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Chapter Title: Appendix B - A Mathematical Analysis of the Relationship of New Credits, Total Credits Outstanding and Net Credit Change

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APPENDIX B

A Mathematical Analysis of the Relationship of New Credits, Total Credits Outstanding and Net Credit Change

A Mathematical Analysis of the Relationship of New Credits, Total Credits Outstanding and Net Credit Change

IN ORDER to make the problem determinate it is assumed that all new contracts contain the same terms, and provide for amortization of the loan according to some definite pattern. As successive payments are made the total discounted value of the remaining future payments falls off. Let p(a) be the fraction of the initial value of the contract which remains at *a* units of time after the contract was made. At the beginning of the period this must be equal to unity, and after some finite time, n, no value remains. Therefore

$$\begin{aligned} p(0) &= 1 \\ p(a) &\equiv 0, \quad a \geqq n \end{aligned}$$

If the terms of the contract call for amortization of the loan at a constant rate per unit of time, so that value falls off linearly, we have

$$\begin{aligned} p(a) &= 1 - \frac{a}{n}, \quad a \leqq n \\ &= 0 \qquad , \quad a \geqq n \end{aligned} \tag{1}$$

This is the case considered in the text. Usually, however, contracts call for instalment payments of equal magnitude. Since each instalment covers both reduction of the value of the loan and interest on the outstanding balance, this requires that the rate of amortization of the loan be small at first when interest charges are large. As the value falls off, interest charges decrease so that amortization may proceed ever more rapidly. Actually we must have

$$\begin{aligned} p(a) &= \frac{(1+i)^n - (1+i)^a}{(1+i)^n - 1}, \quad a \leqq n \\ &\equiv 0, \qquad\qquad\qquad\qquad a \geqq n \end{aligned} \tag{2}$$

where i is the rate of interest per unit of time.[1] In the limit, as the

[1] This formula is derived as follows. The amount by which value is reduced plus interest charges on outstanding balances must equal a constant, or

interest rate becomes negligible, this "sinking fund" formula reduces to the straight line amortization formula given above.[2] In the accompanying Chart B-I examples of the straight line and sinking fund formulas are shown.

Unless statement is made to the contrary, the present analysis makes no special assumptions concerning the amortization function p(a) except that it can never be increasing. For some purposes it is convenient to consider only discrete intervals of time so that *a* ranges over integral values exclusively. In such cases the continuous function p(a) can be replaced by the discrete series

$$p_o(=1),\ p_1,\ p_2,\ \cdots,\ p_j,\ \cdots,\ p_n,\ 0,\ \cdots$$

At any time, t, the total volume of outstanding credits, O_t, equals the sum of credits of all ages; that is, new credits plus credits left from the previous period, plus credits left from two periods back, . . . , plus credits from n periods back. Hence

$$\begin{aligned} O_t &= 1 \cdot c_t + p_1 c_{t-1} + p_2 c_{t-2} + \cdots + p_n c_{t-n} \\ &= \sum_0^n p_j c_{t-j} \end{aligned} \tag{3}$$

where c_t, c_{t-1}, . . . represent respectively new credits at time t, $t - 1$, . . . The net change in outstanding credits is equal to the volume of new credits minus repayments, or

$$\begin{aligned} O_t - O_{t-1} &= c_t - (p_o - p_1)c_{t-1} - (p_1 - p_2)c_{t-2} - \cdots \\ &\quad - (p_{n-1} - p_n)c_{t-n} - (p_n - 0)c_{t-1-n} \\ &= c_t - \sum_0^n w_j c_{t-1-j} \end{aligned} \tag{4}$$

where w_o, w_1, $\cdots$, w_n represent respectively $(p_o - p_1)$, $(p_1 - p_2)$,

[2] In the limit this becomes an indeterminate form which must be evaluated by L'Hospital's rule as follows:

$$\lim_{i \to 0} \frac{(1+i)^n - (1+i)^a}{(1+i)^n - 1} = \frac{0}{0} = \frac{\lim\limits_{i \to 0} n(1+i)^{n-1} - a(1+i)^{a-}}{\lim\limits_{i \to 0} \quad n(1+i)^{n-1}}$$

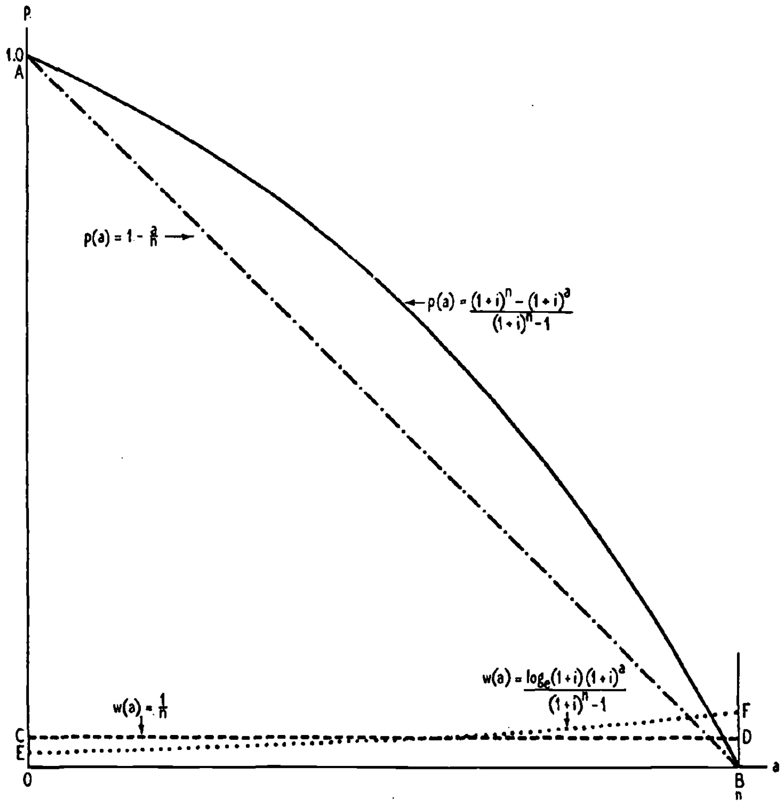
$$= \frac{n-a}{n} = 1 - \frac{a}{n}$$

$-p'(a) + rp(a) = c$, where $r = \log_e(1 + i)$ is the force of interest. This differential equation plus the boundary conditions that $p(0) = 1$ and $p(n) = 0$, determine the function written above.

In the case of discrete time periods we have, instead of a differential equation, the difference equation $-\Delta p_j + ip_j = c$, where $\Delta p_j = p_{j+1} - p_j$. This leads to the same final equation, now defined only for the integral values of a.

CHART B-I

THE "STRAIGHT LINE" AND "SINKING FUND" AMORTIZATION FUNCTIONS



. . . , $(p_n - p_{n+1})$. Since the value of each contract decreases with time, each of the w's is positive. Moreover,

$$\sum_{0}^{n} w_j = 1 - p_1 + p_1 - p_2 + p_2 - \cdots - p_n + p_n - 0 + 0 \cdots$$
$$= 1$$

Therefore the second expression on the right-hand side of equation 4 represents a weighted average of new credits over the previous n + 1 periods. This fact will aid in the determination of lags between the two series.

As an alternative to equation 3, total credits outstanding can be expressed as the difference between new credits cumulated from some date and repayments cumulated from the same date, plus outstanding credits at the date from which cumulation is made.

$$O_t = O_m + C_t - \sum_0^n w_j C_{t-1-j} \tag{5}$$

where C_t represents the new credits cumulated from m until t as follows:

$$C_t = c_{m+1} + c_{m+2} + \cdot \cdot \cdot + c_{t-1} + c_t = \sum_{m+1}^{t} c_k \tag{6}$$

In the continuous case, when $p = p(a)$ and $-p'(a) = w(a)$, we have corresponding to equations 3, 4, 5 and 6

$$O(t) = \int_0^n p(a)\, c(t-a)\, da \tag{3'}$$

$$O'(t) = c(t) - \int_0^n w(a)\, c(t-a)\, da = \int_0^n p(a)\, c'(t-a)\, da \tag{4'}$$

$$O(t) = O(m) + C(t) - \int_0^n w(a)\, C(t-a)\, da \tag{5'}$$

$$C(t) = \int_m^t c(v)\, dv \tag{6'}$$

Integration and differentiation take the place of summation and differencing.

From a consideration of the above relations the following generalizations can be readily deduced. (a) If new credits are always constant, so will be outstanding credits. (b) If new credits always increase (decrease) linearly, so will outstanding credits; a similar theorem holds if new credits follow any polynomial law. (c) An exponential rate of increase in new credits will result in the same exponential rate of increase in outstanding credits. (d) A monotonically increasing (decreasing) new credits series will result in an ever-increasing (decreasing) series of outstanding credits. (e) If new credits vary periodically, so also will outstanding credits, and with the same period.

These theorems admit of simple mathematical proof. In what follows, unprimed small Greek letters will represent constants characteristic of new credits, and primed Greek letters will represent corresponding constants typifying total outstanding credits.

(a) When $c(t) \equiv \alpha$

$$O(t) \equiv \alpha \int_0^n p(a) da = \alpha'$$

(b) When $c(t) \equiv \beta + \gamma t$

$$O(t) = \int_0^n (\beta + \gamma t - \gamma a)p(a)da = \beta' + \gamma' t$$

$$\text{where } \beta' = \beta \int_0^n p(a)da - \gamma \int_0^n ap(a)da$$

$$\text{and } \gamma' = \gamma \int_0^n p(a)da$$

(c) When $c(t) \equiv \delta e^{rt}$

$$O(t) = \delta \int_0^n e^{rt}e^{-ra}p(a)da = \delta' e^{rt}$$

$$\text{where } \delta' = \delta \int_0^n e^{-ra}p(a)da$$

(d) When $c'(t) > 0$

$$O'(t) = \int_0^n p(a)c'(t-a)da > 0$$

since $p(a) > 0$

(e) If $c(t) \equiv c(t-\theta)$

$$O(t) = \int_0^n p(a)c(t-a)da = \int_0^n p(a)c(t-\theta-a)da = O(t-\theta)$$

All of the above theorems have assumed that new credits follow the above laws for all time. Actually, if new credits follow such a law only after some time, say t^o, the above theorems will hold for outstanding credits after time $t^o + n$; that is, with a lag equal to the maximum length of the contract.

A further generalization can be made concerning the relation between the mean of the new credits series and the mean of the total outstanding credit series. From the well-known statistical theorem that the mean of a sum is equal to the sum of the means, we have

$$\overline{O} = \sum_0^n p_j \bar{c}_{t-j} \tag{7}$$

where $\overline{O}$ represents the mean of outstanding credits, and $\bar{c}_{t-j}$ is the mean of new credits lagged j periods. Aside from a minor adjustment at the ends of the statistical series, each of the respective different series $c_t, c_{t-1}, \cdots, c_{t-j}, \cdots$ contains the same terms, differing only by a time lag. Hence each has the same mean equal to $\bar{c}$. Therefore,

$$\overline{O} = \bar{c} \sum_0^n p_j \tag{8}$$

Because of the linearity of the relation between O and c, relations similar to those in equations 3, 4 and 5 hold between the deviations from the trend of O and the trend deviations of c. Let us write

$$c_t = c_t' + c_t'' \tag{9}$$

where c_t' = trend value of new credits, and c_t'' = deviations from trend. Then

$$O_t = O_t' + O_t'' \tag{10}$$

where

$$O_t' = \sum_0^n p_j c_{t-j}'$$

$$O_t'' = \sum_0^n p_j c_{t-j}''$$

For the most part we can omit considerations of trend, and deal only with deviations from trend. These will form a quasi-periodic oscillatory series. In the text it is stated that the amplitude of fluctuation of outstanding credits is less than that of new credits. This can now be proved. As a measure of amplitude, the standard deviations of the respective series suggest themselves. These are not entirely satisfactory since the amount of new credits per unit of time depends upon the length of the time unit and is not dimensionally comparable with total credits outstanding. But if we divide each standard deviation by the mean of the corresponding series, units are standardized and a valid measure of amplitude is derived.

The O series is a linear sum of the c series. By a well-known statistical theorem the square of its standard deviation can be written

$$\sigma_o^2 = \sum_0^n \sum_0^n p_j p_k \sigma_{c_{t-j}} \sigma_{c_{t-k}} r_{c_{t-j}, c_{t-k}} \tag{11}$$

where $r_{c_{t-j}, c_{t-k}}$ represents the serial correlation coefficient between new credits lagged j and k units respectively. Aside from terminal adjustments the respective series $c_t, c_{t-1}, \ldots$ contain the same terms and have a common standard deviation, σ_c. Therefore

$$\sigma_o^2 = \sigma_c^2 \sum_0^n \sum_0^n p_j p_k r_{c_{t-k}, c_{t-j}} < \sigma_c^2 \sum_0^n \sum_0^n p_j p_k \tag{12}$$

since some of the correlation coefficients will be less than unity. But

$$\left(\frac{\sigma_o}{\overline{O}}\right)^2 = \frac{\sigma_o^2}{\bar{c}^2 (\sum_0^n p_j)^2} = \frac{\sigma_o^2}{\bar{c}^2 \sum_0^n \sum_0^n p_j p_k} < \left(\frac{\sigma_c}{\bar{c}}\right)^2 \tag{13}$$

Thus our theorem is proved. It could also be shown that the net change in outstanding credit expressed in standard units is a weighted average of net changes in new credits also expressed in standard units. Hence the changes in outstanding credits will be less extreme than those of the new credits series.

We now consider the timing of the respective series. No restriction is placed upon the series of new credits except that its oscillations should be fairly simple; more specifically, its peaks and troughs should always be separated by a zero value of the series (a trend value of the unadjusted series). Let U_c, L_c, T_c represent respectively upper turning points, lower turning points, and trend or zero values. The series need not be perfectly regular or symmetric, but we must have the following succession: . . . $U_c\, T_c\, L_c\, T_c\, U_c\, T_c\, L_c$ . . . Continuous rather than discrete series are investigated.

No restrictions are placed upon the form of the amortization function p(a) except that the maximum length of the contract must be less than the distance between both successive trend values and successive turning points. (For a regular and symmetric cycle, such as a sine curve, this means that the maximum length of the contract must not exceed one-half the length of a complete cycle.)

Under these mild conditions the following definite theorem can be enunciated: *the peaks and troughs of total credits outstanding will lag behind the respective peaks and troughs of the new credits series; the lag of the turning points cannot, however, be larger than the interval between such a turning point and the succeeding trend value.* Thus the O series will have a maximum after the new credits series has begun to fall but before it reaches its zero or trend value. For a regular and symmetric series the time lag cannot exceed one-fourth the length of a complete cycle.

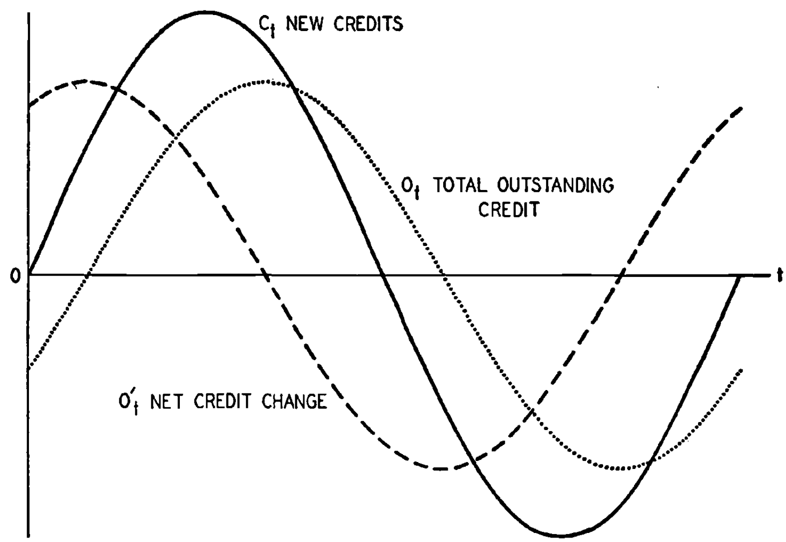
Employing analogous notation for the turning points and trend values of the O series, we must have the following timing relations:

$$\begin{array}{cccccccccccc} U_c & & T_c & & L_c & & T_c & & U_c & & T_c & & L_c & & T_c \\ & U_O & & & & L_O & & & & U_O & & & & L_O & \end{array}$$

That outstanding credit must be rising when new credits are at a peak follows from the fact that new credits are increasing monotonically in the previous interval. When new credits are equal to zero (at the trend value), repayments are still positive because new credits were necessarily positive in the previous interval. Therefore when new credits are at the trend value, outstanding credits are falling. Hence at some intermediate point a maximum must have been reached. A similar argument holds for a minimum.

A few further generalizations can be made. If the new credits series is symmetrical around its turning points or trend values, the respective lags cannot exceed one-half the maximum length of the contract. In considering different types of contracts it will be found that those which provide for rapid amortization during the early life of the contract will be characterized by shorter lags at the turning points, other things being equal. The sharpness of the new credits peaks and troughs does not seem to affect the lag so much as the skewness at the peak; that is, the more rapidly new credits fall off

CHART B-II
IDEAL RELATIONSHIP OF NEW CREDITS, TOTAL OUTSTANDING CREDIT AND NET CREDIT CHANGE[a]



[a] All three curves represent deviations from trend values of the variables.

compared to the rise approaching the peak, the sooner will the outstanding credit series turn down.

For many problems it is important to know the relationship between new credits and the rate of change of outstanding credits (net credit change). This turns out to be precisely the same formal relationship as the acceleration principle postulates between the rate of increase of consumption and net investment. Ordinarily the turning points of new credits lag *behind* the turning points of net credit change, just as the acceleration principle requires gross investment to fall *after* consumption has failed to maintain its rate of increase. The above statement has never been adequately substantiated in the literature. It is certainly true for sine curves and symmetric regular periodic curves; and if some special assumptions are made concerning amortization (replacement), it will hold for all possible behavior of the curves. But it is not valid in all conceivable cases.

All of the propositions enunciated previously are strictly true under the hypotheses assumed. In particular, the maximum length of contract has been assumed never to exceed the length of the cycle in new credits. If we are considering short cycles (such as seasonal fluctuations in new credits) or goods of extreme durability (such as housing), this may involve a serious restriction. Empirical experimentation with hypothetical examples suggests that in a wide variety of cases the turning points of outstanding credits will follow those of new credits by considerably less than a quarter cycle, even when the maximum length of contract greatly exceeds the length of the cycle. If, however, the new credits series is quite oscillatory, with many maxima and minima, it is likely that the outstanding credit series—because it "averages out" fluctuations in new credits—may have fewer maxima and minima than new credits.

In Chart B-II is depicted the typical relationship of new credits, total outstanding credit and net credit change, when new credits vary according to a perfect sine curve and when the contract calls for straight line amortization over a period equal to one-fourth of the cycle. The lag of outstandings behind new credits is approximately one-twelfth of the total cycle. Net credit change leads total outstanding credit by one-fourth of the cycle, and leads new credits by about one-sixth of the cycle.

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