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## ON THE USE OF SURVEY SAMPLE WEIGHTS IN THE LINEAR MODEL\*

BY RICHARD D. PORTER

*If individuals have different coefficients in a linear model, then the choice of regression technique for estimating population averages depends on the sample design. We examine various estimators of the random coefficient model for panel data, where the random component arises from the random selection of individuals out of a finite population.*

### 1. INTRODUCTION

#### 1.1. *Problem*

Sample surveys such as the Current Population Survey are a rich source of economic data. If the sample is drawn according to the principles of sample survey theory, each member will have an attached weight. For example, suppose there are two strata A and B and that a sample is drawn in which members in A are sampled at a rate 6:1,000 (six per thousand population individuals in A) whereas members in B are sampled at a rate of 3:1,000. Then to compute a population total, say the total wage bill for the population as a whole, it is sensible to give twice as much weight to an earnings measurement in B as to an earnings measurement in A, that is, the weights will be proportional to the inverse of the probability of being selected. But when different classes or strata are sampled at different rates, should the associated weights be used in estimating a behavioral econometric model? And how should they be used? In practice we usually have more information about the method by which the sample was drawn than just sampling weights for each observation. We also know the type of sampling procedure (such as simple random sampling with replacement, simple random sampling without replacement, stratified random sampling, single-stage Cluster sampling, multi-stage sampling) as well as detailed probability descriptions of the procedure. We often know the probability that any unit will be drawn as well as the joint probability that any pair of units will be drawn. As before, this information about the sampling design can be incorporated into estimates of population totals, standard error estimates for the estimated population totals, and so forth. But what use should we make of this information in estimating a behavioral econometric model?

In the econometric literature, opinions divide. Some authors advocate that the sample weights be used in linear econometric models in a way which is similar to the use of weights in computing finite population totals: they recommend using weighted least-squares.<sup>1</sup> Other writers argue that such sample survey

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<sup>1</sup> See Klein and Morgan (1951), Klein (1953, pp. 305-313), Hu and Stormsdorfer (1970), and Cohen, Rea, and Lerman (1970, pp. 193-194).

information is irrelevant for econometric models.<sup>2</sup> Most econometric textbook authors do not discuss this issue.<sup>3</sup>

### 1.2. Homogeneous Coefficients: The Choice of the Regression Technique Does Not Depend on the Sample Design

If the coefficients in the behavioral model are homogeneous throughout the population, then the sample design does not affect the validity of the usual (least-squares) estimates. To pursue this point consider the following example.

Suppose there are  $q$  possible samples of size  $n$  that can be drawn from a population of size  $N$  according to the sampling design chosen and that the probability of selecting each sample is known. To represent this probability model for sampling we construct a random variable  $S$  taking on  $q$  distinct values  $s_1, s_2, \dots, s_q$  with associated probabilities  $p_1, p_2, \dots, p_q$ . Let the regression model for any sample, say the  $s$ th, be given by

$$(1) \quad \mathbf{y}_s = X_s \boldsymbol{\beta} + \mathbf{u}_s,$$

where  $X_s$  is a  $n \times k$  matrix of regressors,  $\mathbf{y}_s$  is a  $n \times 1$  vector of regressands,  $\boldsymbol{\beta}$  is a fixed  $k \times 1$  vector of unknown coefficients and  $\mathbf{u}_s$  is a  $n \times 1$  vector of unobserved disturbances. We treat  $X_s$  as fixed so that the only source of variation in  $\mathbf{y}_s$  is due to the variation in the disturbance vector  $\mathbf{u}_s$ . We postulate that  $\mathbf{u}_s$  is generated by a classical probability mechanism which is independent of the sampling design and exhibits the usual properties

$$(2) \quad E_c(\mathbf{u}_s | X_s) = \mathbf{0} \quad \text{for all } s,$$

$$(3) \quad E_c(\mathbf{u}_s \mathbf{u}_s' | X_s) = \sigma^2 I \quad \text{for all } s,$$

where  $E_c$  denotes the expectator operator. We distinguish the operator by the subscript  $c$ , where  $c$  stands for the classical probability mechanism generating the disturbances. Assume  $X_s$  has full column rank for all  $s$  so that the least-squares estimator of  $\boldsymbol{\beta}$ , namely

$$(4) \quad \mathbf{b}(s) = (X_s' X_s)^{-1} X_s' \mathbf{y}_s$$

exists.

To evaluate properties of  $\mathbf{b}$  remember that we must take into account two sources of random variation: that caused by the random selection of individuals and that caused by the random variation in the disturbance vector. Since the unconditional expectation  $E\mathbf{b}(s)$  is the sum of the conditional expectations, we have

$$(5) \quad E[\mathbf{b}(s)] = \sum_{i=1}^q E_c[\mathbf{b}(s) | S = s_i] p_i,$$

<sup>2</sup> See Cramer (1971, p. 143), Fleischer and Porter (1970, pp. 99–111), and Roth (1971). I became aware of several of these references by reading Roth's memorandum, Roth (1971).

<sup>3</sup> See e.g., Dhrymes (1970), Goldberger (1964), Goldberger (1968), Johnston (1963), Kmenta (1971), Malinvaud (1966), Theil (1971), Zellner (1971). A notable exception is Klein's pioneering textbook, Klein (1953); Champernowne—Champernowne (1969)—takes up survey sampling theory but does not relate it to the regression model.

where  $E_c[\mathbf{b}(s)|S = s_i]$  represents the conditional expected value of  $\mathbf{b}(s)$  given the event  $S = s_i$ . Given our specification for  $\mathbf{u}$  we can show that  $\mathbf{b}(s)$  is an unbiased estimator of  $\beta$ . Inserting (4) and (1) into (5) and simplifying gives

$$(6) \quad E[\mathbf{b}(s)] = \sum_{i=1}^q E_c[\beta + (X'_s X_s)^{-1} X'_s \mathbf{u}_s | S = s_i] p_i = \beta \sum_{i=1}^q p_i = \beta.$$

The crucial relations used to derive (6) are (a)  $X_s$  is fixed for a given sample and (b)  $E_c[\mathbf{u}_s | S = s_i] = \mathbf{0}$ . The assumption that  $\mathbf{u}$  does not depend on the sampling procedure is critical for establishing (b).

If we restrict our analysis to be conditioned upon the particular  $X$  matrix which is drawn, then the Gauss-Markov theorem holds and the least-squares estimator will be a best linear unbiased estimator (BLUE) of  $\beta$ .<sup>4</sup> Indeed, it would appear that  $\mathbf{b}(s)$  will have these optimal properties when we also allow for sampling variations.<sup>5</sup>

The implication of the foregoing analysis is that for homogeneous populations we are not obliged to incorporate the structure of the sampling plan into our regression analysis. Of course, the sample design is important regardless of whether coefficients are homogeneous or heterogeneous.

### 1.3. Outline of the Paper

In the rest of the paper we adopt the assumption that the coefficients differ across individuals. Then it appears that the choice of the regression technique depends on the sample design so we explore some procedures for combining the information on the sample design with the specification of the behavioral model to obtain estimates of certain population parameters. In Section 2 we review some results from sample survey theory. We employ these results in Section 3 to form estimators for the *random coefficient* regression model based on panel data. Here the "random" component in the coefficient arises solely from the random selection of individuals. Although this problem has been intensively studied recently,<sup>6</sup> the analysis has implicitly proceeded under the assumption of random sampling from an infinite population. We consider the more usual sampling design in which sampling is done without replacement from a finite population with unequal probabilities. See Konijn (1962) for a related contribution when the data source is a single cross section.<sup>7</sup>

<sup>4</sup> See, e.g. Theil (1971, p. 119).

<sup>5</sup> The proof follows the standard proof of the Gauss-Markov theorem, Theil (1971, pp. 119-120). The proof consists of showing that the covariance matrix of the least squares estimator, say  $V$ , is

$$V = \sum_{i=1}^q p_i \sigma^2 (X'_{si} X_{si})^{-1},$$

while any other linear unbiased estimator, say  $A_s y_s$ —where  $A_s$  may be functionally dependent on  $s$ —has a covariance matrix equal to

$$V + \sum_{i=1}^q p_i (A'_{si} A_{si}).$$

<sup>6</sup> See Rao (1965), Zellner (1966), Swamy (1968), (1970), (1971), (1972), Theil (1971), Lindley and Smith (1972) and Schmalensee (1972).

<sup>7</sup> I am grateful to Professor Zellner for bringing Konijn's valuable study to my attention.

## 2. SAMPLING FROM FINITE POPULATIONS

In this section we review some elements of sampling theory from finite populations.<sup>8</sup> The object of this theory is descriptive: to estimate finite population totals or averages.

### 2.1. Simple Random Sampling Without Replacement

We start with the concept of an ordered *random sample*. Let the finite population being sampled consist of  $N$  items, numbered  $1, 2, \dots, N$ . An ordered sample from this population is an arrangement of the items in a particular order. For example, if the population consists of three elements  $\{1, 2, 3\}$ , there are six possible ordered samples of size two:  $(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)$ .<sup>9</sup> When each of these ordered samples appears with equal frequency in repetitive sampling, the sample is called an *ordered random sample*. Define the product  $N(N-1)\dots(N-n+1) = \pi(N, n)$ . Probabilities are herein computed in accord with the equivalence law of ordered random sampling:

*Theorem 1* (The Equivalence Law of Ordered Random Sampling)

If an ordered random sample of size  $n$  is drawn from a population of size  $N$ , then on any particular one of the  $n$  draws, each of the  $N$  items has the same probability  $1/N$  of appearing.

*Proof.* See Hodges and Lehmann (1970, pp. 55-59).

The theorem generalizes to more than one item in a general way but we need consider only:

*Theorem 2*

Any pair of items, say  $I$  and  $J$ , has the same probability  $1/\binom{N}{2}$  of appearing on any 2 specified draws. (Note that we do not indicate the *order* in which  $I$  and  $J$  appear on the two specified draws.)

*Proof.* Without loss of generality suppose that the two draws are the first and the second. If  $I$  appears on the first and  $J$  appears on the second, the remaining items can be drawn in  $\pi(N-2, n-2) = (N-2)(N-3)\dots(N-n+1)$  ways; alternatively,  $J$  may appear on the first and  $I$  on the second in  $\pi(N-2, n-2)$  ways. Thus, the probability of  $\{I, J\}$  on draws 1 and 2 is  $2\pi(N-2, n-2)/\pi(N, n) = 1/\binom{N}{2} = 2/(N)(N-1)$ .

Suppose we are not interested in an *ordered* random sample but in an *unordered* random sample. We can obtain an unordered random sample by first drawing an ordered random sample and then disregarding the order.<sup>10</sup>

Let  $y$  designate the variable which we are measuring in the population;  $y$  may be a scalar or a vector. For the present we will let  $y$  be a scalar. The value of  $y$  for the first item in the population is  $y_1$ , for the second  $y_2$ , and so forth. If we consider

<sup>8</sup> Hodges and Lehmann (1970, Sections 2.3, 4.3, 7.2, 9.1 and 10.3), Kendall and Stuart (1966, Chapters 39-40), and Cochran (1963) are useful introductions to the sampling theory. We draw on them in this section.

<sup>9</sup> Note that we use braces " $\{ \}$ " when the order is irrelevant and parentheses " $( )$ " when the order becomes important.

<sup>10</sup> See Hodges and Lehmann (1970, p. 54).

a random drawing of one item, say  $\bar{y}$ , from this population, its expected value and variance are

$$(7) \quad E[\bar{y}] = \sum_{i=1}^N y_i \Pr(\bar{y} = y_i) = \sum_{i=1}^N y_i(1/N) \equiv \mu$$

$$(8) \quad \text{Var}[\bar{y}] = \sum_{i=1}^N (y_i - \mu)^2(1/N) \equiv \sigma^2.$$

Note that the population mean and variance,  $\mu$  and  $\sigma^2$ , are generated by a very simple probability mechanism: the random drawing of one item from this population.

It will simplify matters if we adopt the following notational conventions. Let  $p_i(r)$  be the probability that the  $i$ th person is selected on the  $r$ th draw. Let  $p_{ij}(r, s)$  be the probability that the  $i$  and  $j$ th individuals are selected on the  $r$  and  $s$ th draws respectively. Let  $\bar{N} = \{1, 2, \dots, N\}$  and  $\bar{n} = \{1, 2, \dots, n\}$ . As a shorthand we will write

$$\sum w_i = \sum_{i=1}^N w_i, \quad \sum_{i \neq j} w_{ij} = \sum_{j=1}^N \sum_{i=1, i \neq j}^N w_{ij}, \quad \sum w_i = \sum_{i=1}^n w_i,$$

and

$$\sum w_{ij} = \sum_{i=1}^n \sum_{j=1, j \neq i}^n w_{ij}.$$

We next draw an ordered random sample, say  $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)$ , from this population. By Theorem 1, each  $\bar{y}$  has the same probability distribution:

$$(9) \quad p_i(r) = 1/N \quad \text{for all } i \in \bar{N} \text{ and } r \in \bar{n}.$$

Consequently for each  $r \in \bar{n}$

$$(10) \quad E(\bar{y}_r) = \sum y_i N^{-1} = \mu$$

$$(11) \quad \text{Var}(\bar{y}_r) = \sum (y_i - \mu)^2 N^{-1} = \sigma^2.$$

In view of the proof of Theorem 2 we have

$$(12) \quad p_{ij}(r, s) = 1/N(N-1) \quad \text{for all } r, s \in \bar{n}, r \neq s \text{ and } i, j \in \bar{N}, i \neq j.$$

Thus the covariance between  $\bar{y}_r$  and  $\bar{y}_s$  is equal for all  $r$  and  $s$ . If  $C$  is this common covariance,  $C$  satisfies

$$\text{Var}(\sum \bar{y}_i) = n\sigma^2 + (n^2 - n)C.$$

When  $n = N$ ,  $\sum \bar{y}_i$  is a constant with zero variance so  $N\sigma^2 + N(N-1)C = 0$  and

$$(13) \quad C = -\sigma^2/(N-1).$$

We now consider the problem of estimating  $\mu$ . It is convenient to cast this problem in the format of a linear model. Let  $\varepsilon_i$  be a variable defined by  $\varepsilon_i \equiv y_i - \mu$ , for  $i$  in  $\bar{N}$ . If we observe the entire population,  $\mu$  is known exactly; this implies that

$\varepsilon_i, i = 1, 2, \dots, N$  are known quantities. However the sample values  $\tilde{\varepsilon}_r \equiv \tilde{y}_r - \mu$ ,  $r$  in  $\bar{n}$ , are random variables with the following properties:

$$(14) \quad p_i(r) = N^{-1} \quad \text{for all } i \in \bar{N} \text{ and all } r \in \bar{n}$$

$$p_{ij}(r, s) = 1/N(N-1) \quad \text{for all } i, j \in \bar{N}, i \neq j \text{ and } r, s \in \bar{n}, r \neq s.$$

Our sample  $(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_s)$  thus belongs in the following setup:

$$(15) \quad \tilde{\mathbf{y}} = \mathbf{1}\mu + \tilde{\boldsymbol{\varepsilon}}$$

$$(16) \quad E(\tilde{\boldsymbol{\varepsilon}}) = \mathbf{0}$$

$$(17) \quad E\tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}}' = \sigma^2\boldsymbol{\Omega}, \boldsymbol{\Omega} = (1 - \rho)\mathbf{I} + \rho\mathbf{I}\mathbf{I}'$$

where  $\boldsymbol{\Omega}$  is a  $n \times n$  matrix,  $\rho \equiv -(N-1)^{-1}$  and  $\mathbf{y}' = (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)$ ,  $\tilde{\boldsymbol{\varepsilon}}' = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \dots, \tilde{\varepsilon}_n)$ ,  $\mathbf{I}' = (1, 1, \dots, 1)$  are  $1 \times n$  vectors. For the model of (15), (16), and (17) the best linear unbiased estimator (BLUE) of  $\mu$  is, of course, the Aitken generalized least-squares estimator

$$(18) \quad \hat{\mu} = (\mathbf{I}'\boldsymbol{\Omega}^{-1}\mathbf{I})^{-1}\mathbf{I}'\boldsymbol{\Omega}^{-1}\tilde{\mathbf{y}}.$$

Let

$$(19) \quad r = \{1 + (n-1)\rho\}.$$

One can easily verify that<sup>11</sup>

$$(20) \quad \boldsymbol{\Omega}^{-1} = \frac{1}{(1-\rho)r} [r\mathbf{I} - \rho\mathbf{I}\mathbf{I}']$$

so that

$$\mathbf{I}'\boldsymbol{\Omega}^{-1} = r^{-1}\mathbf{I}'$$

$$\mathbf{I}'\boldsymbol{\Omega}^{-1}\mathbf{I} = r^{-1}n.$$

Thus

$$(21) \quad \hat{\mu} = rn^{-1}r^{-1}\mathbf{I}'\tilde{\mathbf{y}} = \frac{\sum y_i}{n} = (\mathbf{I}\mathbf{I})^{-1}\mathbf{I}'\tilde{\mathbf{y}}.$$

That is, the Aitken estimator and the ordinary least-squares estimator are identical in this case.

## 2.2. Simple Random Sampling Without Replacement With Unequal Probabilities

We now relax the assumption that all individuals have an equal chance of being selected on each draw and permit probabilities of being drawn to differ between individuals and from drawing to drawing. Most sample designs are special cases of this scheme.<sup>12</sup> As before let  $p_{ij}(r, s)$  be the probability that the  $i$  and  $j$ th individuals are selected on the  $r$  and  $s$ th draws respectively in a sample of size  $n$  from a population of size  $N$ ;  $i$  and  $j$  range from 1 to  $N$  and  $r$  and  $s$  from

<sup>11</sup> This result is well known. See, for example, Kendall and Stuart (1966, p. 167).

<sup>12</sup> See Kendall and Stuart (1966, p. 177 ff).

1 to  $n$  where  $i \neq j$  and  $r \neq s$ . The probability that the  $i$ th person is selected on the  $r$ th draw,  $p_i(r)$ , is

$$(22) \quad p_i(r) = \sum_{\substack{s=1 \\ r \neq s}}^n \sum_{\substack{j=1 \\ i \neq j}}^N p_{ij}(r, s)/(n-1).$$

Since someone is always selected at the  $r$ th drawing,

$$(23) \quad \sum_{i=1}^N p_i(r) = 1.$$

Let  $\pi_i$  be the probability that the  $i$ th person is selected in the sample,

$$(24) \quad \pi_i = \sum_{r=1}^n p_i(r).$$

Finally, let  $\pi_{ij}$  be the joint probability that the  $i$ th and  $j$ th persons are selected in the sample,

$$(25) \quad \pi_{ij} = \sum_{\substack{r=1 \\ r \neq s}}^n \sum_{s=1}^N p_{ij}(r, s).$$

Since  $p_{ij}(r, s) = p_{ji}(s, r)$ , we have, of course, that  $\pi_{ij} = \pi_{ji}$ .

For our purposes, it will suffice to characterize the sampling design in terms of  $\pi_i$  and  $\pi_{ij}$ . From (23) and (24) we find

$$(26) \quad \sum \pi_i = n.$$

From (22), (24), and (25) we get

$$(27) \quad \sum_{j=1}^N \pi_{ij} = (n-1)\pi_i$$

$$(28) \quad \sum_{\substack{i=1 \\ i \neq j}}^N \sum_{j=1}^N \pi_{ij} = n(n-1).$$

Before, we were careful to distinguish between the labelling of observations in the sample and that in the population. The second person in our sample will not usually be the second person in the population. However, now we will label the sample observations in the order in which they are drawn and *not* distinguish between the order in the sample and the order in the population. As long as we are considering symmetric functions of sample observations this notational convention will not lead us astray.

A result we shall often call upon is the following:

*Theorem 3*

Suppose a sample of size  $n$ ,  $y_1, y_2, \dots, y_n$  is drawn from a population of size  $N$ . Then for any function  $g$

$$(29) \quad E[\sum g(y_i)] = \sum \pi_i g(y_i)$$

$$(30) \quad E[\sum g(y_i, y_j)] = \sum \pi_{ij} g(y_i, y_j).$$



*Proof.* So that there is no ambiguity let us first write out (29) and (30) fully:

$$(29') \quad E \left[ \sum_{i=1}^n g(y_i) \right] = \sum_{i=1}^N \pi_i g(y_i)$$

$$(30') \quad E \left[ \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n g(y_i, y_j) \right] = \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \pi_i g(y_i, y_j).$$

To prove (29) note that

$$(31) \quad E[\sum g(y_i)] = \sum_i E g(y_i).$$

But by definition  $E g(y_i) = \sum_j g(y_j) p_j(i)$ . Thus

$$E[\sum g(y_i)] = \sum_i \sum_j g(y_j) p_j(i) = \sum_j g(y_j) \sum_i p_j(i) = \sum_j g(y_j) \pi_j.$$

This proves (29); equation (30) follows by a similar argument.

We can use Theorem 3 to obtain a linear unbiased estimator of the population mean,  $\mu$ ,

$$(32) \quad \mu = \frac{1}{N} \sum_{i=1}^N y_i.$$

Suppose the same weight  $v_i$  is to be assigned to an individual whenever he is selected. A linear estimator will have the form

$$(33) \quad \hat{\mu} = \sum_{i=1}^n v_i y_i,$$

with the weights to be determined by the unbiasedness condition. Using (29) with  $g(y_i) = v_i y_i$  we find

$$(34) \quad E[\hat{\mu}] = \sum_{i=1}^N (v_i y_i) \pi_i.$$

Then equating coefficients in (32) and (34) we must have

$$(35) \quad \hat{\mu} = \frac{1}{N} \sum_{i=1}^n \frac{y_i}{\pi_i}.$$

### 3. SURVEY SAMPLING AND THE RANDOM COEFFICIENT REGRESSION MODEL FOR PANEL DATA

#### 3.1. Introduction

Recently, there has been renewed interest in the random coefficient regression model.<sup>13</sup> A specification leading to a random coefficient regression model occurs in the survey sampling framework. Suppose the population consists of  $N$  individuals

<sup>13</sup> See the references in footnote 6. Also see Hildreth and Houck (1968). Swamy (1971), (1972) provides an extensive bibliography on this literature.

and let the economic relationship for the  $i$ th unit be given by

$$(36) \quad y_i = X_i \beta_i + u_i, i \in \bar{N},$$

where  $y_i$  is a  $T \times 1$  vector of observations on the dependent variable,  $X_i$  is a  $T \times K$  matrix of observations with rank  $K$  on  $K$  independent variables,  $\beta_i$  is a  $K \times 1$  vector of *non-random* coefficients and  $u_i$  is a  $T \times 1$  vector of disturbance terms with mean zero for each  $i$ .

It is convenient to think of  $T$  as the number of time periods so that, for example, the  $t$ th element of  $y_i$  and  $u_i$  refer to the  $t$ th period. We allow for heterogeneity across individuals: each unit has its own coefficient vector.

The random coefficient model arises when a sample is drawn from a population. At the beginning of the first sampling period  $n$  individuals are randomly selected out of the population. In  $T$  successive periods the *same*  $n$  individuals are sampled. Assembling the observations on the  $n$  individuals for  $T$  periods we have<sup>14</sup>

$$(37) \quad \begin{aligned} y_1 &= X_1 \beta_1 + u_1 \\ y_2 &= X_2 \beta_2 + u_2 \\ &\vdots \\ y_n &= X_n \beta_n + u_n. \end{aligned}$$

The random selection of individuals determines the random coefficient model for the system in (37). Let the population coefficient vector of interest be given by<sup>15</sup>

$$(38) \quad \beta = \frac{1}{N} \sum \beta_i.$$

We will develop various estimators for  $\beta$  under two sampling schemes: simple random sampling without replacement and random sampling without replacement with unequal probabilities.

### 3.2. Simple Random Sampling Without Replacement

In simple random sampling the units are drawn without replacement with equal probabilities. We shall make the following specification initially for the system of observations in (37) which came from the population in (36).

#### Assumption 3.1:

1. The number of units sampled ( $n$ ) and the number of time periods ( $T$ ) are such that  $n > K$  and  $T > K$ .
2. For each unit  $i$  in the population, the independent variables are fixed in repeated samples on  $y_i$ . The rank of  $X \equiv [X_1', X_2', \dots, X_n']'$  is  $K$  for every possible sample drawn.
3. The disturbance vectors  $u_i$  ( $i \in \bar{N}$ ) are independently distributed each having mean zero. The variance-covariance matrix of  $u_i = \sigma_{ii} I_T$ .
4. The  $n$  units are drawn by simple random sampling without replacement from the population of  $N$  units.

<sup>14</sup> As in Section 2 we *do not* distinguish between the labeling order in the sample and the population.

<sup>15</sup> We could carry out the analysis for other population concepts such as  $\beta^* = \sum w_i \beta_i$ , where  $w_i$  are known weights.

As was stressed in the introduction, there are two different sources of random variation in this model, one being the behavioral random error, the  $\mathbf{u}$  vectors, and the other being the variation in  $\beta$  vectors caused by the random selection of individuals. In evaluating expectations of random variables it will often be convenient to distinguish these two sources of variation. We shall use the shorthand  $S$  to denote the summation over individual units, i.e., the variation caused by sampling. And we shall let  $c$  denote the integration over the behavioral random errors, the  $\mathbf{u}$ 's.

Since the method of sampling is simple random sampling, the results reviewed in Section 2 apply directly to the  $\beta$ 's. In particular, from (10) we have

$$(39) \quad E_S(\beta_i) = \beta, \quad i \in \bar{n}.$$

We shall define the variance-covariance matrix for the population by

$$(40) \quad \Delta = \sum \frac{(\beta_i - \beta)(\beta_i - \beta)'}{N}.$$

We assume that  $\Delta$  is positive definite. The sampling errors

$$(41) \quad \delta_i = \beta_i - \beta \quad i \in \bar{n} \text{ have zero mean values.}$$

Using (11) we have

$$(42) \quad E(\delta_i \delta_i') = \Delta, \quad i \in \bar{n}.$$

Finally, the matrix version of (13) is (43):

$$(43) \quad E(\delta_i \delta_j') = -\frac{\Delta}{N-1} \quad i, j \in \bar{n}, \quad i \neq j.$$

For the model of (3.1) we shall consider two estimates. The first will be the simple average of the least-squares estimators of each unit in the sample. The second estimator is an approximate Aitken estimator.

#### *Average Least-Squares Estimator*

Let  $\mathbf{b}$  be the first estimator,

$$(44) \quad \mathbf{b} = \frac{1}{n} \sum \mathbf{b}_i,$$

where

$$(45) \quad \mathbf{b}_i = (X_i' X_i)^{-1} X_i' y_i.$$

Considering the variation in  $\mathbf{u}_i$  above we have the usual result that

$$(46) \quad E_c(\mathbf{b}_i | S) = \beta_i$$

where  $E_c(\mathbf{b}_i | S)$  denotes the conditional expected value of  $\mathbf{b}_i$  given the  $i$ th unit is drawn. From (39) and (46) we obtain

$$(47) \quad E(\mathbf{b}) = \frac{1}{n} \sum E_S E_c[\mathbf{b}_i | S] = \frac{1}{n} \sum E_S(\beta_i) = \beta.$$

That is,  $\mathbf{b}$  is an unbiased estimator of  $\beta$ .

Next we determine the variance-covariance matrix for  $\mathbf{b}$  and an estimator of it. The error between  $\mathbf{b}$  and  $\beta$  is

$$(48) \quad \mathbf{b} - \beta = n^{-1}[\sum(\delta_i + (X_i'X_i)^{-1}X_i'u_i)].$$

It will simplify notation to introduce  $P_i$  by

$$(49) \quad P_i = \sigma_{ii}(X_i'X_i)^{-1}.$$

The variance-covariance matrix of  $\mathbf{b}$ , say  $S_{bb}$ , is

$$S_{bb} = E(\mathbf{b} - \beta)(\mathbf{b} - \beta)'$$

Evaluating  $S_{bb}$  we find

$$(50) \quad S_{bb} = \frac{\Delta(N-n)}{n(N-1)} + \frac{1}{Nn} \sum P_i.$$

To obtain an estimate of  $S_{bb}$  we shall first evaluate the matrix  $S_b$ ,

$$(51) \quad S_b = \sum \mathbf{b}_i \mathbf{b}_i' - \frac{1}{n}(\sum \mathbf{b}_i)(\sum \mathbf{b}_i)'$$

Substituting

$$\mathbf{b}_i = \beta_i + (X_i'X_i)^{-1}X_i'u_i$$

into (51) and taking expectations gives

$$(52) \quad E[S_b] = (n-1)\Delta + (n-1)\frac{\Delta}{N-1} + \frac{(n-1)}{N} \sum P_i.$$

Let

$$(53) \quad M_i = I - X_i(X_i'X_i)^{-1}X_i'$$

$$(54) \quad \mathbf{e}_i = M_i y_i.$$

As is well known

$$(55) \quad s_{ii} = \frac{\mathbf{e}_i' \mathbf{e}_i}{T-K}$$

is an unbiased estimator of  $\sigma_{ii}$  so that

$$(56) \quad \frac{1}{n} \sum s_{ii}(X_i'X_i)^{-1}$$

is an unbiased estimator of  $(1/N) \sum P_i$ . In view of (52) and (56), an unbiased estimator of  $\Delta$  is

$$(57) \quad \hat{\Delta} = \left[ \frac{S_b}{n-1} - \frac{1}{n} \sum \hat{P}_i \right] \frac{N-1}{N},$$

where

$$(58) \quad \hat{P}_i = s_{ii}(X_i'X_i)^{-1}.$$

Thus an unbiased estimate of  $S_{bb}$  will be

$$(59) \quad \hat{S}_{bb} = \hat{\Delta} \frac{(N-n)}{n(N-1)} + \frac{1}{n^2} \sum \hat{p}_i.$$

A possible operational difficulty with the estimator for  $\Delta$ ,  $\hat{\Delta}$ , is that it may not be positive definite or even positive semi-definite. A necessary condition for  $\hat{\Delta}$  to be positive semi-definite is  $n > K$ .<sup>16</sup> However, this difficulty does not extend to the estimator for  $S_{bb}$ .

#### An Approximate Aitken Estimator

Assuming that the estimate of  $\Delta$  is positive definite, we can create an estimator for  $\beta$  which uses more of the model specification than the average least square estimator,  $b$ . This Aitken estimator has the property that it will be dependent on the particular  $X$  matrix which is drawn. To form this estimator of  $\beta$  we follow Swamy (1971, Chapter 4), and write the sample system of  $nT$  observations (37) together as

$$(60) \quad \mathbf{y} = X\beta + D(X)\delta + \mathbf{u},$$

where

$$\begin{aligned} \mathbf{y} &= (y'_1, y'_2, \dots, y'_n)' \\ X &= [X'_1, X'_2, \dots, X'_n]' \\ D(X) &= \begin{bmatrix} X_1 & 0 & 0 & \dots & 0 \\ 0 & X_2 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & & X_n \end{bmatrix}^{17} \\ \delta &= (\delta'_1, \delta'_2, \dots, \delta'_n)' \\ \mathbf{u} &= (u'_1, u'_2, \dots, u'_n)' \end{aligned}$$

Conditional on  $X$  the  $nT \times 1$  disturbance vector for (60),  $D(X)\delta + \mathbf{u}$ , has the following variance-covariance matrix

$$(61) \quad E\{[D(X)\delta + \mathbf{u}]\{D(X)\delta + \mathbf{u}\}'|X\} \equiv H(\theta) = \begin{bmatrix} X_1\Delta X'_1 + \sigma I & -zX_1\Delta X'_2 & \dots & -zX_1\Delta X'_n \\ -zX_2\Delta X'_1 & X_2\Delta X'_2 + \sigma I & \dots & -zX_2\Delta X'_n \\ \vdots & & & \vdots \\ -zX_n\Delta X'_1 & \dots & & X_n\Delta X'_n + \sigma I \end{bmatrix},$$

where  $z = 1/(N-1)$  and  $\sigma = (1/N) \sum \sigma_{ii}$ . The matrix  $H(\theta)$  is a symmetric  $nT \times nT$  matrix. It is functionally dependent on  $X$ ,  $z$  and an unknown  $\frac{1}{2}[K(K+1)+2]$

<sup>16</sup> See Schmalensee (1972, p. 6) for a proof of this result for Swamy's specification of the random coefficient model, Swamy (1971, Chapter 4). That proof carries over to our specification.

<sup>17</sup> The zeroes in  $D$  are  $T \times K$  null matrices.

vector of parameters,  $\theta$ , containing the distinct elements of  $\Delta$  and  $\sigma$  arranged in a particular order. It can readily be shown that  $H(\theta)$  has an inverse.<sup>18</sup> Conditional on  $X$ , the BLUE of  $\beta$  is the Aitken estimator,

$$(62) \quad \mathbf{b}(\theta) = (X'H(\theta)^{-1}X)^{-1}X'H(\theta)^{-1}\mathbf{y}.$$

Since  $\Delta$  and  $\sigma$  are unknown,  $\mathbf{b}(\theta)$  is not operational. We can, however, form an approximate Aitken estimator by substituting unbiased estimates for  $\Delta$  and  $\sigma$ . Thus let  $H(\hat{\theta})$  be the  $nT \times nT$  matrix formed by substituting  $\hat{\Delta}$  for  $\Delta$  and  $s = (1/n) \sum s_{ii}$  for  $\sigma$  into  $H(\theta)$ . The approximate Aitken estimator is

$$(63) \quad \mathbf{b}(\hat{\theta}) = (X'H(\hat{\theta})^{-1}X)^{-1}X'H(\hat{\theta})^{-1}\mathbf{y}.$$

We conjecture that under fairly general conditions  $\mathbf{b}(\hat{\theta})$  will have desirable asymptotic properties.<sup>19</sup>

### 3.3. Random Sampling Without Replacement With Unequal Probabilities

We now generalize from simple random sampling to random sampling without replacement with unequal probabilities. We again consider two estimators: a simple weighted average of the least-squares estimators and an approximate Aitken estimator.

We make the following assumption

*Assumption 3.2:*

(1)–(3) the same as Assumption 3.1 (1)–(3).

(4) Sampling is done without replacement with unequal probabilities.  $\pi_i$  will be the overall probability that the  $i$ th unit is drawn and  $\pi_{ij}$  the joint probability that the  $i$  and  $j$ th units are drawn.

#### Weighted Average of Least Squares

From (35) it follows that a natural estimator for  $\beta$  is a simple weighted average of the least-squares estimators, where the weights are inversely proportional to the probability of being selected in the sample. That is, consider the estimator  $\mathbf{b}^*$ ,

$$(64) \quad \mathbf{b}^* = \frac{1}{N} \sum \frac{\mathbf{b}_i}{\pi_i}.$$

Using (29) and (46) we find

$$NE(\mathbf{b}^*) = E_s \left[ \sum \frac{E_c(\mathbf{b}_i|S)}{\pi_i} \right] = E_s \left[ \sum \frac{\beta_i}{\pi_i} \right] = \sum \beta_i$$

<sup>18</sup> See appendix.

<sup>19</sup> See Swamy (1971), (1972) for a discussion of large sample properties when  $N$  is infinite. His analysis needs to be modified for our work. However, much of his analysis does carry over to the present problem. For  $T$  sufficiently large with  $n$  fixed, we can treat  $\mathbf{b}_i$  ( $i = 1, 2, \dots, n$ ), as if they were sample of size  $n$  from the population of  $\beta$ 's, i.e.,  $(\beta_1, \beta_2, \dots, \beta_n)$ . Then we can combine the result with the central limit results of Hajek (1960) for finite populations, to get the full set of asymptotic properties of  $\mathbf{b}(\hat{\theta})$ . Also, see Theil (1971, p. 399). If  $\mathbf{u}$  and  $\delta$  are symmetrically distributed about the null vector, then we can use the type of argument developed by Kakwani (1967) to show that  $\mathbf{b}(\hat{\theta})$  is an unbiased estimator of  $\beta$ .

so that  $\mathbf{b}^*$  is an unbiased estimator of  $\beta$ . Let  $S_{b^*b^*}$  be the variance-covariance matrix for  $\mathbf{b}^*$ . Evaluating  $S_{b^*b^*}$  we find

$$(65) \quad S_{b^*b^*} = \frac{1}{N^2} \left[ \sum \frac{P_i}{\pi_i} + \sum \frac{\beta_i \beta'_i (1 - \pi_i)}{\pi_i} + \sum \frac{\beta_i \beta'_j (\pi_{ij} - \pi_i \pi_j)}{\pi_i \pi_j} \right].$$

By inspection of (65) we recognize that an unbiased estimator of  $S_{b^*b^*}$  is

$$(66) \quad \hat{S}_{b^*b^*} = \frac{1}{N^2} \left[ \sum \frac{(1 - \pi_i) \mathbf{b}_i \mathbf{b}'_i}{\pi_i^2} + \sum \frac{\mathbf{b}_i \mathbf{b}'_j (\pi_{ij} - \pi_i \pi_j)}{\pi_i \pi_j \pi_j} + \sum \frac{\hat{P}_i}{\pi_i} \right].$$

#### *An Approximate Aitken Estimator*

We now develop an approximate Aitken estimator for this model. As before the analysis is conditioned on  $X$ .

To construct the Aitken procedure we would like to write an observation at, say, the  $r$ th draw as

$$(67) \quad \mathbf{y}_r = X_r \beta + \mathbf{v}_r$$

where the disturbance  $\mathbf{v}_r$  satisfies

$$(68) \quad E[\mathbf{v}_r | X] = \mathbf{0}.$$

However, for random sampling without replacement with unequal probabilities,

$$\mathbf{v}_r = X_r(\beta_r - \beta) + \mathbf{u}_r$$

and

$$(69) \quad E[\mathbf{v}_r | X] = X_r \left( \sum_i \beta_i p_i(r) - \beta \right).$$

Note that the expected value of  $\mathbf{v}_r$  will not vanish unless  $p_i(r) = 1/N$ , i.e., we engage in simple random sampling. To avoid this problem we transform each draw in the following way. If the  $l$ th unit in the population is chosen on the  $r$ th draw write

$$e_r = N^{-1} p_l(r)^{-1}$$

and let

$$\tilde{\mathbf{y}}_r = \mathbf{y}_r e_r, \quad \tilde{\beta}_r = \beta e_r, \quad \tilde{\mathbf{u}}_r = \mathbf{u}_r e_r.$$

The transformed representation of the  $r$ th draw is then

$$(70) \quad \tilde{\mathbf{y}}_r = X_r \tilde{\beta}_r + \tilde{\mathbf{u}}_r$$

and the expected value of  $\tilde{\beta}_r = \beta$ . The difficulty with this particular transformation is that the variance-covariance matrix for the transformed system of  $n$  draws depends on the draw-by-draw probabilities, the  $p_i(r)$  and  $p_{ij}(r, s)$  terms. To circumvent this complication we assume that the sample design satisfies the following equations,<sup>20</sup>

<sup>20</sup> If we interpret all quantities as referring to a particular stratum then whenever the number sampled ( $n$ ) within a stratum is small relative to the number of units in the stratum ( $N$ ), equations (71) and (72) are likely to be adequate approximations (within the stratum). See Cochran (1962, p. 260-262) for a description of a common method for selecting units with unequal probabilities but without replacement which will approximately satisfy these equations within a stratum. In this case the approximate Aitken estimator developed in the text will be defined for each stratum. An estimate of the overall population mean for all strata taken together can then be formed by suitably averaging the estimates from the different strata.

$$(71) \quad p_i(r) = \frac{\pi_i}{n} \quad \text{for all } r \in \bar{n} \text{ and } i \in \bar{N}$$

$$(72) \quad p_{ij}(r, s) = \frac{\pi_{ij}}{n(n-1)} \quad \text{for all } r, s \in \bar{n}, r \neq s \text{ and } i, j \in \bar{N}, i \neq j.$$

We now analyze the transformed system of equations having the form of equation (70) for all  $r \in \bar{n}$ , where  $e_r = n/\pi_i N$  when the  $l$ th unit is chosen at the  $r$ th draw. The following results will be useful in this analysis. From (71) and (72) we can easily show that for draws  $r$  and  $s$ ,  $r \neq s$ ,

$$(73) \quad E_S(z_r) = \frac{1}{n} \sum \pi_i z_i$$

$$(74) \quad E_S(z_r z_s) = \frac{1}{n} \frac{1}{(n-1)} \sum \pi_{ij} z_i z_j.$$

From (73) we find

$$(75) \quad E_S[\tilde{\beta}_r] = \sum_i \tilde{\beta}_i p_i(r) = \sum_i \frac{\tilde{\beta}_i \pi_i}{e_i n} = \frac{1}{N} \sum_i \tilde{\beta}_i = \beta.$$

Let  $\delta_r$  be the sampling error in the transformed random coefficient  $\tilde{\beta}_r$ ,

$$(75) \quad \delta_r = \tilde{\beta}_r - \beta, \quad r \in \bar{n}.$$

By construction

$$E_S[\delta_r] = 0 \quad r \in \bar{n}.$$

Each  $\delta_r$  will have the same variance-covariance matrix, say  $\tilde{\Delta}$ ,

$$\tilde{\Delta} = E\delta_r \delta_r' = E\tilde{\beta}_r \tilde{\beta}_r' - \beta\beta'.$$

Evaluating  $\tilde{\Delta}$  gives

$$(76) \quad \tilde{\Delta} = \sum \frac{\beta_i \beta_i' (n - \pi_i)}{N^2 \pi_i} - \frac{1}{N^2} \sum \beta_i \beta_i'.$$

We assume  $\tilde{\Delta}$  is positive definite. By inspection of (76) we infer that an unbiased estimator of  $\tilde{\Delta}$  is

$$(77) \quad \hat{\Delta} = \sum \frac{\mathbf{b}_i \mathbf{b}_i' (n - \pi_i)}{N^2 \pi_i^2} - \sum \frac{\hat{P}_i (n - \pi_i)}{N^2 \pi_i^2} - \frac{1}{N^2} \sum \frac{\mathbf{b}_i \mathbf{b}_i'}{\pi_i}.$$

The covariance between  $\delta_r$  and  $\delta_s$ , say  $\tilde{\Delta}_c$ , will be identical for all  $r \neq s$  and satisfy

$$\tilde{\Delta}_c = -\frac{\tilde{\Delta}}{N-1} \quad \text{for all } r \text{ and } s \in \bar{n}, r \neq s.$$

Using the foregoing results, the system of  $nT$  observations may be written as

$$(78) \quad \tilde{\mathbf{y}} = X\beta + D(X)\delta + \tilde{\mathbf{u}}$$



where

$$\bar{\mathbf{y}} = (\bar{\mathbf{y}}'_1, \bar{\mathbf{y}}'_2, \dots, \bar{\mathbf{y}}'_n)'$$

$$\bar{\mathbf{u}} = (\bar{\mathbf{u}}'_1, \bar{\mathbf{u}}'_2, \dots, \bar{\mathbf{u}}'_n)'$$

$$\bar{\delta} = (\bar{\delta}'_1, \bar{\delta}'_2, \dots, \bar{\delta}'_n)'$$

and  $X$  and  $D(X)$  are given beneath (60). Given  $X$  the disturbance in (78) has a variance-covariance matrix  $G(\boldsymbol{\varphi})$

$$(79) \quad G(\boldsymbol{\varphi}) = \begin{bmatrix} X_1 \bar{\Delta} X'_1 + \bar{\sigma} I & X_1 \bar{\Delta}_c X'_2 & \dots & X_1 \bar{\Delta}_c X'_n \\ X_2 \bar{\Delta}_c X'_1 & X_2 \bar{\Delta} X'_2 + \bar{\sigma} I & \dots & X_2 \bar{\Delta}_c X'_n \\ \vdots & \vdots & \ddots & \vdots \\ X_n \bar{\Delta}_c X'_1 & \dots & \dots & X_n \bar{\Delta} X'_n + \bar{\sigma} I \end{bmatrix}$$

where  $\boldsymbol{\varphi}$  contains the distinct unknown parameters elements of  $\bar{\Delta}$  and  $\bar{\sigma}$ , with  $\bar{\sigma} = (n/N^2) \sum (\sigma_{ii}/\pi_i)$ .

If  $\Delta$  and  $\bar{\sigma}$  were known,

$$\mathbf{b}(\boldsymbol{\varphi}) = (X'G(\boldsymbol{\varphi})^{-1}X)^{-1}X'G(\boldsymbol{\varphi})^{-1}\mathbf{y}^*$$

would be the BLUE of  $\boldsymbol{\beta}$ . An approximate Aitken estimator may be formed by substituting  $\hat{\Delta}, \hat{\Delta}_c = \hat{\Delta}/(N-1)$ , and  $\hat{\bar{\sigma}} = (n/N^2) \sum (s_{ii}/\pi_i^2)$  for  $\bar{\Delta}, \bar{\Delta}_c$  and  $\bar{\sigma}$  into  $G(\boldsymbol{\varphi})$  to obtain  $G(\hat{\boldsymbol{\varphi}})$ ; the estimator is

$$(80) \quad \mathbf{b}(\hat{\boldsymbol{\varphi}}) = (X'G(\hat{\boldsymbol{\varphi}})^{-1}X)^{-1}X'G(\hat{\boldsymbol{\varphi}})^{-1}\bar{\mathbf{y}}.$$

If  $\hat{\Delta}$  is not positive definite (or at least positive semidefinite) we face a negative variance problem.<sup>21</sup> There does not appear to be an easy solution to the negative variance problem. One can never be sure whether or not the result arises because of a model misspecification or is just an anomaly of a given sample.

#### An Extension

It is not difficult to see how these results may be generalized to permit contemporaneous correlation between  $\mathbf{u}$ 's in the population. That is, consider

*Assumption 3.3.*

(1), (2), (4) same as corresponding conditions in Assumption 3.2.

(3) The disturbance vectors  $\mathbf{u}_i$ , ( $i \in \bar{N}$ ) each have mean zero and  $E\mathbf{u}_i\mathbf{u}'_j = \sigma_{ij}I$  for all  $i$  and  $j$ .

The correct unbiased estimator of  $\bar{\Delta}$  becomes

$$(81) \quad \hat{\Delta} = \sum \frac{\mathbf{b}_i\mathbf{b}'_i(n - \pi_i)}{N^2\pi_i^2} - \sum \frac{\hat{P}_i(n - \pi_i)}{N^2\pi_i^2} - \frac{1}{N^2} \sum \frac{\mathbf{b}_i\mathbf{b}'_j}{\pi_{ij}} + \frac{1}{N^2} \sum \frac{s_{ij}\hat{P}'_iX'_j\hat{P}_j}{s_{ii}s_{jj}\pi_{ij}}$$

where

$$s_{ij} = \mathbf{y}'_i M_i M_j \mathbf{y}_j / \text{trace}(M_i M_j)$$

<sup>21</sup> See Swamy (1971) and Schmalensee (1972) for discussions of this problem and additional references.

The matrix  $G(\Phi)$  and therefore  $G(\hat{\Phi})$  changes also for Assumption 3.3. The  $i$ th block diagonal matrix is still

$$X_i \bar{\Delta} X_i' + \bar{\sigma} I$$

but the  $i$ -jth off-diagonal matrix becomes

$$X_i \bar{\Delta}_c X_j' + \bar{\sigma}_c I, \quad \text{where } \bar{\sigma}_c = \frac{1}{n(n-1)} \sum \pi_{ij} \sigma_{ij} e_i e_j.$$

#### 4. FUTURE EXTENSIONS

In this paper we explore the consequences of using information on the design of a sample survey to estimate population averages in a linear model. An analysis of the sampling properties of the alternative estimators considered awaits further study.

Finally, we treat the sample design as being given exogenously. It may prove illuminating to relax this assumption and rank alternative sample designs on the basis of their precision in estimating population averages in a linear model.

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#### APPENDIX: INVERSE OF $H(\theta)$

$H(\theta)$  may be written as

$$(1) \quad H(\theta) = D[Z \otimes \Delta]D' + \sum \otimes I \\ = R + DBD'$$

where  $\otimes$  is the Kronecker product symbol,

$$R = \sum \otimes I_T, \quad \sum = \sigma I_n, \quad B = Z \otimes \Delta,$$

and  $Z = (z_{ij})$  is an equicorrelated matrix with  $z_{ii} = 1$ ,

$$z_{ij} = -z, \quad i \neq j.$$

Since  $\Delta$  is positive definite (by assumption)  $\Delta^{-1}$  exists. The inverse of  $Z$  is readily found, see Rao (1965, p. 53, problem 2(ii)).

Now

$$(2) \quad R^{-1} = \sum^{-1} \otimes I_T$$

$$(3) \quad B^{-1} = Z^{-1} \otimes \Delta^{-1}.$$

Finally, using a result given in Rao (1965, p. 29, problem 29), we find

$$(4) \quad (R + DBD')^{-1} = R^{-1} - R^{-1}D(D'R^{-1}D)^{-1}D'R^{-1} \\ + R^{-1}D(D'B^{-1}D)^{-1}((D'R^{-1}D)^{-1} + B)^{-1}(D'R^{-1}D)^{-1}DR^{-1}.$$

Inspecting the r.h.s. of (4) we note that in view of (2) and (3), the largest matrix to be inverted is  $nK$  by  $nK$ . If  $\Delta$  is positive semidefinite,  $H(\theta)$  is also nonsingular.

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