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Explaining International Fertility Differences

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Abstract

Why do fertility rates vary so much across countries? Why are European fertility rates so much lower than American fertility rates? To answer these questions we extend the Becker-Barro framework to incorporate the decision to accumulate human capital (that determines earnings) and health capital (that determines life expectancy). We find that cross-country differences in productivity and taxes can explain the observed differences in fertility and mortality.

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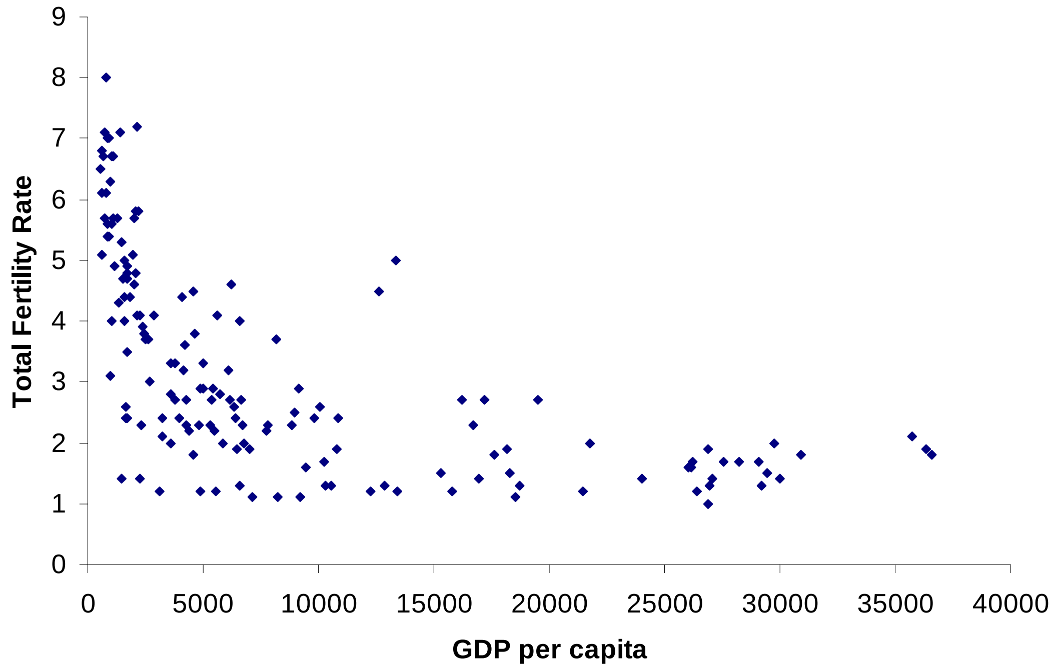

Figure 1: Income and Fertility, 2000

1 Introduction

The Question: Fertility and mortality rates vary considerably across countries. While the average family in the U.S. has 2.1 children and has a life expectancy at age 1 of over 78,

- the average family in Niger has 7.4 children and life expectancy is only 51,
- the average European family has 1.5 children and life expectancy is 78

Our main objective in this study is to understand what role economic forces play in the fertility decisions of the typical Niger and European families, and in the allocation of resources that affect life expectancy.

The Motivation: Differences in fertility and mortality rates have a very large impact on output per worker. Our previous work (Manuelli and Seshadri (2005))

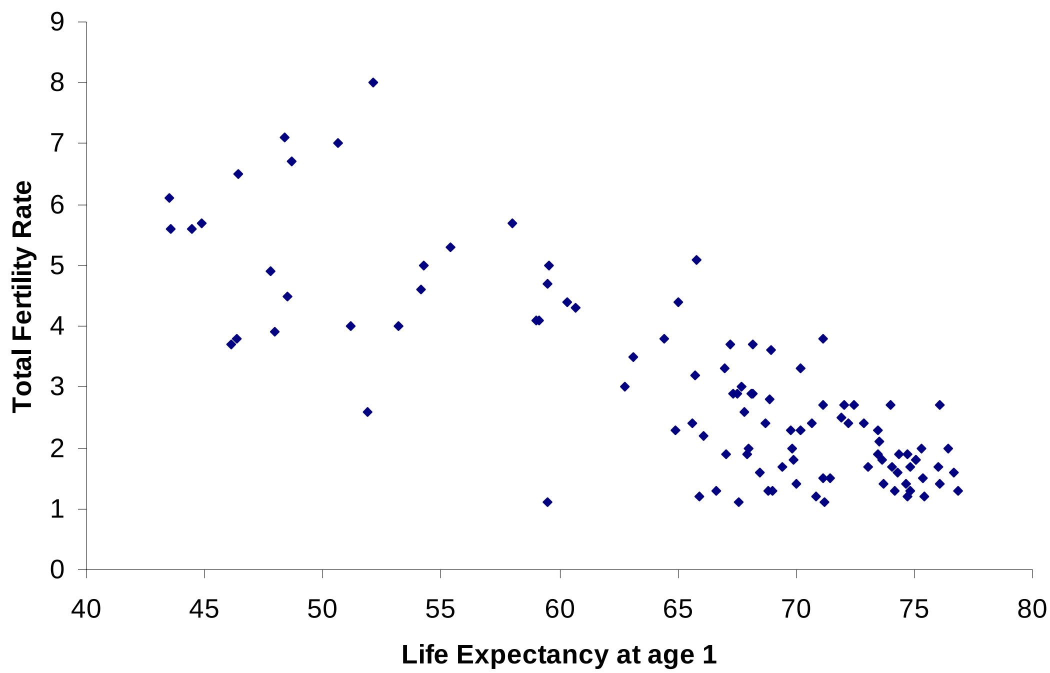

Figure 2: Life Expectancy and Fertility, 2000

suggests that if countries in the bottom decile of the world income distribution were 'endowed' with the US demographics, output per worker in these poor countries would more than double. The natural next step is to study how economic forces affect demographic variables. Moreover, our approach allows for the evaluation of alternative policies to increase output per worker (i.e. human capital vs. health capital subsidies).

The Methodology: There are two general features shared by most papers that endogeneize fertility. First, among the studies that produce quantitative estimates, it is common to assume a two or three period overlapping generations set-up in which parents care about the quantity and quality of children. We depart from this way of modeling by incorporating the full life-cycle of the individual's utility maximization.

Second, we depart from the literature in the way we model human capital. The seminal contribution by Becker, Murphy and Tamura (1990) uses a human capital production function in which the return to human capital increases with the stock of human capital. Lucas (2002) takes a similar approach and assigns an important role to endogenous human capital accumulation. Both papers view shifts in the production function for human capital as the underlying reason for the long term fertility decline. We follow Ben-Porath (1967) and Mincer (1974) in formulating the human capital accumulation decision over the course of the life-cycle in order to explain the joint behavior of the quantity and quality of children across countries.

In addition, we add a second form of capital —health capital— that determines life expectancy. Thus, as in Galor (2005) and Acemoglu and Robinson (2006) we consider the interaction between mortality and development.

Our formulation allows us to look at the joint predictions of the model for life expectancy, fertility rates (a measure of quantity of children), and years of schooling (which proxies for the quality of a child). This permits a direct test of the quantity quality trade-off.

The Mechanism: As in all versions of the Becker-Barro model, in the steady state —and in the absence of financial market imperfections— there is a positive relationship between the interest rate and fertility. This relationship is one useful tool to understanding the impact of the exogenous sources of variation that we consider: Retirement age, TFP and taxes. It is instructive to analyze the impact of each of these in the context of our model.

a. *Changing TFP*: When total factor productivity goes up, wages rise. This leads parents to invest more in the human capital of their progeny (and in their own human capital). This increases the marginal cost of having children relative to consumption. From a macro perspective, the increase in TFP results in a higher demand for capital (due to a wealth effect driven by higher consumption in the retirement period). In equilibrium, the capital-human capital ratio increases. This increase is sufficiently strong to bring the interest rate (and fertility) down. Thus, our framework is able to capture the quantity-quality

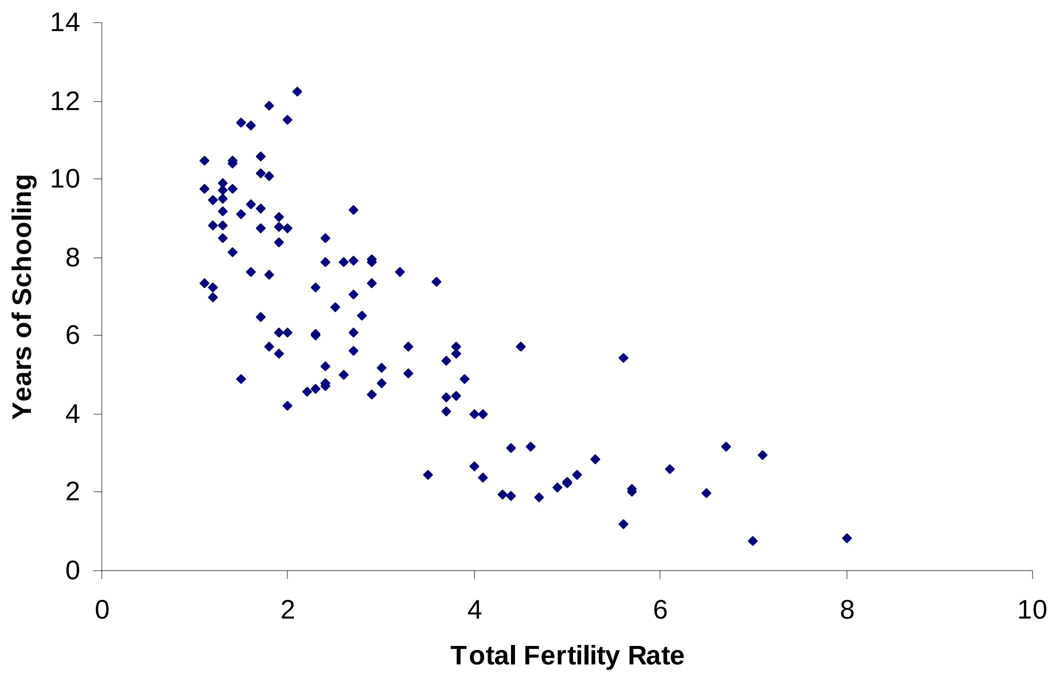

Figure 3: The Quantity-Quality Trade-off, 2000

trade-off despite having decreasing returns to scale in the production of human capital.

c. *Changing Taxes*: When the tax rate on labor income goes up, individuals decrease their investments in human capital. Thereby the capital-output ratio rises and the equilibrium interest rate decreases. Consequently, fertility declines.

In comparing rich and poor nations, TFP is higher in the US relative to the poor nations and so is life expectancy. These two effects alone account for the large fertility differential between the United States and poorer nations. Furthermore, taxes on labor income in Europe are higher than in the United States. This leads to a lower fertility rate in Europe (as well as the lower schooling level). Quantitatively, we show that these three forces combined, explain international differences in fertility rates.

2 Economic Environment:

In this section we describe the basic model. We present an economic environment with imperfect altruism and we study the choices of quantity and quality of children, as well as lifespan. We show that, with perfect markets, "quality" is determined using the usual investment criteria in models with human capital, while "quantity" is also determined by comparing costs and benefits. In the last section, we display how the endogenous demographich variables affect economy wide aggregates.

2.1 The Individual Household Problem

The representative household is formed at age I (age of independence). At age B, e^f children are born. The period of 'early childhood' (defined by the assumption that children are not productive during this period) corresponds to the (parent) age B to $B+6$. The children remain with the household (and as such make no decisions of their own) until they become independent at (parent) age $B+I$. The parent retires at age R, and dies at age T.

Let a denote an individual's age. Each parent chooses his own consumption, $c(a)$, as well as consumption of each of his children, $c_k(a)$, during the years that they are part of his household, $a \in [a, I)$, to maximize his utility. We adopt the standard Barro-Becker approach, and we specify that parent's utility depends on his own consumption, as well as the utility of his children. In addition to consumption, the parent chooses the amount of market goods to be used in the production of new human capital, $x(a)$, and the fraction of the time allocated to the formation of human capital, $n(a)$ (and, consequently, what fraction of the available time to allocate to working in the market, $1 - n(a)$) for him and each of his children while they are still attached to his household. The parent also decides to make investments in early childhood, which we denote by x_E (e.g. medical care, nutrition and development of learning skills), that determine the level of each child's human capital at age 6, $h_k(6)$, or h_E for short, and the amount of market goods, g_k, allocated to the production of health capital. While human capital is used to produce income, health capital is used to produce life expectancy. Finally, the parent chooses how much to bequeath to each children at the time they leave the household, b_k. We assume that each parent has unrestricted access to capital markets, but that he cannot commit his children to honor his debts. Thus, we restrict b_k to be non-negative.

The utility function of a parent who has h units of human capital, and a bequest equal to b at age I is given by

$$\begin{aligned} V^P(h,b,g) &= \int_I^{T(g)} e^{-\rho(a-I)} u(c(a))da + e^{-\alpha_0+\alpha_1 f} \\ & \int_0^I e^{-\rho(a+B-I)} u(c_k(a))da + e^{-\alpha_0+\alpha_1 f} e^{-\rho B} V^k(h_k(I), b_k, g_k) \end{aligned} \quad (1)$$

Thus, the contribution to the parent's utility of an a year old child still attached to him is $e^{-\alpha_0+\alpha_1 f} e^{-\rho(a+B-I)} u(c_k(a))$, since at that time the parent is $a + B$ years old. In this formulation, $e^{-\alpha_0+\alpha_1 f}$ captures the degree of altruism. If $\alpha_0 = 0$, and $\alpha_1 = 1$, this is a standard infinitively-lived agent model. Positive values of α_0, and values of α_1 less than 1 capture the degree of imperfect altruism. The term $V^k(h_k(I), b_k, g_k)$ is the utility of a child at the time he becomes independent.

Each parent maximizes $V^P(h,b,g)$ subject to two types of constraints: the budget constraint, and the production function of human capital. The former is given by

$$\begin{aligned} & \int_I^{T(g)} e^{-r(a-I)}c(a)da + e^f \int_0^I e^{-r(a+B-I)}c_k(a)da + \int_I^R e^{-r(a-I)}x(a)da + \quad (2) \\ & e^f \int_6^I e^{-r(a+B-I)}x_k(a)da + e^f e^{-rB}b_k + e^f e^{-r(B+6-I)}x_E + e^f e^{-r(B-I)}g_k \\ \leq \quad & \int_I^R e^{-r(a-I)}wh(a)(1-n(a))da + e^f \int_6^I e^{-r(a+B-I)}[wh_k(a)(1-n_k(a))]da + b. \end{aligned}$$

We adopt Ben-Porath's (1967) formulation of the human capital production technology, augmented with an early childhood period. We assume that

$$\begin{aligned} \dot{h}(a) &= z_h[n(a)h(a)]^{\gamma_1}x(a)^{\gamma_2} - \delta_h h(a), \qquad a \in [I,R) && (3) \\ \dot{h}_k(a) &= z_h[n_k(a)h_k(a)]^{\gamma_1}x_k(a)^{\gamma_2} - \delta_h h_k(a), \qquad a \in [6,I) && (4) \\ & h_k(6) = h_B x_E^{v}, && (5) \\ & h(I) \quad \textit{given}, \qquad 0 < \gamma_i < 1, \qquad \gamma = \gamma_1 + \gamma_2 < 1, \end{aligned}$$

The technology to produce human capital of each child at the beginning of the potential school years, $h_k(6)$ or h_E is given by (5). Our formulation captures the idea that nutrition and health care are important determinants of early levels of human capital, and those inputs are, basically, market goods. Equation (3) correspond to the standard human capital accumulation model initially developed by Ben-Porath (1967).

The function $T(g)$ gives the mapping between expenditures on health — which we assume take place at the time of birth— and life expectancy. We assume that this function is increasing concave and bounded.

In the steady state, it is possible to separate the optimal consumption decision from the optimal human capital accumulation decision. In particular, the optimal choice of bequests requires that (in the interior case)

$$e^{-\alpha_0+\alpha_1 f}e^{-\rho B}\frac{\partial V^k(h_k(I),b_k,g_k)}{\partial b} = \Phi e^f e^{-rB},$$

where Φ is the Lagrange multiplier associated with the budget constraint. Since in the steady state it must be the case that

$$\frac{\partial V^k(h_k(I), b_k, g_k)}{\partial b} = \frac{\partial V^P(h, b, g)}{\partial b} = \Phi,$$

it follows that

$$r = \rho + [\alpha_0 + (1 - \alpha_1)f]/B. \quad (6)$$

Thus, as in all the Becker-Barro type of models there is a one to one mapping between the fertility rate and the interest rate: High fertility countries are also high interest rate countries. However, the elasticity of the market discount factor with respect to the fertility rate is $(1 - \alpha_1)/B$ which can be small.

If the non-negativity constraint on bequests is not binding then, in the steady state, the standard separation result obtains: any allocation that maximizes utility should also maximize income. In our case, the result is a little more delicate as income early in life is appropriated by parents and, even though the model resembles and infinite horizon model, the notion of income that is maximized is just lifetime income. Thus, it follows that to determine equilibrium investment all that is necessary is to maximize the present discounted value of income. (See Manuelli and Seshadri (2005), and also the Appendix).

An intuitive (and heuristic) argument that shows the correspondence between the utility maximization and the income maximization problem is as follows: Suppose that parents (who make human capital accumulation decisions for their children until age I) do not choose the maximize the present value of income of their children (only part of which they keep). In this case, and since $b_k > 0$, the parent could increase the utility of each child by adopting the income maximizing human capital policy and adjusting the transfer to finance this change. It follows that the cost to the parent is the same and the child is made better off. Since the parent appropriates the income generate by child labor, one might wonder if it is not in the best interest of the parent to take the child out of school early and send him to work. However, this cannot be optimal as the parent can choose the optimal —from the point of view of the child— human capital policy and change the bequest as necessary. Since the parent's income is unchanged and the child is better off, this results in an increase in the utility of the parent.

In the unconstrained case, it is possible to fully characterize the solution to the income maximization problem. The main features of the solution are (see Manuelli and Seshadri (2005) for details or the Appendix):

1. The optimal allocation of time implies that $n(a) = 1$ (the individual spends all his time producing human capital) for a finite number of years. This period, whose length we denote by s, corresponds to the number of years of schooling.

2. For $a > 6 + s$, the individual is working but he continues to invest —at lower rates— in human capital.

3. Higher wages increase schooling and investment in human capital.

It follows then that the optimal choice of "quality" —interpreted as the level of human capital per child— depends on aggregate (or macro) factors. Given the interest rate, retirement age and wage rate, human capital is independent of fertility decisions (in the unconstrained case). Thus, any effect of fertility upon quality is driven by general equilibrium effects.

In order to characterize the solution to the household problem, we need to describe the optimal choice of consumption (this is standard) and the optimal choice of fertility. The first order condition corresponding to the optimal choice of f is

$$\begin{aligned} & \alpha_1 e^{-\alpha_0+\alpha_1 f} e^{-\rho B}[\int_0^I e^{-\rho(a-I)} u(c_k(a))da + V^k(h_k(I), b_k, g_k)] \qquad (7) \\ = \quad & \Phi e^f e^{-rB}[\int_0^I e^{-r(a-I)} c_k(a)da + e^{-r(6-I)} x_E + b_k + g_k e^{rI} \\ & - \int_6^I e^{-r(a-I)}(wh_k(a)(1-n_k(a)) - x_k(a))da]. \end{aligned}$$

The interpretation is simple. The left hand side corresponds to the marginal benefit of a child. It is given by his utility multiplied by the effective discount factor. The right hand side corresponds to the cost —measured in utility units— of an additional child. This cost is the sum of consumption expenditures, investment in early childhood capital and health capital, bequests and net income. Note that both the costs and the potential benefits (i.e. if net income is positive) are considered only during the period that the child spends attached to his parent.

In the steady state, it must be the case that

$$V^k(h_k(I), b_k, g_k) = V(h, b, g), \qquad h_k(I) = h,\ b = b_k,\ g_k = g.$$

Moreover, in the steady state, the effective discount factor, $-\alpha_0 + \alpha_1 f - \rho B$, equals $f - rB$. Using these steady state restrictions, (7) is

$$\begin{aligned} & \frac{\alpha_1}{1 - e^{f - rB}} \left[\int_0^{T(g)} e^{-\rho(a-I)} u(c(a)) da \right] \frac{1}{u'(c(I))} \qquad (8) \\ = & \left[\int_0^I e^{-r(a-I)} [c(a) - (wh(a)(1 - n(a)) - x(a))] da + e^{-r(6-I)} x_E + b + e^{rI} g_k \right]. \end{aligned}$$

The left hand side of (8) is the goods equivalent of the an infinite sequence of life cycle utility using the effective discount factor $(f - rB)$ to discount future flows multiplied by the marginal valuation of an additional child (α_1). The right hand side contains the same cost items that we discussed before: consumption and expenditures in human and health capital and transfers net of child labor.

The optimal choice of health capital satisfies

$$e^{-\rho(T(g)-I)} u(c_k(T(g)) T'(g) = u'(c(I)) e^{rI}. \qquad (9)$$

The second order condition requires that the left hand side of (9) be a decreasing function of g. Given that this is satisfied (more on this later), the condition implies that decreases in the marginal utility of income —for example driven by increases in productivity— result in increases in health capital and longer life expectancy.

To obtain a more precise characterization of the solution, we assume that the utility function is isoelastic and given by,

$$u(c) = \frac{c^{1-\theta}}{1-\theta}.$$

The optimal choice of consumption is given by

$$c(a) = c(I)e^{\frac{r-\rho}{\theta}(a-I)}, \qquad a \in [0, T(g)] \tag{10a}$$

In order to compute the right hand side of (8), we need the equilibrium values of the endogenous variables. In the Appendix we present the solution to the income maximization problem.

For $a \geq 0$ let net income be defined as $y(a) = wh(a)(1-n(a)) - x(a)$. Given our demographic structure, $y(a)$ satisfies

$$y(a) = \begin{cases} 0 & 0 \leq a < 6 \\ -x_E & a = 6 \\ -x(a) & 6 < a \leq 6+s \\ wh(a)(1-n(a)) - x(a) & 6+s < a \leq R \\ 0 & R < a \leq T(g) \end{cases}$$

Similarly, let $h^e(a)$ be the effective supply of human capital by an individual of age a. It follows that,

$$h^e(a) = \begin{cases} 0 & 0 \leq a < 6+s \\ h(a)(1-n(a)) & 6+s < a \leq R \\ 0 & R < a \leq T(g) \end{cases}.$$

Finally, $\tilde{x}(a) = h^e(a) - y(a)$.

Manipulation of the first order conditions and imposing that, In the steady state, $b_k = b$ in the budget constraint (2) implies that (8) is given by

$$\frac{\alpha_1 + \theta - 1}{1-\theta} c(I) e^{-\frac{r-\rho}{\theta} I} \frac{e^{\lambda(r)T(g)} - 1}{\lambda(r)} = g - P(y, r), \tag{11}$$

where

$$\lambda(x) \equiv \frac{r-\rho}{\theta} - x,$$

For any function $m(a)$ and discount factor x, the present value operator is defined by

$$L(m, x) \equiv \int_0^{T(g)} e^{-xa} m(a) da$$

Of course, in order for an equilibrium to exist it must be the case that $\theta+\alpha_1 > 1$. This condition —first derived by Becker and Barro— simply says that, at the margin, each household wants to have some children. If this condition is violated, each household's utility increases as consumption per child is increased and the number of children decreases. In addition, it must be the case that the (present discounted value) of investments in life extension must exceed the (present discounted value) of net labor income.

Equation (11) highlights the factors that affect the cost and benefits of an additional child. The net benefit —the left hand side of (11)— is given by the adjusted present discounted value of consumption. The marginal cost depends positively on expenditures in health and negatively on the present discounted valuye of net labor income.

Given the specific utility function, the appropriate version of (9) is

$$c(I)e^{-\frac{r-\rho}{\theta}I}e^{\lambda(r)T(g)}T'(g) = (1-\theta), \tag{12}$$

The second order condition for a maximum requires

$$\lambda(r)(T'(g))^2 + T''(g) \leq 0.$$

This formulation is intuitively plausible. The second order condition guarantees that the function $e^{\lambda(r)T(g)}T'(g)$ is decreasing in g. Then, an increase in TFP presumably leads to an increase in $c(I)$. This, in turn, lowers the cost of acquiring more g and life expectancy increases.

2.2 Equilibrium

Given the individual decision on human capital accumulation and investment as a function of age, all we need is to compute the age structure of the population to determine aggregate human capital. Since the capital-human capital ratio is pinned down by the condition that the marginal product of capital equal the cost of capital, this suffices to determine output per worker.

Demographics Since we consider only steady states, we need to derive the stationary age distribution of this economy. Let $N(a,t)$ be the number of people of age a at time t. Thus, our assumptions imply

$$N(a,t) = e^f N(B, t-a)$$

and

$$N(T(g), t) = 0, \qquad t > T(g)$$

It is easy to check that in the steady state

$$N(a,t) = \phi(a) e^{\eta t}, \tag{13}$$

where

$$\phi(a) = \eta \frac{e^{-\eta a}}{1 - e^{-\eta T(g)}}, \tag{14}$$

and $\eta = f/B$ is the growth rate of population.

Equilibrium From (6) it follows that if the bequest constraint is not binding, the interest rate is given by

$$r = \rho + \frac{\alpha_0}{B} + (1-\alpha_1)\eta. \tag{15}$$

Optimization on the part of firms implies that

$$p_k(r + \delta_k) = zF_k(\kappa, 1), \tag{16}$$

where κ is the physical capital - human capital ratio. The wage rate per unit of human capital, w, is,

$$w = zF_h(\kappa, 1). \tag{17}$$

Aggregate output and consumption per person satisfy

$$\begin{aligned} & \int_0^{T(g)} c(a)\phi(a)da + \phi(0)g \qquad\qquad (18) \\ = \quad & [zF(\kappa,1) - (\delta_k + \eta)\kappa p_k] \int_0^{T(g)} h^e(a)\phi(a)da - \int_0^{T(g)} \tilde{x}(a)\phi(a)da, \end{aligned}$$

An alternative description of (18) is given by

$$c(I)e^{-\frac{r-\rho}{\theta}I}\frac{e^{\lambda(\eta)T(g)}-1}{\lambda(\eta)} = -g + L(y,\eta) + (r-\eta)\kappa L(h^e,\eta). \tag{19}$$

For this to be an equilibrium, we need to verify that, at the candidate solution, $b > 0$.

The system formed by equations (11) —after substituting in (47a), (47b) and (47c)—, (12), and (18) define a solution for the triplet $(c(I), g, f)$ once the relationship between fertility and prices —as captured in (15), (16), (17)— is taken into account.

In order to informally discuss the role of TFP it is convenient to use (12) to eliminate the initial level of concumption. Then, the net marginal benefit of a an additional child (as a function of g and r) be denoted $\bar{B}$ is given by

$$\bar{B}(g,r) = (\alpha_1 + \theta - 1)\frac{1-e^{-\lambda(r)T(g)}}{\lambda(r)T'(g)} = g - L(y,r). \tag{20}$$

Similarly, aggregate consumption as a function of g and r, denoted $\bar{C}$ satisfies

$$\bar{C}(g,r) = (1-\theta)\frac{1-e^{-\lambda(\eta)T(g)}}{\lambda(\eta)T'(g)}e^{(r-\eta)T(g)} = -g + L(y,\eta) + (r-\eta)\kappa L(h^e,\eta). \tag{21}$$

Given the equilibrium wage rate and (15), (21) can be used to solve for g. Then, substituting this value into (20) gives a condition that determines η.

Even though the expressions are very complicated functions of the deep parameters, it is useful to use them to understand under what conditions TFP increases can decrease fertility. Assuming that the all expressions are differentiable functions of TFP (z), it follows from (20) that if an increase in TFP will result in lower fertility it must be that, at the original fertility (interest rate) level, the marginal benefit of a child increases by less than the marginal cost.

This corresponds to

$$\frac{\partial\bar{B}(g,r)}{\partial g}\frac{dg}{dz} < \frac{dg}{dz} - \frac{dL(y,r)}{dz}. \tag{22}$$

Since increases in TFP (and wages) increase the present discounted value of net labor earnings, $dL(y,r)/dz > 0$. Increases in TFP increase g (i.e. $dg/dz > 0$)

since life expectancy is a normal good. It follows that a necessary condition for (22) to hold is that

$$\frac{\partial \bar{B}(g,r)}{\partial g} < 1,$$

which is equivalent to

$$e^{-\lambda(r)T(g)} + \frac{T''(g)}{\lambda(r)(T'(g)^2}(e^{-\lambda(r)T(g)} - 1) < \frac{1}{\alpha_1 + \theta - 1}. \tag{23}$$

This expression shows that the impact of TFP changes upon fertility is a quantitative issue. Simple algebra shows that the left hand side of (23) is greater than one.[1] Thus, for values of $\alpha_1 + \theta$ close to 2, the inequality will be violated. In order for increases in TFP to induce decreases in the number of children, it must be the case that the degree of imperfect altruism is sufficiently low (i.e. α_1 small).

Even if the necessary condition holds (22) requires that the response of health capital to TFP be, in some sense, large than the response of the present value of y to the same change. However, the later is likely to be small if the degree of returns to scale in the human capital production function is close to one (as it is in our calibration).

[Note: More analysis to be added later.]

3 Calibration

We use standard functional forms for the utility function and the final goods production function. As indicated before, the utility function is assumed to be of the CRRA variety

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, 0 < \theta < 1.$$

The production function is assumed to be Cobb-Douglas

$$F(k,h) = zK^{\alpha}H^{1-\alpha}.$$

[1] To see this, note that the second order condition requires $\lambda(r)(T'(g))^2 + T^{'^{'}}(g) < 0$.

We assume that the mapping between health expenditures and life expectancy is

$$T(g) = \bar{T}(1 - e^{-\mu g}), \qquad \mu > 0.$$

Our calibration strategy involves choosing the parameters so that the steady state implications of the model economy presented above is consistent with observations for the United States (circa 2000). Thus, we calibrate the model to account for contemporaneous observations in the U.S. We then vary the exogenous demographic variables and choose the level of TFP for other countries so that the model's predictions for output per worker match that for the chosen country. Consequently, while output per worker for other countries are chosen so as to match output per worker by construction, the model makes predictions on years of schooling, age earnings profiles and the amount of goods inputs used in the production of human capital.

There are some parameters that are standard in the macro literature. Thus, following Cooley and Prescott (1995), the discount factor is set at $\rho = 0.04$ and the depreciation rate is set at $\delta_k = .06$. Capital's share of income is set at 0.33. Less information is available on the fraction of job training expenditures that are not reflected in wages. There are many reasons why earnings ought not to be equated with $wh(1-n)-x$. First, some part of the training is off the job and directly paid for by the individual. Second, firms typically obtain a tax break on the expenditures incurred on training. Consequently, the government (and indirectly, the individual through higher taxes) pays for the training and this component is not reflected in wages. Third, some of the training may be firm specific, in which case the employer is likely to bear the cost of the training, since the employer benefits more than the individual does through the incidence of such training. Finally, there is probably some smoothing of wage receipts in the data and consequently, the individual's marginal productivity profile is likely to be steeper than the individual's wage profile. For all these reasons, we equate earnings with $wh(1 - n) - \pi x$ and set $\pi = 0.5$. The parameter α_1 determines the degree of curvature in the altruism function of the individual. Note that this also determines the real interest rate. We proceed by choosing the level of α_1 in the United States so as to match a fertility rate (corresponding to $2 \times e^f$ in the model) of 2.1. Finally, we assume that $B = 25$.

Our theory implies that it is only the ratio $h_B^{1-\gamma}/(z_h^{1-\upsilon} w^{\gamma_2-\upsilon(1-\gamma_1)})$ that matters for the moments of interest. Consequently, we can choose z, p_k (which determine w) and h_B arbitrarily and calibrate z_h to match a desired moment. The calibrated value of z_h is common to all countries. Thus, the model does not assume any cross-country differences in an individual's 'ability to learn.' This leaves us with 10 parameters, $\alpha_0, \delta_h, z_h, \gamma_1, \gamma_2, \upsilon, \alpha_1$,$\theta, \bar{T}$ and μ.

The moments we seek in order to pin down these parameters are:

1. Earnings at age R/Earnings at age 55 of 0.8. Source: SSA
2. Earnings at age 50/Earnings at age 25 of 2.17. Source: SSA
3. Years of schooling of 12.08. Source: Barro and Lee
4. Schooling expenditures per pupil (primary and secondary) relative to GDP per capita of 0.214. Source: OECD, Education at a Glance, 2003.
5. Pre-primary expenditures per pupil relative to GDP per capita of 0.14. Source: OECD, Education at a Glance, 2003.
6. Fertility Rate of 2.1. Source: UNDP
7. Lifetime Intergenerational Transfers/GDP of 4.5%. Gale and Scholz, 1994
8. Capital output ratio of 2.52. Source: NIPA
9. Health Expenditures/GDP of 10%
10. Life Expectancy of 78 years

Theory implies that when bequests are in the interior, the human capital allocations that result from the solution to the parents problem correspond to the allocations that result from the simpler income maximization problem. Consequently, proceed in two steps since the 10 equations in 10 unknowns are 'block-separable'. For a given real interest rate and the wage rate, we calibrate the parameters $\delta_h, z_h, \gamma_1, \gamma_2$ and ν so as to match moments 1, 2, 3, 4 and 5. Thus, we use the properties of the age-earnings profile to identify the parameters of the production function of human capital. This, of course, follows a standard tradition in labor economics. We then choose the other five parameters so as to match moments 6,7, 8, 9 and 10.

The calibration requires us to solve a system of 8 equations in 8 unknowns and we obtained a perfect match. The resulting parameter values are

Parameter	α_0	δ_h	z_h	γ_1	γ_2	ν	α_1	θ	$\bar{T}$	μ
Value	0.24	0.018	0.361	0.63	0.3	0.55	0.65	0.62	101.2	0.71

Of some interest are our estimates of α_0 and γ_i. Since the first one is positive, it implies that agents are imperfectly altruistic. Our estimate of γ_2 is fairly large, and indicates that, in order for the model to be consistent with both average schooling in the U.S. as well as the pattern of the age-earnings in the data, market goods have to enter in the production function of human capital. Also, $\alpha_1 + \theta > 1$, which is necessary for the existence of an equilibrium with positive fertility.

4 Results

Before turning to the results, we first describe the data so as to get a feel for the observations of interest. We start with the countries in the PWT 6.1 and put them in deciles according to their output per worker, y. Next, we combine them with observations on years of schooling (s), expenditures per pupil relative to output per worker (x_s), life expectancy (T), and the total fertility rate ($2 \times e^f$) for each of these deciles. The population values are displayed in the following table.

Table 1: World Distribution					
	Relative output	Schooling		Life Exp	Fertility
		Years	Expenditures		TFR*(1-inf)
Decile	$\frac{y}{y^{US}}$	s	x_s	T	$2 \times e^f$
90-100	0.921	10.93	3.8	78	1.74
80-90	0.852	9.94	4.0	76	2.1
70-80	0.756	9.72	4.3	73	2.28
60-70	0.660	8.70	3.8	71	2.50
50-60	0.537	8.12	3.1	69	2.82
40-50	0.437	7.54	2.9	64	3.37
30-40	0.354	5.88	3.1	57	3.92
20-30	0.244	5.18	2.7	54	4.76
10-20	0.146	4.64	2.5	51	5.32
0-10	0.052	2.45	2.8	46	5.66

Table 1 illustrates the wide disparities in incomes across countries. The United States possesses an output per worker that is about 20 times as high as countries in the bottom decile. Further notice that years of schooling also varies systematically with the level of income — from about 2 years at the bottom deciles to about 11 at the top. The quality of education as proxied by expenditures on primary and secondary schooling as a fraction of GDP also seems to increase with the level of development. This measure should be viewed with a little caution as it includes only public inputs and not private inputs (including the time and resources that parents invest in their kids). Next, notice that demographic variables also vary systematically with the level of development - higher income countries enjoy greater life expectancies and lower fertility rates. More important, while demographics vary substantially at the lower half of the income distribution, they do not move much in the top half.

4.1 Accounting for International Differences in Fertility

We now examine the ability of the model to simultaneously match the cross country variation in output per capita and years of schooling. To be clear, we choose the level of TFP in a particular country so as to match output per worker.[2] We then see if the predictions for the fertility rate, life Expectancy and schooling are in accordance with the data.

Table 2: Fertility, Life Expectancy and Schooling - Data and Model

Decile	$\frac{y}{y^{US}}$	TFP	s		x_s		$2 \times e^f$		T	
			Data	Model	Data	Model	Data	Model	Data	Model
90-100	0.921	0.99	10.93	11.24	3.8	3.73	1.74	2.22	78	76
80-90	0.852	0.98	9.94	10.56	4.0	3.86	2.1	2.31	76	74
70-80	0.756	0.96	9.72	10.11	4.3	3.94	2.28	2.49	73	72
60-70	0.660	0.94	8.70	8.92	3.8	4.12	2.50	2.73	71	68
50-60	0.537	0.92	8.12	7.96	3.1	4.54	2.82	2.99	69	67
40-50	0.437	0.89	7.54	6.44	2.9	4.13	3.37	3.71	64	64
30-40	0.354	0.86	5.88	5.52	3.1	3.83	3.92	3.98	57	59
20-30	0.244	0.83	5.18	4.24	2.7	3.46	4.76	4.58	54	56
10-20	0.146	0.81	4.64	2.94	2.5	2.88	5.32	5.22	51	53
0-10	0.052	0.73	2.45	2.12	2.8	2.29	5.66	5.82	46	48

Table 2 presents the predictions of the model and the data. The model is able to capture reasonably well the variation across countries in the quantity of children as captured by the fertility rate and the quality of children, as captured by years of schooling. As we move from the bottom to the top decile of the world income distribution, fertility in the model decreases from 5.82 to 2.22 which compares very favorably with that observed in the data. The variation in life expectancy from 48 years to 76 is also in line with the data. Furthermore, the model also captures the variation in schooling quantity and quality across countries.

[2] We assume that $R = \min\{64, T\}$.

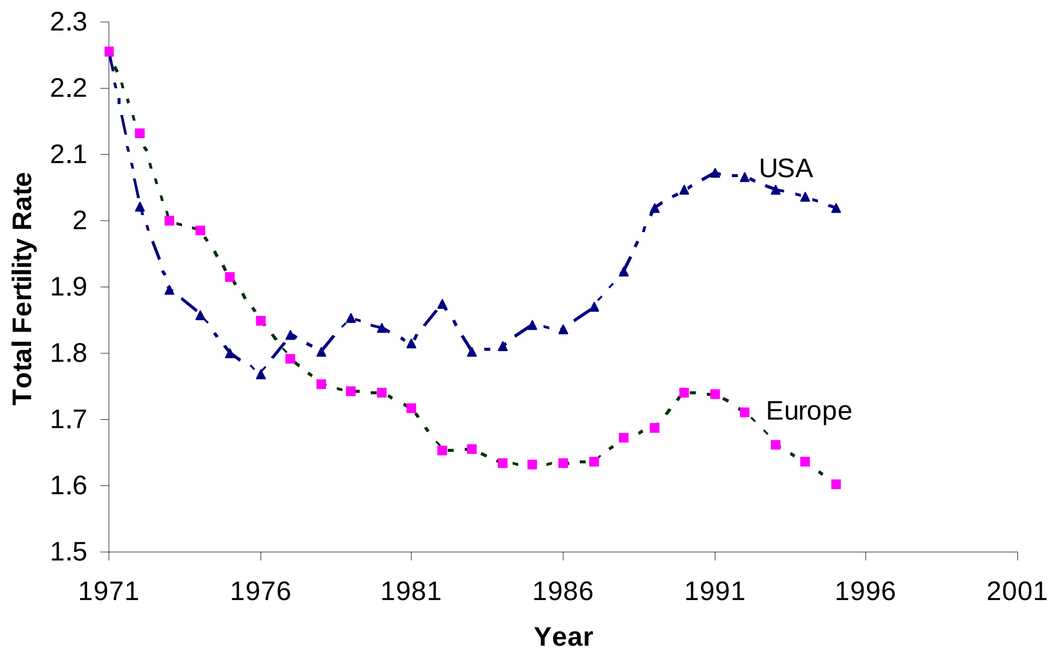

Figure 4: US-Europe Fertility Differences, 1970-1995

5 Accounting for US-Europe Fertility Differences

The previous section demonstrated the ability of the model to capture the variation in fertility rates across the different stages of development. Despite the model's ability to capture the variation in fertility rates across the world distribution of income, there is one glaring failure - the inability to capture the low fertility rate observed in many European countries. Indeed this feature of the data has been puzzling - why would the US and European nations, which presumably are at similar stages of economic development, have dramatically different fertility rates? In this section we examine the ability of the model to generate such differential behavior in fertility rates using differences in tax rates on labor income as a way of explaining these differences.

Figure 4 shows the marked divergence in the fertility rates of the United States and the European nations starting around 1976.[3] While American fertility rates increased by more than 17% over the next two decades, European fertility rates fell by a little more than 11%. At the same time, while taxes on labor income in the United States virtually stayed constant, tax rates on labor income in most European countries as well as Japan and Canada went up. Prescott (2003) argues that these higher taxes explain the lower hours worked in Europe relative to the US. Davis and Henrekson (2004) present evidence in support of the negative effect of taxes on labor supply. Table 3 presents data on tax rates (as reported in Prescott, 2003) and total fertility rates. The remarkable aspect of the Table is the degree to which the % change in tax rates and the % change in fertility rates are negatively related.

Table 3: Taxes and Fertility, G7 - Data

Country	Tax rate on labor income			Total Fertility Rate			GDP per worker		
	1975	1995	% change	1975	1995	% change	1975	1995	% change
Germany	0.52	0.59	**13**	1.53	1.33	**-13**	0.76	1.07	41
France	0.49	0.59	**20**	2.28	1.68	**-26**	0.81	1.13	39
Italy	0.41	0.64	**56**	2.33	1.19	**-49**	0.81	1.27	57
Canada	0.44	0.52	**18**	1.97	1.64	**-17**	0.93	1.13	22
U.K.	0.45	0.44	**-2**	1.79	1.73	**-3**	0.71	1.00	42
Japan	0.25	0.37	**48**	2.17	1.42	**-35**	0.53	0.92	75
U.S.A.	0.40	0.40	**0**	1.92	2.04	**6**	1.00	1.41	41

To formalize the effects of taxes on fertility, imagine adding a tax on labor income (τ^h) and capital income (τ^k) into the baseline model. The effective prices that the consumer faces are $\widetilde{r} = r(1-\tau^k)$ and $\widetilde{w} = w(1-\tau^h)$. Further, assume that the revenues from these taxes are rebated back to individuals in a lump-sum fashion so that an individual $q(a)$ at age a.

[3] The European nations included in Figure 4 are Finland, France, Germany, Hungary, Italy, Netherlands, Norway, Portugal, Sweden, Switzerland and the UK.

In order to describe the equilibrium in a model with taxes and transfers, let

$$y^{\tau}(a) = \begin{cases} 0 & 0 \le a < 6 \\ -x_E & a = 6 \\ -x(a) & 6 < a \le 6+s \\ w(1-\tau^h)h(a)(1-n(a)) - x(a) & 6+s < a \le R \\ 0 & R < a \le T(g) \end{cases}$$

and let the transfer received by an individual of age a be denoted $q(a)$. The analog of (20) is

$$\bar{B}(g,r) = g - L(y^{\tau}, r) - L(q,r),$$

feasibility is given by

$$\bar{C}(g,r) = -g + L(y,\eta) + \left(\frac{r-(1-\tau^k)\eta}{1-\tau^k}\right)\kappa L(h^e,\eta),$$

and the government budget constraint satisfies.

$$(\tau^k r\kappa + \tau^h w)L(h^e,\eta) = L(q,\eta).$$

At a heuristic level, it is possible to describe how changes in tax rates, in particular τ^h) affect the benefits and costs of an additional child. As before, assume that all expressions are differentiable, then the appropriate condition is

$$\left(\frac{\partial \bar{B}(g,r)}{\partial g} - 1\right)\frac{dg}{d\tau^h} < -\frac{dL(y^{\tau},r)}{d\tau^h} - \frac{dL(q,r)}{d\tau^h}.$$

If tax increases result in lower expenditures on health capital ($dg/d\tau^h < 0$), and since a necessary condition for TFP increases to decrease fertility is that $\partial\bar{B}(g,r)/\partial g - 1 < 0$, then it must be the case that increases in τ^h have a sufficiently large (negative) impact on $dL(y^{\tau},r)/d\tau^h$ so that the right hand side of the previous expression is positive.[4]

To evaluate the quantitative effects of changes in the tax rates on human capital, we hold τ_k fixed in what ensues. In order to re-calibrate the model to

[4] Here we assume the "standard" case in which increases in the tax rate result in increases in revenue and, through the government budget constraint, this increases transfers. Thus, $\partial L(q,r)/\partial\tau^h > 0$.

match the targets for the United States, we set $\tau_k = 0.3$ and $\tau_h = 0.4$.[5] The parameter values change slightly. Now, imagine changing the tax rate on labor income and solving for the new steady state.

Table 4: Effect of Taxes of Fertility - Model	
τ_h	$2 \times e^f$
0.40	2.10
0.45	1.77
0.50	1.61
0.55	1.42
0.60	1.22

Table 4 indicates that taxes have a powerful effect on fertility. What happens when τ_h rises? An increase in τ_h reduces the effective wage rate thereby leading to a reduction in human capital investment. Hence, the marginal cost of giving birth to children declines. However, the reduced wage rate, also implies lower consumption for the parent. A rise in the tax rate, increases marginal cost relative to consumption and consequently fertility declines. An alternative way to see this is to think about the impact of taxes on aggregate physical and human capital. When τ_h rises, the stock of human capital falls. Furthermore, since the proceeds are rebated back to the consumer in a lump-sum fashion, the individual does not have much of a need to access the capital market in order to smooth the receipts across his life-cycle. Consequently, the stock of physical capital falls by less than that of human capital. This implies that the capital output rises and the real interest rate falls. Consequently, the fertility rate must also fall.

5.1 The Effect of Social Security on Fertility

The analysis above illustrates the rather powerful effect that taxes have on fertility rates. Recall that the proceeds of the taxes were re-distributed back to consumers in a lump-sum fashion regardless of their age. In this section, we

[5] Our quantitative results do not hinge on $\tau_k = 0.3$.

examine the implications of redistributing to retirees, i.e. examining the impact of payroll taxes that fund social security. Imagine starting from a situation wherein the tax rate on labor income is 40% (and the tax rate on capital income is 30%) and all the proceeds are redistributed in a lump-sum fashion to individuals between ages I and R. Now consider increasing the tax rate on labor income from τ_h to τ_h' and redistributing the proceeds associated with $(\tau_h\prime - \tau_h)wH$ to fund lump sum payments of equal amounts to individuals between the ages of $R+1$ and T.

It can be shown that, at the initial prices, a switch corresponds to a decrease in the present value of transfers. This is the case if the ionterest rate, r, is greater than the growth rate of population, η. To see this, let revenue be fixed at $\bar{R}$. Assume that payments are constant. Then, when everyone receives a constant stream, it follows that this level, denoted q, is just the solution to

$$\bar{R} \equiv [\tau^h zF_h(\kappa, 1) + \tau^k zF_k(\kappa, 1)\kappa]L(h^e, \eta) = q\frac{1 - e^{-\eta T}}{\eta}.$$

The relevant present value of that object is

$$L(q, r) = \bar{R}\frac{1 - e^{-rT}}{r}\frac{\eta}{1 - e^{-\eta T}}.$$

Consider now a policy that, holding revenue constant, distributes the same amount of revenue only among the retired. The relevant instantaneous payment, q', is

$$q' = \bar{R}\frac{\eta}{e^{-\eta R} - e^{-\eta T}},$$

while the corresponding present value of benefits is

$$L(q', r) = \bar{R}\frac{e^{-rR} - e^{-rT}}{r}\frac{\eta}{e^{-\eta R} - e^{-\eta T}}.$$

A simple calculation shows that $L(q', r) < L(q, r)$. This, implies that, at the original prices, the marginal cost of a child increases and hence, fertility should decrease even more! Thus, given taxes, any redistribution in the direction of the “old” has a negative effect on fertility. [Note: Compare this with BDJ]

Table 5 displays the results.

Table 5: Effect of Social Security on Fertility - Model	
τ_h	$2 \times e^f$
0.40	2.10
0.45	1.95
0.50	1.84
0.55	1.76
0.60	1.66

Notice that social security has a negative effect on fertility rates. This is driven by the way it is financed —using labor income taxes— and not by the timing of payments. The effects at play are the same as in the previous section. When the tax rate on human capital rises, the stock of human capital declines since the incentive to accumulate human capital declines. Since the proceeds of the higher tax rate are re-distributed only to retirees, this reduces savings for retirement and consequently the stock of physical capital. The results indicate that the negative effect on human capital exceeds that on physical capital - thereby the capital output ratio rises and the fertility rate falls.

Notice that social security leads to a decrease in fertility rates only because the presence of human capital makes the supply of labor elastic. Absent human capital —or if the social security program was financed using lump-sum taxes— there is only one effect at play: a rise in social security receipts would lead to a fall in the stock of capital, which would lead to a fall in the capital output ratio and hence raise the fertility rate. Indeed, this is the argument in Boldrin et. al. (2005). Thus the addition of human capital, and the major role played by taxation into the Barro-Becker model, implies that more generous social security regimes financed by higher taxes on labor income have a negative net effect on fertility.

6 The U.S.: 1900 vs. 2000

In this section we use the calibrated model to predict life expectancy, fertility and schooling for the U.S. in 1900. To be precise, we take our base calibrated model (with taxes) as a good description of the (steady state) of the U.S. economy circa 2000. We view (this is preliminary) the U.S. economy circa 1900 to be in a steady state. The only differences between 2000 and 1900 are the tax rates (which are assumed to be 0 in 1900) and TFP (which is chosen to match output per worker).

The results of the experiment are in Table 6

Table 6: US: 1900 and 2000 - Data and Model

Period	$\frac{y}{y^{US}}$	TFP	s		g/y		$2 \times e^f$		T	
			Data	Model	Data	Model	Data	Model	Data	Model
1900	0.19	0.86	5.4	4.83	na	0.04	3.8	4.32	52	48
2000	1	1	12.08	12.08	.10	0.10	2.1	2.1	78	78

The model does a reasonable job of accounting for the changes over the last century. It suggests that "true" technological change need not have been very large, as TFP in 2000 is only 16% higher than in 1900. Of course, the model also implies that small changes in TFP induce large changes in effective human capital. The predictions of the model for fertility, schooling and life expectancy are not perfect but they are reasonably close. If anything, it predicts —relative to the data— larger responses in the endogenous variables as a result of changes in TFP.

One limitation of this analysis is that it assumes that the U.S. economy was at a steady state in both 1900 and 2000. In future versions we will look at the transition.

7 Technological Progress in the Health Sector (to be added)

In this section (to be competed) we explore the impact of improvements in medical technology. In the paper we assumed that productivity growth in the medical technology match those of the goods producing technology. In this section we will consider the effects of increasing μ —which corresponds to higher productivity of market inputs in the production of life expectancy— and of increasing h_B —which we take to capture the productivity of market inputs in producing early childhood human capital.

Qualitatively (add details) it can be shown that decreases in μ increase life expectancy and fertility. On the other hand, increases in h_B decrease fertility as the make the shadow price of "quality of children" lower. Thus, we expect a combination of the two changes to be able to reproduce the increases in life expectancy, decreases in fertility and increases in schooling observed in the data.

8 Inheritance Taxes

In this section we explore the role of inheritance taxes. Assume that if a parent transfers b units of consumption to each child, the cost is $e^{\tau}b$. Thus, the inheritance tax rate is $e^{\tau}-1$. In this case, the relevant version of (6) is

$$r = \rho + [\alpha_0 + \tau + (1-\alpha_1)f]/B.$$

If the government rebates the proceeds back to the individuals, then it must be that

$$(e^{\tau}-1)\phi(B)be^{f} = \int_0^{T(g)} \tilde{q}(a)\phi(a)da.$$

In this setting, we can think about what is going on holding η constant, but letting r change as τ changes. [More to follow]

9 Conclusions

This paper integrates a life-cycle model of human and physical capital accumulation where life expectancy is endogenous with the Becker-Barro framework. This permits an interesting trade-off between the quantity and quality of children and the quantity and quality of life. The model is able to capture the wide variation in fertility rates seen across the income distribution. Further, the model suggests that a substantial part of the lower fertility rates in Europe are due to the higher labor income tax rates.

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10 Appendix

Proposition 1 *Assume that $r = \rho + [\alpha_0 + (1-\alpha_1)f]/B$, then the solution to the optimal human capital accumulation corresponding to the maximization of (1) subject to (2)-(5) is identical to the solution of the following income maximization problem*

$$\max \int_6^R e^{-r(a-6)}[wh(a)(1-n(a)) - x(a)]da - x_E \tag{24}$$

subject to

$$\dot{h}(a) = z_h[n(a)h(a)]^{\gamma_1} x(a)^{\gamma_2} - \delta_h h(a), \qquad a \in [6, R), \tag{25}$$

and

$$h(6) = h_E = h_B x_E^{\upsilon} \tag{26}$$

with h_B given.

Proof of Proposition 1. : We show that the first order conditions corresponding to both problems coincide. Since the problems are convex, this suffices to establish the result. Consider first the first order conditions of the income maximization problem given the stock of human capital at age 6, $h(6) = h_E$. Let $q(a)$ be the costate variable. A solution satisfies

$$\begin{aligned} whn &\leq q\gamma_1 z_h (nh)^{\gamma_1} x^{\gamma_2}, \qquad \textit{with equality if } n < 1, && \text{(27a)} \\ x &= q\gamma_2 z_h (nh)^{\gamma_1} x^{\gamma_2}, && \text{(27b)} \\ \dot{q} &= rq - [q\gamma_1 z_h (nh)^{\gamma_1} x^{\gamma_2} h^{-1} - \delta_h] - w(1-n), && \text{(27c)} \\ \dot{h} &= z_h (nh)^{\gamma_1} x^{\gamma_2} - \delta_h h, && \text{(27d)} \end{aligned}$$

where $a \in [6, R]$. The transversality condition is $q(R) = 0$.

Let Φ be the Lagrange multiplier associated with the budget constraint (2). Then, the relevant (for the decision to accumulate human capital) problem solved by a parent is

$$\max \Phi\{\int_I^R e^{-r(a-I)}[wh(a)(1-n(a))-x(a)]da$$
$$+e^f\int_B^{B+I} e^{-r(a-I)}[wh_k(a)(1-n_k(a))-x_k(a)]da$$
$$-e^f e^{-rB}b_k - e^f e^{-r(B+6)}x_E\} + e^{-\alpha_0+\alpha_1 f}e^{-\rho B}V^k(h_k(B+I),b_k),$$

where, in this notation, a stands for the parent's age. It follows that the first order conditions corresponding to the choice of $[h(a), n(a), x(a), q_p(a)]$ are identical to those corresponding to the income maximization problem (27), including the transversality condition $q_p(R) = 0$ for $a \in [I, R]$. It follows that $q_p(a) = q(a)$.Simple algebra shows that the first order conditions corresponding to the optimal choices of $[h_k(a), n_k(a), x_k(a), q_k(a)]$ also satisfy (27) for $a \in [6, I)$. However, the appropriate transversality condition for this problem is

$$q_k(B+I) = e^{-[\alpha_0+(1-\alpha_1)f]}e^{-(\rho-r)B}\frac{1}{\Phi}\frac{\partial V^k(h_k(B+I),b_k)}{\partial h_k(B+I)}.$$

However, given (6), and the envelope condition

$$\frac{\partial V^k(h_k(B+I),b_k)}{\partial h_k(B+I)} = \Phi_k q_p(I),$$

evaluated at the steady state $\Phi = \Phi_k$, it follows that

$$q_k(B+I) = q_p(I).$$

Thus, the program solved by the parent (for $a \in [I, R]$) is just the continuation of the problem he solves for his children for $a \in [6, I)$. It is clear that if (6) does not hold, then there is a 'wedge' between how the child values his human capital after he becomes independent, $q_p(I)$, and the valuation that his parent puts on the same unit if human capital, $q_k(B + I)$. ■

Proposition 2 *There exists a unique solution to the income maximization problem. The number of years of schooling, s, satisfies*

1.

$$F(s) = \frac{h_B^{1-\gamma}}{z_h^{1-\upsilon}w^{\gamma_2-\upsilon(1-\gamma_1)}}\left(\frac{\upsilon}{r+\delta_h}\right)^{(1-\gamma)\upsilon}\left(\frac{\gamma_2^{\gamma_2}\gamma_1^{(1-\gamma_2)}}{r+\delta_h}\right)^{-(1-\upsilon)}, \quad (28)$$

where

$$F(s) \equiv m(6+s)^{1-\upsilon(2-\gamma)}e^{(1-\gamma)(\delta_h+r\upsilon)s}$$
$$\left[1-\frac{r+\delta_h}{\gamma_1}\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r+\delta_h(1-\gamma_1)}\frac{1-e^{-\frac{\gamma_2 r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}s}}{m(s+6)}\right]^{\frac{(1-\gamma)(1-\upsilon(1-\gamma_1))}{1-\gamma_1}},$$

and

$$m(a)=1-e^{-(r+\delta_h)(R-a)},$$

provided that

$$m(6)^{1-\upsilon(2-\gamma)} > \frac{h_B^{1-\gamma}}{z_h^{1-\upsilon}w^{\gamma_2-\upsilon(1-\gamma_1)}}\left(\frac{\upsilon}{r+\delta_h}\right)^{(1-\gamma)\upsilon}\left(\frac{\gamma_2^{\gamma_2}\gamma_1^{(1-\gamma_2)}}{r+\delta_h}\right)^{-(1-\upsilon)}.$$

Otherwise the privately optimal level of schooling is 0.

2. *The level of human capital at the age at which the individual finishes his formal schooling is given by*

$$h(s+6)=\left[\frac{\gamma_2^{\gamma_2}\gamma_1^{\gamma_1}z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right]^{\frac{1}{1-\gamma}}\frac{\gamma_1}{r+\delta_h}m(6+s)^{\frac{1}{1-\gamma}} \tag{29}$$

For simplicity, we prove a series of lemmas that simplify the proof of Proposition (2). Moreover, It is convenient to define several functions that we will use repeatedly.

Let

$$C_h(z_h,w,r)=\left[\frac{\gamma_2^{\gamma_2}\gamma_1^{\gamma_1}z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right]^{\frac{1}{1-\gamma}},$$

and

$$m(a)=1-e^{-(r+\delta_h)(R-a)}.$$

The following lemma provides a characterization of the solution in the post schooling period.

Lemma 3 *Assume that the solution to the income maximization problem stated in Proposition 1 is such that $n(a)=1$ for $a\leq 6+s$ for some s. Then, given $h(6+s)$ the solution satisfies, for $a\in[6+s,R)$,*

$$x(a)=\left(\frac{\gamma_2 w}{r+\delta_h}\right)C_h(z_h,w.r)\left[1-e^{-(r+\delta_h)(R-a)}\right]^{\frac{1}{1-\gamma}}, \qquad a\in[6+s,R), \tag{30}$$

$$
\begin{aligned}
h(a) &= e^{-\delta_h(a-6-s)}\{h(6+s) + \frac{C_h(z_h, w.r)}{\delta_h} e^{-\delta_h(6+s-R)} \\
&\quad \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} (1 - x^{\frac{r+\delta_h}{\delta_h}})^{\frac{\gamma}{1-\gamma}} dx\}, \qquad a \in [6+s, R),
\end{aligned} \tag{31}
$$

and

$$
q(a) = \frac{w}{r+\delta_h}[1 - e^{-(r+\delta_h)(R-a)}], \qquad a \in [6+s, R). \tag{32}
$$

Proof of Lemma 3. : Given that the equations (27) hold (with the first equation at equality), standard algebra (see Ben-Porath, 1967 and Haley, 1976) shows that (32) holds. Using this result in (27b) it follows that

$$
x(a) = \left[\frac{\gamma_2^{\gamma_2}\gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right]^{\frac{1}{1-\gamma}} \left(\frac{\gamma_2 w}{r+\delta_h}\right) \left[1 - e^{-(r+\delta_h)(R-a)}\right]^{\frac{1}{1-\gamma}},
$$

which is (30). Next substituting (30) and (32) into (27d) one obtains a non-linear non-homogeneous first order ordinary differential equation. Straightforward, but tedious, algebra shows that (31) is a solution to this equation. ■

The next lemma describes the solution during the schooling period.

Lemma 4 *Assume that the solution to the income maximization problem stated in Proposition 1 is such that $n(a) = 1$ for $a \leq 6+s$ for some s. Then, given $h(6) = h_E$ and $q(6) = q_E$, the solution satisfies, for $a \in [6, 6+s)$,*

$$
x(a) = \left(h_E^{\gamma_1} q_E \gamma_2 z_h\right)^{\frac{1}{1-\gamma_2}} e^{\frac{r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}(a-6)}, \qquad a \in [6, 6+s) \tag{33}
$$

and

$$
\begin{aligned}
h(a) &= h_E e^{-\delta_h(a-6)}[1 + \left(h_E^{-(1-\gamma)} q_E^{\gamma_2}\gamma_2^{\gamma_2} z_h\right)^{\frac{1}{1-\gamma_2}} \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r + \delta_h(1-\gamma_1)} \\
&\quad (e^{\frac{\gamma_2 r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}(a-6)} - 1)]^{\frac{1}{1-\gamma_1}}, \qquad a \in [6, 6+s)
\end{aligned} \tag{34}
$$

Proof of Lemma 4. : From (27b) we obtain that

$$
x(a) = (q(a)h(a)^{\gamma_1})^{\frac{1}{1-\gamma_2}} (\gamma_2 z_h)^{\frac{1}{1-\gamma_2}}. \tag{35}
$$

Since we are in the region in which the solution is assumed to be at a corner, (27a) implies

$$
h(a) \leq \left(\frac{\gamma_1}{w}\right)^{\frac{1-\gamma_2}{1-\gamma}} (\gamma_2^{\gamma_2} z_h)^{\frac{1}{1-\gamma}} q(a)^{\frac{1}{1-\gamma}} \tag{36}
$$

In order to better characterize the solution we now show that the shadow value of the total product of human capital in the production of human capital grows at a constant rate. More precisely, we show that For $a \in [6, 6+s)$, $q(a)h^{\gamma_1}(a) = q_E h_E^{\gamma_1} e^{[r+\delta_h(1-\gamma_1)](a-6)}$. To see this, let $M(a) = q(a)h^{\gamma_1}(a)$. Then,

$$\dot{M}(a) = M(a)[\frac{\dot{q}(a)}{q(a)} + \gamma_1 \frac{\dot{h}(a)}{h(a)}].$$

However, it follows from (27c) and (27d) after substituting (35) that

$$\begin{aligned} \frac{\dot{h}(a)}{h(a)} &= z_h h(a)^{\gamma_1 - 1} x(a)^{\gamma_2} - \delta_h, \qquad a \in [6, 6+s) \\ \frac{\dot{q}(a)}{q(a)} &= r + \delta_h - \gamma_2 z_h h(a)^{\gamma_1 - 1} x(a)^{\gamma_2}, \qquad a \in [6, 6+s). \end{aligned}$$

Thus,

$$\frac{\dot{q}(a)}{q(a)} + \gamma_1 \frac{\dot{h}(a)}{h(a)} = r + \delta_h(1-\gamma_1).$$

The function $M(a)$ satisfies the first order ordinary differential equation

$$\dot{M}(a) = M(a)[r + \delta_h(1-\gamma_1)]$$

whose solution is

$$M(a) = M(6)e^{[r+\delta_h(1-\gamma_1)](a-6)}$$

which establishes the desired result.

Using this result the level of expenditures during the schooling period is given by

$$x(a) = \left(h_E^{\gamma_1} q_E \gamma_2 z_h\right)^{\frac{1}{1-\gamma_2}} e^{\frac{r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}(a-6)}, \qquad a \in [6, 6+s).$$

Substituting this expression in the law of motion for $h(a)$ (equation (27d), the equilibrium level of human capital satisfies the following first order non-linear, non-homogeneous, ordinary differential equation

$$\dot{h}(a) = \left(h_E^{\gamma_1\gamma_2} q_E^{\gamma_2} \gamma_2^{\gamma_2} z_h\right)^{\frac{1}{1-\gamma_2}} e^{\frac{\gamma_2[r+\delta_h(1-\gamma_1)]}{(1-\gamma_2)}(a-6)} h^{\gamma_1}(a) - \delta_h h(a).$$

It can be verified, by direct differentiation, that (34) is a solution. ∎

The next lemma describes the joint determination, given the age 6 level of human capital h_E, of the length of the schooling period, s, and the age 6 shadow price of human capital, q_E.

Lemma 5 *Given h_E, the optimal shadow price of human capital at age 6, q_E, and the length of the schooling period, s, are given by the solution to the following two equations*

$$q_E = \left[\frac{\gamma_1^{\gamma_1(1-\gamma_2)}\gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} w^{(1-\gamma_1)(1-\gamma_2)}}{(r+\delta_h)^{(1-\gamma_2)}}\right]^{\frac{1}{1-\gamma}} h_E^{-\gamma_1} \quad (37)$$
$$e^{-(r+\delta_h(1-\gamma_1))s} m(s+6)^{\frac{1-\gamma_2}{1-\gamma}},$$

and

$$q_E^{\frac{\gamma_2}{1-\gamma_2}} h_E^{\frac{\gamma_1\gamma_2}{1-\gamma_2}} e^{-\delta_h(1-\gamma_1)s}\left(\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r+\delta_h(1-\gamma_1)}\right)(\gamma_2^{\gamma_2} z_h)^{\frac{1}{1-\gamma_2}} \quad (38)$$
$$[e^{\frac{\gamma_2 r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}s} - 1] + h_E^{1-\gamma_1} e^{-\delta_h(1-\gamma_1)s}$$
$$= \left(\frac{\gamma_1^{(1-\gamma_2)}\gamma_2^{\gamma_2}}{(r+\delta_h)}\right)^{\frac{1-\gamma_1}{1-\gamma}} (z_h w^{\gamma_2})^{\frac{1-\gamma_1}{1-\gamma}} [m(s+6)]^{\frac{1-\gamma_1}{1-\gamma}}.$$

Proof of Lemma 5. To prove this result, it is convenient to summarize some of the properties of the optimal path of human capital. For given values of (q_E, h_E, s) the optimal level of human capital satisfies

$$h(a) = h_E e^{-\delta_h(a-6)}[1 + \left(h_E^{-(1-\gamma)} q_E^{\gamma_2}\gamma_2^{\gamma_2} z_h\right)^{\frac{1}{1-\gamma_2}} \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r+\delta_h(1-\gamma_1}} \quad (39)$$
$$(e^{\frac{\gamma_2 r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}(a-6)} - 1)]^{\frac{1}{1-\gamma_1}}, \; a \in [6, 6+s)$$

$$h(a) = e^{-\delta_h(a-s-6)}\{h(6+s) + \frac{C_h(z_h, w, r)}{\delta_h} e^{-\delta_h(6+s-R)} \quad (40)$$
$$\int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} (1 - x^{\frac{r+\delta_h}{\delta_h}})^{\frac{\gamma}{1-\gamma}} dx\}, \qquad a \in [6+s, R).$$

Moreover, during at age $6+s$, (36) must hold at equality. Thus,

$$h(6+s) = \left(\frac{\gamma_1}{w}\right)^{\frac{1-\gamma_2}{1-\gamma}} (\gamma_2^{\gamma_2} z_h)^{\frac{1}{1-\gamma}} q(6+s)^{\frac{1}{1-\gamma}}.$$

Using the result in Lemma 4 in the previous equation, it follows that

$$q(6+s) = \frac{\left(h_E^{\gamma_1} q_E\right)^{\frac{1-\gamma}{1-\gamma_2}} e^{\frac{1-\gamma}{1-\gamma_2}(r+\delta_h(1-\gamma_1))(6+s)}}{\left(\frac{\gamma_1}{w}\right)^{\gamma_1} (\gamma_2^{\gamma_2} z_h)^{\frac{\gamma_1}{1-\gamma_2}}}. \quad (41)$$

Since

$$q(6+s) = \frac{w}{r+\delta_h}[1 - e^{-(r+\delta_h)(R-s-6)}],$$

it follows that

$$q_E = \left[\frac{\gamma_1^{\gamma_1(1-\gamma_2)}\gamma_2^{\gamma_1\gamma_2}z_h^{\gamma_1}w^{(1-\gamma_1)(1-\gamma_2)}}{(r+\delta_h)^{(1-\gamma_2)}}\right]^{\frac{1}{1-\gamma}} h_E^{-\gamma_1} e^{-(r+\delta_h(1-\gamma_1))s} m(s+6)^{\frac{1-\gamma_2}{1-\gamma}},$$

which is (37). Next, using (39) evaluated at $a = 6+s$, and (36) at equality (and substituting out $q(6+s)$) using either one of the previous expressions we obtain (38). ■

We now discuss the optimal choice of h_E. Since q_E is the shadow price of an additional unit of human capital at age 6, the household chooses x_E to solve

$$\max q_E h_B x_E^{\upsilon} - x_E.$$

The solution is

$$h_E = \upsilon^{\frac{\upsilon}{1-\upsilon}} h_B^{\frac{1}{1-\upsilon}} q_E^{\frac{\upsilon}{1-\upsilon}}. \tag{42}$$

Proof of Proposition 2. Uniqueness of a solution to the income maximization problem follows from the fact that the objective function is linear and, given $\gamma < 1$, the constraint set is strictly convex. Even though existence can be established more generally, in what follows we construct the solution. To this end, we first describe the determination of years of schooling. Combining (37) and (38) it follows that

$$h_E = e^{\delta_h s} m(s+6)^{\frac{1}{1-\gamma}} (z_h w^{\gamma_2})^{\frac{1}{1-\gamma}} \left(\frac{\gamma_2^{\gamma_2}\gamma_1^{(1-\gamma_1)}}{r+\delta_h}\right)^{\frac{1}{1-\gamma}} \left[1 - \frac{r+\delta_h}{\gamma_1}\frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r+\delta_h(1-\gamma_1)}\frac{1-e^{-\frac{\gamma_2 r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}s}}{m(s+6)}\right]^{\frac{1}{1-\gamma_1}}. \tag{43}$$

Next, using (37) in (42), h_E must satisfy

$$h_E = h_B^{\frac{1}{1-\upsilon(1-\gamma_1)}} \upsilon^{\frac{\upsilon}{1-\upsilon(1-\gamma_1)}} \left(\frac{\gamma_1^{\gamma_1(1-\gamma_2)}\gamma_2^{\gamma_1\gamma_2}}{(r+\delta_h)^{1-\gamma_2}}\right)^{\frac{\upsilon}{(1-\gamma)(1-\upsilon(1-\gamma_1))}} \left(z_h^{\gamma_1} w^{(1-\gamma_1)(1-\gamma_2)}\right)^{\frac{\upsilon}{(1-\gamma)(1-\upsilon(1-\gamma_1))}} e^{-\frac{\upsilon(r+\delta_h(1-\gamma_1))}{1-\upsilon(1-\gamma_1)}s} m(s+6)^{\frac{\upsilon(1-\gamma_2)}{(1-\gamma)(1-\upsilon(1-\gamma_1))}}. \tag{44}$$

Finally, (43) and (44) imply that the number of years of schooling, s, satisfies

$$m(s+6)^{1-\upsilon(2-\gamma)} e^{(1-\gamma)(\delta_h + r\upsilon)s} \tag{45}$$

$$\left[1 - \frac{r+\delta_h}{\gamma_1} \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_2 r + \delta_h(1-\gamma_1)} \frac{1 - e^{-\frac{\gamma_2 r + \delta_h(1-\gamma_1)}{(1-\gamma_2)} s}}{m(s+6)}\right]^{\frac{(1-\gamma)(1-\upsilon(1-\gamma_1))}{1-\gamma_1}}$$

$$= \frac{h_B^{1-\gamma}}{z_h^{1-\upsilon} w^{\gamma_2 - \upsilon(1-\gamma_1)}} \left(\frac{\upsilon}{r+\delta_h}\right)^{(1-\gamma)\upsilon} \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{(1-\gamma_2)}}{r+\delta_h}\right)^{-(1-\upsilon)}.$$

As in the statement of the proposition, let the left hand side of (45) be labeled $F(s)$. Then, an interior solution requires that $F(0) > 0$, or,

$$m(6)^{1-\upsilon(2-\gamma)} > \frac{h_B^{1-\gamma}}{z_h^{1-\upsilon} w^{\gamma_2 - \upsilon(1-\gamma_1)}} \left(\frac{\upsilon}{r+\delta_h}\right)^{(1-\gamma)\upsilon} \left(\frac{\gamma_2^{\gamma_2} \gamma_1^{(1-\gamma_2)}}{r+\delta_h}\right)^{-(1-\upsilon)}.$$

Inspection of the function $F(s)$ shows that there exists a unique value of s, say $\bar{s}$, such that $F(s) > 0$, for $s < \bar{s}$, and $F(s) \leq 0$, for $s \geq \bar{s}$. It is clear that $\bar{s} < R - 6$. Hence, the function $F(s)$ must intersect the right hand side of (45) from above. The point of intersection is the unique value of s that solves the problem. ■

It is convenient to collect a full description of the solution as a function of aggregate variables and the level of schooling, s.

Solution to the Income Maximization Problem It follows from (27a), and the equilibrium values of the other endogenous variables, the time allocated to human capital formation is 1 for $a \in [6, 6+s)$, and

$$n(a) = \frac{m(a)^{\frac{1}{1-\gamma}}}{e^{-\delta_h(a-s-6)} m(6+s)^{\frac{1}{1-\gamma}} + \frac{(r+\delta_h)e^{-\delta_h(a-R)}}{\gamma_1 \delta_h} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} (1 - x^{\frac{r+\delta_h}{\delta_h}})^{\frac{\gamma}{1-\gamma}} dx}, \tag{46}$$

for $a \in [6+s, R]$.

The amount of market goods allocated to the production of human capital is given by

$$x(a) = \left(\frac{\gamma_2 w}{r+\delta_h}\right) C_h(z_h, w, r) m(6+s)^{\frac{1}{1-\gamma}} e^{\frac{r+\delta_h(1-\gamma_1)}{(1-\gamma_2)}(a-s-6)}, \ a \in [6, 6+s), \tag{47a}$$

$$x(a) = \left(\frac{\gamma_2 w}{r+\delta_h}\right) C_h(z_h, w, r) m(a)^{\frac{1}{1-\gamma}}, \qquad a \in [6+s, R). \tag{47b}$$

$$x_E = \upsilon \left[\frac{\gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} w^{(1-\gamma_1)(1-\gamma_2)}}{(r+\delta_h)^{(1-\gamma_2)}}\right]^{\frac{1}{1-\gamma}} \frac{m(6+s)^{\frac{(1-\gamma_2)}{1-\gamma}}}{e^{(r+\delta_h(1-\gamma_1))s}} \tag{47c}$$

The level of human capital of an individual of age a in the post-schooling period (i.e. $a \geq 6+s$) is given by

$$h(a) = C_h(z_h, w, r)\{e^{-\delta_h(a-s-6)} \frac{\gamma_1}{r+\delta_h} m(6+s)^{\frac{1}{1-\gamma}} + \frac{e^{-\delta_h(a-R)}}{\delta_h} \tag{48}$$

$$\int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} (1 - x^{\frac{r+\delta_h}{\delta_h}})^{\frac{\gamma}{1-\gamma}} dx\}, \qquad a \in [6+s, R).$$

The stock of human capital at age 6, h_E, is

$$h_E = \upsilon^{\upsilon} h_B \left[\frac{\gamma_1^{\gamma_1(1-\gamma_2)} \gamma_2^{\gamma_1\gamma_2} z_h^{\gamma_1} w^{(1-\gamma_1)(1-\gamma_2)}}{(r+\delta_h)^{(1-\gamma_2)}}\right]^{\frac{\upsilon}{1-\gamma}} \tag{49}$$

$$e^{-\upsilon(r+\delta_h(1-\gamma_1))s} m(6+s)^{\frac{\upsilon(1-\gamma_2)}{1-\gamma}}$$

while the supply of human capital to the market by an individual of age a (for $a \geq 6+s$) is

$$h(a)(1-n(a)) = C_h(z_h, w, r) w\{\gamma_1 e^{-\delta_h(a-6-s)} \frac{m(6+s)^{\frac{1}{1-\gamma}}}{r+\delta_h} \tag{50}$$

$$-\gamma_1 \frac{m(a)^{\frac{1}{1-\gamma}}}{r+\delta_h} + \frac{e^{-\delta_h(a-R)}}{\delta_h} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} [(1 - x^{\frac{r+\delta_h}{\delta_h}}]^{\frac{\gamma}{1-\gamma}} dx\}$$

where

$$C_h(z_h, w, r) = \left[\frac{\gamma_2^{\gamma_2} \gamma_1^{\gamma_1} z_h w^{\gamma_2}}{(r+\delta_h)^{\gamma}}\right]^{\frac{1}{1-\gamma}}.$$

It follows that income during the working years satisfies

$$y(a) = C_h(z_h, w, r) w\{\gamma_1 e^{-\delta_h(a-6-s)} \frac{m(6+s)^{\frac{1}{1-\gamma}}}{r+\delta_h} - (\gamma_1+\gamma_2) \frac{m(a)^{\frac{1}{1-\gamma}}}{r+\delta_h} +$$

$$\frac{e^{-\delta_h(a-R)}}{\delta_h} \int_{e^{\delta_h(6+s-R)}}^{e^{\delta_h(a-R)}} [(1 - x^{\frac{r+\delta_h}{\delta_h}}]^{\frac{\gamma}{1-\gamma}} dx\}$$

during the working life (i.e. $6+s < a < R$). This expression shows that net income depends in a complicated way on the interest rate but it is independent of life expectancy.

Note. Health in the Utility Function: An Example Consider this example: the utility function is given by

$$u(c,g) = \frac{\left(c^{\zeta} g^{1-\zeta}\right)^{1-\theta}}{1-\theta}.$$

Assume that to reach this level of "health" the cost, in terms of goods is $g^0 g^{1+\psi}$. In this case, the optimal choice of g requires that

$$\int_0^T e^{-\rho(a-I)} u_g(c(a), g) da = \Phi(1+\psi) g^0 g^{\psi}.$$

For the particular example, this corresponds to

$$\frac{1-\zeta}{g} \int_0^T e^{-\rho(a-I)} \left(c(a)^{\zeta} g^{1-\zeta}\right)^{1-\theta} da = \Phi(1+\psi) g^0 g^{\psi}.$$

It follows that utility is given by

$$\int_0^T e^{-\rho(a-I)} \frac{\left(c(a)^{\zeta} g^{1-\zeta}\right)^{1-\theta}}{1-\theta} da = \frac{\Phi(1+\psi) g^0 g^{1+\psi}}{(1-\zeta)(1-\theta)}.$$

Since the optimal choice of fertility requires that

$$\left[\int_0^T e^{-\rho(a-I)} \frac{\left(c(a)^{\zeta} g^{1-\zeta}\right)^{1-\theta}}{1-\theta} da\right] / \Phi = g^0 g^{1+\psi} - P(y; 0, R; \hat{r})$$

it is necessary that

$$P(y; 0, R; \hat{r}) = g^0 g^{1+\psi} - \frac{(1+\psi) g^0 g^{1+\psi}}{(1-\zeta)(1-\theta)} > 0.$$

Thus, a sufficient (and necessary) condition is

$$(1-\zeta)(1-\theta) > (1+\psi)$$

which is inconsistent with our assumptions. However, an alternative specification of the utility function might possibly work. To see this consider preferences give by,

$$u(c,g) = \frac{c^{1-\theta}}{1-\theta} + \upsilon_g\left(\frac{g^{1-\zeta}}{1-\zeta} + G\right),$$

where, as before, we assume that $0 < \theta < 1$, but $\zeta > 1$ (and $G > 0$). In this case, the optimal (from an individual point of view) choice of g satisfies (details missing).