Leverage Management

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Abstract

An asset manager trades off the benefits of higher leverage against the costs of adjusting leverage in order to mitigate expected losses due to insolvency. We explicitly calculate optimal dynamic incentive-compatible leverage policies in simple versions of this problem.

1 Introduction

We characterize optimal leverage management by an asset manager. Positions in a risky asset are financed at a constant interest rate. Subject to proportional transactions costs and solvency constraints (or minimum capital requirements), the fund manager adjusts leverage so as to maximize the present value of fees, which are paid over time at a constant fraction of the liquidation value of current funds under management. The investor liquidates the fund for exogenous reasons at a Poisson arrival time. We explicitly solve the optimal investment problem, assuming constant proportional transactions costs and *iid* asset returns. Under natural conditions, there is a constant critical leverage ratio, above which no trade is optimal. Below the critical leverage ratio, the manager buys assets as necessary to force leverage up to the critical level. The fee structure is shown to be incntive compatible, in that the manager invests optimally on behalf of the investor.

We focus solely on the impact of transactions costs and a given incentive-compatible management fee on the leverage policy of fund managers. We do not explore optimal incentive contracts in more general settings. For recent examples, see OuYang (2003) and the extensive literature that he cites, which does not focus on leverage.

2 Fund Investment

Cash may be borrowed or lent at a constant continuously compounding interest rate r. An asset is marked to market at a price S_t satisfying

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t,$$

for a constant mean appreciation rate μ and constant volatility $\sigma > 0$, where B is a standard Brownian motion that is a martingale with respect to a probability space $(\Omega, \mathcal{F}, P)$ and filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$ satisfying the usual conditions.

The ask (purchase) price of the asset at time t is γS_t; the bid (sale) price is δS_t, for constants γ and δ with $0 < \delta < 1 < \gamma$. The cumulative market values of assets purchased and sold by time t are W_t and U_t, respectively, for increasing adapted right-continuous left-limits processes W and U chosen by the fund manager. The fund manager receives incentive fees at a rate given by a constant fraction $a > 0$ of the current liquidation value of the fund, and finances the fee by liquidating both cash and the risky asset in proportion to the fund's positions. As we shall see, this fee structure and fee financing policy is incentive compatible, under conditions on the investor's liquidation timing. (See Hugonnier and Kaniel (2004) for incentive compatibility for certain cases of endogenous changes in investments by the investor.) The fund is initiated with a position x in cash and y in assets. Given an investment policy (U, W), the market values $Y_t^{U,W}$ of asset and $X_t^{U,W}$ of cash held by the fund at time t therefore satisfy

$$\begin{aligned} dX_t^{U,W} &= rX_t^{U,W}\,dt - aX_t^{U,W}\,dt + \delta\,dU_t - \gamma\,dW_t, && (2.1)\\ X_{0-}^{U,W} &= x, \\ dY_t^{U,W} &= Y_t^{U,W}(\mu\,dt + \sigma\,dB_t) - aY_t^{U,W}\,dt - dU_t + dW_t, && (2.2)\\ Y_{0-}^{U,W} &= y. \end{aligned}$$

This allows for an instantaneous shift in the portfolio at any time, including for example time zero, at which the initial risky asset position changes from y to $Y_0^{U,W} = y - \Delta U_0 + \Delta W_0$, where ΔU_t and ΔW_t denote the jumps of U and W at time t.

The investment policy (U, W) determines an insolvency time

$$\tau(U, W) = \inf\{t \geq 0 : X_t^{U,W} + \delta Y_t^{U,W} \leq 0\}, \tag{2.3}$$

at which the only feasible policy that avoids a net negative liquidation value is to immediately liquidate the fund, leaving $X_t^{U,W} = Y_t^{U,W} = 0$ for $t > \tau(U, W)$.

3 Incentives

A risk-neutral investor relies on the asset manager for access to the investment strategy, and forces liquidation of the strategy at a Poisson arrival time T, independent of the risky asset returns (that is, independent of B), with constant mean arrival rate[1] λ, assuming that the fund has not already been liquidated at $\tau(U, W)$ because of insolvency. In any case, the investor receives $X_T^{U,W} + \delta Y_T^{U,W}$ at T, and values the investment policy (U, W) at

$$\Pi(x, y, U, W) = E\left[e^{-rT}(\delta Y_T^{U,W} + X_T^{U,W})\right].$$

[1]To be precise, T is exponentially distributed, independent of B, such that $\{1_{\{T \geq t\}} - 1_{\{T<t\}}\lambda t : t \geq 0\}$ is an $(\mathcal{F}_t)$-martingale.

The fund manager's valuation of the fees is

$$J(x,y,U,W) = E\left(\int_0^T e^{-rt} a\left(\delta Y_t^{U,W} + X_t^{U,W}\right) dt\right). \tag{3.1}$$

The incentives of the fund manager and investor are aligned, according to the following result.

Proposition 3.1 *For all* (x,y,U,W)*,*

$$\Pi(x,y,U,W) = \frac{\lambda}{a} J(x,y,U,W).$$

That is, the stipulated fee structure has the property that the manager and investor have identical ordinal preferences over investment policies. A proof is given in the appendix. The same proof of alignment of incentives extends easily to the case of a bounded intensity process $\{\lambda_t : t \geq 0\}$ for the investor's liquidation time[2] T, with a management fee paid at a rate given by a fraction of the liquidation value $X_t^{U,W} + \delta Y_t^{U,W}$ that is of the fund of the form $a_t = k\lambda_t$, for a constant $k > 0$. The associated extended definitions of $X^{U,W}$ and $Y^{U,W}$ are given by the obvious adjustments of (2.1)-(2.2) for a_t in place of a. If λ is correlated with risky asset returns, however, the analysis of optimal leverage is substantially more complicated, and we have not examined this case.

Incentive compatibility also extends to cases in which the fund manager's fee is in the form of a two-part tarriff, one part paid at an ongoing rate given by fraction a_0 of the current valuation $\delta Y_t + X_t$, the other given by a fraction b of the final liquidation value $\delta Y_T + X_T$, paid in a lump sum at time T, in which case one can apply our current analysis of optimal leverage for the all-in effective ongoing fee rate $a = a_0 + b\lambda$.

Our results also apply, with an appropriate adjustment of coefficients, if the investor makes partial liquidations, whose fractional sizes are independent of risky-asset returns, at independent Poisson arrival times $T_1, T_2, \ldots$. In general, the independence of the liquidation timing of the investor and fund performance is unrealistic, although of increasing relevance if the fund

[2]This means that liquidation by the investor occurs at the stopping time $T = \inf\{t : \int_0^t \lambda_s \, ds = Z\}$, where Z is an standard exponential variable independent of B, so that T is the first event time of a non-explosive counting process with intensity λ.

constitutes a small fraction of the investor's financial resources, with returns somewhat unrelated to the investor's other resources. Our results also extend to injection of new capital into the funds under management provided this occurs independently of returns, *and* provided that the new funds are required to be allocated so as to maintain the pre-injection asset-debt ratio of the fund. This highly unrealistic restriction implies that the fund manager can ignore the potential impact of anticipated capital injections on the investment policy. We have not analyzed the implications of anticipated capital injections beyond this artificially restricted case.

If the investor is risk-averse, with utility of the form $E[e^{-rT}u(\delta Y_T + X_T)]$ for some concave $u(\,\cdot\,)$, then an incentive-compatible fee is obtained by payments to the fund manager at a rate per unit time given by a fraction of $u(\delta Y_t + X_t)$, paid continually. With the particular case $u(w) = w^\eta$, for some $\eta \in (0, 1)$, our analysis extends in a straightforward way, exploiting the homogeneity of this utility, and similarly explicit solutions arise. (The Hamilton-Jacobi-Bellman equation in this case is similar to that of Davis and Norman (1990).) We avoid this case mainly because such non-linear incentive structures are rare in practice.

4 Optimal Strategy

The fund manager's problem is to find an investment policy (U^*, W^*) achieving the optimum

$$V(x, y) = \sup_{U,W} \; J(x, y, U, W). \tag{4.1}$$

Inspired by Davis and Norman (1990), we will exploit homogeneity of $V(\,\cdot\,)$, conjecturing its parametric form by analysis of the Hamilton-Jacobi-Bellman (HJB) equation associated with (4.1), and then verifying this conjecture, as well as the optimality of the policies associated with the HJB equation, by relatively standard martingale arguments.

We assume throughout that

$$r - a < \mu - a < r + \lambda. \tag{4.2}$$

Without the condition $r < \mu$, investment in the risky asset is unattractive (in the absence of hedging motives). Without the condition $\mu - a < r + \lambda$, investment in the risky asset alone allows infinite value $V(x, y) = +\infty$, whenever $x + \delta y > 0$.

4.1 The HJB Equation

We anticipate that levering the fund is optimal, so begin with the case $x < 0$ and $y > 0$, for an asset-debt ratio $\xi = -y/x$. We conjecture a constant critical asset-debt ratio $\xi_0 > 0$ with the property that, for $\xi < \xi_0$, no trade is optimal, and for $\xi \geq \xi_0$, it is optimal to buy assets instantly, as necessary to maintain an asset-debt ratio below ξ_0. We also conjecture that the fund manager sells assets only when forced to, that is, when $X_t + \delta Y_t = 0$, at which time the entire portfolio must be liquidated in order to maintain solvency. We define the asset-debt ratio intervals $H = [\delta^{-1}, \xi_0)$ and $I = [\xi_0, \infty)$. Because $P(T > t) = e^{-\lambda t}$, we can write

$$J(x, y, U, W) = E\left(\int_0^\infty e^{-\rho t} a\left(\delta Y_t^{U,W} + X_t^{U,W}\right) dt\right), \tag{4.3}$$

where $\rho = \lambda + r$.

For $-y/x$ in the no-trade interval H, assuming sufficient smoothness (which will be verified), the HJB equation associated with (4.1) implies that

$$(r-a)xV_x(x,y)+(\mu-a)yV_y(x,y)+\frac{\sigma^2}{2}y^2V_{yy}(x,y)-\rho V(x,y)+a(\delta y+x) = 0. \tag{4.4}$$

Based on the homogeneity of V, we let

$$f(\xi) = \frac{V(x,y)}{-x} = V\left(-1, \frac{y}{-x}\right), \tag{4.5}$$

from which we recover

$$V(x,y) = (-x)f(\xi). \tag{4.6}$$

Plugging (4.5) into (4.4), we conjecture that in H,

$$\xi^2 f''(\xi) + \frac{2(\mu - r)}{\sigma^2}\xi f'(\xi) - \frac{2(\rho - r + a)}{\sigma^2} f(\xi) + \frac{2a(\xi - 1)}{\sigma^2} = 0. \tag{4.7}$$

Solving, we conjecture that, in H, we have $f(\xi) = g(\xi; C_1, C_2)$, where, for coefficients C_1 and C_2 to be determined, $g(\,\cdot\,; C_1, C_2) : H \to \mathbb{R}$ is defined by

$$g(\xi; C_1, C_2) = C_1\xi^{w_1} + C_2\xi^{w_2} + \frac{a}{\rho - \mu + a}\delta\xi - \frac{a}{\rho - r + a}, \tag{4.8}$$

where

$$w_1 = \frac{-b + \sqrt{b^2 + 8\sigma^2(\rho - r + a)}}{2\sigma^2} \tag{4.9}$$

$$w_2 = \frac{-b - \sqrt{b^2 + 8\sigma^2(\rho - r + a)}}{2\sigma^2}, \tag{4.10}$$

for $b = 2(\mu - r) - \sigma^2$. In the asset-debt interval I in which buying assets is conjectured, the reduced value function $f(\,\cdot\,)$ is conjectured to be linear in ξ, with

$$f(\xi) = \kappa\left(\xi - \frac{1}{\gamma}\right), \tag{4.11}$$

for a coefficient κ to be determined. Given that the fund must be liquidated when $X_t + \delta Y_t = 0$, the usual smooth-fit conditions would then suggest that

$$g\left(\frac{1}{\delta}; C_1, C_2\right) = 0, \tag{4.12}$$

$$g(\xi_0; C_1, C_2) = \kappa\left(\xi_0 - \frac{1}{\gamma}\right), \tag{4.13}$$

$$g'(\xi_0; C_1, C_2) = \kappa, \tag{4.14}$$

$$g''(\xi_0; C_1, C_2) = 0. \tag{4.15}$$

With (4.8), these 4 equations (4.12)-(4.15) are to be solved for the 4 unknowns: C_1, C_2, κ, and ξ_0. There need not be explicit solutions, although we will give conditions for the existence of solutions, and show how to compute them. We say that ξ_0 solves (4.12)-(4.15) if there exist C_1, C_2, and κ such that $(C_1, C_2, \kappa, \xi_0)$ solve (4.12)-(4.15).

Provided

$$\frac{\gamma}{\delta} \geq \frac{\lambda + a}{\lambda + r - \mu + a},$$

it turns out ξ_0 is infinity or has no solution. Intuitively, this means that one should never increase leverage. In this case, we conjecture that

$$f(\xi) = C_2\xi^{w_2} + \frac{a}{\rho - \mu + a}\delta\xi - \frac{a}{\rho - r + a},$$

with the boundary condition,

$$f\left(\frac{1}{\delta}\right) = 0.$$

So, for this somewhat degenerate case,

$$C_2 = -\delta^{w_2} a \left(\frac{1}{\rho - \mu + a} - \frac{1}{\rho - r + a} \right),$$

and we conjecture that

$$f(\xi) = -\delta^{w_2} a \left(\frac{1}{\rho - \mu + a} - \frac{1}{\rho - r + a} \right) \xi^{w_2} + \frac{a}{\rho - \mu + a} \delta \xi - \frac{a}{\rho - r + a}. \tag{4.16}$$

4.2 Optimal Trading Strategy

We next verify the proposed optimal control, which is to buy the risky asset minimally, and as necessary to maintain $\xi \leq \xi_0$, where ξ_0 solves (4.12)-(4.15). To be precise, for each (x, y), we say that (U, W) enforces an upper bound ξ_0 on the asset-debt ratio if $-Y_t^{U,W}/X_t^{U,W} \leq \xi_0$ for all t, and that (U, W) is minimal in this regard if $U = 0$ and $W_t \leq \hat{W}_t$ for all t, for any other policy $(\hat{U}, \hat{W})$ that enforces the upper bound ξ_0. The existence of a minimally enforcing policy follows from the existence of solutions to standard Skorohod problems. The solution is $U_t = 0$ for all t and *(i)* $W(0) = 0$ if $-y/x \geq \xi_0$, and otherwise $W(0)$ solves $-(y + W(0))/(x - \gamma W(0)) = \xi_0$, and *(ii)* for $t > 0$, reflection of $-Y_t^{U,W}/X_t^{U,W}$ at ξ_0.

There are two subtleties. First, as we mentioned earlier, for some parameters (for example, high transactions costs), no trade is optimal, or there is no solution for the critical asset-debt ratio ξ_0. Second, for some parameter settings (for example, sufficiently low volatility σ), the value $J(x, y, U, W)$ can be made infinite, even under the parameter restriction $r - a < \mu - a < r + \lambda$. To be more precise, define

$$\gamma_L = \frac{w_1 w_2}{(w_1 - 1)(w_2 - 1)} \left(\frac{w_1(w_1 - 1)}{w_2(w_2 - 1)} \right)^{\frac{1}{w_1 - w_2}} \delta,$$

and

$$\gamma_H = \frac{\lambda + a}{\lambda + r - \mu + a} \delta.$$

If $\gamma \geq \gamma_H$ there is no solution to (4.12)-(4.15) for ξ_0. If $\gamma_L < \gamma < \gamma_H$, there is a solution ξ_0 to (4.12)-(4.15). Finally, if $\gamma < \gamma_L$, $J(x, y, U, W)$ can be made infinite. We will have further discussion of these issues later in this section.

A policy (U^*, W^*) is optimal at (x, y) if $J(x, y, U^*, W^*) = V(x, y)$. Our main optimality result is as follows.

Theorem 4.1 *If $\gamma \geq \gamma_H$, then the optimal policy is no trade ($U^* = W^* = 0$). If $\gamma_L < \gamma < \gamma_H$, there is a unique solution ξ_0 to (4.12)-(4.15), and, for any (x, y), the unique optimal policy (U^*, W^*) minimally enforces a fixed lower bound on the fund leverage $\delta Y_t/(\delta Y_t + X_t)$.*

The upper bound ξ_0 on the ratio of assets to debt is equivalent to a lower bound $\delta\xi_0/(\delta\xi_0 - 1)$ on leverage. Figure 1 shows, for a particular illustrative parametric example, that as the transactions cost coefficient γ gets small, the lower bound on leverage increases. Figure 2 shows that the leverage lower-bound decreases with σ. Adapting terminology from the credit-risk literature, we let $\delta\xi/\sigma(\delta\xi - 1)$ be the "distance to insolvency," meaning the number of standard deviations that assets would need to instantly fall in order to reach insolvency. Figure 3 shows the relationship between the policy maximal distance to insolvency and volatility σ. In Figure 4, we see how the market value of the fund management fees responds to changes in transactions costs.

Theorem 4.1 is proved through the following propositions.[3] If $\gamma_L < \gamma < \gamma_H$, we let $\hat{V}(x, y) = -xg(y/x; C_1, C_2)$ for $-y/x$ in H, and $\hat{V}(x, y) = \kappa(-y/x - \delta/\gamma)$ for $-y/x$ in I, where $(C_1, C_2, \kappa, \xi_0)$ solve (4.12)-(4.15).

Proposition 4.2 *If $\gamma_L < \gamma < \gamma_H$, then $\hat{V}(x, y) \geq J(x, y, U, W)$ for any (x, y, U, W).*

Proposition 4.3 *If $\gamma_L < \gamma < \gamma_H$, then, for any (x, y), $\hat{V}(x, y) = J(x, y, U^*, W^*)$, where (U^*, W^*) minimally enforces the upper bound ξ_0 on the asset-debt ratio.*

4.3 Optimal Startup

At the inception of the investment, there are in all four possible ways to initiate the fund with the investor's cash: (i) stay in cash; (ii) use some of

[3] Note for authors: We may need additional propositions and proofs for the case of $\gamma > \gamma_H$, but this task is easy given our proofs of the following two propositions.

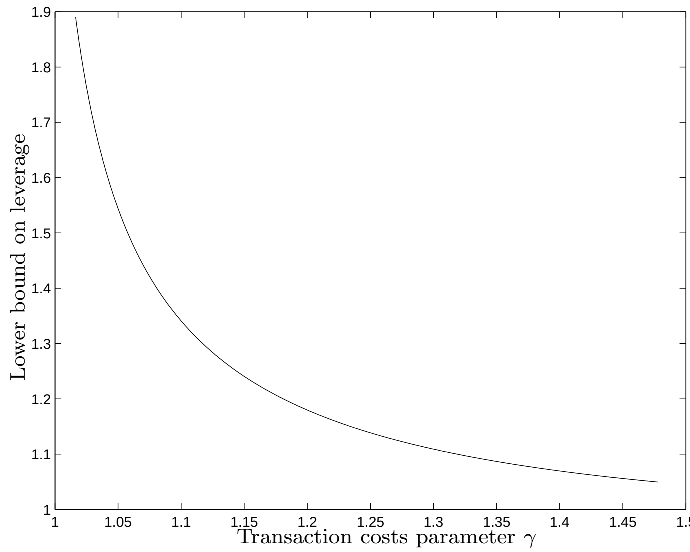

Figure 1: The lower bound $\delta\xi_0/(\delta\xi_0 - 1)$ on leverage, as it depends on the transactions cost parameter γ. Parameters: $\delta = 0.98; \sigma = 0.4; r = 0.02; \lambda = 0.05; \mu = 0.05; a = 0.005$.

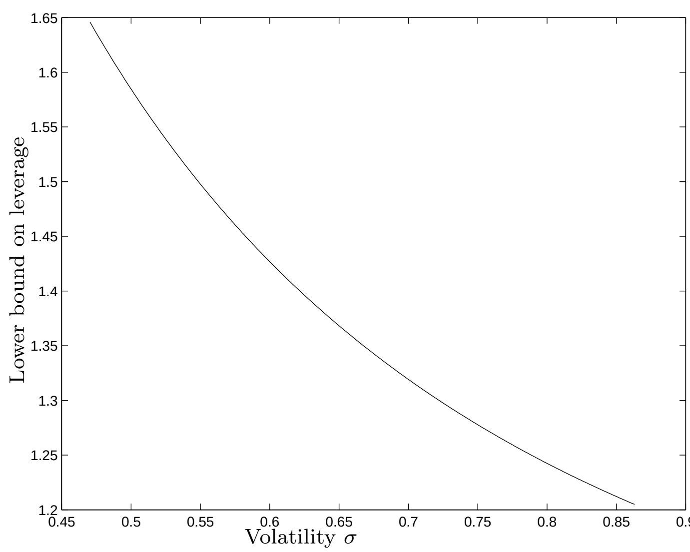

Figure 2: The lower bound $\delta\xi_0/(\delta\xi_0 - 1)$ on leverage, as it depends on the volatility parameter σ. Parameters: $\delta = 0.98; \gamma = 1.02; r = 0.02; \lambda = 0.05; \mu = 0.05; a = 0.005$.

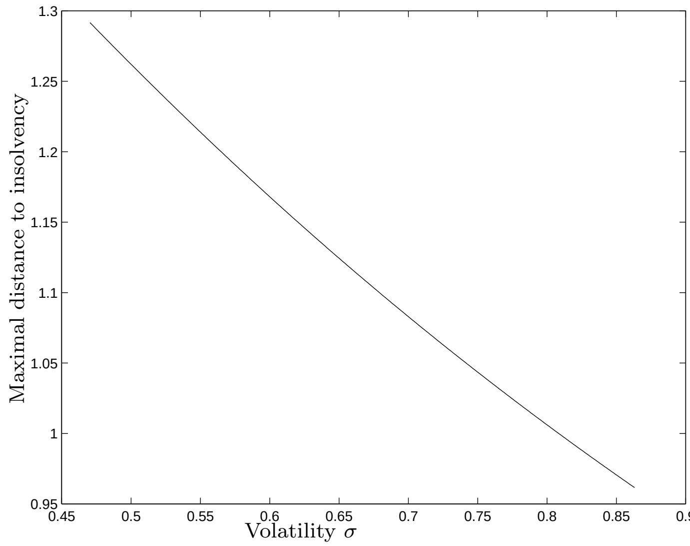

Figure 3: Policy maximal distance to insolvency, $(\delta\xi_0 - 1)/(\sigma\delta\xi_0)$, as it depends on volatility. Parameters: $\delta = 0.98; \gamma = 1.02; r = 0.02; \lambda = 0.05; \mu = 0.05; a = 0.005$.

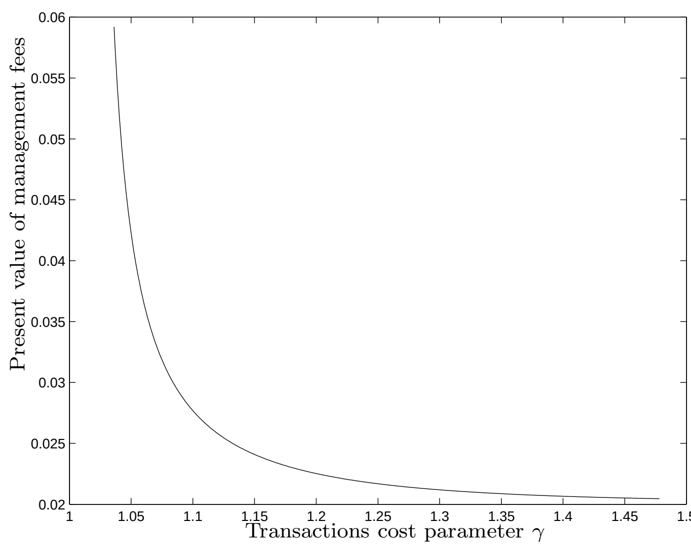

Figure 4: Market value of a fund manager with 1.1 dollars of assets and 1 dollar debt, as it depends on transactions costs parameter. Parameters: $\delta = 0.98; \sigma = 0.4; r = 0.02; \lambda = 0.05; \mu = 0.05; a = 0.005$.

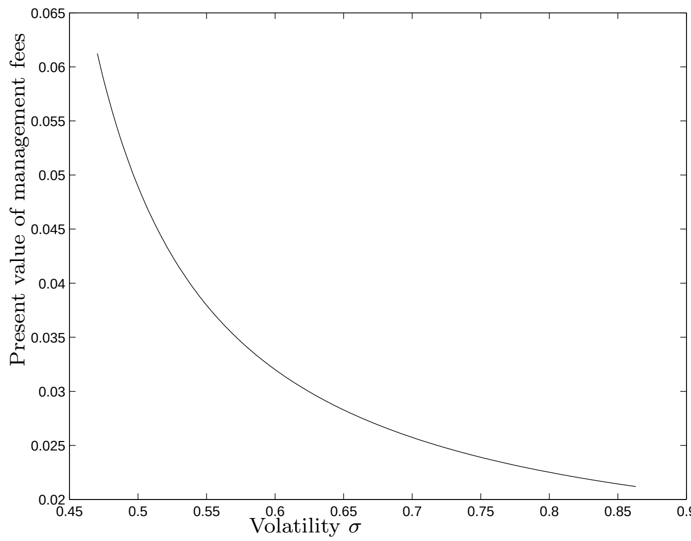

Figure 5: Market value of a fund manager with 1.1 dollars of assets and 1 dollar debt, as it depends on volatility parameter. Parameters: $\delta = 0.98; \gamma = 1.02; r = 0.02; \lambda = 0.05; \mu = 0.05; a = 0.005$.

the cash to buy the risky asset; (iii) invest all of the cash in risky assets; (iv) invest all of the cash, as well as borrowed cash in risky assets.

If $\gamma \geq \gamma_H$, it is optimal to stay in cash; if $\gamma < \gamma_H$, then the portfolio should be brought immediately to the optimal leverage $-Y(0)/X(0) = \xi_0$. If, however, staying in cash is in fact optimal, then it would not have been optimal for the investor to pay a fee for fund management (assuming that the investor has access to risk-free investments). So, the case of interest is $\gamma < \gamma_H$, implying leverage.

4.4 Remarks

4.4.1 Sub-Optimal Strategies

Equations (4.12)-(4.14) hold for any strategy, optimal or not.

4.4.2 Infinite Current Value

Theorem 4.4 *If $\gamma \leq \gamma_L$, then $V(x, y) = +\infty$.*

In this case, speaking intuitively, even though $\mu - a < \rho$, the fund manager prefers to push up leverage without bound.

5 Numerics

We can rewrite (4.12)-(4.15) as

$$\begin{aligned}
C_1 \left(\frac{1}{\delta}\right)^{w_1} + C_2 \left(\frac{1}{\delta}\right)^{w_2} + \frac{a}{\rho - \mu + a} - \frac{a}{\rho - r + a} &= 0 \\
C_1 \xi_0^{w_1} + C_2 \xi_0^{w_2} + \frac{a}{\rho - \mu + a} \delta \xi_0 - \frac{a}{\rho - r + a} &= \kappa \left(\xi_0 - \frac{1}{\gamma}\right) \\
C_1 w_1 \xi_0^{w_1 - 1} + C_2 w_2 \xi_0^{w_2 - 1} + \frac{a}{\rho - \mu + a} \delta &= \kappa \\
C_1 w_1 (w_1 - 1) \xi_0^{w_1 - 2} + C_2 w_2 (w_2 - 2) \xi_0^{w_2 - 2} &= 0.
\end{aligned}$$

For given parameters $r, \mu, \rho, \sigma, a, \delta, \gamma$, we can solve for the four unknowns C_1, C_2, κ, ξ_0 numerically.

6 Conclusion

In this paper, we solve for a risk-neutral fund manager's optimal leverage policy. The assumptions are: fixed percentage management fee, fixed percentage trading friction, one risky asset with constant volatility and expected return, a constant risk-free rate, and a risk-neutral investor. We studied an asset-management fee under which the incentives of the investor and the fund manager are aligned. For all cases in which transactions costs are small enough to justify fund management, positive leverage is optimal, and leverage is adjusted whenever it drops below a critical level, in order to maintain at least the critical leverage.

7 Appendix: Proofs

7.1 Proof of Proposition 3.1

We have

$$\begin{aligned}
\Pi(x,y,U,W) &= E\left[e^{-rT}(\delta Y_T - X_T)\right] \\
&= E\left[E\left[e^{-rT}(\delta Y_T - X_T) \;\middle|\; T\right]\right] \\
&= \int_0^\infty e^{-\lambda t}\lambda E[e^{-rt}(\delta Y_t - X_t)]\,dt \\
&= \lambda \int_0^\infty e^{-(\lambda+r)t} E(\delta Y_t - X_t)\,dt \\
&= \lambda \int_0^\infty e^{-\rho t} E(\delta Y_t - X_t)\,dt. \qquad (7.1)
\end{aligned}$$

We also have

$$\begin{aligned}
J(x,y,U,W) &= E\left[\int_0^T e^{-rt} a(\delta Y_t - X_t)\,dt\right] \\
&= E\left[a\int_0^\infty e^{-rt} 1_{\{T>t\}}\,(\delta Y_t - X_t)\,dt\right] \\
&= a\int_0^\infty E\left[1_{\{T>t\}}\right] e^{-rt} E[\delta Y_t - X_t]\,dt \\
&= a\int_0^\infty e^{-\lambda t} e^{-rt} E[\delta Y_t - X_t]\,dt \\
&= a\int_0^\infty e^{-\rho t} E[\delta Y_t - X_t]\,dt. \qquad (7.2)
\end{aligned}$$

It follows that

$$\Pi(x,y,U,W) = \frac{\lambda}{a} J(x,y,U,W).$$

Lemma 7.1 *The constants w_1 and w_2 satisfy $w_1 > 1, w_2 < 0$,*

$$0 < -\frac{w_1}{w_2 - 1} < 1,$$

and

$$0 < -\frac{w_1 - 1}{w_2} < 1.$$

Proof We have

$$\begin{aligned}
&\rho > \mu - a \\
\Rightarrow\ &\rho - r + a > \mu - r \\
\Rightarrow\ &8\sigma^2(\rho - r + a) + (\sigma^2 - 2(\mu - r))^2 > 8\sigma^2(\mu - r) + (\sigma^2 - 2(\mu - r))^2 \\
\Rightarrow\ &\sqrt{8\sigma^2(\rho - r + a) + (\sigma^2 - 2(\mu - r))^2} > \sigma^2 + 2(\mu - r) \\
\Rightarrow\ &\sqrt{8\sigma^2(\rho - r + a) + (\sigma^2 - 2(\mu - r))^2} + (\sigma^2 - 2(\mu - r)) > 2\sigma^2 \\
\Rightarrow\ &w_1 = \frac{(\sigma^2 - 2(\mu - r)) + \sqrt{(\sigma^2 - 2(\mu - r))^2 + 8\sigma^2(\rho - r + a)}}{2\sigma^2} > 1.
\end{aligned}$$

Similarly we can conclude that $w_2 < 0$.

We also have

$$-\frac{w_1}{w_2 - 1} = \frac{(\sigma^2 - 2(\mu - r)) + \sqrt{(\sigma^2 - 2(\mu - r))^2 + 8\sigma^2(\rho - r + a)}}{(\sigma^2 + 2(\mu - r)) + \sqrt{(\sigma^2 - 2(\mu - r))^2 + 8\sigma^2(\rho - r + a)}},$$

and it follows

$$0 < -\frac{w_1}{w_2 - 1} < 1.$$

Finally,

$$-\frac{w_1 - 1}{w_2} = \frac{(-\sigma^2 - 2(\mu - r)) + \sqrt{(\sigma^2 - 2(\mu - r))^2 + 8\sigma^2(\rho - r + a)}}{(-\sigma^2 + 2(\mu - r)) + \sqrt{(\sigma^2 - 2(\mu - r))^2 + 8\sigma^2(\rho - r + a)}}.$$

Note that $\rho + a > \mu$, we have

$$0 < -\frac{w_1 - 1}{w_2} < 1.$$

Lemma 7.2 *Suppose that $\gamma_L < \gamma < \gamma_H$, and let*

$$\xi_{0c} = \frac{1}{\delta}\left(\frac{w_2(w_2 - 1)}{w_1(w_1 - 1)}\right)^{\frac{1}{w_1 - w_2}}.$$

There exists a unique $\xi_0 > \xi_{0c}$ that solves (4.12)-(4.15). For this ξ_0, the associated constant C_2 is strictly negative.

Proof 7.2

First, we show that

$$\frac{w_1 w_2}{(w_1-1)(w_2-1)}\left(\frac{w_1(w_1-1)}{w_2(w_2-1)}\right)^{\frac{1}{w_1-w_2}} < \frac{\rho-r+a}{\rho-\mu+a}.$$

Some algebra leads to

$$\frac{w_1 w_2}{(w_1-1)(w_2-1)} = \frac{\rho-r+a}{\rho-\mu+a}.$$

Also, we have shown that

$$0 < -\frac{w_1}{w_2-1} < 1,$$

$$0 < -\frac{w_1-1}{w_2} < 1.$$

Because

$$\frac{1}{w_1-w_2} > 0,$$

we have

$$\left(\frac{w_1(w_1-1)}{w_2(w_2-1)}\right)^{\frac{1}{w_1-w_2}} < 1.$$

The desired result follows.

Plug (4.8) into (4.15), we have:

$$C_1 = -C_2 \xi_0^{w_2-w_1} \frac{w_2(w_2-1)}{w_1(w_1-1)}. \tag{7.3}$$

Further, plug (4.8) into (4.12), we have:

$$-C_2 \xi_0^{w_2-w_1} \frac{w_2(w_2-1)}{w_1(w_1-1)} \left(\frac{1}{\delta}\right)^{w_1} + C_2 \left(\frac{1}{\delta}\right)^{w_2} + \frac{a}{\rho-\mu+a} - \frac{a}{\rho-r+a} = 0.$$

Plug in the value of C_1 from (7.3), we have:

$$C_2 = \frac{-\left(\frac{a}{\rho-\mu+a} - \frac{a}{\rho-r+a}\right)}{1 - \frac{w_2(w_2-1)}{w_1(w_1-1)}(\xi_0 \delta)^{w_2-w_1}} \delta^{w_2}. \tag{7.4}$$

We have assumed that $\xi_0 > \xi_{0c}$, so

$$1 - \frac{w_2(w_2-1)}{w_1(w_1-1)}(\xi_0\delta)^{w_2-w_1} > 0,$$

this gives $C_2 < 0$.

The boundary conditions (4.13) and (4.14) give

$$g(\xi_0^-, C_1, C_2) = \kappa(\xi_0 - \frac{1}{\gamma}) = g_1(\xi_0^-, C_1, C_2)\left(\xi_0 - \frac{1}{\gamma}\right).$$

After some algebra, we can get:

$$\frac{1}{\gamma} = \frac{\frac{a}{\rho-r+a} + C_2\xi_0^{w_2}(w_2-1)\frac{w_1-w_2}{w_1}}{\frac{a\delta}{\rho-\mu+a} + C_2 w_2 \xi_0^{w_2-1}\frac{w_1-w_2}{w_1-1}}. \tag{7.5}$$

Now, we can study the change of the value γ as ξ_0 changes. Define:

$$\begin{aligned} F(\xi_0,\gamma) &= -\frac{a}{\rho-r+a} - C_2(\xi_0)\xi_0^{w_2}(w_2-1)\frac{w_1-w_2}{w_1} \\ &\quad + \frac{1}{\gamma}\left(\frac{a\delta}{\rho-\mu+a} + C_2(\xi_0)w_2\xi_0^{w_2-1}\frac{w_1-w_2}{w_1-1}\right). \end{aligned}$$

We observe from (7.5) that $F(\xi_0,\gamma) \equiv 0$. Plugging in C_2 as in (7.4),

$$\begin{aligned} F(\xi_0,\gamma) = &-\frac{a}{\rho-r+a} + \frac{\frac{a}{\rho-\mu+a} - \frac{a}{\rho-r+a}}{1-\frac{w_2(w_2-1)}{w_1(w_1-1)}(\xi_0\delta)^{w_2-w_1}}\delta^{w_2}\xi_0^{w_2}(w_2-1)\frac{w_1-w_2}{w_1} \\ &+ \frac{1}{\gamma}\left(\frac{a\delta}{\rho-\mu+a} - \frac{\frac{a}{\rho-\mu+a} - \frac{a}{\rho-r+a}}{1-\frac{w_2(w_2-1)}{w_1(w_1-1)}(\xi_0\delta)^{w_2-w_1}}\delta^{w_2}w_2\xi_0^{w_2-1}\frac{w_1-w_2}{w_1-1}\right). \end{aligned}$$

We first calculate $\partial F/\partial\gamma$:

$$\frac{\partial F}{\partial \gamma} = -\frac{1}{\gamma^2}\left(\frac{a\delta}{\rho-\mu+a} + C_2(\xi_0)\delta^{w_2}w_2\xi_0^{w_2-1}\frac{w_1-w_2}{w_1-1}\right). \tag{7.6}$$

We know $\rho > \mu - a, \mu > r, w_1 > 1, w_2 < 0$ and $C_2 < 0$, we can conclude that $\partial F/\partial\gamma < 0$.

In the following, we calculate $\partial F/\partial \xi_0$. After some complicated algebra, we get

$$\begin{aligned}\frac{\partial F}{\partial \xi_0} &= C_2\frac{(1-w_2)(w_1-w_2)}{w_1}\xi_0^{w_2-1}\left(1+\frac{\frac{(w_2-w_1)(w_2-1)}{w_1(w_1-1)}(\delta\xi_0)^{w_2-w_1}}{1-\frac{w_2(w_2-1)}{w_1(w_1-1)}(\delta\xi_0)^{w_2-w_1}}\right.\\ &\quad \left.-\frac{1}{\gamma}w_2\xi_0^{-1}\frac{1}{w_1-1}\left(1+\frac{\frac{w_2(w_2-w_1)}{w_1(w_1-1)}(\delta\xi_0)^{w_2-w_1}}{1-\frac{w_2(w_2-1)}{w_1(w_1-1)}(\delta\xi_0)^{w_2-w_1}}\right)\right)\\ &= \frac{C_2\frac{(1-w_2)(w_1-w_2)}{w_1}\xi_0^{w_2-1}}{1-\frac{w_2(w_2-1)}{w_1(w_1-1)}(\delta\xi_0)^{w_2-w_1}}\left(1-\frac{w_2-1}{w_1-1}(\delta\xi_0)^{w_2-w_1}\right.\\ &\quad \left.-\frac{1}{\gamma}w_2\xi_0^{-1}\frac{1}{w_1(w_1-1)}\left(w_1-w_2(\delta\xi_0)^{w_2-w_1}\right)\right). \qquad (7.7)\end{aligned}$$

In (7.7), we have $C_2<0, w_1>1, w_2<0$ and $1-(w_2(w_2-1))/(w_1(w_1-1))>0$, so we conclude that $\partial F/\partial \xi_0>0$.

By definition, $F(\xi_0,\gamma)\equiv 0$, therefore

$$\frac{\partial F}{\partial \xi_0}+\frac{\partial F}{\partial \gamma}\frac{d\gamma}{d\xi_0}=0.$$

In addition, recall that $\partial F/\partial\gamma<0$ and $\partial F/\partial \xi_0>0$, we can conclude

$$\frac{d\gamma}{d\xi_0}>0.$$

Note here $d\gamma/d\xi_0$ has an analytical expression, being $\partial F/\partial \xi_0$ as in (7.7) divided by $\partial F/\partial\gamma$ as in (7.6). It is easy to see that if $\xi_0>\xi_{0c}$, $d\gamma/d\xi_0$ exists.

As $\xi_0\to\xi_{0c}^+$, and substituting in (7.4), we have $C_2\to-\infty$. We know that

$$\frac{1}{\gamma}=\frac{\frac{1}{C_2}\frac{a}{\rho-r+a}+\xi_0^{w_2}(w_2-1)\frac{w_1-w_2}{w_1}}{\frac{1}{C_2}\frac{a\delta}{\rho-\mu+a}+w_2\xi_0^{w_2-1}\frac{w_1-w_2}{w_1-1}}. \qquad (7.8)$$

Therefore,

$$\lim_{\xi_0\to\xi_{0c}^+}\frac{1}{\gamma}\to\frac{\xi_{0c}(w_2-1)(w_1-1)}{w_1w_2}=\frac{1}{\gamma_L}. \qquad (7.9)$$

On the other extreme, as $\xi_0\to\infty$, from (7.4), we have

$$C_2\to-\left(\frac{a}{\rho-\mu+a}-\frac{a}{\rho-r+a}\right)\delta w_2.$$

Plugging in (7.8), we conclude:

$$\lim_{\xi_0\to\infty}\frac{1}{\gamma}=\left(\frac{1}{\delta}\right)\frac{\rho-\mu+a}{\rho-r+a}=\frac{1}{\gamma_H}. \tag{7.10}$$

Recall in the range $\xi_0>\xi_{0c}$, as ξ_0 increases, $\gamma(\xi_0)$ monotone and continuously increases. As ξ_0 changes from ξ_{0c} to ∞, γ changes from γ_L to γ_H. So for a particular $\gamma\in(\gamma_L,\gamma_H)$, there exists some $\xi_0\in(\xi_{0c},\infty)$ that solves (4.12)-(4.15). ∎

Lemma 7.3 *If $\gamma_L<\gamma<\gamma_H$, then the function $f(\cdot)$ defined by (4.8)-(4.15) is concave in $[1/\delta,\infty)$. Moreover, $C_1>0$.*

Proof 7.3 In $[\xi_0,\infty)$, $f''(\xi)=0$. Because $f(\cdot)$ is C_1, we need only prove that $f''(\xi)\le 0$ on $[1/\delta,\xi_0)$. We have

$$\begin{aligned} f''(\xi) &= C_1w_1(w_1-1)\xi^{w_1-2}+C_2w_2(w_2-1)\xi^{w_2-2}\\ &= C_1w_1(w_1-1)\xi^{w_2-2}\left(\xi^{w_1-w_2}+\frac{C_2w_2(w_2-1)}{C_1w_1(w_1-1)}\right). \end{aligned}$$

It is clear that the equation $f''(\xi)=0$ has only one possible strictly positive solution in ξ. We know that $f''(\xi_0)=0$, hence

$$\xi_0^{w_1-w_2}+\frac{C_2w_2(w_2-1)}{C_1w_1(w_1-1)}=0,$$

plus the properties of w_1 and w_2 in Lemma 7.1, implying that C_1 and C_2 are of opposite signs. We also know that $C_2<0$ from Lemma 7.2, so $C_1>0$.

Because

$$\xi^{w_1-w_2}+\frac{C_2w_2(w_2-1)}{C_1w_1(w_1-1)}$$

is increasing with respect to ξ, and because

$$\xi_0^{w_1-w_2}+\frac{C_2w_2(w_2-1)}{C_1w_1(w_1-1)}=0,$$

we know that, for $\xi\in[\frac{1}{\delta},\xi_0)$, we have

$$\xi^{w_1-w_2}+\frac{C_2w_2(w_2-1)}{C_1w_1(w_1-1)}<0.$$

From Lemma 7.1, $w_1>1$. Because we also have $C_1>0$, we conclude that, for $\xi\in[1/\delta,\xi_0)$, we have $f''(\xi)<0$. Therefore $f(\xi)$ is concave on $[1/\delta,\infty)$. ∎

Lemma 7.4 *If $\gamma_L < \gamma < \gamma_H$, then $\delta V_x - V_y \leq 0$ for $\xi \in [1/\delta, \infty)$.*

Proof For $\gamma_L < \gamma < \gamma_H$, $f(\cdot)$ is concave and C^2 on $[1/\delta, \infty)$. So, for $\xi \in (1/\delta, \infty)$, there exists some ξ_m in $(1/\delta, \xi)$ such that

$$f'(\xi_m) = \frac{f(\xi) - f(\frac{1}{\delta})}{\xi - \frac{1}{\delta}}.$$

Since $f(\xi)$ is concave, $f'(\xi) \leq f'(\xi_m)$. So,

$$\begin{aligned} & f'(\xi) \leq \frac{f(\xi) - f(\frac{1}{\delta})}{\xi - \frac{1}{\delta}} = \frac{f(\xi)}{\xi - \frac{1}{\delta}} \\ \Rightarrow \quad & \delta\left(-f(\xi) + \xi f'(\xi)\right) - f'(\xi) \leq 0 \\ \Rightarrow \quad & \delta V_x - V_y \leq 0. \end{aligned}$$

$\blacksquare$

Lemma 7.5 *If $\gamma_L < \gamma < \gamma_H$, then $V_y - \gamma V_x \leq 0$ for $\xi \in [1/\delta, \infty)$.*

Proof In the asset-debt interval I, $f(\xi) = \kappa(\xi - 1/\gamma) = f'(\xi)(\xi - 1/\gamma)$, so $V_y - \gamma V_x = 0$.

In the asset-debt interval H, $f(\xi) \leq \kappa(\xi - 1/\gamma)$. Otherwise, there is some ξ_m such that $f(\xi_m) > \kappa(\xi - 1/\gamma)$, which is not possible because a concave curve cannot have point that is above its tangent. So, in the interval H,

$$f'(\xi) \geq \kappa = \frac{\kappa(\xi - \frac{1}{\gamma})}{\xi - \frac{1}{\gamma}} \geq \frac{f(\xi)}{\xi - \frac{1}{\gamma}} \Rightarrow V_y - \gamma V_x \leq 0.$$

The desired result follows. $\blacksquare$

Lemma 7.6 *Suppose that $\gamma_L < \gamma < \gamma_H$. We define functions G and D by*

$$-xG(\xi) = D(x, y) = V_x r x + V_y \mu y + \frac{1}{2} V_{yy} \sigma^2 y^2 - \rho V + a(\delta y + x).$$

Then $xG(\xi) = D(x, y) \leq 0$.

Proof In the interval H, from (4.4), $D(x,y)=0$. In the interval I, we have $f(\xi)=\kappa(\xi-\delta/\gamma)$. Plugging this into $D(x,y)$, we have:

$$\begin{aligned} -xG(\xi) &= -x\left(\frac{\kappa r\delta}{\gamma}+\kappa\mu\xi+a(\delta\xi+1)-\kappa\rho\left(\xi-\frac{\delta}{\gamma}\right)\right) \\ &= -x\left((\kappa(\mu-\rho)+\delta a)\xi+\frac{\kappa r\delta}{\gamma}+\frac{\rho\kappa\delta}{\gamma}+a\right). \end{aligned}$$

Since $f(\,\cdot\,)$ is C^2, $G(\,\cdot\,)$ is continuous. So,

$$G(\xi_0^+)=G(\xi_0^-)=0.$$

That gives

$$(\kappa(\mu-\rho)+\delta a)\xi_0+\frac{\kappa r\delta}{\gamma}+\frac{\rho\kappa\delta}{\gamma}+a=0,$$

therefore

$$(\kappa(\mu-\rho)+\delta a)<0.$$

So $G'(\xi)<0$. From continuity, $G(\xi_0)=G(\xi_0^-)=0$, this implies that $G(\xi)\leq 0$ in the interval I. The result follows. ∎

Lemma 7.7 *Suppose that $\gamma_L<\gamma<\gamma_H$. Then, for the optimal control (U^*,W^*), we have, for any fixed time S,*

$$E\left[\int_0^S e^{-2\rho t}V_y\sigma^2\left(Y_t^{U^*,W^*}\right)^2 dt\right]<\infty.$$

Proof Recall that $V_y=f'(\xi)\leq f'(1/\delta)<\infty \Rightarrow$. Thus, for some constant K,

$$E\left[\int_0^S e^{-2\rho t}V_y\sigma^2\left(Y_t^{U^*,W^*}\right)^2 dt\right]<KE\left[\int_0^S \left(Y_t^{U^*,W^*}\right)^2 dt\right].$$

In the following part of the proof, we introduced a new stochastic process (X^N,Y^N), having the same outcomes as (X^*,Y^*) for $t<\tau$, but we take (X^N,Y^N) to the debt-asset process associated with never selling assets in order to decrease debt, even to avoid the insolvency condition $\delta Y_t^N<X_t^N$.

Rigorously, consider a stochastic process $(X^M((W^M)), Y^M((W^M)))$, with the associated control W^M, which satisfies

$$\begin{aligned} dX_t^M &= rX_t^M\,dt - \gamma dW_t^M, \quad X(0-) = x \\ dY_t^M &= \mu Y_t^M\,dt + \sigma Y_t^M\,dB_t - aY_t^M\,dt + dW_t^M, \quad Y(0-) = y. \end{aligned}$$

We define W^N to be the minimally enforcing policy which satisfies: *(i)* $W^N(0) = 0$ if $-y/x \geq \xi_0$, and otherwise $W^N(0)$ solves $-(y + W^N(0))/(x - \gamma W^N(0)) = \xi_0$, and *(ii)* for $t > 0$, reflection of $-Y_t^{W^N}/X_t^{W^N}$ at ξ_0. We also define $X^N = X^M(W^N)$ and $Y^N = Y^M(W^N)$. One trivial conclusion from the uniqueness of this stochastic process is that if the stopping time in (2.3) satisfies $\tau(\omega) > S$, then $Y_t^*(\omega) = Y_t^N(\omega)$ for any $t < S$, where Y^* is the asset process associated with the candidate optimal control (U^*, W^*).

Consider the stochastic process Z defined by

$$Z_t = \frac{(Y_t^N)^{\gamma\xi_0}}{-X_t^N}. \tag{7.11}$$

For $-Y_t^N/X_t^N < \xi_0$, we know that $dW^N = 0$, so

$$\begin{aligned} dZ_t &= \frac{(Y_t^N)^{\gamma\xi_0 - 1}}{-X_t^N}\gamma\xi_0 dY_t^N + \frac{(Y_t^N)^{\gamma\xi_0}}{(X_t^N)^2}dX_t^N \\ &= -\frac{(Y_t^N)^{\gamma\xi_0}}{X_t^N}\left((\gamma\xi_0\mu - r)\,dt + \gamma\xi_0\sigma\,dB_t\right) \\ &= Z_t\left((\gamma\xi_0\mu - r)\,dt + \gamma\xi_0\sigma\,dB_t\right). \end{aligned}$$

For $-Y_t^N/X_t^N = \xi_0$, because W_t^N is non-decreasing and continuous, we have:

$$\begin{aligned} dZ_t &= \frac{(Y_t^N)^{\gamma\xi_0 - 1}}{-X_t^N}\gamma\xi_0 dY_t^N + \frac{(Y_t^N)^{\gamma\xi_0}}{(X_t^N)^2}dX_t^N \\ &= \frac{(Y_t^N)^{\gamma\xi_0 - 1}}{-X_t^N}\gamma\xi_0\left(\mu Y_t^N\,dt + \sigma Y_t^N\,dB_t + dW_t^N\right) + \frac{(Y_t^N)^{\gamma\xi_0}}{(X_t^N)^2}\left(rX_t^N\,dt - \gamma dW_t^N\right) \\ &= Z_t\left((\gamma\xi_0\mu - r)\,dt + \gamma\xi_0\sigma\,dB_t\right). \end{aligned}$$

So we conclude that

$$dZ_t = Z_t\left((\gamma\xi_0\mu - r)\,dt + \gamma\xi_0\sigma\,dB_t\right). \tag{7.12}$$

Hence Z is a geometric Brownian motion.

For a specific ω, define $\tau_0(\omega) = \sup\{t, dW_t^N(\omega) \neq 0, t \leq S\}$. Then from τ_0 to S, $dW_t^N \equiv 0$. So we have $X_t^N = X_{\tau_0}^N e^{r(t-\tau_0)}$.

$$\begin{aligned}\max_{t\leq S} Z_t(\omega) &\geq Z_{\tau_0} = \left(-X_{\tau_0}^N\right)^{\gamma\xi_0-1} \xi_0^{\gamma\xi_0} = e^{-r(S-\tau_0)(\gamma\xi_0-1)} \left(-X_S^N\right)^{\gamma\xi_0-1} \xi_0^{\gamma\xi_0} \\ &\geq e^{-rS(\gamma\xi_0-1)}(-X_S^N)^{\gamma\xi_0-1}\xi_0^{\gamma\xi_0}.\end{aligned}$$

So,

$$-X_S^N(\omega) \leq C\left(\max_{t\leq S} Z_t(\omega)\right)^{\frac{1}{\gamma\xi_0-1}},$$

for some constant C.

Recall, from the definition of Z_t,

$$Y_t^N(\omega) = (-X_t(\omega)Z_t(\omega))^{\frac{1}{\gamma\xi_0}} \leq \left(C\left(\max_{t\leq S} Z_t(\omega)\right)^{\frac{1}{\gamma\xi_0-1}} Z_t(\omega)\right)^{\frac{1}{\gamma\xi_0}}$$

For a specific ω, for $Y_t(\omega)$, if $\tau > S$, then by definition, $Y_t(\omega) = Y_t^N(\omega)$. On the other hand, if insolvency happened before S, then $Y_S(\omega) = 0 \leq Y_S^N(\omega)$. So we have:

$$\begin{aligned}&Y_S(\omega) \leq Y_S^N(\omega) \\ \Rightarrow\ &(Y_S(\omega))^2 \leq \left(Y_S^N(\omega)\right)^2 \\ \Rightarrow\ &E\left[Y_S^2\right] \leq E\left[\left(Y_S^N\right)^2\right] \\ \Rightarrow\ &E\left[Y_S^2\right] \leq E\left[\left(C\left(\max_{t\leq S} Z_t\right)^{\frac{1}{\gamma\xi_0-1}} Z_S\right)^{\frac{2}{\gamma\xi_0}}\right].\end{aligned}$$

For convenience we define

$$H(S) = E\left[\left(C\left(\max_{t\leq S} Z_t\right)^{\frac{1}{\gamma\xi_0-1}} Z_S\right)^{\frac{2}{\gamma\xi_0}}\right].$$

We have proved that Z_t is a geometric Brownian motion with positive drift. So $H(\cdot)$ is some continuous increasing function. Using Fubini's theorem:

$$\begin{aligned}&E\left[\int_0^S \left(Y_t^{\{U^*,W^*\}}\right)^2 dt\right] = \int_0^S E\left[\left(Y_t^{\{U^*,W^*\}}\right)^2\right] dt \\ &\leq \int_0^S E\left[H(t)\right] dt \leq \int_0^S \left[H(S)\right] dt \leq SH(S) < \infty.\end{aligned}$$

The desired result follows. ∎

Lemma 7.8 *For $\gamma > \gamma_H$, we have: (i) $f(\cdot)$, defined by (4.16), is concave; $\delta V_x - V_y \leq 0$ for $\xi \geq 1/\delta$; $V_y - \gamma V_x \leq 0$ for $\xi \geq 1/\delta$; $V_x rx + V_y \mu y + 1/2 V_{yy}\sigma^2 y^2 - \rho V + a(\delta y + x) \leq 0$ for $\xi \geq 1/\delta$; and for the candidate optimal control (U^*, W^*) and any fixed time S,*

$$E\left[\int_0^S e^{-2\rho t} V_y \sigma^2 (Y_t^{U^*,W^*})^2 \, dt\right] < \infty.$$

Proof We prove the five claims as follows.
1. [Concavity] We have

$$f''(\xi) = C_2 w_2 (w_2 - 1)\xi^{w_2 - 2}.$$

and

$$C_2 = -\delta^{w_2} a \left(\frac{1}{\rho - \mu + a} - \frac{1}{\rho - r + a}\right) < 0.$$

Therefore $f''(\xi) < 0$ for $\xi \in [\frac{1}{\delta}, \infty)$, so $f(\xi)$ is concave in the leverage interval H.

2. [$\delta V_x - V_y \leq 0$ for $\xi \geq \frac{1}{\delta}$.] For $\gamma \geq \gamma_H$, $f(\xi)$ is concave, and is C^2 in $[1/\delta, \infty)$. $\Rightarrow \forall \xi \in (1/\delta, \infty)$, there exists ξ_m in $(1/\delta, \xi)$ s.t.

$$f'(\xi_m) = \frac{f(\xi) - f(\frac{1}{\delta})}{\xi - \frac{1}{\delta}}.$$

Since $f(\xi)$ is concave, $f'(\xi) \leq f'(\xi_m)$. So,

$$f'(\xi) \leq \frac{f(\xi) - f(\frac{1}{\delta})}{\xi - \frac{1}{\delta}} = \frac{f(\xi)}{\xi - \frac{1}{\delta}}.$$

Therefore $\delta\left(-f(\xi) + \xi f'(\xi)\right) - f'(\xi) \leq 0$, meaning $\delta V_x - V_y \leq 0$.

3. [$V_y - \gamma V_x \leq 0$ for $\xi \geq 1/\delta$.] We have

$$\begin{aligned}
& V_y - \gamma V_x \\
= \;& f'(\xi) - \gamma(-f(\xi) + \xi f'(\xi)) \\
= \;& C_2\left(w_2 \xi^{w_2 - 1} + \gamma(1 - w_2)\xi^{w_2}\right) + \frac{a}{\rho - \mu + a}\delta - \gamma \frac{a}{\rho - r + a} \\
= \;& C_2\left((1 - \gamma\xi) w_2 \xi^{w_2 - 1} + \gamma \xi^{w_2}\right) + \frac{a}{\rho - \mu + a}\delta - \gamma \frac{a}{\rho - r + a}
\end{aligned}$$

We know $\gamma > 1, \xi > 1 \Rightarrow 1-\gamma\xi < 0$, gives $C_2((1-\gamma\xi)w_2\xi^{w_2-1}+\gamma\xi^{w_2}) < 0$. On the other hand, $\gamma \geq \gamma_H$, gives $\delta a/(\rho-\mu+a)-\gamma a/(\rho-r+a) < 0$. So,

$$V_y - \gamma V_x < 0.$$

4. $[V_x rx + V_y \mu y + \frac{1}{2}V_{yy}\sigma^2 y^2 - \rho V + a(\delta y + x) \leq 0$ for $\xi \geq \frac{1}{\delta}.]$ From (4.4), $D(x,y) = 0$, and the rest follows.

5. For the candidate optimal control (U^*, W^*), and for any fixed time S,

$$E\left[\int_0^S e^{-2\rho t} V_y \sigma^2 (Y_t^{U^*,W^*})^2 \, dt\right] < \infty.$$

Recall that $V_y = f'(\xi) \leq f'(1/\delta) < \infty$, implying that, for some constant K,

$$E\left[\int_0^S e^{-2\rho t} V_y \sigma^2 \left(Y_t^{\{U^*,W^*\}}\right)^2 \, dt\right] < KE\left[\int_0^S \left(Y_t^{\{U^*,W^*\}}\right)^2 \, dt\right]$$

Consider another stochastic process $\{X^N(t), Y^N(t)\}$ with the system equation:

$$\begin{aligned} dX^N(t) &= rX^N(t)\,dt, \quad X^N(0^-) = x. \\ dY^N(t) &= \mu Y^N(t)\,dt + \sigma Y^N(t) dB(t), \quad Y^N(0^-) = y. \end{aligned}$$

One trivial conclusion from the uniqueness of this stochastic process is that if the stopping time in 2.3 satisfies $\tau(\omega) > S$, then $Y_t^*(\omega) = Y_t^N(\omega)$ for any t, where Y_t^* is the stochastic process associated with the optimal control $\{U^* \equiv 0, W^* \equiv 0\}$.

For a specific ω, for $Y_t(\omega)$, if $\tau > S$, then by definition, $Y_t(\omega) = Y_t^N(\omega)$. On the other hand, if insolvency happened before S, then $Y_S(\omega) = 0 \leq Y_S^N(\omega)$, so

$$\begin{aligned} Y_S(\omega) \leq Y_S^N(\omega) &\Rightarrow (Y_S(\omega))^2 \leq \left(Y_S^N(\omega)\right)^2 \\ &\Rightarrow E\left[Y_S^2\right] \leq E\left[\left(Y_S^N\right)^2\right]. \end{aligned}$$

For convenience, we define

$$H(S) = E\left[\left(Y_S^N\right)^2\right].$$

From its definition, Y_S^N is a geometric Brownian motion with positive drift. So $H(S)$ is some continuous increasing function. Using Fubini's theorem,

$$E\left[\int_0^S \left(Y_t^{U^*,W^*}\right)^2 dt\right] = \int_0^S E\left[\left(Y_t^{U^*,W^*}\right)^2\right] dt$$
$$\le \int_0^S E[H(t)]\,dt \le \int_0^S E[H(S)]\,dt \le SH(S) < \infty.$$

The desired result follows. ∎

Proof Proposition 4.2

Consider

$$S(t) = e^{-\rho t} V\left(X_t^\pi, Y_t^\pi\right) + \int_0^{t\wedge\tau} e^{-\rho s} a\left(\delta Y^\pi(s) + X^\pi(s)\right) ds. \tag{7.13}$$

For $t < \tau$, Itô's Formula gives us:

$$dS_t = e^{-\rho t}\Big(-\rho V(X_t^\pi, Y_t^\pi)\,dt + V_x(X_t^\pi, Y_t^\pi)\,dX_t + V_y(X_t^\pi, Y_t^\pi)\,dY_t$$
$$+\frac{1}{2}V_{yy}(X_t^\pi, Y_t^\pi)\sigma^2 Y_t^2\,dt + a\left(\delta Y_t^\pi + X_t^\pi\right)\,dt\Big).$$

Pluging in (2.1) and (2.2), we get

$$dS_t = e^{-\rho t}\left(V_x r X_t^\pi + V_y \mu Y_t^\pi + \frac{1}{2}\sigma^2 (Y_t^\pi)^2 - \rho V + a(\delta Y_t^\pi + X_t^\pi)\right) dt$$
$$+e^{-\rho t}\left(V_x\delta - V_y\right) dU + e^{-\rho t}\left(V_y - V_x\gamma\right) dW + e^{-\rho t} V_y \sigma Y_t^\pi\,dB_t.$$

From Lemmas 7.4 to 7.8, plus the fact that W_t and U_t are non-decreasing in t, we can conclude that S is a supermartingale. Because $S \ge 0$, the martingale convergence theorem gives us $S_t \to S_\infty$ where $E[S_\infty] \le E[S(0)]$.

Using Fatou's Lemma, we have $\lim_{t\to\infty} E[S(t)] \ge E[S_\infty]$. Hence $V(x,y) = S_0 \ge \lim_{t\to\infty} E(S_t) \ge E(S_\infty) \ge E[J(x,y,U,W)]$.

Proof Proposition 4.3

1. S^* is a martingale.

For simplicity, define control $\pi = (U, W)$ and $\pi^* = (U^*, W^*)$. Consider

$$S^*(t) = e^{-\rho t} V(X_t^{\pi^*}, Y_t^{\pi^*}) + \int_0^{t\wedge\tau} e^{-\rho s} a\left(\delta Y^{\pi^*}(s) + X^{\pi^*}(s)\right) ds. \tag{7.14}$$

For $t < \tau$, Itô's Formula gives us

$$dS_t^* = e^{-\rho t}\Big(-\rho V(X_t^{\pi^*}, Y_t^{\pi^*})\,dt + V_x(X_t^{\pi^*}, Y_t^{\pi^*})\,dX_t + V_y(X_t^{\pi^*}, Y_t^{\pi^*})\,dY_t + \frac{1}{2}V_{yy}(X_t^{\pi^*}, Y_t^{\pi^*})\sigma^2 Y_t^2\,dt + a\left(\delta Y_t^{\pi^*} + X_t^{\pi^*}\right)\,dt\Big).$$

Plugging in (2.1) and (2.2), we get

$$dS_t = \mu_S\,dt + e^{-\rho t}\left(V_x\delta - V_y\right)\,dU^* + e^{-\rho t}\left(V_y - V_x\gamma\right)\,dW_t^* + e^{-\rho t}V_y\sigma Y_t^{\pi^*}\,dB_t,$$

where

$$\mu_S = e^{-\rho t}\left(V_x r X_t^{\pi^*} + V_y\mu Y_t^{\pi^*} + \frac{1}{2}\sigma^2\left(Y_t^{\pi^*}\right)^2 - \rho V + a\left(\delta Y_t^{\pi^*} + X_t^{\pi^*}\right)\right).$$

In the leverage interval I, from Lemma 7.6, $\mu_S < 0$. But because of the definition of W^*, we do not stay in leverage interval I, so we are integrating μ_s on a set of measure 0 which gives 0. Also $V_y - V_x\gamma = \kappa - \gamma(\kappa/\gamma) = 0$. On the other hand, in the leverage interval H, by the PDE (4.4), $\mu_S = 0$. Also $dW^* = 0$. Therefore the process S^* defined by

$$dS_t^* = e^{-\rho t}V_y\sigma Y_t^{\pi^*}\,dB_t$$

is a local martingale. Lemma 7.7 aprovides the Novikov's condition for this local martingale, and therefore S^* is a martingale.

2. $V(x, y) = E[J(x, y, U^*, W^*)]$.

We have that S^* is a non-negative martingale. Applying the submartingale convergence theorem to the process $-S^*$, we get $S_\infty^*(\omega) = \lim_{t\to\infty} S_t^*(\omega)$ exists almost surely.

We now prove that

$$E\left[e^{-\rho t}V(X_t, Y_t)\right] \to 0. \tag{7.15}$$

We know

$$S(t) = e^{-\rho t}V\left(X_t^{\pi^*}, Y_t^{\pi^*}\right) + \int_0^{t\wedge\tau} e^{-\rho s}a\left(\delta Y^{\pi^*}(s) + X^{\pi^*}(s)\right)\,ds$$

is a Martingale. Also

$$\int_0^{t\wedge\tau} e^{-\rho s}a\left(\delta Y^{\pi^*}(s) + X^{\pi^*}(s)\right)\,ds$$

is an increasing sequence. Therefore $e^{-\rho t}V(X_t^{\pi^*}, Y_t^{\pi^*})$ is a super martingale. We also know that $e^{-\rho t}V(X_t^{\pi^*}, Y_t^{\pi^*}) \geq 0$, so the martingale convergence theorem implies that there is some constant $C \geq 0$ such that $e^{-\rho t}V \to C$ almost surely. If $C > 0$, implies $V \to Ce^{\rho t}$ a.s.. On the other hand, $f(\xi)$ is concave on the range $(1/\delta, \infty)$, there follows

$$f'(\xi)\Big|_{\xi=\left(\frac{1}{\delta}\right)^+}\left(\xi - \frac{1}{\delta}\right) \geq f(\xi),$$

so $(\delta y + x) > kV(x, y)$ for some fixed $k > 0$. So,

$$\lim_{t\to\infty}(\delta Y_t + X_t)e^{-\rho t} > kC$$

a.s. would imply that

$$\int_0^\infty e^{-\rho s}a(\delta Y_t + X_t)\, ds \geq kC\int_0^\infty ds \to \infty$$

a.s., implying that $E[S(t)] \to \infty$, which is a contradition! So $C = 0$.

$$\int_0^{t\wedge\tau} e^{-\rho s}a\left(\delta Y_s^{\pi^*} + X_s^{\pi^*}\right)\, ds \leq \int_0^\tau ae^{-\rho s}\left(\delta Y_s^{\pi^*} + X_s^{\pi^*}\right)\, ds$$

Recall that we defined $Y_t = X_t = 0, \forall t > \tau$, so

$$\int_0^{t\wedge\tau} e^{-\rho s}a(\delta Y_s^{\pi^*} + X_s^{\pi^*})\, ds$$

is dominated by

$$\int_0^\infty ae^{-\rho s}(\delta Y_s^{\pi^*} + X_s^{\pi^*})\, ds$$

From Proposition 4.2, we have for any control $\{U, W\}$,

$$E[\int_0^\infty ae^{-\rho s}(\delta Y_s^{\pi} + X_s^{\pi})\, ds] \leq V(X, Y).$$

In particular,

$$E[\int_0^\infty ae^{-\rho s}(\delta Y_s^{\pi^*} + X_s^{\pi^*})\, ds] \leq V(X, Y) < \infty.$$

Dominated convergence then implies that

$$\lim_{t\to\infty} E\left[\int_0^{t\wedge\tau} e^{-\rho s} a(\delta Y_s^{\pi^*} + X_s^{\pi^*})\, ds\right] = E\left[\int_0^{\tau} e^{-\rho s} a(\delta Y_s^{\pi^*} + X_s^{\pi^*})\, ds\right]. \tag{7.16}$$

From (7.15) and (7.16), we have

$$\lim_{t\to\infty} E[S(t)] \to E\left[\int_0^{\tau} e^{-\rho s} a(Y_s^{\pi^*} + X_s^{\pi^*})\, ds\right] = J(x, y, U^*, W^*).$$

Also since $S(t)$ is a martingale, we have $E[S(t)] = E[S(0)] = S(0)$ is constant, implies $\lim_{t\to\infty} E[S(t)] = J(x, y, U^*, W^*)$. So we have:

$$V(x, y) = S(0) = J(x, y, U^*, W^*).$$

∎

Lemma 7.9 *If $y > -x/\delta$, then, as $\gamma \to \gamma_L^+$, $V(x, y) \to \infty$.*

Proof Plugging (7.3) into (4.8), we have

$$\begin{aligned} f(\xi) &= -C_1\xi^{w_1} + C_2\xi^{w_2} + \frac{a}{\rho-\mu+a}\delta\xi - \frac{a}{\rho-r+a} \\ &= -C_2\left(\frac{w_2(w_2-1)}{w_1(w_1-1)}\xi_0^{w_2-w_1}\xi^{w_1} - \xi^{w_2}\right) + \frac{a}{\rho-\mu+a}\delta\xi - \frac{a}{\rho-r+a}. \end{aligned}$$

We note that

$$\frac{a}{\rho-\mu+a}\delta\xi - \frac{a}{\rho-r+a}$$

does not depend on γ, and can therefore be treated as a constant in our analysis.

From Lemma 7.1, $-w_2/(w_1-1) > 1$ and $-(w_2-1)/w_1 > 1$, so

$$\frac{w_2(w_2-1)}{w_1(w_1-1)} > 1.$$

Letting $\alpha = (w_2(w_2-1))/(w_1(w_1-1)) - 1$, we have $\alpha > 0$, which does not depend on γ.

From (7.9) and the monotone relationship between γ and ξ_0, we conclude that, as $\gamma \to \gamma_L^+$, $\xi_0 \to \xi_{0c}$, and hence $C_2 \to -\infty$.

For $\xi = \xi_0$,

$$\begin{aligned} f(\xi_0) &= -C_2\left((\alpha+1)\xi_0^{w_2-w_1}\xi_0^{w_1} - \xi_0^{w_2}\right) + Const \\ &= -C_2\alpha\xi_0^{w_2} + k_1, \end{aligned}$$

for a constant k_1. As $\gamma \to \gamma_L$, $C_2 \to -\infty$, and $\xi_0 \to \xi_{0c} > 1/\delta$, this means that

$$\lim_{\gamma\to\gamma_L} f(\xi_0) \to \infty.$$

Therefore

$$\lim_{\gamma\to\gamma_L} V\left(X, (-\xi_0 X)\right) \to \infty.$$

In proposition 4.2, we have proved that

$$V(x,y) \geq J(x,y,U,W).$$

In particular, for the special case of a policy $\{U_0, W_0\}$ that brings the ratio $-Y^{U_0,W_0}/X^{U_0,W_0}$ to ξ_0 at $t = 0$, then follow the optimal control (U^*, W^*) from that point, we have

$$V(x,y) \geq J(x,y,U_0,W_0).$$

Applying Proposition 4.3, we have

$$V(x,y) \geq J(x,y,U_0,W_0) = J(x_{0^+}, y_{0^+}, U^*, W^*) = V(x_{0^+}, y_{0^+}).$$

For $y_0 < \xi_0(-x_0)$, one sells the asset and pays back debt so as to get to $y_{0^+} = \xi_0(-x_{0^+})$. Knowing that $\delta(y_0 - y_{0^+}) = (x_{0^+} - x_0)$, we can solve the equations and get

$$x_{0^+} = -\frac{y_0 + \frac{x_0}{\delta}}{\xi_0 - \frac{1}{\delta}} < 0,$$

and $y_{0^+} = -\xi_0 x_{0^+}$.

For $y_0 > \xi_0(-x_0)$, at $t = 0$ one buys asset and increases debt so as to get to $y_{0^+} = \xi_0(-x_{0^+})$. Knowing that $\gamma(y_0 - y_{0^+}) = (x_{0^+} - x_0)$, we can solve the equations and get

$$x_{0^+} = -\frac{y_0 + \frac{x_0}{\gamma}}{\xi_0 - \frac{1}{\gamma}} < 0,$$

and $y_{0^+} = -\xi_0 x_{0^+}$.

So, we have $V(x,y) \geq V(x_{0^+}, -\xi_0 x_{0^+})$, where $x_{0^+} < 0$. We know that $\lim_{\gamma\to\gamma_L} V(x, (-\xi_0 x)) \to \infty$, so $\lim_{\gamma\to\gamma_L} V(x,y) \to \infty$. ∎

Proof 4.4 Consider a fund manager using the following particular control (U^p, W^p). The fund manager creates two funds. In the first, he pretends that the cost of buying one dollar's worth of asset is some $\gamma_p > \gamma_L$, follows the associated optimal control. If the fund manager could "burn" a certain amount of money every time he buys the risky asset, this would be a feasible policy. Since he cannot burn money, when buying the asset, he can instead store the funds that he would have burned in the second fund, in the risk-free security.

When the insolvency condition $Y_1 = -X_1/\delta$ is achieved in the first fund, that fund is liquidated. The current market value of management fees from the first account is denoted $V^p(x, y)$. From Lemma 7.9, by choosing the γ_p arbitrarily close to γ_L, we can achieve arbitrarily high $V^p(x, y)$. The second account is always positive, so the sum of the current value of management fees from both accounts is greater than $V^p(x, y)$. Since $V^p(x, y)$ can be arbitrarily high, the supremum of $J(x, y, U, W)$ is $+\infty$. ∎

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