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CHAPTER II

THE CONCEPT OF LONG TERM INTEREST RATES

THEORETICALLY, a rate of interest is a measure of an exchange relation between present economic goods and future economic goods of so nearly the same kind as to be, for the purposes of the exchange relation, considered identical. In actual practice, the concept is almost invariably purely monetary. Though interest rates the world over are continually being expressed in terms of the convertible or inconvertible currencies of various countries and in terms of metallic monetary standards such as gold or silver, they are seldom expressed in terms of any such non-monetary commodities as wheat or cotton or even in terms of any 'composite' commodity whose price might be assumed to fluctuate with 'the general level of commodity prices'.

There are two essential elements in the interest concept. To derive any *rate* of interest from a stated set of facts, we must know (1) what is the ratio of the future quantity of money or other good, in which the rate is to be expressed, to the present quantity for which it is being exchanged, and (2) what is the length of time elapsing between the 'present' and 'future' of the particular problem. For example, if a lender gives up a present $10,000 in exchange for a promised payment of $11,025 two years from now and if he actually receives the $11,025 at the expiration of the two years, the rate of interest which he will have received during the two-year period will be 10¼ per cent *biennially,* or 10¼ per cent *per two-year period.*

If the lender had obtained the $11,025 at the end of the two-year period by lending $10,000 for a payment of $10,500 at the end of the first year, and then lending this $10,500 for a payment of $11,025 at the end of the second year, he would be able to say not only that he had realized 10¼ per cent *biennially* during the two-year period but also that he had realized 5 per cent *annually* during each of the one-year periods. However, only by assuming that he had obtained the same rate in each of the two years could he accurately describe the rate during the two-year period of the first case as 5 per cent *per annum.* Unless such an assumption be made, the 5 per cent figure is a mere 'average'. It tells us nothing about the rates which either the lender or the borrower should consider that he had actually realized in the separate years. For example, if the lender could have obtained only 4 per cent per annum for a one-year loan, he must logically consider that he is obtaining more than 5 per cent per annum for the second year. Moreover, this same reasoning applies in its fullness to even such an apparently clear-cut case as that in which a present $10,000 is exchanged for $500 payable one year hence and $10,500 payable at the end of two years. If the lender could have obtained only 4 per cent per annum for a one-year loan, he must think of the $500 payment as made up of $400 interest and $100 payment on the principal sum, and of the $600 difference between $10,500 and $9,900 as one year's interest on a loan of $9,900. 'Long time interest rates' are always mere 'averages' of short time rates.

At 5 per cent per annum, compounded annually, $10,000 would, in two years, grow into $11,025. The $10,000 is the 'present value' of $11,025 due two years hence with interest at 5 per cent per annum, compounded annually. The 'present value' of a specified sum of future money, due in a specified time, and upon the assumption of a particular uniform rate of interest until the payment of the future sum and a particular 'compounding period', is such a sum of present money as would grow into the specified future sum, in the specified time, at the specified rate of interest and with the specified 'compounding period'. The concept is purely mathematical. The question whether the assumptions are, in fact, legitimate or absurd has nothing to do with the problem of calculating the 'present value'. If 6 per cent per annum had been assumed as the rate of interest, instead of 5 per cent, the 'present value' of the $11,025 due two years hence would have been $9,812.22+ instead of $10,000. If 100 per cent per annum had been the assumed rate, the 'present value' would have been $2,756.25 instead of either $9,812.22+ or $10,000. Having made these preliminary observations, we are in position to discuss the meaning that must be attached to the 'yield' of a 'bond'.

In the modern economic world the commonest examples of 'long time interest rates' are furnished by the 'yields' of long term 'bonds'.

The typical bond is a promise to make a series of periodic 'interest' payments (usually one every six months) and a payment of a 'principal' sum at 'maturity'.[1] The 'yield' of a bond selling at a specified price is that rate of interest which, if it be assumed in order to obtain the 'present values' of the various future payments, will make the sum of such 'present values' equal the specified price of the bond.

If the reader will examine a 'bond table', he will find that if a 4 per cent $100 bond, interest payable semi-annually, maturing in 2½ years, sells for $97.68, it 'yield 5 per cent per annum'. However, since ordinary bond tables give, as the annual yield, *twice the semi-annual yield,* this '5 per cent per annum' means that the yield is 2½ per cent per six months' period, compounded semi-annually.[1a] But exactly what does this semi-annual yield of 2½ per cent mean? Like most mathematical questions, this may be correctly answered in many ways, but two seem peculiarly enlightening.

The price paid for the bond ($97.68) equals the sum of the 'present values' of the five $2 'interest' payments and the $100 'principal' payment. The 'present value' of the $2 interest payment due six months hence is $\frac{\$2.00}{1.025}$ or, to the nearest cent, $1.95. Similarly the present value of the $2 interest payment due one year hence is $\frac{\$2.00}{(1.025)^2}$, or $1.90, and the present value of the interest payment due eighteen months hence is $\frac{\$2.00}{(1.025)^3}$, or $1.86. The present values, to the nearest cent, of the five $2.00 interest payments are: $1.95, $1.90, $1.86, $1.81,

1 The semi-annual payments made to the investor are semi-annual payments and nothing more. To term them 'interest payments' is somewhat misleading, but the terminology is so thoroughly established, and in general so well understood, that to speak of 'dividend payments' or to introduce some other term would probably be more disturbing than to keep to the established usage. Similarly the 'principal' of a bond is universally understood to mean the 'face' of the bond or the amount payable at maturity (excluding the last coupon) and not the amount originally lent or the amount later invested in the bond by any subsequent purchaser.

1a The usual practice of the makers of bond books is to calculate the yields in terms of the 'compounding period' and to assume that the compounding period equals the time between interest payments. This yield is then multiplied by the number of compounding periods in a year and presented as a yield *per annum*. This is a harmless convention—if understood. Of course 2½ per cent compounded semi-annually amounts to $100(1.025^2-1)$ or 5.0625 per cent, and not 5 per cent, compounded annually.

$1.77. Similarly, the present value of the principal payment of $100, due in 2½ years, is $88.39. The total of these six present values is $97.68, and this is therefore the price paid for the bond.

Another way of looking at the problem, which some persons find even more enlightening, is the following: the buyer pays $97.68 for the bond. If he is to receive 2½ per cent semi-annually on his investment, there will be owing to him, at the end of six months, 2½ per cent of $97.68, or $2.44. However, he accepts $2 (the 'interest' payment called for by the first 'coupon') and leaves the extra 44 cents with the borrower to draw 2½ per cent semi-annually. The borrower then owes him $97.68 + $0.44, or $98.12. This now bears interest at 2½ per cent semi-annually. And so forth. The procedure can be clearly shown in a table.

Value of bond at time of purchase (price paid)	$97.68
Accrued interest (2½ per cent of 97.68)	+ 2.44
Value of bond just before payment of first coupon	100.12
Payment of second coupon	— 2.00
Value of bond immediately after payment of first coupon	98.12
Accrued interest (2½ per cent of $98.12)	+ 2.45
Value of bond just before payment of second coupon	100.57
Payment of second coupon	— 2.00
Value of bond immediately after payment of second coupon	98.57
Accrued interest (2½ per cent of $98.57)	+ 2.46
Value of bond just before payment of third coupon	101.03
Payment of third coupon	— 2.00
Value of bond immediately after payment of third coupon	99.03
Accrued interest (2½ per cent of $99.03)	+ 2.48
Value of bond just before payment of fourth coupon	101.51
Payment of fourth coupon	— 2.00
Value of bond immediately after payment of fourth coupon	99.51
Accrued interest (2½ per cent of $99.51)	+ 2.49

The amount the holder of the bond receives at maturity—$100.00 principal plus $2 interest (called for by the fifth coupon) 102.00

From the above illustrations the reader will notice that, though the present value of a distant future payment is of course less than the present value of a near payment, there is, in terms of dollars, only one 'yield' for the bond.[2] The 'yield' is a *single rate of interest* such that the present value of all the future payments, if they were calculated by assuming this rate (with the *semi-annual* compounding convention), would equal the price paid for the bond. It is a technical mathematical concept.[3]

In the illustration of the bond maturing in 2½ years, bought at $97.68 and paying $2 semi-annually, which we have been using, a naive and simple way of looking at the rate of interest would be to state that for two years the buyer receives 2.047 per cent semi-annually on his investment of $97.68,[3a] and then for six months receives 4.422 per cent semi-annually on his investment (still $97.68).[4] Finally, at the expiration of the last six months, he also receives the return of his loan, namely $97.68. Or, using the semi-annual compounding convention of the bond tables, the bond would be thought of as paying 4.094 per cent per annum for two years and then 8.844 per cent per annum for six months. However, neither of these figures is the 'yield' of the bond. The bond has only one yield, namely, 5 per cent per annum. The 'yield' is a species of 'average'.[5]

[2] Assuming, of course, that the compounding period is stated—as, for example, quarterly, semi-annually, or annually. In our discussion we are assuming semi-annual compounding. See note 1a.

[3] It should be noted that the various amounts given in the preceding illustration as "Value of bond immediately after payment of ——— coupon" are prices at which the bond would yield 2½ per cent per annum to maturity. An examination of a table showing the prices at which a 4 per cent bond would yield 5 per cent per annum will show that, for maturities of 2½ years, 2 years, 1½ years, 1 year and ½ year, the prices are $97.68, $98.12, $98.57, $99.03, $99.51.

[3a] $2 is 2.047+ per cent of $97.68.

[4] $102 — $97.68 = $4.32, which is 4.422 per cent of $97.68.

[5] The 'yield' per annum of a single payment loan (no 'interest' payments) is a simple function of the geometric averages of the various 'accumulation factors' for the separate compounding periods, whatever those factors or the rates of which they are functions may be assumed to be. For example, if the compounding period be a year, the 'yield' per annum of a single payment loan due in three years and carrying 4 per cent interest the first year, 5 the second, and 6 the third year is $100\ (\sqrt[3]{1.04 \times 1.05 \times 1.06} - 1)$ or a shade less than 5 per cent.

On the other hand, the buyer, in making up his own mind as to what he would be willing to pay for the bond, might use, as his *personal* rates of interest, 4 per cent per annum for the first six months, 4½ per cent for the next six months, then 5 per cent, 5½ per cent, and 6 1/10 per cent. Using these particular rates he would find that he could afford to pay just $97.68 for the bond. Four per cent, 4½ per cent, 5 per cent, 5½ per cent and 6 1/10 per cent would be the rates of interest that he considered appropriate and that he was using for the successive half-yearly periods, but they would not be the 'yield' of the bond. There would be only one 'yield' to the bond, namely, 5 per cent per annum (2½ per cent per half-year).

Though we have been emphasizing that there is only one 'yield' to a bond, it does not follow, as we have also suggested above, that because there is only one 'yield' there is only one *rate of interest*. Indeed, there is clear-cut evidence that this is not true. For many economic purposes the 'yield' of a bond must be considered as an *average* of various rates of interest used during successive future periods.[6]

Variations in the 'yield' of loans of the same grade but of different maturities would seem not only to offer conclusive evidence that 'yield' should be thought of as an average, but also to throw some light on the *implicit* interest rates for the successive years. Both municipalities and corporations often offer 'serial' bonds with a large choice of maturity, the various maturities having different 'yields'. For example, on May 15, 1930, the City of Detroit, Michigan, offered to the public $9,350,000 of 4¼ per cent bonds of which not less than $227,000 matured each May 15 from May 15, 1931 to May 15, 1960. The 'yields' at which the various maturities were offered were: 1931, 3.50 per cent; 1932, 4.00 per cent; 1933, 4.10 per cent; 1934, 4.20 per cent; 1935 to 1960 inclusive, 4.25 per cent.

If the above 'yields' were properly adjusted to the market, and if costs of underwriting are excluded, the City of Detroit could, on May 15, 1930, borrow for one year at 3.50 per cent. Unless the city would have had to pay, on May 15, 1931, more than 4 per cent to borrow

(Footnote [5] concluded)

When there are 'interest payments' the 'average' is of a less simple and unweighted kind than in the case of a single-payment loan. But it is essentially an average.

[6] The different rates of interest for the successive periods covered by the bond must, of course, be such that they give the same total present value as would be obtained by assuming the uniform rate of interest called the 'yield'.

for *another* year, it would have been cheaper to make two separate successive loans, each running one year, than to borrow for two years at 4 per cent, as the second maturity proposes. Indeed, a little computation will prove that unless the city would have had to pay on May 15, 1931 as high a rate as 4.524 per cent, it would have been as cheap to borrow twice, each time for one year (once on May 15, 1930 at 3.50 per cent, and again on May 15, 1931 at 4.524 per cent), as it was to borrow once for two years at 4 per cent.[7]

If the schedule of 'yields' has any logical foundation it must mean that the Detroit municipal authorities or their banking advisers considered (whether quite consciously or not) 3.50 per cent a 'proper' rate of interest to use during the first year and 4.524 per cent a 'proper' rate to use during the second year. Assuming then these two rates of interest, we may from the 'yield' (4.10 per cent) of the bond maturing in three years (May 15, 1933) discover the implicit rate of interest assumed to be proper the third year. Proceeding in this manner we find that the 'yields' for the successive maturities of these Detroit bonds implicitly involve a set of interest rates for the successive years. These implicit interest rates are 3.500 per cent, 4.524 per cent, 4.311 per cent, 4.529 per cent, 4.475 per cent, 4.247 per cent, 4.250 per cent

[7] If a 4¼ per cent bond having two years to run 'yields' 4 per cent, it sells for $100.476. How such a bond may just as well be considered as giving a return of 3.50 per cent per annum during the first year and 4.524 per cent per annum during the second year, as 4 per cent per annum during both years, is shown in the following table (the error of one cent on a thousand dollar bond results from dropping decimals):

Price paid for the bond	$100.476
Accrued interest (1.75 per cent of $100.476)	1.758
Value of bond just before payment of 1st coupon	102.234
Payment of 1st coupon	2.125
Value of bond immediately after payment of 1st coupon	100.109
Accrued interest (1.75 per cent of $100.109)	1.752
Value of bond just before payment of 2nd coupon	101.861
Payment of 2nd coupon	2.125
Value of bond immediately after payment of 2nd coupon	99.736
Accrued interest (2.262 per cent of $99.736)	2.256
Value of bond just before payment of 3d coupon	101.992
Payment of 3d coupon	2.125
Value of bond immediately after payment of 3d coupon	99.867
Accrued interest (2.262 per cent of $99.867)	2.259
The amount the holder of the bond receives at maturity—$100.00 principal plus $2.125 interest called for by the 4th coupon	$102.126

(for the seventh year and for each succeeding year up to May 15, 1960).

On the same date (May 15, 1930) that the City of Detroit offered to the public the bonds just discussed, the New York Central Railroad Company offered a series of 4½ per cent Equipment Trust Certificates with the same maturities as the Detroit bonds.[8] The 'yields' at which the various maturities were offered were: 1931, 4.00 per cent; 1932, 4.20 per cent; 1933, 4.35 per cent; 1934, 4.40 per cent; 1935–45 inclusive, 4.50 per cent. The interest rates for the successive years implicit in these 'yields' are: 4.000 per cent, 4.412 per cent, 4.668 per cent, 4.564 per cent, 4.949 per cent, 4.500 per cent (for the sixth year and for each succeeding year up to May 15, 1945). A comparison of these figures with the corresponding figures for the Detroit city bonds shows that the two series are not very similar. While the railroad offers a full one-half per cent per annum more on the one-year notes, its two-year notes yield only one-fifth per cent per annum more than the city's two-year notes. As a result of these facts the implicit rate of discount for the second year is actually *less* for the railroad than for the city. Both the railroad and the city implicit interest series are quite irregular. For example, the city series shows a sharp peak in the second year and the railroad series a sharp peak in the fifth year. For the third year the city series is lower than in either the second or fourth year while the railroad series is higher than in either the second or fourth year. Similarly, the fourth year shows a maximum for the city series and a minimum for the railroad series.

By June 1931 the New York Central Railroad was offering more of this same series of 4½ per cent Equipment Trust Certificates (dated May 15, 1930 and maturing serially May 15, 1932 to May 15, 1945). The 'yields' at which the various maturities were offered, however, were startlingly different from what they had been in May 1930. For the successive maturities the 'yields' were:

YEAR	PER CENT	YEAR	PER CENT	YEAR	PER CENT
1932	2.00	1937	3.625	1942	3.85
1933	3.00	1938	3.70	1943	3.90
1934	3.50	1939	3.70	1944	3.90
1935	3.50	1940	3.70	1945	3.95
1936	3.625	1941	3.80		

[8] Except that May 15, 1945 is the last maturity of the New York Central bonds, and May 15, 1960 the last maturity of the Detroit bonds.

This series shows irregularities in the implicit interest rates quite similar to those inherent in the two preceding illustrations. For example, the implicit interest rate for the fourth year is much lower than that for either the third or fifth year.

The successive short term interest rates that are implicit in the 'yields' of serial bonds at the issue prices seem only by accident ever to be other than quite erratic. For example, while the mere fact that the future was unknown might explain why the Detroit authorities were willing to pay higher rates than 3.50 per cent on the longer term bonds, if the payment of such higher rates were necessary to complete their financing with the maturities they desired, it can hardly explain the curious ups and downs shown by the sequence of the various implicit short term (annual) rates. Any rational decision as to what should be the 'yields' assigned to the successive maturities in a group of serial bonds logically involves a conscious forecast of successive short term interest rates. It does not seem possible that the erratic short term rates implicit in the Detroit serial bonds (or in either of the two New York Central emissions) were the result of a detailed and definite set of forecasts of future short term rates or even that they were the result of the superimposing of a forecast of the future financial condition of the city (or the railroad) on any reasonable forecast of general market short term rates. In practice, the city authorities probably decided first upon the maturities and the amount to come due on each maturity and then made a rough-and-ready guess of the various 'yields' that would suffice to sell the bonds. We cannot even say that the 'yields' are estimates by the issuing group of the current appraisal of future short term rates by the bond-buying public. The rates themselves offer almost conclusive evidence that no such appraisal is made by either the issuing group or the bond-buying public. The primary reason that implicit short term rates are nearly always *erratic* would seem to be that they are almost never the result of conscious forecasting. Though they are mathematically implicit in the various 'yields' no recognition is given to that fact.[9]

[9] The reader must not, from the above discussion, assume that we consider 'yield' to be a useless concept. Though it must be thought of as an average, it is an average of which we cannot discover the individual items. The fact that a train makes a 100 mile run at an 'average' speed of 40 miles an hour is a piece of real information even if we know nothing about its speed at various times and places. We have seen that even the 'implicit' rates which may be obtained from the 'yields' of serial

If future rates for the highest grade of six-month obligations were being accurately forecast, a bond of the highest grade would, theoretically, realize in each future half-yearly period between coupons the same return as that carried by six-month obligations at the beginning of the period. The *price* of the bond must fluctuate in such a manner as to attain this objective. If in a tight short term money market in which six-month obligations of the highest grade are selling on a 7 per cent per annum basis, a 4 per cent bond be selling at par, its *price* at the end of the six-month period must have *risen* to $101.50, if it is to show a return of 7 per cent per annum for the six-month period. This, of course, means a *fall* in the 'yield' during the six months. To preserve the theoretical relationship between present long term and future short term interest rates, the 'yields' of bonds of the highest grade should *fall* during a period in which short term rates are higher than the yields of the bonds and *rise* during a period in which short term rates are lower.[10] Now experience is more nearly the opposite. The forecasting of short term interest rates by long term interest rates is, in general, so bad that the student may well begin to wonder whether, in fact, there really is any attempt to forecast.

However, an examination of the courses of 'time' and 'call' money rates offers almost conclusive evidence that forecasting is really attempted and that at least one reason it is so badly done is that it is so difficult. Both 'time loans' and 'call loans' are loans made to stock brokers with stocks and/or bonds as collateral. The only outstanding

(Footnote [9] concluded)
bonds are largely mathematical deductions from economic material which cannot bear the strain of such analysis. Furthermore, even if we knew the forecasts of future short term rates implicit in the 'yield' of a bond, we would, for many purposes, prefer the average. Not only has it the advantage of brevity that is possessed by all averages, but it also has a lack of ambiguity that the individual items could not possess. We must remember that, while the 'yield' is the same for the buyer as for the seller of the bond, the individual estimates of future short term rates may be different for each buyer and seller in the market. Even the implicit rates derived from serial bonds are, at best, only short term rates in the minds of the corporation's officials. We must not forget that any particular maturity may fail to sell, or, if the series is sold as a unit, the prices that later emerge in the open market may be quite different from those of the original issue.

[10] In general, though less accurately, the *prices* should *rise* in periods of high short time interest rates and *fall* in periods of low short time rates. Fall or rise in 'yield' is, of course, not necessarily associated with rise or fall in price. If a bond selling above par is to retain a constant 'yield' it must fall in price continually. In a similar manner a bond selling below par must rise in price.

difference between the two types of loan is the length of time they run. 'Call loans' run 24 hours; 'time loans' run from one to six months. Now, if it were actually known that money placed on 'call' for the next ninety days would yield exactly 6 per cent per annum, no bank or other lender would place money on 'time' for that period at a lower rate than 6 per cent per annum. Generally the lenders would insist upon a little more than 6 per cent to recompense them for having their funds in a less liquid condition. As periods of high call rates are periods of disturbed monetary conditions, this differential would be greater when the next ninety days are to show high call rates than when they are to show low call rates.

In line with these facts, 90-day time loan rates would, theoretically, always be as high as or higher than an *average* (of the type described in note 5) of call rates for the succeeding ninety days. In periods preceding low call rates, 90-day time rates would range only a little higher than the average call rate for the next ninety days but, in periods preceding high call rates, time rates would range appreciably higher than the average call rate for the next ninety days. Moreover, unless the movements of the differential were very erratic, 90-day time rates would, week by week and month by month, show the same ups and downs as the *average* of call loan rates for the next ninety days. Furthermore, as they would move with an average of *future* call loan rates, they would reach maxima and minima distinctly earlier than call loan rates. In general, we would expect 90-day time loan rates to reach maxima and minima about 45 days (or 1½ months) before call loan rates. What are the facts?

In the first place, a comparison of 90-day time loan rates with *averages* of call loan rates for the next ninety days shows that the time rates usually range higher than the call *averages,* as theory would lead us to expect. However, the relation of the magnitudes of the differentials to the levels of the call rates is not what we might anticipate under good forecasting. When the future call averages are low, the time rates almost always range much higher than those averages; when the future call averages are high, the time rates range little if any higher than the averages. When the future call averages are extraordinarily high the time rates are commonly *lower* than the averages. Seldom do the time rates correctly forecast a period of extraordinarily high call rates. Even when they reach as high a maximum as the call averages, the maximum usually occurs too late to constitute any forecast. Over and over again, in a period immediately preceding high call rates, it was possible to borrow on time and relend on call (during the 'time' period) at a large profit.

An examination of a chart on which are plotted 90-day time money rates and the averages of call money rates for each succeeding ninety days reveals little evidence of good forecasting. When 4- or 6-month time money rates are similarly compared with the proper averages of future call money rates, even less evidence of good forecasting is forthcoming. This applies not only to time money *levels* but also to the timing of movements and the positions of maxima and minima. Time loan rates fail to forecast call loan rates because neither borrowers nor lenders of money on 'time' know much more than nothing at all about the future course of call loan rates.

But this is not the whole story. Before the Federal Reserve system went into operation both call and time loan rates showed pronounced *seasonal* fluctuations. The existence of these seasonal fluctuations was almost universally recognized and their chief characteristics were fairly well known. It was admitted that both call money rates and time money rates contained two elements—a seasonal and a non-seasonal. Under such circumstances, would it not be natural to believe that the poor forecasting of call money rates by time money rates was the result of poor forecasting of the non-seasonal element in the call money rates and to expect that the time money *seasonal* would, upon examination, be found actually to forecast the call money *seasonal?*

At last we have arrived at something that was really known about future short time interest rates, and we find the theory that forecasting is necessarily attempted is at last upheld by the data. The time money *seasonal* shows unmistakable evidences of attempted forecasting of the call money *seasonal,* as may be seen by comparing the monthly seasonal for time money rates (Chart 20 [11]) with a three-month moving average of the monthly seasonal for call money rates.[12] It is true that the lag of the three-month average call money seasonal is usually closer to one month than it is to the one month and a half which the theory would in general demand. However, the essential thing is that there is a distinct lag; the time money seasonal moves *before* the call money seasonal.[13]

[11] For the figures see Appendix A, Table 22.

[12] For the figures see Appendix A, Table 21.

[13] This may be clearly seen from Chart 20 where the two seasonals are presented.

Here we have evidence of definite and relatively successful forecasting. The chief trouble seems to be, not that the time money seasonal does not move early enough, but that it does not move far enough. Its fluctuations are too small. Year after year the fluctuations of the *three-month moving average* of the call money monthly seasonal are greater than the fluctuations of the time money monthly seasonal. The borrowers and lenders of time money seemed loath to adjust their rates completely to what they knew of the call money seasonal. This is somewhat strange because profits could have been made by those who noticed the discrepancy. Before the Federal Reserve system went into effect stock brokers should have borrowed more heavily on call for the first eight months of the year and more heavily on time for the last four months of the year.

If, from call and time loan rates we eliminate the seasonal fluctuations, and then compare the two resulting series, we find the forecasting even worse than for the two original, unadjusted series. Bankers and brokers acted as if they knew virtually nothing about future cyclical or other non-seasonal movements of call money rates. They did know something about the *seasonal* fluctuations. What they knew about they were able to forecast, at least approximately; what they did not know about they were unable to forecast at all—except by accident.

In much of the preceding discussion of the relations that, theoretically, would exist between long and short term interest rates we have implicitly made one fundamental assumption which in actual practice may or may not be warranted: *the assumption of payment.* In connection with any loan there are always two rates of interest which may or may not be the same: first, there is the *promised* or *hypothetical* yield, which can be calculated at the time the loan is made or the bond is purchased, but which may never materialize; second, there is the *realized* or *actual* yield which cannot be known until the last payment has been made. If a 4 per cent bond, maturing in 30 years, be purchased at 90 and held for 22½ years, and if, during that time, forty-five $2.00 payments be made but no payments of any kind thereafter, the *promised* yield is 4.62 per cent but the *realized* yield is zero per cent. Only on the assumption of absolute certainty of payment is it legitimate to say that the *promised* yield of a bond should logically be an accurate forecast of (completely determined by) the course of future short term interest rates. In actual practice, a forecast that is quite distinct from any forecast of short term interest rates is introduced into the determination of the *promised* yield—the forecast of the degree of certainty of the future payments. The *realized* yield is not, of course, a forecast at all, as it does not come into existence until after the event.[14]

Realized yield concerns the real though unknown future; *promised* yield concerns a hypothetical future which may or may not materialize. It is a mere forecast. However, though the *realized* yield has, in this sense, a reality that the *promised* yield does not possess, it is the *promised* yield that is almost invariably referred to when the word 'yield' is used without designating its meaning. The 'yield' of a bond is the *promised* yield. This fact must never be forgotten. Its recognition clears up many theoretical difficulties.

In calculating the 'yield' of a bond the assumption is made that all future interest payments and the principal payment will be made on the dates specified in the bond. Of course, such an assumption is necessarily absurd in the case of a perpetuity—such as Canadian Pacific debenture 4's or any 'preferred' stock. The chance that all future payments will be made is negligibly small for any extremely long term bond, such as West Shore 4's of 2361. The importance of this condition from a practical standpoint may, of course, easily be overemphasized. If West Shore 4's of 2361 are bought to yield 5 per cent per annum to maturity, the price paid will be $80.00 for each $100 face value of the bond. This $80.00 present payment may be distributed as follows: $73.23 is paid for the interest payments of the first fifty years, $6.20 for the interest payments of the next fifty years, and only 57 cents for all succeeding interest payments and the payment of the principal sum —on the assumption of a uniform interest rate of 5 per cent per annum for all future inter-coupon periods.

'Certainty of payment' is for most purposes a purely psychological concept. Only to the extent that it is an opinion in the minds of buyers and sellers can it affect the price of bonds. Security in the opinion of buyers and sellers is commonly spoken of as though it were security in fact. Security *in fact* can be known only when the future has be-

[14] In the light of *promised* and *realized* rates of interest, the concept of 'pure interest' (as a *promised* rate) is seen to be a merely psychological concept. The 'pureness' is necessarily a forecast rather than a fact.

come the past. As the future cannot be known security is always relative; absolute security is a pseudo idea. An actual bond (before maturity) can never be absolutely secure *in fact.* So many buyers and sellers of bonds may *think* of it as absolutely secure that its market price may act as though it were extremely (though not necessarily 'absolutely') secure. In general, the more buyers and sellers who consider a bond to be absolutely secure or nearly so, the lower will be its 'yield'. There is, however, no point at which one can stop and say 'this is absolute security'.

How arbitrary and unreal, from an economic standpoint, may be the mathematically necessary *assumption of payment* is illustrated by the variation in the 'yields' of bonds containing identical promises as to future payments—that is, bonds carrying the same 'coupon rate' and having the same maturity. We immediately realize that, for bonds having the highest 'yields', such 'yields' are merely 'promised' and will probably never be 'realized'. From an economic standpoint they are primarily indexes of lack of confidence in the certainty of the future payments rather than indexes of how those payments would at present be valued—if there were perfect assurance that they would be paid on the promised dates.

The *assumption of payment* (which must be made before the 'yield' can be calculated) is seen to be, in such cases, if not an assumption demonstrably contrary to fact, at least of very dubious validity. 'Promised' yield is not necessarily 'realized' yield.

The concept of 'pure' or 'riskless' interest is metaphysical. The practical contrast is not between 'pure' and 'impure' but between 'promised' or 'expected' and 'actual' or 'realized'. It is quite quixotic to attempt to divide the 'promised' (or even 'realized') return from a bond into 'interest' and 'profits' or something else. Moreover, such a division is unnecessary for either theoretical or historical treatment. Bonds and other interest-bearing obligations may be classified according to their ('promised') yields without introducing the concept of 'pure interest', and the economic significance of such yields may be studied without deciding what the rate of 'riskless' yield would be. All rates of interest are of economic importance. The movements of the yields of second grade bonds sometimes have a much more direct bearing on changes in economic conditions than the movements of the yields of first grade bonds. For example, the yields of bonds of superlative quality may actually *fall* during a period of great business disturbance and distrust—while the yields of second grade bonds are *rising*. The existence at any time of an abnormally large volume of bonds selling at prices that show extremely high 'yields' is almost certain to be of great economic significance, even though it is not necessarily any evidence that 'long term interest rates as such' are extremely high.

Sometimes distrust of all securities becomes so great that 'investment' deteriorates into 'hoarding'. Many erstwhile investors now demand actual cash, in extreme cases actual specie. Even the highest grade bonds are no longer acceptable. The hoarder demands what he believes to be 'absolute security'. He will accept zero or even negative interest (rent of a safe deposit box). However, such a condition differs only in degree and not in kind from the more commonly occurring flight from the lower grade long term securities into the highest grade short term obligations.

Generally speaking, the relative economic importance of securities of various grades varies with their total market values. If any large proportion of the total market value of securities outstanding in a community is by most persons considered almost absolutely safe, fluctuations in the yield of those securities are, of course, of great economic importance. On the other hand, in a community where there are almost no investments that are generally considered superlatively safe, fluctuations in the yield of such investments are of only academic interest. While movements of the yield of securities considered superlatively safe might be of great importance in a community such as England in the last years of the nineteenth century, it would have little significance in a community such as California in the 1850's. When money in California was commonly lending at 18 to 24 per cent per annum, some few individuals were undoubtedly satisfied to invest in securities yielding them less than 6 per cent per annum. Fluctuations in the yield of such securities were of little economic significance in that community at that time.

The destructive effects of a fall in the prices of bonds are not necessarily dependent on whether the bonds were originally (before the fall) considered high or low grade. The chief reasons that usually make a fall in the prices of high grade bonds more serious than a fall in the prices of low grade bonds are two. In the first place, the total market value of such bonds outstanding (before the fall) is usually much greater than the total value of the low grade bonds. In the second place, banks usually invest more heavily in high grade bonds than in low grade bonds, and anything that affects the solvency or even liquidity of the banks is always peculiarly serious. In periods when banks are carrying a large volume of low grade bonds, a fall in the prices of those bonds may be almost as serious as a fall in the prices of high grade bonds.

Not only the economic importance of the yield of investments that are considered superlatively safe but also the yield itself is affected by changes in the volume of such securities available. The yields of securities of even as high a grade as United States Liberty Bonds and Treasury Certificates have in the past often been unmistakably responsive to the larger fluctuations in the amount outstanding. This is not, to any appreciable extent, the result of a general belief that certainty of payment is affected more than negligibly by such fluctuations. It merely illustrates the fact that an increase in the volume of even such securities does not automatically create new purchasers—except at lower prices—any more than an increase in the supply of a commodity creates new purchasers—except at lower prices.[15]

Furthermore, the volume of investment funds demanding the highest degree of safety is affected by changing opportunities for earnings in less secure investments. The relation of the yield of the highest grade investments to the yield of other investments is always important. Seldom do many persons demand security at any price. Usually, many are willing to take risks with the hope of larger returns than they could obtain from investments that they consider 'absolutely secure'. If the speculative opportunities connected with investments that are believed to have some element of risk seem to increase, the proportion of the investment funds of the country that will demand 'absolute security' will probably decline. The yield of 'absolutely secure' investments will advance. If the opportunities connected with investments recognized as having some element of risk seem to decline or if the risks seem to have increased, the proportion of the investment funds of the country demanding great or 'absolute' security will probably increase.

The evil effects of a pronounced *rise in the yield* of any class of

[15] In spite of partial offsetting by the possibility of 'discounting' at the Federal Reserve banks.

bond are, for some purposes,[16] more easily understood if we speak, as we did a few paragraphs back, in terms of a *fall in price*. For example, the great damage is done by the fall in the *prices of bonds already on the market,* not by the rise in the rates of interest that corporations that wish to engage in new borrowing will have to pay. The effect of the fall in the prices of outstanding obligations is in the present. The effect of the higher yields of the new bonds is in the future. A fall in the prices of bonds actually outstanding immediately affects the financial position of all their holders, while the drain on the resources of a borrowing company, that results from a rise in the rate it must pay on a new issue, will extend over the life of the bond. The first has a concentrated and immediate effect; the effect of the second is spread out thinly over many future years.

A pronounced fall in the price of bonds actually outstanding is serious not only because it destroys present purchasing power, but also because it leads to one of the vicious circles of the business cycle. If the bonds have been used as collateral for loans, that collateral must be increased or a part of the loan must be repaid. If it be repaid by selling some of the bonds, such 'distress selling' tends to lower the price of the bonds just as directly as does the forced selling of any commodity. Contrary to ordinary economic assumptions, things are being sold, not because they are dear, but because they are cheap.[17]

From a theoretical standpoint it would seem that major fluctuations in the yields (or prices) of bonds of the highest grade should be relatively more important in periods of prosperity than in periods of de-

[16] In discussing the action of bonds in the business cycle it sometimes seems easier to think in terms of *price* than in terms of *yield.* Why should we not substitute price for yield in all our discussions? Probably the simplest way to answer this question is to point out that 'yield' may often be a better way to measure price than prices themselves. It measures a corrected rather than a raw price. It may be considered as the reciprocal of an adjusted price—a price that has been corrected for varying coupon rates and maturities. Though it is highly desirable to remember the implications involved in 'yield', those implications do not need to frighten us from using the concept. It is not only extremely useful but almost necessary.

[17] This vicious circle is, of course, made still more vicious by those who sell, because they become afraid that prices may go so low that they would eventually be forced to sell—or merely because they believe prices are going lower.

Economists have usually underemphasized the importance of price *movements* as compared with price *levels* in inducing purchases or sales. In the speculative markets, commodities and securities are as often bought because their prices have been going up, or sold because their prices have been going down, as because their prices are low or high.

pression. As bonds of the highest grade are those bonds which are generally so *considered,* there are naturally more of them in periods of prosperity than in periods of depression. A rise in the yield of bonds of the highest grade occurring in the midst of a period of prosperity should be of greater significance than a fall in their yield in a period of depression. Of course this reasoning is somewhat complicated by the fact that a rise in yield (or fall in price) always exerts positive pressure, while the effects of a fall in yield (or rise in price) are largely negative; it creates opportunities rather than necessities or compulsions. An examination of the historical facts strongly supports the thesis that a rise in the yield of interest-bearing obligations of the highest grade—whether they be of long or short maturity—has greater power to terminate a period of prosperity than has a fall in their yields to initiate such a period.

We have seen that, if 'promised' rates were 'realized' and if long term rates accurately forecast short term rates, it would be relatively unimportant to an investor whether he bought long or short term securities. If he bought short term when he really needed long, he would have to be continually reinvesting; and, if he bought long when he needed short, he would have to sell. But both the short and the long term returns 'realized' would be the same whether they were obtained from a succession of short term investments or from a long term investment with possibilities of sale. The price fluctuations of a long term bond would be exactly sufficient to adjust the successive implicit short term rates of the bond to the future rates for future short term loans—no more and no less. The *price* fluctuations of the bond would therefore be unaffected by the interval to maturity. A 4 per cent bond selling at $90 must rise to $91.15 in six months if the return is to be 7 per cent per annum for those six months—whether the bond matures in five years or a century.

Of course, bond *prices* do not move this way in the actual market. Not only do they tend to fall rather than rise in periods of short term stringency, but also the more distant their maturity the greater are their *price* fluctuations. The *price* fluctuations of the highest grade bonds maturing in ten years tend to be appreciably greater than the price fluctuations of those maturing in two or three years. But the increase in *price* fluctuation resulting from an increase in time to maturity is not as great as it would be with a constancy of *yield* fluctuation. The longer the maturity the smaller the yield fluctuations—though, because the *price* fluctuations *increase* with an increase in time to maturity (rather than remain constant), the decrease in extent of fluctuation in yield with lengthening of time to maturity is not nearly so great as it would be if the long term rates accurately forecast the short term rates.[18] The longer the maturity of a bond the greater are the *price* fluctuations, and hence the greater are the fluctuations in the actual short time return realized by buying at the beginning and selling at the end of the short time period. If we define the 'ninety-day yield of a bond' on a particular date as the return that would have been realized if the bond had been bought on that date and sold (without commissions) ninety days later, we find that the 'ninety-day yields' of even the highest grade long term bonds have usually fluctuated much more violently than ninety-day time money rates—usually more violently than even ninety-day averages of call money rates.

An important reason why bond yields (and prices) fluctuate as much as they do is that few buyers of long term bonds buy them with the intention of holding them to maturity. They expect to sell them at some indefinite time in the future. Now to determine what the selling price will be at any particular future date requires something more than even absolute assurance that all interest payments and the principal payment will be met on the dates specified in the bond, and exact knowledge (if it were attainable) of future short time interest rates for the entire life of the bond. The buyer must know what will be the *opinion* of buyers and sellers concerning these matters on that future date—*and whether the potential future buyers will also be not only willing but able to pay.* They can not be forced to buy. Unlike short time loans, long time loans are not 'self liquidating'. Prior to its distant maturity, nobody has to buy or retire a particular long term bond at a particular time or go into bankruptcy. This is why it is so peculiarly inappropriate for banks to place any large percentage of their demand funds in long term bonds.

The fact that long term bonds are bought and sold and not necessarily or even usually held to maturity makes us realize again the artificiality of the concept of security in the case of 'promised' yield. Mere length of time introduces an element of real insecurity in all long

[18] The movements of time and call money rates offer an exception to this generalization. Time money rates have fluctuated *less* than they would have if they had accurately forecast call money rates.

term loans. Only short term loans can be even imagined to be 'absolutely secure'. Who can make even a good guess as to what a particular long term bond will be selling for two years from now? Yet such a guess is an essential element of the 'security' of any short term loan that is to be made by buying the long term bond now and selling it two years hence.

We have, so far in this chapter, been discussing the subject of long time interest rates without asking the question: how much longer term is one loan than another? For a study of the relations between long and short time interest rates, it would seem highly desirable to have some adequate measure of 'longness'. Let us use the word 'duration' to signify the essence of the time element in a loan. If one loan is essentially a longer term loan than another we shall speak of it as having greater 'duration'.

Now the promise contained in a loan is either a promise to make one and only one future payment or a promise to make more than one future payment. If two loans are made at the same rate of interest, and if each loan involves a promise to make one future payment only, the loan whose future payment is to be made earlier is clearly a shorter term loan than the other. For example, if $100 be lent for one year at 5 per cent per annum, the only payment to be $105 at the end of the year, and if another $100 be lent for two years at 5 per cent per annum, the only payment to be $110.25 at the end of the two years, the first loan is clearly a shorter term loan than the second. If, on the other hand, either or both loans involve a promise to make more than one future payment, or if the rates of interest ascribed to the two loans are not the same, it may be extremely difficult to decide which is essentially the longer term loan.

It is clear that 'number of years to maturity' is a most inadequate measure of 'duration'. We must remember that the 'maturity' of a loan is the date of the last and final payment only. It tells us nothing about the sizes of any other payments or the dates on which they are to be made. It is clearly only one of the factors determining 'duration'. Sometimes, as in the case of a low coupon, short term bond, it may be overwhelmingly the most important factor. At other times, as in the case of a long term, diminishing annuity, its importance may be so small as to be almost negligible. Because of its nature, length of time to maturity is not an accurate or even a good measure of 'duration'. 'Duration' is a reality of which 'maturity' is only one factor.

Whether one bond represents an essentially shorter or an essentially longer term loan than another bond depends not only upon the respective 'maturities' of the two bonds but also upon their respective 'coupon rates'—and, under certain circumstances, on their respective 'yields'. Only if maturities, coupon rates and yields are identical can we say, without calculation, that the 'durations' of two bonds are the same.

If two bonds have the same maturity and the same yield but one has a higher coupon rate than the other, the one having the higher coupon rate represents an essentially shorter term loan than the other. For example, if each bond is selling on a 5 per cent basis, a 6 per cent bond maturing in 25 years necessarily represents an essentially shorter term loan than a 4 per cent bond maturing in 25 years. This may easily be seen by comparing a $400 face value 6 per cent bond maturing in 25 years with a $500 face value 4 per cent bond maturing in 25 years. On both bonds the total of all future payments, both principal and interest, is $1,000. But on the 6 per cent bond the payments are $12 each six months for 24½ years, and then a final payment of $412, while on the 4 per cent bond the payments are $10 each six months for 24½ years, and then a final payment of $510. It is plain that the $1,000 is being paid earlier on the 6 than on the 4 per cent bond. Though both have the same 'maturity', the 6 per cent bond represents a loan of shorter 'duration' than the 4 per cent bond.

The difference in 'duration' of the two bonds is manifest in their prices. As the payments are made earlier on the 6 per cent bond, its price (if the 'yields' of the two bonds are the same) is necessarily higher. For example, as each bond 'yields' 5 per cent, the price of the $400 face value 6 per cent bond will be $456.72, while the price of the $500 face value 4 per cent bond will be only $429.10.

We see, then, that if two bonds have the same yield and the same maturity but different coupon rates, the bond having the higher coupon rate represents the loan of shorter 'duration'. Instead of examining in a similar manner the case in which the two bonds have the same coupon rate and the same maturity but different yields, and the case in which they have the same coupon rate and the same yield but different maturities, we shall now consider directly the general problem of how to measure 'duration'. Let us approach this problem by considering the maturity of a bond as a function of the maturities of the separate loans of which it may be said to consist.

It would seem almost natural to assume that the 'duration' of any loan involving more than one future payment should be some sort of a weighted average of the maturities of the individual loans that correspond to each future payment. Two sets of weights immediately present themselves—the *present* and the *future* values of the various individual loans.

Future value weighting seems clearly inadmissible. It gives absurdly long 'durations'. If $2,000 be lent at 5 per cent per annum in the form of two loans, one of $1,000 at 5 per cent per annum[19] payable in one lump sum of $1,050 at the end of one year, and one of $1,000 at 5 per cent per annum payable in one lump sum of $131,501.26 at the end of 100 years, the 'average maturity' or 'duration' of the two loans, if calculated by taking an arithmetic average of the two maturities, using the *present* values as weights, is 50½ years. If the *future* values ($1,050 and $131,501.26) be used as weights, the 'average maturity' is found to be more than 99 years.

In this illustration, the *present* values (or amounts lent) were equal. Let us examine a case in which the *future* values are equal. If $959.98 be lent at 5 per cent per annum in the form of two loans, one of $952.38 at 5 per cent per annum payable in one lump sum of $1,000 at the end of one year, and one of $7.60 at 5 per cent per annum payable in one lump sum of $1,000 at the end of 100 years, the 'average maturity' or 'duration' of the two loans, if calculated by taking an arithmetic average of the two maturities, using the *present* values ($952.38 and $7.60) as weights, is about 21½ *months.* If the *future* values be used as weights, the average maturity is 50½ years.

How absurd it seems to think of a loan of $2,000 made up of two loans each of $1,000, one maturing in one year and one in 100 years, as having a 'duration' of over 99 years. And how absurd to think of a loan of $1,000 made up of two loans, one of $952.38 maturing in one year and the other of $7.60—less than 1 per cent of the larger loan—maturing in 100 years, as having a 'duration' of 50½ years.[20]

[19] In the present discussions, we have not followed the 'semi-annual compounding' convention. For simplicity of treatment, we have assumed throughout that payments are made *annually* and compounding is done *annually.*

[20] If one billion dollars were to be lent as a single payment loan at 5 per cent per annum for one year, and one cent as a single payment loan at 5 per cent per annum

But are not the results obtained by using *present* values as weights also open to criticism? If the 'durations' obtained by using *future* value weighting seem unmistakably too long, does not at least one of the 'durations' obtained from *present* value weighting seem very short?

Moreover, if the average maturity of two equal *future* payments be assumed to be the arithmetic average of the two maturities with the *present* values of the future payments as weights, some seemingly paradoxical results may appear. For example, if the yield be 5 per cent and if the two future payments be $1 at the end of one year and $1 at the end of 10 years, the average maturity will be about 4½ years. If the dates of payment be one year and 27 years, the average maturity will be about 6.7 years. But if they be one year and 50 years the average maturity will be only 5.1 years, and if the dates of payment be one year and 100 years the average maturity will be appreciably less than 2 years! In this particular illustration, the average maturity has a maximum when the second payment is made in about 27 years! However, these results do not seem so ridiculous if we remember that, as the date of payment of the second $1 becomes arithmetically more and more distant, *its present value, or the amount actually lent,* becomes geometrically smaller and smaller. In the limiting case, in which the second $1 is paid at infinity, the 'average maturity' is one year, but the amount of the loan for which the second dollar is to be paid is zero. The argument for present value weighting seems strong.[21]

(Footnote [20] concluded)
for 520 years, *future* value weighting would give the composite loan a duration of about 260 years.

[21] The actuaries have proposed and solved a problem that must not be confounded with ours. It is termed the problem "of finding the *equated time* for a number of sums due at different times, or, in other words, the average date at which, on the basis of an agreed rate of interest, all the sums might be paid without theoretical advantage or disadvantage to either party" (British) *Institute of Actuaries Text-Book, Part I,* pp. 24 and 25.

The answer is a date such that, if the *sum* of all the *present* values of the different future payments was compounded to that date at the rate of interest used to obtain those individual present values, it would amount to the sum of all the future payments. This is a neat and symmetrical answer to the problem proposed, and it gives better results in practice than the common method of 'equating time', which is based on *future* weighting, but it seems an unreal answer to an unreal question. It is quite logical in assuming that the present value of the single future payment must equal the present value of the sum of the individual future payments, but it seems to beg the question when it also assumes that the *future* value of the single payment at the date of its payment must equal the sum of the individual

Now, if present value weighting be used, the 'duration' of a bond is an average of the durations of the separate single payment loans into which the bond may be broken up. To calculate this average the duration of each individual single payment loan must be weighted in proportion to the size of the individual loan; in other words, by the ratio of the present value of the individual future payment to the sum of all the present values, which is, of course, the price paid for the bond.[22]

Let F = the 'face' value of the bond in dollars, i.e. the 'principal' sum in dollars;

I = the number of dollars paid semi-annually, i.e. the number of dollars called for by one 'coupon';

P = the number of dollars paid for the bond, i.e. the 'price' in dollars;

n = the number of half years the bond has to run, i.e. the number of half years to 'maturity';

R = the semi-annual *rate* of the 'yield', e.g. if the bond is selling to yield 4 per cent per annum, R = 1.02 (under the semi-annual convention of the bond tables);

Q = the ratio of the face value of the bond to a coupon payment, i.e., $Q = \frac{F}{I}$;

D = the 'duration' of the bond—in half years;

Then

$$D = \frac{\frac{I}{R} + \frac{2I}{R^2} + \frac{3I}{R^3} + \dots + \frac{nI}{R^n} + \frac{nF}{R^n}}{\frac{I}{R} + \frac{I}{R^2} + \frac{I}{R^3} + \dots + \frac{I}{R^n} + \frac{F}{R^n}}.$$

(Footnote 21 concluded)
future payments each taken at its particular date of payment. This assumption overweights the time importance of distant payments.

22 In terms of the symbols of the next paragraph,

$$P = \frac{I}{R} + \frac{I}{R^2} + \frac{I}{R^3} + \dots + \frac{I}{R^n} + \frac{F}{R^n} = \frac{I}{R-1} - \frac{\frac{I}{R-1} - F}{R^n}$$

Summing the terms in the numerator, and in the denominator, of this fraction and substituting QI for F, we find that

$$D = \frac{R}{R-1} - \frac{QR + n(1+Q-QR)}{R^n - 1 - Q + QR}.$$

An examination of this expression for the value of D shows that the larger the value of Q the greater the duration; in other words, the smaller the 'coupon' payments are relatively to the face value of the bond the greater is the duration of the bond. Furthermore, the larger the value of R the smaller the duration. D increases with n, though, if R be greater than $1 + \frac{1}{Q}$, in other words if the bond be selling below par, D reaches a maximum before n reaches infinity, declining gradually thereafter to $\frac{R}{R-1}$, the value reached when n equals infinity.

When $Q = 0$, in other words, when the series of future payments constitutes a mere annuity without any 'principal' payment whatever, $D = \frac{R}{R-1} - \frac{n}{R^n - 1}$. When Q equals infinity, in other words, if the loan is single payment, $D = n$.

If $R = 1$, in other words if the 'yield' of the bond be zero,

$$D = \frac{\frac{n^2+n}{2} + nQ}{n + Q}.$$ (note 23)

Unity is the limiting value of D as R approaches infinity.

[23] $$\frac{R}{R-1} - \frac{QR + n(1+Q-QR)}{R^n-1-Q+QR}$$

$$= \frac{R\,(R^n-1-Q+QR) - (R-1)\,[QR + n(1+Q-QR)]}{(R-1)\,(R^n-1-Q+QR)}$$

which, when 1 is substituted for R, takes the indeterminate form of $\frac{0}{0}$. However, the fraction may easily be evaluated by the ordinary methods of the calculus. The first derivative of the numerator divided by the first derivative of the denominator is, if $R = 1$, still indeterminate. However, taking second derivatives, we get

$$\frac{n\,(n+1)\,R^{n-1} + 2nQ}{n\,(n+1)\,R^{n-1} + 2Q - n\,(n-1)\,R^{n-2}}.$$

Letting $R = 1$ in this expression we obtain the value for D given in the text.

When n = 1, D = 1. When n equals infinity, as when a bond (such as Canadian Pacific debenture 4's) has no maturity date, $D = \frac{R}{R-1}$. But, if R be greater than $1 + \frac{1}{Q}$, in other words if the bond be selling below par, D will attain a maximum value before n reaches infinity.[24] However, unless R be very large, the value of n making D a maximum will be large and the maximum value of D will be very little greater than the value associated with an infinite value for n.[25]

A short table presenting the relations between time to maturity and duration, for a 4, a 5, and a 6 per cent bond each selling at par, will illustrate the ordinary characteristics of the duration concept (p. 51).

The concept of 'duration' throws a flood of light on the fluctuations of bond yields in the actual market. Not merely do the yields of long term bonds tend to fluctuate much less violently than the yields of short term bonds or the rates on short term loans, such as are represented by commercial paper, but the relation between maturity and violence of fluctuation in yield is much as we would expect to find it from our analysis of the nature of 'duration'. While there is a great difference between the amplitude of the fluctuations in yield of bonds of

[24] The explanation of seeming paradoxes of this type has already been discussed.

[25] Equating to zero the derivative of D with respect to n leads to an insoluble equation; but an approximate solution is that, for other than extremely large values of R, D will reach a maximum when n is a shade greater than $\frac{R}{R-1} + \frac{QR}{QR-Q-1}$. For example, if a 4 per cent bond be selling on a 6 per cent basis (3 per cent per half year on the semi-annual compounding convention), $\frac{R}{R-1} + \frac{QR}{QR-Q-1}$ will equal 134⅓, and this is approximately the value of n (in half years) that will, in fact, make D a maximum. But this maximum value of D, when n equals 134⅓, is less than 34⅔ half years and when n equals infinity D equals 34⅓ half years, a decline of less than two months in its value.

A higher yield will, of course, give a maximum value for D with a smaller value for n and the difference between the maximum value of D and its value when n equals infinity will be increased. For example, if the 4 per cent bond be selling on an 8 per cent basis, $\frac{R}{R-1} + \frac{QR}{QR-Q-1}$ will equal 78. When D is actually a maximum, n lies between 78 and 79 half years. The maximum value of D is then slightly less than 27⅛ half years but the value of D when n equals infinity is only 26 half years, a difference of a little more than half a year.

DURATION IN YEARS [26]

YEARS TO MATURITY	4 PER CENT BOND	5 PER CENT BOND	6 PER CENT BOND
1	.990	.987	.985
3	2.857	2.823	2.790
6	5.393	5.257	5.126
10	8.339	7.989	7.662
15	11.422	10.727	10.094
25	16.026	14.536	13.254
50	21.970	18.765	16.273
100	25.014	20.353	17.120
Infinity	25.5	20.5	17.167

extremely short maturity and of those having ten years or so to run, and an appreciable difference between the fluctuations in yield of the latter and of bonds having forty-five or fifty years to run, there is virtually no discernible difference between the action of these last bonds and the action of those having a hundred years or more to run.

The concept is, of course, full of theoretical difficulties. It is easy to think of the 'duration' of a bond as increasing while the time to maturity is decreasing, if 'long term interest rates' are declining during the period. It would seem only logical, for the purposes of our problem, to think of time not in terms of years or months but in terms of its relation to the growth of capital. But in all our illustrations we have, for purposes of computation, used as 'yield' the yield of *the individual bond* whose 'duration' we were discussing. This amounts to assuming that 'duration' is lengthened by mere increase of security as well as by a true decrease in the 'preference for present over (assured) future money'.

But this assumption leads us into one of the quagmires of 'pure' interest. Are the promised future payments of a low grade bond really

[26] If the interest were payable and compounded annually, instead of semi-annually, the durations would be slightly greater than those given above, the difference increasing with increases in the time to maturity. For infinite maturities they would be one-half year greater, that is 26, 21 and 17⅔ years instead of 25½, 20½ and 17 1/6 years.

If the ordinary concept of 'equated time' (see note 21) were used to calculate duration, no maximum values would appear. A bond with an infinite maturity like British Consols or Canadian Pacific debenture 4's would have an infinite duration. A 6 per cent bond selling at par and maturing in 10 years would have a duration of 7.95 years (instead of 7.66 as in the text table), if it matured in 25 years its duration would be 15.50 years (instead of 13.25), for 50 years its duration would be 23.45 (instead of 16.27), for 100 years 32.92 (instead of 17.12), and if it matured in 200 years its duration would be 43.39 years instead of less than 17 1/6 years as in the text table.

discounted at higher rates than the promised future payments of a high grade bond, or is the difference in 'yield' traceable not to any difference in rates of discount but to a difference in what is discounted, this being, in the case of an ultra high grade bond, the actually promised payments, but, in the case of a low grade bond, the mathematical 'expectations' that result from multiplying each promised payment by the assumed probability that it will be met? [27]

Another difficulty connected with the problem will be merely mentioned. We have made the assumption that the rate of interest for each future six month period is the rate corresponding to the 'yield' of the bond. Now the reader realizes that this assumption may easily be contrary to fact. However, we drew attention, earlier in this chapter, to the insuperable difficulties connected with any attempt to discover the real rates of discount for each half-yearly period in the future. If we knew these future discount rates we might then be able to state that two bonds which, *at different dates,* each had the same number of years to run, the same coupon rate and the same 'yield' had quite different durations.

If, for example, the 'yield' of the earlier bond involved a set of relatively high discount rates for the years of the immediate future and low discount rates for the succeeding years to maturity, while this condition was reversed for the later bond, the earlier bond would have a longer duration than the later bond. Because the coupon rates, yields and maturities are identical, the prices of the two bonds will be the same. In other words, the sum of the present values of the future payments will be the same. Hence that bond in which the earlier payments are relatively heavily discounted, and therefore the 'weights' applicable to the shorter constituent maturities are relatively light, will have a longer duration.

The difficulties connected with the problem of arriving at a completely satisfactory concept of 'duration' are, indeed, extremely great. Any proposed solution almost necessarily involves some paradoxes. We have tried to open the reader's eyes to the existence of the problem. The logical atmosphere in which the analysis has had to be carried on may seem to have been somewhat rarefied at times; but we believe that, if the reader has followed the arguments carefully, he will at least not

[27] But see Ch. III, note 8.

accuse the writer of being like the good Puritan knight who, in religious controversy,

> ". . . could raise scruples dark and nice,
> And after solve 'em in a trice
> As if Divinity had catch'd
> The itch, on purpose to be scratch'd."