95% forecast range in his sample varies around 2% per annum from 1990 to 2001. Furthermore, Welch (2000) shows that academic financial economists' forecasts of 30-year equity premium range from 2% to 13%.[8]

[8]Heterogeneous beliefs could also be extracted from financial market data. As shown in our model, heterogeneous beliefs lead to trading among market participants in bonds and financial derivatives. As a result, one can estimate the degree of belief dispersion from the observed trading volume and price patterns. While market data are more reliable than surveys, the extraction of beliefs relies on

The substantial amount of belief dispersion displayed in the various survey data invites future studies of the impacts of belief dispersion on other economic variables such as bond yields and yield volatility. Our model suggests that if the belief dispersion among agents unexpectedly increases at one instant, bond yields would decrease and yield volatility would increase. Furthermore, these effects are stronger for bonds with longer maturities. Thus, over time, we expect yields of risk-free long-term bonds to move negatively with agents' belief dispersion about future inflation or GDP growth. We also expect the conditional volatility of the yields to move positively with the belief dispersion.

Belief dispersion is often taken for granted as a symptom of greater uncertainty. However, these are two distinct concepts. Belief dispersion captures the interpersonal variation in expectations, while uncertainty represents the intrapersonal variation. Zarnowitz and Lambros (1987) clarify this conceptual difference, and empirically examine it using survey data from the Survey of Professional Forecasters. Since this survey also asks respondents to supplement their point estimates with estimates of the probability that GDP and the implicit price deflator will fall into various ranges, Zarnowitz and Lambros measure uncertainty from these probability estimates. By comparing the uncertainty measure with measures of interpersonal forecast dispersion, they only find weak evidence that uncertainty and belief dispersion are positively correlated.

Several empirical studies, e.g., Levi and Makin (1979), Bomberger and Frazer (1981), and Zarnowitz and Lambros (1987), have examined the relationship between dispersion of inflation forecasts in survey data and interest rates. Most of these studies focus on short rates instead of bond yields (or long rates), with the exception of Bomberger and Frazer (1981). Consistent with our model, this study finds that 3 to 5 year rate and 10 year rate are both negatively related to the dispersion in inflation forecasts. Furthermore, the existing studies typically do not differentiate effects caused by belief dispersion from those caused by uncertainty. Our model predicts the existence of belief dispersion effects, even after controlling for uncertainty. The task of identifying the belief dispersion effects is left for future empirical studies.

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specific model assumptions and is subject to potential mis-specifications.

Our model also has potential implications for monetary policies. A monetary authority usually only directly controls the overnight interest rate. For the overnight interest rate to affect long term interest rates and other prices, the links rely almost entirely on market expectations for the future course of short-term rates. Many monetary economists have pointed out the importance of managing market expectations in monetary policies, e.g., Blinder (1998), Bernanke (2004), Svensson (2004), and Woodford (2005). Consistent with this view, our model shows that dispersion in market expectations can directly affect long-term interest rates and increase their volatility. If the objective of monetary authorities is to stabilize prices, our model suggests that they should pay close attention to dispersion in market expectations, and reduce this dispersion at their capacity. This argument thus supports the establishment of a reliable communication channel between the monetary authority and the market. As observed by Bernanke (2004), this practice would increase policy transparency and achieve a closer alignment between market expectations and the policymakers' views. Our model points out an additional dimension of consideration– the reduction in dispersion of market expectations and the subsequent speculative trading.

4 Conclusion

In this paper, we provide an equilibrium model to analyze the effects of agents' belief dispersion on bond yields. Our model shows that the price of a bond is the wealth weighted average of bond prices in homogeneous economies, in each of which only one type of agent is present. Since bond prices in homogeneous economies are convex functions of agents' beliefs about future economic growth rates, belief dispersion increases bond prices and decreases bond yields. This effect is stronger for longer maturity bonds. Furthermore, the relative-wealth fluctuation caused by agents' speculative trading amplifies shocks to the economy and therefore increases yield volatility. Taken together, these results highlight the importance of incorporating belief dispersion into economic analysis of bond yields.

A Appendix

A.1 Proof of Lemma 1

According to the learning process described in equation (4), the belief dynamics of groups i and K are

$$\begin{aligned} d\hat{l}_t^i &= -\lambda_l(\hat{l}_t^i - \bar{l})dt + \frac{\lambda_f \gamma_t}{\sigma_f^2}\left[df_t + \lambda_f(f_t - \hat{l}_t^i)dt\right], \\ d\hat{l}_t^K &= -\lambda_l(\hat{l}_t^K - \bar{l})dt + \frac{\lambda_f \gamma_t}{\sigma_f^2}\left[df_t + \lambda_f(f_t - \hat{l}_t^K)dt\right]. \end{aligned}$$

Taking the difference between these equations, we obtain

$$dg_t^i = -\left(\lambda_l + \frac{\lambda_f^2}{\sigma_f^2}\gamma_t\right) g_t^i dt.$$

A.2 Proof of Theorem 1

The market clearing conditions require that the aggregate investment to the risky technology is equal to the total wealth in the economy:

$$\sum_{i=1}^{N} \theta_I^i(t) W_t^i = W_t.$$

By substituting agents' investment strategy in equation (10) and dividing both sides by W_t, we obtain that

$$\frac{f_t - r_t}{\sigma_I^2}\sum_{i=1}^{N}\omega_t^i = 1.$$

Since $\sum_{i=1}^{N}\omega_t^i = 1$, we have that $r_t = f_t - \sigma_I^2$.

The market clearing conditions also require that the aggregate investment to the security f is zero:

$$\sum_{i=1}^{N} \theta_f^i(t) W_t^i = 0.$$

By substituting agents' investment strategy in equation (10) and dividing both sides by W_t, we obtain that

$$\sum_{i=1}^{N}\omega_t^i \frac{\hat{\mu}_f^i - r_t}{\sigma_f^2} = \sum_{i=1}^{N}\omega_t^i \frac{\mu_f(t) - \lambda_f(f_t - \hat{l}_t^i) - r_t}{\sigma_f^2} = 0.$$

Thus,

$$\mu_f(t) = r_t + \sum_{i=1}^{N} \omega_t^i \lambda_f (f_t - \hat{l}_t^i).$$

Since the risky technology is the only storage tool in the economy and every agent consumes a fraction β of her wealth, the aggregate wealth fluctuates according to

$$\frac{dW_t}{W_t} = dI_t/I_t - \beta dt = (f_t - \beta)\, dt + \sigma_I dZ_I(t).$$

According to equation (10), agents in group i put $\frac{\hat{\mu}_f^i - r_t}{\sigma_f^2}$ fraction of their wealth in security f. Using equations (8) and (12), we can expand the dollar amount of their position as

$$\frac{\hat{\mu}_f^i - r_t}{\sigma_f^2} W_t^i = \frac{\lambda_f}{\sigma_f^2} \left[\hat{l}_t^i - \sum_{j=1}^{N} \omega_t^j \hat{l}_t^j \right] W_t^i.$$

A.3 Proof of Proposition 1

By substituting agents' consumption and investment strategies into equation (9), we obtain the following wealth process for group-i agents:

$$\begin{aligned} \frac{dW_t^i}{W_t^i} &= \left[r_t - \beta + \left(\frac{f_t - r_t}{\sigma_I} \right)^2 + \left(\frac{\hat{\mu}_f^i - r_t}{\sigma_f} \right)^2 \right] dt \\ &\quad + \frac{f_t - r_t}{\sigma_I} dZ_I(t) + \frac{\hat{\mu}_f^i - r_t}{\sigma_f} d\hat{Z}_f^i(t). \end{aligned} \tag{21}$$

We obtain group-K agents' wealth dynamics from equation (21):

$$\begin{aligned} \frac{dW_t^K}{W_t^K} &= \left[r_t - \beta + \left(\frac{f_t - r_t}{\sigma_I} \right)^2 + \left(\frac{\hat{\mu}_f^K - r_t}{\sigma_f} \right)^2 \right] dt \\ &\quad + \frac{f_t - r_t}{\sigma_I} dZ_I(t) + \frac{\hat{\mu}_f^K - r_t}{\sigma_f} d\hat{Z}_f^K(t). \end{aligned} \tag{22}$$

According to equation (5),

$$d\hat{Z}_f^i(t) - d\hat{Z}_f^K(t) = -\frac{\lambda_f}{\sigma_f} \left(\hat{l}_t^i - \hat{l}_t^K \right) dt = -\frac{\lambda_f}{\sigma_f} g_t^i dt.$$

By substituting $d\hat{Z}^i_f(t)$ above into equation (21), we obtain

$$\begin{aligned} \frac{dW^i_t}{W^i_t} &= \left[r_t - \beta + \left(\frac{f_t - r_t}{\sigma_I}\right)^2 + \left(\frac{\hat{\mu}^i_f - r_t}{\sigma_f}\right)^2 \right] dt \\ & \quad -\frac{\lambda_f}{\sigma_f^2} g^i_t \left(\hat{\mu}^i_f - r_t\right) dt + \frac{f_t - r_t}{\sigma_I} dZ_I(t) + \frac{\hat{\mu}^i_f - r_t}{\sigma_f} d\hat{Z}^K_f(t). \qquad (23) \end{aligned}$$

Under group-K agents' probability measure, applying Ito's lemma to η^i_t we obtain

$$\frac{d\eta^i_t}{\eta^i_t} = \frac{dW^i_t}{W^i_t} - \frac{dW^K_t}{W^K_t} + \left(\frac{dW^K_t}{W^K_t}\right)^2 - \left(\frac{dW^K_t}{W^K_t}\right)\left(\frac{dW^i_t}{W^i_t}\right).$$

By substituting $\frac{dW^i_t}{W^i_t}$ and $\frac{dW^K_t}{W^K_t}$ in equations (22) and (23) into the equation above and by using an additional fact that

$$\hat{\mu}^i_f - \hat{\mu}^K_f = \lambda_f g^i_t,$$

we obtain that

$$\frac{d\eta^i_t}{\eta^i_t} = \frac{\lambda_f}{\sigma_f} g^i_t d\hat{Z}^K_f(t).$$

A.4 Proof of Lemma 2

For any random variable X_T with $E^i[X_T] < \infty$, we can define $Y_T = \frac{W^i_T}{W^i_t} X_T$. Suppose there is a financial security which is a claim to the cash flow Y_T. Then investor i's valuation for this security is

$$E^i_t\left[\frac{u'(c^i_T)}{u'(c^i_t)} Y_T\right] = E^i_t\left[\frac{c^i_t}{c^i_T} Y_T\right] = E^i_t\left[\frac{W^i_t}{W^i_T} Y_T\right],$$

where the second equality follows from his consumption rule $c^i_t = \beta W^i_t$. Similarly, investor K's valuation for this security is $E^K_t\left[\frac{W^K_t}{W^K_T} Y_T\right]$. In the absence of arbitrage, investor i and K should have the same valuation, that is

$$E^i_t\left[\frac{W^i_t}{W^i_T} Y_T\right] = E^K_t\left[\frac{W^K_t}{W^K_T} Y_T\right].$$

Substituting the expression of Y_T into the above equation, we obtain

$$E^i_t\left[X_T\right] = E^K_t\left[\frac{\eta^i_T}{\eta^i_t} X_T\right].$$

A.5 Proof of Theorem 2

We can derive the stochastic discount factor from group-K agents' marginal utility. Group-K agents' consumption is

$$c_t^K = \beta W_t^K = \frac{\omega_t^K}{\sum_{i=1}^N \omega_t^i}\beta W_t = \frac{\beta W_t}{\sum_{i=1}^N \eta_t^i}.$$

The implied stochastic discount factor is

$$\begin{aligned} M_t &= e^{-\beta t}\frac{u'(c_t^K)}{u'(c_0^K)} = e^{-\beta t}\frac{W_0^K}{W_t^K} = e^{-\beta t}\frac{W_0}{W_t}\frac{\sum_{i=1}^N \eta_t^i}{\sum_{i=1}^N \eta_0^i} \\ &= e^{-\beta t}\frac{W_0}{W_t}\sum_{i=1}^N\left(\frac{\eta_0^i}{\sum_{i=1}^N \eta_0^i}\frac{\eta_t^i}{\eta_0^i}\right) \\ &= \left(\sum_{i=1}^N \omega_0^i \frac{\eta_t^i}{\eta_0^i}\right) e^{-\beta t}\frac{W_0}{W_t}. \end{aligned}$$

Direct algebra substitutions provide that

$$\frac{M_T}{M_t} = \left(\sum_{i=1}^N \omega_t^i \frac{\eta_T^i}{\eta_t^i}\right) e^{-\beta(T-t)}\frac{W_t}{W_T}.$$

Thus, at time t, the price of a financial security that pays off X_T at time T is

$$\begin{aligned} P_t &= E_t^K\left[\frac{M_T}{M_t}X_T\right] \\ &= E_t^K\left[\left(\sum_{i=1}^N \omega_t^i \frac{\eta_T^i}{\eta_t^i}\right) e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right] \\ &= \sum_{i=1}^N \omega_t^i E_t^K\left[\frac{\eta_T^i}{\eta_t^i}e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right]. \end{aligned}$$

Since $\frac{\eta_T^i}{\eta_t^i}$ is the Randon-Nikodyn derivative of group-i agents' probability measure with respect to the measure of group-K agents (Lemma 2),

$$E_t^K\left[\frac{\eta_T^i}{\eta_t^i}e^{-\beta(T-t)}\frac{W_t}{W_T}X_T\right] = E_t^i\left[\frac{M_T}{M_t}X_T\right].$$

Thus,

$$P_t = \sum_{i=1}^N \omega_t^i P_t^i,$$

where

$$P_t^i = E_t^i \left[\frac{M_T}{M_t} X_T \right]$$

is the price of the security in a homogeneous economy where only group-i agents are present.

A.6 Proof of Proposition 2

When only group-i agents are present in the economy, their bond valuation is given by

$$B^i = E_t^i \left[\frac{M_T^H}{M_t^H} \right] = E_t^i \left[e^{-\beta(T-t)} \frac{W_t}{W_T} \right].$$

The bond price must be a function of the following variables:

$$B^i = B^H \left(f_t, \hat{l}_t^i, \tau, \gamma_t \right),$$

where f_t and $\hat{l}_t^i$ are random factors, while τ and γ_t are deterministic. Applying Ito's lemma provides

$$\begin{aligned} \frac{dB^H}{B^H} &= \left[-\lambda_f (f_t - \hat{l}_t^i) \frac{B_f^H}{B^H} - \lambda_l (\hat{l}_t^i - \bar{l}) \frac{B_l^H}{B^H} + \frac{1}{2} \sigma_f^2 \frac{B_{ff}^H}{B^H} + \frac{1}{2} \frac{\lambda_f^2 \gamma_t^2}{\sigma_f^2} \frac{B_{ll}^H}{B^H} + \lambda_f \gamma \frac{B_{fl}^H}{B^H} \right] dt \\ &+ \left[\sigma_f \frac{B_f^H}{B^H} + \frac{\lambda_f \gamma_t}{\sigma_f} \frac{B_l^H}{B^H} \right] dZ_f^{'}(t) + \left[\frac{d\gamma_t}{dt} \frac{B_\gamma^H}{B^H} - \frac{B_\tau^H}{B^H} \right] dt \end{aligned}$$

The bond price has to satisfy the following relationship with the stochastic discount factor:

$$E_t^i \left(\frac{dB^H}{B^H} \right) + E_t^i \left(\frac{dM^H}{M^H} \right) + E_t^i \left(\frac{dB^H}{B^H} . \frac{dM^H}{M^H} \right) = 0.$$

This is equivalent to the following differential equation:

$$-\lambda_f (f_t - \hat{l}_t^i) \frac{B_f^H}{B^H} - \lambda_l (\hat{l}_t^i - \bar{l}) \frac{B_l^H}{B^H} + \frac{1}{2} \sigma_f^2 \frac{B_{ff}^H}{B^H} + \frac{\lambda_f^2 \gamma_t^2}{2\sigma_f^2} \frac{B_{ll}^H}{B^H} + \lambda_f \gamma_t \frac{B_{fl}^H}{B^H} - f_t + \sigma_I^2 - \frac{B_\tau^H}{B^H} + \frac{d\gamma_t}{dt} \frac{B_\gamma^H}{B^H} = 0. \tag{24}$$

We conjecture the following solution

$$B^H \left(f_t, \hat{l}_t^i, \tau, \gamma_t \right) = e^{-a(\tau) f_t - b(\tau) \hat{l}_t^i - c(\tau, \gamma_t)}.$$

By substituting the conjectured solution and $\frac{d\gamma_t}{dt}$ in equation (6) into the differential equation in (24), we obtain

$$
\begin{aligned}
&\left[a'(\tau)+\lambda_f a(\tau)-1\right] f_t+\left[b'(\tau)-\lambda_f a(\tau)+\lambda_l b(\tau)\right] \hat{l}_t^i \\
&+\left[\frac{\partial c(\tau,\gamma_t)}{\partial \tau}+\left(\frac{\lambda_f^2}{\sigma_f^2}\gamma_t^2+2\lambda_l\gamma_t-\sigma_l^2\right)\frac{\partial c(\tau,\gamma_t)}{\partial \gamma_t}\right. \\
&\qquad \left. -\lambda_l \bar{l} b(\tau)+\frac{1}{2}\sigma_f^2 a^2(\tau)+\frac{\lambda_f^2\gamma_t^2}{2\sigma_f^2}b^2(\tau)+\lambda_f\gamma_t a(\tau)b(\tau)+\sigma_I^2\right]=0
\end{aligned}
$$

For the equation to hold, each of the square bracket terms must be zero.

Thus, $a(\tau)$ and $b(\tau)$ satisfy the following differential equations

$$
\begin{aligned}
&a'(\tau)+\lambda_f a(\tau)-1=0, \\
&b'(\tau)-\lambda_f a(\tau)+\lambda_l b(\tau)=0,
\end{aligned}
$$

subject to the boundary conditions

$$a(0)=b(0)=0.$$

Solving these equations provides

$$
\begin{aligned}
a(\tau) &= \frac{1}{\lambda_f}\left(1-e^{-\lambda_f\tau}\right), \\
b(\tau) &= \frac{1}{\lambda_l}\left(1-e^{-\lambda_l\tau}\right)+\frac{1}{\lambda_f-\lambda_l}\left(e^{-\lambda_f\tau}-e^{-\lambda_l\tau}\right).
\end{aligned}
$$

$c(\tau,\gamma_t)$ satisfy the following partial differential equation

$$
\begin{aligned}
&\frac{\partial c(\tau,\gamma_t)}{\partial \tau}+\left(\frac{\lambda_f^2}{\sigma_f^2}\gamma_t^2+2\lambda_l\gamma_t-\sigma_l^2\right)\frac{\partial c(\tau,\gamma_t)}{\partial \gamma_t} \\
&\qquad -\lambda_l \bar{l} b(\tau)+\frac{1}{2}\sigma_f^2 a^2(\tau)+\frac{\lambda_f^2\gamma_t^2}{2\sigma_f^2}b^2(\tau)+\lambda_f\gamma_t a(\tau)b(\tau)+\sigma_I^2=0
\end{aligned}
$$

subject to the boundary condition

$$\forall\gamma>0, \quad c(0,\gamma)=0.$$

If the variance of agents' belief is equal to its stationary level $\gamma_t = \bar{\gamma}$. This implies $\frac{d\gamma_t}{dt} = 0$. Hence the above partial differential equation collapses into the following ordinary differential equation

$$\frac{dc(\tau, \bar{\gamma})}{d\tau} = \lambda_l \bar{l} b(\tau) - \frac{1}{2}\sigma_f^2 a^2(\tau) - \frac{\lambda_f^2 \bar{\gamma}^2}{2\sigma_f^2} b^2(\tau) - \lambda_f \bar{\gamma} a(\tau) b(\tau) - \sigma_I^2,$$

subject to $c(\tau, \bar{\gamma}) = 0$. This equation has the following explicit solution:

$$c(\tau, \bar{\gamma}) = \int_0^\tau \left[\lambda_l \bar{l} b(s) - \frac{1}{2}\sigma_f^2 a^2(s) - \frac{\lambda_f^2 \bar{\gamma}^2}{2\sigma_f^2} b^2(s) - \lambda_f \bar{\gamma} a(s) b(s) - \sigma_I^2 \right] ds.$$

A.7 Proof of Proposition 3

Define l_t^* as the wealth-weighted belief in the heterogeneous economy:

$$l_t^* = \sum_{i=1}^N \omega_t^i \, \hat{l}_t^i.$$

Since each group's wealth share ω_t^i is non-negative and all the wealth shares sum up to one, mathematically we can treat the wealth share distribution just as a probability distribution. Then, the convexity of an exponential function implies that

$$\sum_{i=1}^N \omega_t^i \exp\left[-b(\tau)\hat{l}_t^i\right] \geq \exp\left[-b(\tau)\sum_{i=1}^N \omega_t^i \hat{l}_t^i\right] = \exp\left[-b(\tau) l_t^*\right].$$

The inequality holds strictly if $\hat{l}_t^1, \cdots, \hat{l}_t^N$ are not equal to each other. Using this inequality, the price of a τ-year bond in a heterogeneous economy satisfies

$$\begin{aligned} B_t &= \sum_{i=1}^N \omega_t^i \exp\left[-a(\tau) f_t - b(\tau)\hat{l}_t^i - c(\tau, \gamma_t)\right] \\ &= \exp\left[-a(\tau) f_t - c(\tau, \gamma_t)\right] \sum_{i=1}^N \omega_t^i \exp\left[-b(\tau)\hat{l}_t^i\right] \\ &\geq \exp\left[-a(\tau) f_t - c(\tau, \gamma_t)\right] \exp\left[-b(\tau) l_t^*\right] = B^H(f_t, l_t^*, \tau, \gamma_t). \end{aligned}$$

The expression in the last line above is exactly the bond price in a homogeneous economy whereby all agents hold the average belief l^* of the heterogeneous economy.

A.8 Proof of Proposition 4

According to Theorem 2, the equilibrium bond price is

$$B_t = \sum_{j=1}^{N} \frac{\eta_t^j}{\sum_{i=1}^{N} \eta_t^i} B^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right).$$

The bond yield

$$Y_t(\tau) = -\frac{1}{\tau} \log B_t.$$

To compute the conditional volatility of the bond yield, we examine the diffusion terms of $\log B_t$ by applying Ito's lemma. To save space, we skip all the drift terms in the equations below.

$$\begin{aligned} d\log B_t &\propto \frac{1}{B_t} dB_t \\ &\propto \sum_{j=1}^{N} \frac{B^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right)}{B_t} d\frac{\eta_t^j}{\sum_{i=1}^{N} \eta_t^i} + \frac{1}{B_t} \sum_{j=1}^{N} \frac{\eta_t^j}{\sum_{i=1}^{N} \eta_t^i} dB^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right) \\ &\propto \sum_{j=1}^{N} \frac{B^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right)}{B_t} \omega_t^j \left[\frac{d\eta_t^j}{\eta_t^j} - \sum_{i=1}^{N} \omega_t^i \frac{d\eta_t^i}{\eta_t^i} - a(\tau) df_t - b(\tau) d\hat{l}_t^j\right] \end{aligned}$$

By substituting $\frac{d\eta_t^j}{\eta_t^j}$, df_t, and $d\hat{l}_t^j$ into the expression above, we obtain

$$d\log B_t \propto \sum_{j=1}^{N} \frac{B^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right)}{B_t} \omega_t^j \frac{\lambda_f}{\sigma_f} \left[\hat{l}_t^j - \sum_{i=1}^{N} \omega_t^i \hat{l}_t^i - a(\tau)\frac{\sigma_f^2}{\lambda_f} - b(\tau)\gamma_t\right] dZ_f^1$$

We define

$$l_t^* = \sum_{i=1}^{N} \omega_t^i \ \hat{l}_t^i \tag{25}$$

as the wealth weighted average belief of agents, and

$$l_t^{**} = \sum_{i=1}^{N} \frac{B^H\left(f_t, \hat{l}_t^i, \tau, \gamma_t\right)}{B_t} \omega_t^i \ \hat{l}_t^i. \tag{26}$$

as the average belief weighted by different groups' contribution to the bond price. By using the fact that $\sum_{j=1}^{N} \frac{B^H\left(f_t, \hat{l}_t^j, \tau, \gamma_t\right)}{B_t} \omega_t^j = 1$. we obtain that

$$d\log B_t \propto \frac{\lambda_f}{\sigma_f} \left[l_t^{**} - l_t^* - a(\tau)\frac{\sigma_f^2}{\lambda_f} - b(\tau)\gamma_t\right] dZ_f^1.$$

Thus, the conditional volatility of the bond yield $Y_t(\tau)$ is

$$\frac{|a(\tau)\sigma_f^2 + \lambda_f b(\tau)\gamma_t + \lambda_f\left(l_t^* - l_t^{**}\right)|}{\tau\sigma_f}. \tag{27}$$

If $l_t^* - l_t^{**} \geq 0$, then the bond yield volatility increases with $l_t^* - l_t^{**}$.

Based on the definitions of l_t^* and l_t^{**},

$$\begin{aligned} l_t^* - l_t^{**} &= \sum_{i=1}^{N} \omega_t^i\ \hat{l}_t^i \left[1 - \frac{B^H\left(f_t, \hat{l}_t^i, \tau, \gamma_t\right)}{B_t}\right] \\ &= \frac{1}{B_t}\sum_{i=1}^{N} \omega_t^i\ \hat{l}_t^i \left[B_t - B^H\left(f_t, \hat{l}_t^i, \tau, \gamma_t\right)\right] \\ &= \frac{\exp\left[-a(\tau)f_t - c(\tau,\gamma_t)\right]}{B_t}\sum_{i=1}^{N}\sum_{j=1}^{N} \omega_t^i\omega_t^j\ \hat{l}_t^i \left[e^{-b(\tau)\hat{l}_t^j} - e^{-b(\tau)\hat{l}_t^i}\right]. \end{aligned}$$

By symmetry, if we swap i and j in the summations of the previous equation, it should remain the same:

$$l_t^* - l_t^{**} = \frac{\exp\left[-a(\tau)f_t - c(\tau,\gamma_t)\right]}{B_t}\sum_{i=1}^{N}\sum_{j=1}^{N} \omega_t^i\omega_t^j\ \hat{l}_t^j \left[e^{-b(\tau)\hat{l}_t^i} - e^{-b(\tau)\hat{l}_t^j}\right].$$

Thus, by taking the average of the last two equations, we have

$$l_t^* - l_t^{**} = \frac{\exp\left[-a(\tau)f_t - c(\tau,\gamma_t)\right]}{2B_t}\sum_{i=1}^{N}\sum_{j=1}^{N} \omega_t^i\omega_t^j\ \left(\hat{l}_t^i - \hat{l}_t^j\right)\left[e^{-b(\tau)\hat{l}_t^j} - e^{-b(\tau)\hat{l}_t^i}\right] \geq 0.$$

The inequality above strictly holds if there is any disagreement among agents. Thus, any belief dispersion among agents will cause $l_t^* - l_t^{**}$ to be positive, therefore increasing the bond yield volatility.

A.9 Proof of Proposition 5

The price of a τ-year bond is

$$B_t = \frac{1}{2\Delta}\int_{l^*-\Delta}^{l^*+\Delta} \exp\left[-a(\tau)f_t - b(\tau)x - c(\tau,\gamma_t)\right] dx.$$

After some algebra, we obtain

$$B_t = B^H(f_t, l_t^*, \tau, \gamma_t) K\left[b(\tau)\Delta\right],$$

where $K(x) = \frac{e^x - e^{-x}}{2x}$. It is direct to verify that $K(x)$ is an increasing function. Thus, the bond price increases with Δ, the dispersion in agents' beliefs. The bond yield is

$$\begin{aligned} Y_t &= -\frac{1}{\tau} \log [B_t] \\ &= Y_t^H - \frac{1}{\tau} \log \left\{ \frac{\exp [b(\tau)\Delta] - \exp [-b(\tau)\Delta]}{2b(\tau)\Delta} \right\}. \end{aligned}$$

To derive the conditional volatility of the bond yield, we need to compute the agents' average belief weighted by their wealth and by their contribution to the bond price, as l_t^* and l_t^{**} in equations (25) and (26). It is direct to show that $l_t^* = l^*$ and

$$\begin{aligned} l_t^{**} &= \int_{l^*-\Delta}^{l^*+\Delta} \frac{B^H(f_t, x, \tau, \gamma_t)}{B_t} \frac{x}{2\Delta} dx \\ &= l^* + \frac{1}{b(\tau)} - \frac{e^{b(\tau)\Delta} + e^{-b(\tau)\Delta}}{e^{b(\tau)\Delta} - e^{-b(\tau)\Delta}} \Delta. \end{aligned}$$

Thus,

$$l_t^* - l_t^{**} = \frac{1}{b(\tau)} - \frac{e^{b(\tau)\Delta} + e^{-b(\tau)\Delta}}{e^{b(\tau)\Delta} - e^{-b(\tau)\Delta}} \Delta.$$

By substituting $l_t^* - l_t^{**}$ into equation (27), we obtain the conditional volatility of the bond yield.

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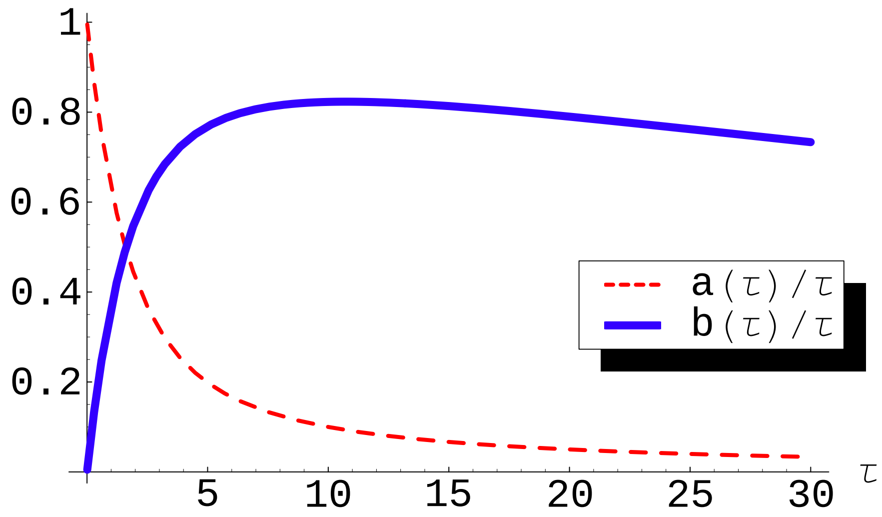

Figure 1: The factor loadings of bond yields in homogeneous economies. $a(\tau)/\tau$ is the loading on f_t, the expected instantaneous return of the risky technology, while $b(\tau)/\tau$ is the loading on $\hat{l}_t^i$, agents' belief about the long-run mean of f_t.

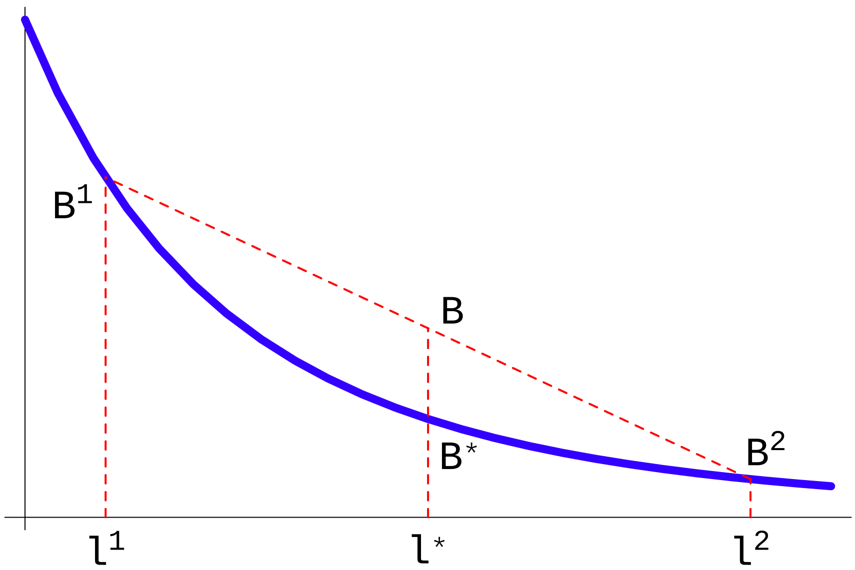

Figure 2: An illustration of the belief dispersion effect on bond prices. This figure plots bond pricing function $B^H(f_t, \hat{l}_t^i, \tau, \gamma_t)$ against $\hat{l}_t^i$. There are two groups, groups 1 and 2, with an equal wealth at time t, and a belief of $\hat{l}_t^1 = l^1$ and $\hat{l}_t^2 = l^2$, respectively. B^i $(i = 1, 2)$ is the bond price that would prevail if the economy is populated by group-i agents only. From equation (18), the bond price in this heterogeneous economy is B, the average of B^1 and B^2. B^* is the bond price that would prevail if all the agents have the average belief $l^* = (l^1 + l^2)/2$.

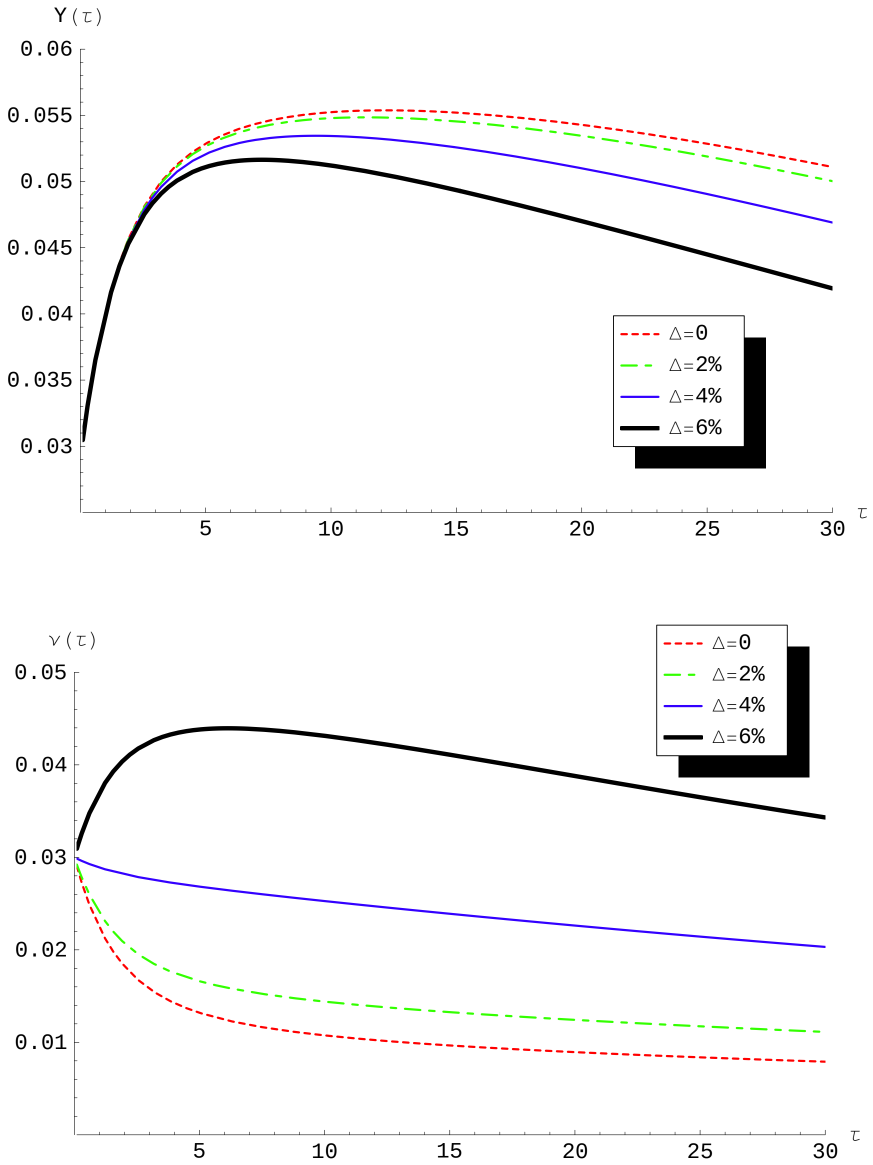

Figure 3: The effects of belief dispersion on yield curve and conditional volatility curve. This figure is based on the following model parameters: $\lambda_f = 1$, $\lambda_l = 0.02$, $\sigma_I = 0.03$, $\sigma_f = 0.03$, $\sigma_l = 0.01$, $f_t = 0.03$, $l^* = 0.06$, $\bar{l} = 0.08$, and $\gamma_t = \bar{\gamma}$. The top panel shows the bond yields, $Y(\tau)$ in equation (19), while the bottom panel shows the conditional yield volatility, $\nu(\tau)$ in equation (20), with respect to different bond maturities between 0 and 30 years, for four different values of Δ, the half distance between the most optimistic and pessimistic beliefs.

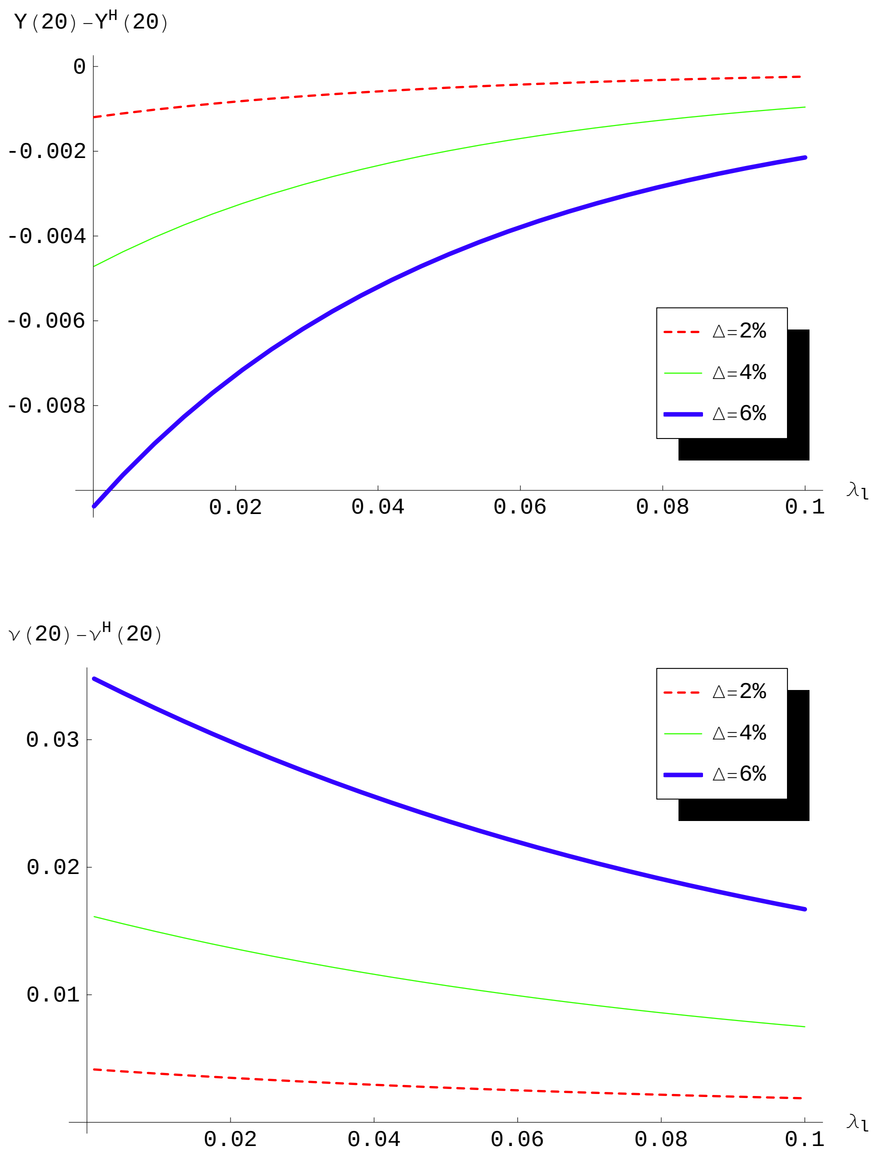

Figure 4: λ_l and the effects of belief dispersion on bond yields and yield volatility. λ_l is the mean reversion parameter of the long run mean of the expected instantaneous return of the risky technology. This figure is based on the following model parameters: $\lambda_f = 1$, $\sigma_I = 0.03$, $\sigma_f = 0.03$, $\sigma_l = 0.01$, $f_t = 0.03$, $l^* = 0.06$, $\bar{l} = 0.08$, and $\gamma_t = \bar{\gamma}$. The top panel shows the impact of belief dispersion on 20-year bond yield, $Y(20) - Y^H(20)$, while the bottom panel shows the impact of belief dispersion on conditional volatility of the yield, $\nu(20) - \nu^H(20)$, with respect to different values of λ_l between 0 and 0.1, for three different values of Δ, the half distance between the most optimistic and pessimistic beliefs.

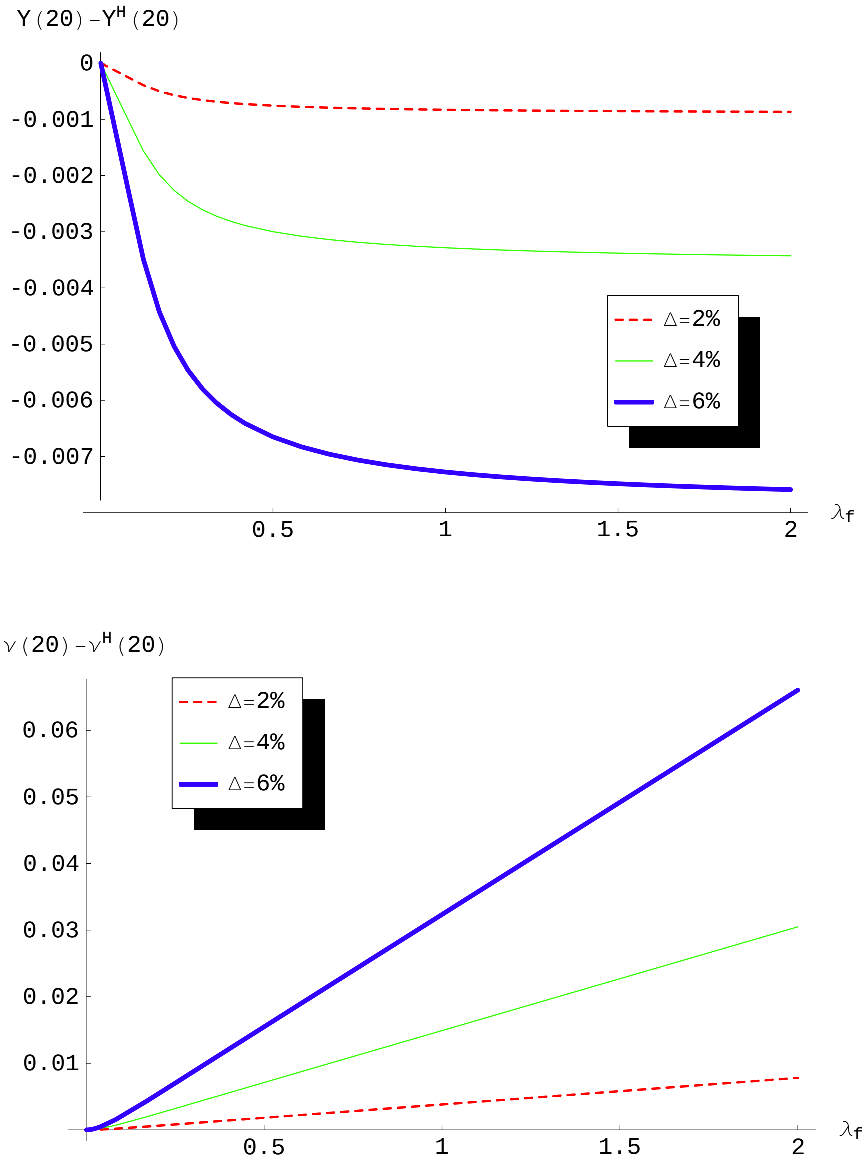

Figure 5: λ_f and the effects of belief dispersion on bond yields and yield volatility. λ_f is the mean reversion parameter of the expected instantaneous return of the risky technology. This figure is based on the following model parameters: $\lambda_l = 0.02$, $\sigma_I = 0.03$, $\sigma_f = 0.03$, $\sigma_l = 0.01$, $f_t = 0.03$, $l^* = 0.06$, $\bar{l} = 0.08$, and $\gamma_t = \bar{\gamma}$. The top panel shows the impact of belief dispersion on 20-year bond yield, $Y(20) - Y^H(20)$, while the bottom panel shows the impact of belief dispersion on conditional volatility of the yield, $\nu(20) - \nu^H(20)$, with respect to different values of λ_f between 0 and 2, for three different values of Δ, the half distance between the most optimistic and pessimistic beliefs.

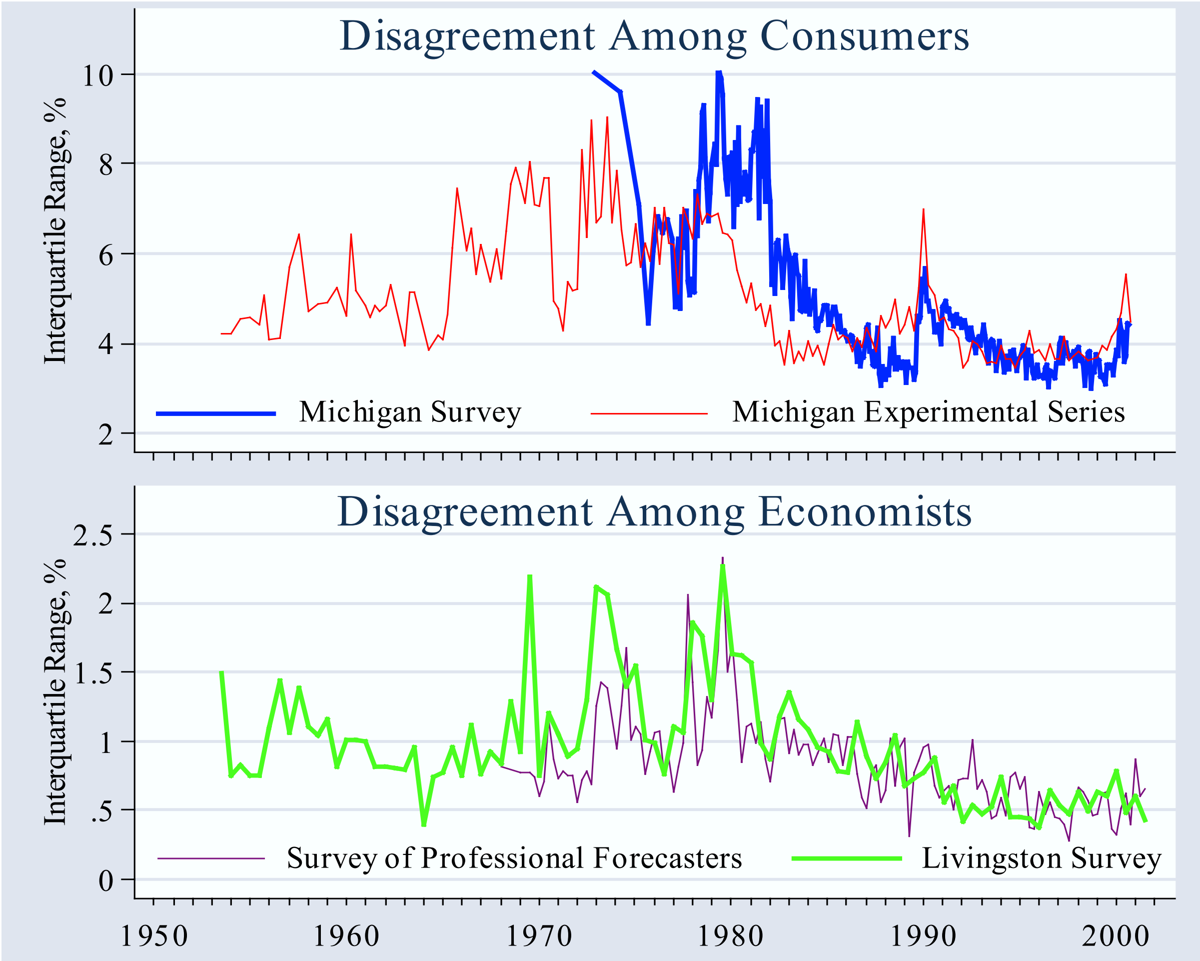

Figure 6: Belief dispersion in surveys of inflation expectations. This figure is extracted from Figure 3 of Mankiw, Reis and Wolfer (2004). The Michigan Survey surveys a cross-section of the general public of their expected price changes over the next 12 months. The Livingston Survey covers economists working in industry of their expectation of the Consumer Price Index over this quarter, in 2 quarters and in 4 quarters. The Survey of Professional Forecasters covers market economists of their expectations of the CPI level in 6 quarters.