Theory and Uses

The Theory of International Comparisons of Real Income and Prices

SIDNEY N. AFRIAT

UNIVERSITY OF WATERLOO, ONTARIO

1.0 CONCEPT OF COMPARISON

1.1 *Framework*

Two or more countries have similar commodities. But prices and quantities differ, and it is required to construct indexes which express a comparison. For a theory of such construction, it is essential to have a prior concept of the intended meaning of the comparison. From such a concept, together with a scheme for the data and a principle by which the data are related to the concept, the theory of construction should follow.

Comparison between two places in a single period is to be viewed in the same abstract framework as comparison between two periods in a single place. Though variables might occur in time and have a corresponding designation, temporal priority has no role. It makes no difference whether the distinctions be of time or place. What is in view is a variety of locations, temporal or geographical or possibly both, where prices and quantities differ, and which, when combined by analysis are to indicate differences of economic situation.

Let there be some k references which are to be compared and are indicated by $t = 1, \ldots, k$. But if there are but two, these can be indicated by $t = 0, 1$ and distinguished as the *base* and *current* reference.

NOTE: This work has been supported by the National Science Foundation under Grant GS 2195.

The distinction between references can be taken to be as between different countries, or different periods of time, or both in conjunction. But in the question of international comparison, there often are k, or in particular two, different countries during the same period.

There are assumed to be some n goods involved. With n as the nonnegative numbers, Ω_n and Ω^n are the spaces of nonnegative column and row vectors of n elements. The basic element of the data is a pair (x, p) formed by a vector $p \in \Omega_n$ of prices p_i and a vector $x \in \Omega^n$ of quantities x_i. Such a pair describes the *demand* of quantities x at prices p. The associated expenditure is $px = \Sigma_i p_i x_i$, and, with $M = px$, $u = M^{-1}p$ defines the associated *budget vector.* It forms with x the pair (x, u) such that $ux = 1$, which can be called the *associated budget.* Often an observed demand is meaningful only through its associated budget. Then discussion is simpler in terms of budgets rather than original observed demands.

By an *expansion set* (S, p) is meant a set of demands $[(x, p): x \in S]$ all associated with the same prices p. The set S could consist of a finite set of points, or it could be a path, in which case (S, p) indicates an *expansion path* associated with prices p. Inherent in the price level concept is the assumption that expansion paths are rays through the origin. In that case, the data for a demand (x, p) amount to data for an expansion path (S, p), where S is the ray $[x\lambda : \lambda \geqq 0]$ through x. A more general model for an expansion path is the linear model, where the paths are lines not necessarily through the origin. In a common special form of this model, the expansion lines associated with different prices converge in a single point, not necessarily the origin.

For a basic scheme, it is postulated that the data consist of a set of demands (x_t, p_t) or, more elaborately, a set of expansions (S_t, p_t), associated with each country $t = 1, \ldots, k$. But in fact a single demand determines an expansion when analysis is based on the concept of a price index and hence on the assumption of homogeneity, which requires that expansion paths be rays through the origin, each determined by any one of its points. Thus again it is expansion data that are available, either explicitly or implicitly.

1.2 *Nature of Comparison*

It is understood that the comparison is to be in real terms. This means, in the first place, that its reference must be exclusively to quantities of the basic goods, independent of the accident that money and prices are part of the data by means of which the comparison is made. The role of money and prices is just to express limitations of opportunities for possessing goods. But these limitations are of no significance if what is limited is not valued. Also value is meaningless if it is not pursued to a maximum within the limited opportunity available. With such optimality, data on choice under limitation communicates information about value which is relevant to comparisons. But value is an attribute of the chooser, and the chooser must therefore be clearly identified. Thus with national measurements it should be decided whether value derives directly from individuals or from the nation as a whole. In the latter case it should be asked in what sense, since national wants are not easily discovered and stated, and have dimensions which are without counterpart for the individual.

Some significant yardstick is presumed in making comparisons between situations. A yardstick that refers to bundles of goods is an ordering of them which expresses their relative value according to a system of wants. This immediately shows an obscurity in the meaning of comparisons between countries where wants are manifestly different, whether this be for the wants of the individual inhabitants or, in any sense conceivable, for that different type of individual represented by a single country. To speak of real national outputs or incomes implies a presupposition of national value, that is, a system of wants described by a utility relation. If this relation is not already on record and available in a form suitable for making the required comparisons, then it can only be inferred from price-quantity data, under the maximum hypothesis. Concerning the mechanism which provides this maximality, and there is only what Schumpeter termed the "maximum doctrine of perfect competition," which is an early concept. It was first criticized, seriously but cautiously, by Marshall and has now altogether lost its suggested meaning. Some such doctrine seems necessary if prices and price indexes are to be relevant to the measurement of a national output that differs from the summation of individual real incomes seemingly permitted by the assumption of homogeneity of utility which underlies the use of price indexes. Such a summation would appear to be permitted mathematically by the homogeneity assumption, but it would have no significance unless called for by some theory. Investigation of real national measures and comparisons seems to have little opportunity for development except for those that refer directly to individuals.

1.3 *General Principles*

A problem to be considered is that of establishing a correspondence between individual incomes in different countries which represent the same real income or purchasing power, that is, which in choice of goods at the prices that prevail would obtain the same real output or utility. Such correspondence is to be established on the basis of a utility order $R \subset \Omega^n \times \Omega^n$ which compares bundles of goods $x, y \in \Omega^n$ according to value, or output of utility, so that the priority xRy signifies x is as good as, or produces at least as much utility as, y. The problem in such a comparison arises because though it is based conceptually on a utility scale no actual scale has been identified. However, the scale is validated by the holders of the incomes themselves. It is with that same scale that they are assumed to make optimal choices under limited budgets. The obverse of this is that data on choices impose a limitation on the scale that is to be applied. With fragmentary observations, this is a loose limitation, but essentially it is all there is with which to proceed.

So far as the idea of a definite structure of wants can be applied at all, it is recognized that wants are related to circumstances and differ between individuals, times, and countries. But the comparisons under consideration essentially involve the notion of a want structure that is common for all, and this universality is what makes the comparison intelligible. Such a yardstick is therefore a purely statistical concept. In such a case, complete generality of the basic model makes no sense. Indeterminacies can just as well be diminished, and in cases even removed, by imposing a special structure on the model. The structure most commonly imposed is homogeneity, which underlies the concept of a price index and with it almost the entire traditional theory of index numbers. It is an important form because it greatly simplifies procedures, and its lack of elaborate overrefinement is appropriate to some applications. But in other applications it effaces structural features which are of crucial significance.

In section 2.0, below, the basic analytical concepts bearing on index number construction are described. A special branch of this subject evolves from the homogeneity assumption, and leads to the theory of price index construction treated in section 3.0. Homogeneity of utility is equivalent to a homogeneous linear form for the expansion functions. Certain objections will be made to this in the present section, and to linear expansion in general. However, a limited generalization of the price index concept will be developed. This is the theory of marginal price indexes, which is presented later. It too is vulnerable to a serious objection, and it is shown how a further extension can overcome this, without sacrificing the practical features that belong to the earlier methods.

1.4 *Analytical Formulation*

The basic concept of index theory is the value-cost function

$$\rho(p, x) = \min\,[py : yRx] \tag{1}$$

which derives from a utility relation R (see section 2.3). It determines the minimum cost at given prices p of a bundle of goods which ranks in R with a given bundle x. Since xRy, because of reflexivity of R,

$$\rho(p, x) \leqq px \tag{2}$$

for all p, x. Certain minimal properties are assumed for R (2.1), which in fact are not at all restrictive for the questions considered, with finite data (2.5), but give a simpler basis for discussion. While $\rho(p, x)$ derives from R, nothing of R is lost in it even when it is prescribed just for one value of p, since then as a function of x it is a utility function representing R:

$$xRy \Leftrightarrow \rho(p, x) \geqq \rho(p, y). \tag{3}$$

Such a particular utility function could be called a *cost gauge* of R. Any one can be constructed from any other:

$$\rho(p, x) = \min\,[py : \rho(p_0, y) \geqq \rho(p_0, x)]. \tag{4}$$

These are the natural yardsticks for the comparisons under consideration, since they deal with value and cost simultaneously, one of them being associated with any price situation; but they all represent a single system of measure by virtue of the transformations between them.

If R is specified for consumers (in practice, it is not), then $M_{10} = \rho(p_1, x_0)$ is the cost at current prices of living at the standard represented by the base consumption x_0. It goes without saying that the notion of costs has no meaning, and the equation has no unique determination, unless the calculation is at *minimum* cost. By (2),

$$M_{10} \leqq p_1 x_0, \tag{5}$$

that is, the "true" cost M, which is true in so far as R is the true utility relation, does not exceed the Laspeyres cost $p_1 x_0$. This is reassuring for traditional doctrine, which has proposed the Laspeyres index to be an upper bound of something. If more is intended than has just been said with (5), which is a vacuous consequence of definition, then it is necessary to be explicit.

Observed demands (x_0, p_0), (x_1, p_1) for the base and current periods, or countries, being available, a formula for M_{10} is required, traditionally in the form $M_{10} = P_{10} M_0$, where P_{10} is a "price index," and $M_0 = p_0 x_0$. Thus with the Laspeyres price index $P_{10} = p_1 x_0 / p_0 x_0$, $M_{10} = p_1 x_0$ as just remarked. But first the question will be considered without regard for the proposed significance of this form. What has mostly been lacking is an explicit recognition of a principle which relates the data to the question. One such principle is that R be such as to show observed cost $M_t = p_t x_t$ not to exceed minimum cost for the value obtained, $M_{tt} = \rho(p_t, x_t)$, that is, the observed demands must be expressed as satisfying a condition which is necessary for optimality, when R is the prevailing utility relation. Then

$$\rho(p_0, x_0) \geqq p_0 x_0, \qquad \rho(p_1, x_1) \geqq p_1 x_1. \tag{6}$$

With (2) this is equivalent to

$$\rho(p_0, x_0) = p_0 x_0, \qquad \rho(p_1, x_1) = p_1 x_1. \tag{7}$$

Any R for which (6) holds can be said to be *compatible* with the data. The further logic of the compatibility relation is shown in section 2.4. Possibly no compatible R exists. If any does the demands can be said to be *consistent.* (A more general theory of consistency is developed in section 2.5.) Then it can be asked, What is the range of values of $M_{10} = \rho_{10}(p_1, x_0)$ for all such R? The range of M_{10} appears to be an interval, which always includes the upper limit $(M_{10})_u$, but not necessarily the lower one $(M_{10})_l$. But the question of attainment of units is unimportant. What we really want to know is whether these are the proverbial bounds of index number lore. The Laspeyres value is identical with $(M_{10})_u$, but the Paasche value not only has no connection with $(M_{10})_l$, but even need not lie in the interval, presumed nonempty since the data are consistent. But it is evident in all arguments involving the Paasche index, and generally whenever the peculiar concept of a price index is dealt with, that an implicit special assumption has been made, namely, the homogeneity of utility. (It is examined in the next section.) Since there are serious objections to that assumption and, also, to limiting the data to base and current demands only, it is interesting to go outside the traditional framework and investigate the determination of $\rho(p_1, x_0)$ in a more general way.

It is simpler to discuss demands (x_t, p_t) through their derived budgets (x_t, u_t), where $u_t = M_t^{-1}p_t$, and $M_t = p_t x_t$, so that $u_t x = 1$. The requirement $M_{tt} = M_t$ for compatibility of R with any set of demands for $t = 1, \ldots, k$ is then

(8) $$\rho_{tt} = 1, \qquad t = 1, \ldots, k$$

where

(9) $$\rho_{rs} = \rho(u_r, x_s), \qquad r, s = 1, \ldots, k.$$

The next step is to determine the range of any ρ_{rs} for all R such that $\rho_{tt} = 1$ for all t. Again the range is an interval, whose limits can be determined. Let

(10) $$D_{rs} = u_r x_s - 1,$$

and define a relation D by

(11) $$rDs \equiv D_{rs} \leqq 0,$$

and Q by

(12) $$rQt \equiv rDsD \ldots Dt \text{ for some } s, \ldots$$

and let

$$(13) \qquad (\rho_{rs})_u = \min [u_r x_t : tQs]$$

$$(\rho_{rs})_l = \min [u_r x : sQt \Rightarrow u_t x \geqq 1].$$

The condition for the consistency of the demands can be stated in three equivalent ways,

$$(14) \quad D_{rs} \leqq 0, D_{st} \leqq 0, \ldots, D_{qr} \leqq 0 \Rightarrow D_{rs} = D_{st} = \cdots = D_{qr} = 0$$

for all $r, s, \ldots, q$ or

$$(15) \qquad rQs \Rightarrow u_s x_r \geqq 0$$

for all r, s or

$$(16) \qquad (\rho_{rs})_l \leqq (\rho_{rs})_u$$

for all r, s. Then, subject to consistency, so that the interval will be nonempty, $(\rho_{rs})_u$, $(\rho_{rs})_l$ are the desired limits. Consistency of the data implies the consistency of these as upper and lower limits, that is, that the one be not less than the other, for all r, s. In fact, by the equivalence of consistency to (16), the converse also is true.

In the special case of a pair of demands (x_0, p_0), (x_1, p_1), the consistency condition becomes

$$(17) \quad p_0 x_1 \leqq p_0 x_0, \quad p_1 x_0 \leqq p_1 x_1 \Rightarrow p_0 x_1 = p_0 x_0, \quad p_1 x_0 = p_1 x_1$$

evidently implied by the Samuelson condition

$$(18) \qquad p_0 x_1 \leqq p_0 x_0, \quad x_1 \neq x_0 \Rightarrow p_1 x_0 > p_1 x_1$$

which can be stated more symmetrically as

$$(19) \qquad p_0 x_1 \leqq p_0 x_0, \quad p_1 x_0 \leqq p_1 x_1 \Rightarrow x_0 = x_1.$$

Further

$$(20) \qquad (\rho_{10})_u = \min [u_1 x_t : u_t x_0 \leqq 1]$$

so that, if $u_1 x_0 \leqq 1$,

$$(21) \qquad (\rho_{10})_u = \min [u_1 x_1, u_1 x_0]$$

$$= \min [1, u_1 x_0] = u_1 x_0$$

and if $u_1 x_0 > 1$,

$$(22) \qquad (\rho_{10})_u = \min [u_1 x_0] = u_1 x_0.$$

Therefore, in any case, $(\rho_{10})_u = u_1 x_0$, which corresponds to the Laspeyres index, as already stated. Also

(23) $$(\rho_{10})_l = \min\,[u_1 x : u_0 x_t \leqq 1 \Rightarrow u_t x \geqq 1].$$

Therefore, if $u_0 x_1 \leqq 1$,

(24) $$(\rho_{10})_l = \min\,[u_1 x : u_0 x \geqq 1,\ u_1 x \geqq 1] \geqq 1,$$

and if $u_0 x_1 > 1$,

(25) $$\begin{aligned}(\rho_{10})_l &= \min\,[u_1 x : u_0 x \geqq 1] \\ &= \min\,[u_{1i}/u_{0i} : u_{0i} > 0],\end{aligned}$$

so certainly $(\rho_{10})_l$ is not related to the Paasche index.

This approach, general though it is in that it depends on no special properties for utility, is nevertheless too rigid. Data may not satisfy the required consistency condition, and in that event the analysis can proceed no further. But the model is in any case unrealistic because there is no provision for error. An extension is shown in sections 2.4 and 2.5 where deviations are explained as error, measured in economic terms of inefficiency. No real economic agents have exact and invariable wants. But assuming any did, none would accurately allocate their expenditure down to the last penny. There would be perhaps a rough tendency toward equilibrium until any effort and cost for improvement would seem to outweigh any plausible benefit. Equilibrium in the larger framework, which takes into account the value and cost of every movement, its sacrifice and gain, would leave disequilibrium in the narrower framework to which analysis is confined. Since the wider framework is unknown, the method must be essentially statistical, dealing with fluctuations in a hypothetical model. Since economic error has the nature of inefficiency, it is appropriate that distance should be measured in an inefficiency sense, instead of, for instance, by a Euclidean sum of squares. A Euclidean distance, however vast, but which corresponded to a difference of a negligible penny would nevertheless be negligible. This argument, which is in opposition to some classical and other statistical techniques usual in econometrics, in particular in production and consumption analysis, is related to the argument in favor of approaching statistics from the viewpoint of decision theory.

To return to the demand consistency condition, in addition to the three equivalent conditions (14), (15), and (16), there are two further equivalents

(26) $$\lambda_r D_{rs} + \lambda_s D_{st} + \cdots + \lambda_q D_{qr} \geqq 0,\ \lambda_r > 0$$

for all $r, s, \ldots$, for some λ_r, or

(27) $$\lambda_r D_{rs} \geqq \phi_s - \phi_r, \qquad \lambda_r > 0,$$

for all r, s for some λ_r, ϕ_r. Moreover, for all λ_r and ϕ_r, (27) implies (26), and for all λ_r, (26) implies (27) for some ϕ_r. With any λ_r, ϕ_r let

(28) $$\phi(x) = \min_t \phi_t + \lambda_t(u_t x - 1).$$

Then (27) is equivalent to the compatibility of the utility function $\phi(x)$ with the given demands, for it is equivalent to

(29) $$\phi(x_t) = \phi_t,\ \min\,[u_r x : \phi(x) \geqq \phi_s] = 1$$

for all r, s, t. Thus, whenever the demands are consistent, that is, compatible with any utility function at all, then they are compatible with a utility function of the form (28), corresponding to a solution of (27). This method, as extended in sections 2.4 and 2.5 to allow different degrees of efficiency and approximation, shows how the utility hypothesis, which is basic to index numbers, can be given constructive realization with any data.

Any utility order R between consumption bundles determines an *adjoint utility order* S between consumption budgets. Thus uSv means there exists x such that $ux \leqq 1$ and for all y if $vy \leqq 1$ then xRy; in other words, there exists a consumption attainable within the budget u which is as good as any attainable within v. Also, any utility function $\phi(x)$ determines an *adjoint utility function*

(30) $$\psi(u) = \max\,[\phi(x) : ux \leqq 1]$$

and if $\phi(x)$ represents R then $\psi(u)$ represents S.

Compatibility of R with (x_0, u_0), where $u_0 x_0 = 1$, is equivalent to $\phi(x_0) = \psi(u_0)$, and to

(31) $$u_0 S v \cdot \Longleftrightarrow \cdot vy \leqq 1 \Rightarrow s_0 S y.$$

Given this,

(32)
$$\begin{aligned}\rho_{10} &= \min\,[u_1x : xRx_0]\\ &= \max\,[\rho : u_1x \leqq \rho \Rightarrow x_0Rx]\\ &= \max\,[\rho : u_0S\rho^{-1}u_1].\end{aligned}$$

Hence, with $\psi(u)$ representing S, it follows that $\rho = \rho_{10}$ is the solution of

(33)
$$\psi(u_0) = \psi(\rho^{-1}u_1).$$

Equivalently, $M = M_{10}$ is the solution of

(34)
$$\psi(M_0{}^{-1}p_0) = \psi(M^{-1}\mathrm{p}_1).$$

With $\phi(x)$ concave, $\psi(M^{-1}p)$ is a concave function of M, so that, for some P_0,

(35)
$$\psi(M^{-1}p_0) - \psi(M_0{}^{-1}p_0) \leqq (M - M_0)/P_0$$

for all M. If $\psi(u_0)$ is differentiable at $u = M_0{}^{-1}p_0$, this implies

(36)
$$(\partial/\partial M_0)\;\psi(M^{-1}p_0) = 1/P_0,$$

so that P_0 appears as the marginal price of utility at the level of expenditure M_0 when commodity prices are p_0. Correspondingly, $1/P_0$ is the marginal utility of money. The average utility of money, or the reciprocal of the average price of utility, is $\psi(M^{-1}p_0)/M_0$. Identity between the average and marginal prices of utility, that is,

(37)
$$M\partial\psi(M^{-1}p)/\partial M = \psi(N^{-1}p),$$

implies that R is homogeneous. Conversely, if R is homogeneous, it can be represented by a linearly homogeneous utility function for which (37) holds. In fact, a necessary and sufficient condition that R be homogeneous, that is

(38)
$$xRy \Rightarrow x\lambda Ry\lambda \qquad (\lambda \geqq 0),$$

is that the utility cost function be factorable thus:

(39)
$$\rho(p, x) = \theta(p)\phi(x)$$

and with this $\theta(p)$ is identified simultaneously with the average price and the marginal price of utility $\phi(x)$. The factorability of utility cost underlies the concept of a level of prices, and hence also that of a price index, which compares levels. There appears to be a lack of explicit recognition that the meaning of a price level, and therefore of all price indexes, depends on homogeneity of utility. Index constructions which depart from this, and which, so to speak, belong to a family which is one rung up the ladder of generality include the "new formula" of Wald (1939), and the "constant utility" index of Klein-Rubin (1947) which, appropriately put, exhibit utility price which is linear but not at the same time homogeneous. They can be seen as extensions of the Fisher and Palgrave formulas, respectively, and have corresponding limitations. Also the Paasche and Laspeyres indexes have correspondents in this family.

With $\phi(x)$, the utility function given by (28), and $\psi(u)$, its dual, compatibility, implied by (27), is equivalent to $\phi(x_t) = \psi(u_t)$. It appears, in (29), that $\phi(x_t) = \phi_t$. Thus ϕ_t appears as the utility at x_t, with a compatible utility function $\phi(x)$. Equivalent to (35) is

(40) $$\psi(\rho^{-1}u_0) - \psi(u_0) \leqq \lambda_0(\rho - 1),$$

where $\rho = M/M_0$, $\lambda_0 = M_0/P_0$. Thus λ_t/M_t appears as the marginal utility of money for $\phi(x)$.

It is curious that (27) holds with $\delta_{rs} = \rho_{rs} - 1$ substituted for $D_{rs} = u_r x_s - 1$. For

$$\phi_t = \phi(x_t) = \psi(u_t), \qquad \psi(\rho_{st}^{-1}u_t) = \psi(u_s),$$

which, with

(41) $$\psi(\rho_{st}^{-1}u_t) - \psi(u_t) \leqq \lambda_t(\rho_{st} - 1),$$

gives

(42) $$\lambda_t\delta_{st} \geqq \phi_s - \phi_t.$$

Since, by (2),

(43) $$\delta_{st} \leqq D_{st}$$

(27) is recovered from (42). A certain duality between the D's and δ's becomes even more specific in further developments. The D's correspond to crosscosts which apply to quantities, essentially Laspeyres indexes, which derive directly from data, and the δ's to crosscosts which apply to their utility, which are the concern of index construction.

1.5 *Concept of the Price Level*

The price level concept has its origin in the arithmetic of the market place, where the product of price and quantity equals exchange value in money, and in the notion that such a scheme of arithmetic can apply just as well for many goods treated as one, so that a product of price and quantity levels, determined from many prices and quantities, equals total exchange value, that is, the sum of products of individual prices and quantities. Thus, by simple market arithmetic

(1) $$p_0x_0 = M_0, \qquad p_1x_1 = M_1$$

and then it is postulated that also

(2) $$P_0X_0 = M_0, \qquad P_1X_1 = M_1,$$

where the P's and X's correspond to levels of prices and quantities. Then

(3) $$P_{10}X_{10} = M_{10},$$

where $P_{10} = P_1/P_0$, and so forth. The idea is then taken further, and it is assumed that the composite price ratio P_{10} can be "approximated" by some kind of average of individual price ratios p_{1i}/p_{0i}. All standard price indexes are apparently expressible as averages—arithmetic, geometric, harmonic, and mixtures—of these, with various weights and exponents. Fisher examined about two hundred of them, and found that their divergences from each other were small compared to the errors inherent in the data. With P_{10} thus "approximated" by almost any formula, it is possible to "deflate" a money ratio M_{10} to determine $X_{10} = P^{-1}M_{10}$ as its "real" correspondent. This is the intelligible scheme for almost any index construction which has had application, starting with Fleetwood in 1708. The Laspeyres index has the merit of almost irreducible simplicity, and the theoretical distinction of being an "upper bound," though of what and to whom requires elaboration.

The essential concept here is that a utility relation R prevails and that its utility cost function $\rho(p, x)$, that is, the minimum cost at prices p of attaining the utility represented by x, can be factored:

(4) $$\rho(p, x) = \theta(p)\phi(x).$$

But this is equivalent to the homogeneity of R, that is

(5) $$xRy \Rightarrow x\lambda Ry\lambda \qquad (\lambda \geqq 0),$$

and implies that both the price and quantity functions $\theta(p)$, $\phi(x)$ are linearly homogeneous,

(6) $$\theta(\mu p) = \mu\theta(p), \qquad \phi(x\lambda) = \phi(x)\lambda,$$

where λ, $\mu \geqq 0$. Clearly all functions which can appear in the factorization (4) can differ only by a constant positive multiplier. The function $\phi(x)$ is fixed entirely by taking $\phi(x_0) = 1$ for any x_0. Thus the pair of *antithetic price and quantity functions* are, to this extent, uniquely determined. While R is represented by a wider class of utility functions, equivalent under increasing transformations, homogeneity in the sense of (5) requires there to exist a subclass of linearly homogeneous utility functions that are equivalent under multiplication by a positive constant; and these are the ones which are relevant.

From (4), and 1.4(2),

(7) $$\theta(p)\phi(x) \leqq px$$

for all (p, x) and the condition for equilibrium is

(8) $$\theta(p)\phi(x) = px,$$

which holds for a demand (x, p) which is compatible with R. Since for all p this holds for some x, it follows that the adjoint of $\phi(x)$ is

(9) $$\psi(u) = [\theta(u)]^{-1}$$

so that

(10) $$\psi(M^{-1}p) = M/\theta(p).$$

With $X = \psi(M^{-1}p)$ as the utility of money M at prices p, and the average price P of utility at that level given by

(11) $$X/M = 1/P,$$

$P = \theta(p)$; so P is fixed when p is fixed and is independent of M. This implies that also

(12) $$\partial X/\partial M = 1/P,$$

that is, the fixed average price necessarily coincides with the marginal price, which is then also fixed. But even if the average price is not fixed, it is possible to have a fixed marginal price. This shows the direction of the first step in generalizing the traditional price index concept, while preserving part of its practical simplicity.

The elementary prices, when they are taken as given in a perfect market, are themselves conceived of as fixed average prices coincident with marginal prices. The traditional price index concept corresponds to the idea that the average price of utility is fixed when elementary prices are fixed. It allows every unit of money to be treated separately and uniformly, aggregates to be treated by simple addition, the utility of a sum being the sum of the equal utilities of the equal units that compose it. Any unit of money in the base period is equivalent in general purchasing power to P_{10} units in the current period, and any M_0 base units are equivalent to

(13) $$M_1 = P_{10}M_0$$

current units, since this is the condition that $X_{10} = 1$, in (3), and $X_0 = X_1$ in (2). It does not matter who or what the money is for; it could be a part or the whole of an individual income or a national income. All amounts of money have only to have a multiplier uniformly applied to them for a general correction for price change to be effected. Such a scheme has great statistical and social convenience because it is put into operation by publication of a single number. However, when the implications of the assumption which underlies the scheme are examined, it appears that no care in the choice of that number can overcome the radical defects. There is a similar, though more favorable, situation even with the next more general scheme, where it is required only that the marginal price of utility be fixed, as will be examined later.

Homogeneity of the utility relation brings about the cost separation (4). Then any x, p in a compatible demand have the relation E defined by

(14) $$xEp \equiv \theta(p)\phi(x) = px.$$

Since then

(15) $$xEp \Rightarrow x\lambda Ep \qquad (\lambda \geqq 0)$$

for all p. Equivalently

(16) $$x \in Ep \Rightarrow x' \subset Ep$$

where $x' = [x\lambda : \lambda \geqq 0]$. It appears then that the expansion locus Ep for any price p is a cone, since if it contains a point then it contains every point on the ray it determines. If, for instance, $\phi(x)$ is quasi-concave then Ep is always a single ray if and only if $\phi(x)$ is strictly quasi-concave. Then, when prices are fixed at p, and as incomes vary, all consumptions lie on the ray Ep through the origin. That is, when prices are fixed the *pattern of consumption is fixed,* that is, the proportions between the goods which enter into consumption are fixed. But it is an overwhelmingly significant fact of experience that the rich, whether individuals or countries, have things that the poor do not have at all, let alone in corresponding proportion. Deliberately to overlook this in a system of calculation that seeks to make general comparisons leaves the significance of such calculation quite obscure, even as to the locus of injustice.

1.6 *National Measurement*

The application of prices and price indexes to measure national output seems to be supported if not by arguments then by urgings from two sides, both objectionable. One is more straightforward and will be remarked upon first. The other has something to do with the "maximum doctrine of perfect competition," so called by Schumpeter, or Adam Smith's teaching of the Invisible Hand, and whatever is to be made of such doctrine.

The homogeneity which is the essential characteristic of the standard index method based on the price level concept pictures a ray in the commodity space, determined only by prices, along which lie the consumptions of all individuals according to their different incomes. Hence the sum of all consumptions, since these lie on a ray, is also a point on that ray, and the income needed to purchase it is identical with the sum of all incomes, since $p\Sigma x = \Sigma px$. It is possible therefore to picture the sum of these incomes, or national income, as the income of a fictitious individual who has the same preferences as all other individuals and who, at the prevailing prices, would therefore spend it as would any others, that is, at the point on the ray corresponding to the consumption bundle that that income would purchase. Thus the nation is to be treated as an individual, and national income as the sum of individual incomes is to be treated like an individual income. National income deflated by a price index, determines the base period income which has the same purchasing power, and it gives a measure of current real national output. This seems to be the logic of the uses in the third category referred to by Chase (1960) in listing the major functions of price indexes, that is, the "deflation of [a] value aggregate to estimate physical quantities." Value means money value at prevailing prices, and physical quantity means quantity level in the utility sense of output.

Again, the average of all consumptions, since these lie on a ray, is also a point on the same ray, and moreover this average, or per capita, consumption corresponds to average or per capita income because $p(1/N)\,\Sigma x = (1/N)\,\Sigma px$. Hence it is possible to picture an "average individual," whose income is the average income, who again has the same preferences as any other individual, and who, at the prices given, would spend it as any one else would, that is, at the point on the ray corresponding to the consumption it can purchase. This puts average or per capita income, like total income, on the same footing as individual income. This seems to be the implicit logic of average comparisons.

The accident that permits individual incomes, average income, and total income all to be treated in the same fashion depends on both linearity and homogeneity. More explicitly, an average of points on a line is on the same line, but so also is a sum of points but generally only if the line passes through the origin. Thus it is necessary both that the locus of consumptions be a line and that the line pass through the origin. Should the locus of consumptions be a line but not pass through the origin, the argument for the "average individual" still holds, but that for the "total individual" loses the basis of its meaning.

This is not a general rejection of comparisons of totals but a description of the implicit logic of a standard procedure. If the concept of the procedure is still acceptable when this logic is made explicit, there can be a requirement that practical procedure be more strictly in accordance with it. There appears to be a reversion to the old concept of general purchasing power in a sense which does not even allow for the plurality of purchasing powers recognized by Keynes (1922). It asks that the economy resemble a molecular structure composed of homogeneous atoms, with reference to which the purchasing power of every penny, described by a particular bundle of commodities, is uniformly determined. But, as Ruggles (1967) says: "The concept of a price index as a measure of the level of prices no longer has significant support among economists." After dealing with the application to real national income, he adds: "Despite this disillusionment with the concepts of price level and economic welfare, the use of price indexes flourishes."

Overall comparison in terms of an "average individual" holds up better conceptually. Moreover, a description of the fictitious average individual, together with a statement of the number of individuals in the national population, conveys all the information that could be communicated about a "total individual," plus something more, namely, population size, and in a form which may have a more direct meaning.

The second approach to the relation of prices to national welfare measurement reinforces this first one, but is more objectionable. The first can be seen to be misleading because it requires that points for individuals lie on a ray and that a sum of points on a ray also lie on that ray, an event which could occur only by chance. It is compelling to view that sum point as associated with an individual, but without a theory to encourage such a summation, to make it conceptually and not only mathematically natural, it is meaningless. Here the Invisible Hand might come to the rescue. According to that doctrine, the freely operating adjustment mechanism of the competitive market automatically brings the economy to a state of maximum welfare or utility output. The prices are therefore proportional to marginal welfares, just like individual marginal utilities. Therefore, with homogeneity granted, the same indexes are applicable. This doctrine could have power beyond index numbers. But it is without quantitative content. Newton's second law of motion—that the force on a unit particle is identical with its acceleration—would be equally meaningless were there not separate ways for determining force and acceleration, such as gravitational or other force theory and kinematics. The maximum doctrine has no separate theory about the determination of welfare, whose maximum must then coincide with the position determined by the market. Its starting point seems to be in the observation that "we owe our bread not to the benevolence of the baker but to his self interest" (Adam Smith), so by adding nonsense to a truism an influential contribution has been made to an economic philosophy.

2.0 ANALYSIS OF VALUE AND COST

2.1 *Utility*

A relation $R \subset \Omega^n \times \Omega^n$ is reflexive and transitive, that is, it is an order in Ω^n, if

(1) $$xRx, xRy \ldots Rz \Rightarrow xRz$$

Equivalently

(2) $$xRy \Leftrightarrow xR \supset yR$$

where $xR = [y : xRy]$, and similarly with Rx. It is complete if $xRy \vee yRx$, and continuous if it is a closed set in $\Omega^n \times \Omega^n$. If R is a complete order in Ω^n then it is continuous if and only if xR, Rx are closed sets in Ω^n. An order R is quasi-concave if the sets Rx are convex.

Any order R in Ω^n can be a *utility relation.* In that case, xRy means that consumption x is as good as, that is, produces as much utility as, y. The *law of disposal* for a utility relation is $x \geqq y \Rightarrow xRy$. An x appears as a point of *oversatiation* if yRx and $y < x$ for some y, and the *law of want* excludes such points. If R is complete and continuous then this implies the law of disposal.

For simplicity, in the present discussion a utility relation is specifically limited to be a complete, continuous order in Ω^n subject to the law of want. It is distinguished as a *normal utility relation* if, moreover, it is quasi-concave.

Any function $\phi(x) \varepsilon \Omega$, where $x \varepsilon \Omega^n$, represents a complete order R in Ω^n where

(3) $$xRy \equiv \phi(x) \geqq \phi(y).$$

It is a *utility function* if R is understood to be a utility relation. The law of disposal for R requires

(4) $$x \leqq y \Rightarrow \phi(x) \leqq \phi(y),$$

and that $\phi(x)$ be nondecreasing; the law of want requires

(5) $$x < y \Rightarrow \phi(x) < \phi(y),$$

and that $\phi(x)$ be semi-increasing. Any complete, continuous order in Ω'' admits representation by a continuous function. An order R which is quasi-concave requires that any function which represents it be quasi-concave, having convex level sets $[y : \phi(y) \geqq \phi(x)]$.

A *normal utility function* represents a normal utility relation and thus, beside being subject to the usually understood limitations, is quasi-concave. A utility function is concave if

(6) $$\phi(x\lambda + y\mu) \geqq \phi(x)\lambda + \phi(y)\mu \qquad (\lambda,\mu \geqq 0,\ \lambda + \mu = 1).$$

This implies that for all $x_0 > 0$ there exists $g_0 \geqq 0$ such that for all $x \geqq 0$,

(7) $$\phi(x) - \phi(x_0) \leqq g_0(x - x_0),$$

and $\phi(x)$ is differentiable at x_0 if and only if such g_0 is unique, in which case the g_0 is the gradient $g(x_0)$ of $\phi(x)$ at x_0. Now a classical utility function is defined by the existence of such $g_0 \geqslant 0$ for all $x_0 \geqq 0$. This implies it is expressible in the form

(8) $$\phi(x) = \min_{x_0} [\phi(x_0) + g_0(x - x_0)]$$

where a $g_0 \geqslant 0$ is determined from any $x_0 \geqq 0$. It is therefore continuous, semi-increasing, and concave, and thus also a normal utility function. A *polyhedral classical utility function,* which is the most important type for empirical analysis, is expressible in the same form, but with x_0 ranging in a finite set, usually corresponding to finite observations, instead of possibly throughout Ω''.

A utility relation R is *homogeneous* if

(9) $$xRy \Rightarrow x\gamma Ry\gamma \qquad (\gamma \geqq 0).$$

A utility function is homogeneous if it represents a homogeneous utility relation. Such a function is equivalent, under transformation by an increasing function, to one which is *linearly homogeneous,* that is, such that

(10) $$\phi(x\gamma) = \phi(x)\gamma \qquad (\gamma \geqq 0).$$

A function with this property is concave if and only if it is quasi-concave.

2.2 *Adjoint Utility*

A *budget constraint* $px \leqq M$ $(x \;\varepsilon\; \Omega^n)$, associated with an expenditure limit $M > 0$ at prices $P \;\varepsilon\; \Omega_n$, is equivalent to a constraint $ux \leqq 1$, where $u = M^{-1}p$. The budget constraints in Ω^n are thus coordinated with the points of Ω_n. Let the relation $W \subset \Omega^n \times \Omega_n$ be defined by $xWu \equiv ux \leqq 1$. Then $Wu = [x : xWu]$ is the *budget set* associated with any $u \;\varepsilon\; \Omega_n$. In such association u can be called a *budget vector,* and otherwise an *exchange vector.* The *primal* space Ω_n has a symmetrical relation with the *dual* space Ω^n, a point of which is a *composition vector* which describes the composition of a consumption, each space being the space of nonnegative homogeneous linear functions defined on the other.

For any $x \;\varepsilon\; \Omega^n$, $u \;\varepsilon\; \Omega_n$, x can be said to be *within, on,* or *under* u according as $ux \leqq 1$, $ux = 1$ or $ux < 1$, or, in a dual sense, the same can be said with x and u interchanged. Thus beside the within-relation just defined there are further relations (I, V) defined by $xIu \equiv ux = 1$, $xVu \equiv ux < 1$.

Any utility relation R in Ω^n has associated with it an *adjoint utility relation* $S = R^*$ in the adjoint space Ω^n, where

$$(1) \qquad uSv \equiv (\vee ux \leqq 1)(\wedge vy \leqq 1)xRy,$$

that is, uSv means there exist x within u which are, according to R, as good as every y within v. Thus uSu means there exists a consumption within the budget u which is as good as any other, that is, which is R maximal. This is always the case if $u > 0$, because of the compactness of Wu and the continuity of R. Thus S is reflexive at every point $u > 0$. Also, because R is a complete order, S is a complete order in the domain where it is reflexive. Thus S is a complete order at least in the interior of Ω_n. Because R is semi-increasing, S is semidecreasing. From the form of the definition of S from R, regardless of properties of R, uS is a closed convex set. By this convexity the order S is quasi-convex. The order R is quasi-concave if Rx is convex, and a necessary and sufficient condition for this is that $R = S^*$, where S^* derives from S by the dual of the formula (1) by which R^* derives from R, that is, the identical formula where budget and composition vectors exchange their roles.

Let $\phi(x)$ be any utility function representing a utility relation R, that is,

$$xRy \Leftrightarrow \phi(x) \geqq \phi(y). \tag{1}$$

Then the *adjoint utility function* $\psi(u) = \phi^*(u)$ is given by

$$\psi(u) = \max\,[\phi(y) : uy \leqq 1], \tag{2}$$

for u wherever this is defined, which is where S is reflexive and is at least in the interior of Ω_n, by the compactness of Wu, if $u > 0$, and the continuity of R. Then the adjoint ψ of the function ϕ, where it is defined, represents the adjoint S of the relation R represented by ϕ, that is,

$$uSv \Leftrightarrow \psi(u) \geqq \psi(v). \tag{3}$$

The function $\psi(u)$, which thus determines the maximum of the utility $\phi(x)$ attainable in the budget set Wu, is continuous, semi-decreasing, and quasi-convex. Since, from the definition

$$vx \leqq 1 \Rightarrow \phi(x) \leqq \psi(v) \tag{4}$$

it follows that

$$\phi(x) \leqq \min\,[\psi(v) : vx \leqq 1], \tag{5}$$

for all x, with equality if and only if

$$\phi(x) = \psi(u), \qquad ux = 1 \text{ for some } u. \tag{6}$$

But this is true for all x if and only if $\phi(x)$ is quasi-concave and thus a normal utility function. Thus a utility function, subject to the given limitations, is quasi-concave, and thus normal, if and only if

$$\phi(x) = \min\,[\psi(v) : vx \leqq 1] \tag{7}$$

for all x. The pair of relations (2) and (7) shows the reciprocal relation between a normal utility function and its adjoint. But in any case

$$\phi^{**}(x) = \min\,[\phi^*(v) : vx \leqq 1] \tag{8}$$

is normal, even if $\phi(x)$ is not, and has the same adjoint $\psi(u) = \phi^*(u)$ as $\phi(x)$. It defines the *normalization* of $\phi(x)$. Any demand observations that admit $\phi(x)$ as compatible with them will also admit $\phi^{**}(x)$. Hence if a utility function is admitted, so also is a normal utility function. In this sense there is equivalent empirical scope between utility functions and normal utility functions and no empirical content to the assumption that a utility function be quasi-concave, that is, have concave contours.

The *profile* corresponding to exchange prices $u \geqslant 0$ of a utility function $\phi(x)$ with adjoint $\psi(u)$ is the function $F(\rho) = \psi(\rho^{-1}u)$, which is increasing since $\phi(x)$ is semi-increasing. Since a utility function and its normalization have the same adjoint, this is also the profile of the normalization. *A necessary and sufficient condition that a utility function be concave is that its levels and profiles be concave.* If $F(\rho)$ is concave, as when the utility function or its normalization is classical, then there exists a $\lambda > 0$ such that

$$\psi(\rho^{-1}u) - \psi(u) \leqslant \lambda(\rho - 1) \tag{9}$$

for all ρ; also $F(\rho)$ is differentiable at $\rho = 1$ if and only such λ is unique, and then $F'(1) = \lambda$. But, with $u = M^{-1}p$, $M(\partial\psi/\partial M)\ (M^{-1}p) = \lambda$; so λ/M appears as the marginal utility of money M at prices p, and $P = M/\lambda$ as the marginal price of utility.

Since, with p fixed, $\psi(M^{-1}p)$ is an increasing function of M, $t = \psi(M^{-1}p)$ has an inverse $M = \sigma(p, t)$, which determines the minimum cost at prices p of attaining the level of utility t. Then, with p fixed, $\Delta M/M \geqq \Delta t/\lambda$, or $\Delta M/\Delta t \geqq P$.

It follows immediately from the definition that $\sigma(p, t)$ is linearly homogeneous as a function of p. Therefore it is concave if and only if it is quasi-concave. But it is quasi-concave, because $\psi(u)$ has this property directly from its definition. Thus $\sigma(p, t)$ is a linearly homogeneous function of p. It is a convex function of t only if the normalization of $\phi(x)$ is concave.

Now setting $t = \phi(x)$, the function $\rho(p, x) = \sigma[p, \phi(x)]$ is obtained. But this function is most basic and is properly introduced directly from the utility relation R, without the auxiliary functions ϕ, ψ, and σ as intermediaries.

2.3 *Utility Cost*

For any utility relation R, with the given limitations,

$$\rho(p, x) = \min\ [py : yRx] \tag{1}$$

exists for all $p \in \Omega_n$, $x \in \Omega^n$ and defines the associated *utility cost function.* It gives the cost at prices p of attaining the standard of utility represented by x. From its definition, for all x, it is a linearly homogeneous concave function of p, determined by the set of homogeneous linear bounds py where yRx. Its linear supports at $p \geqslant 0$ are pz where

$$\rho(p, x) = pz, \qquad zRx. \tag{2}$$

It is differentiable at p if and only if such z is unique, and then the gradient is

$$\rho_p(p, x) = z. \tag{3}$$

From the properties of R as a complete, semi-increasing, continuous order,

$$xRy \Leftrightarrow \rho(p, x) \geqq \rho(p, y), \tag{4}$$

showing that, for all p, $\rho(p, x)$ is a utility function which represents R. Since R is reflexive

$$\rho(p, x) \leqq px. \tag{5}$$

For all p the equality holds for some x, and for all x the equality holds for some p if and only if R is quasi-concave.

A necessary and sufficient condition that R be homogeneous is that $\rho(p, x)$ be factorable, that is, that

$$\rho(p, x) = \theta(p)\phi(x). \tag{6}$$

Then $\phi(x)$ is a linearly homogeneous utility function, unique but for a constant multiplier, which represents R, with adjoint $\psi(u) = [\theta(u)]^{-1}$, and cost function $\sigma(p, t) = \theta(p)t$.

The functions $\theta(p)$, $\phi(x)$ can be called *antithetic price and quantity functions.* Both are linearly homogeneous, and $\theta(p)$ is concave. If $\phi(x)$ is not concave, its normalization is concave and has the same antithesis, and is compatible with every demand that is compatible with $\phi(x)$. Hence nothing essential is lost if $\phi(x)$ is replaced by its normalization, or simply assumed to be concave, so that it is identical with its normalization.

From (5) and (6), antithetic functions satisfy the functional inequality

$$\theta(p)\phi(x) \leqq px \tag{7}$$

for all p, x where the equality holds for all p for some x and for all x for some p. It follows that

(8) $$\theta(p) = \min_x px[\phi(x)]^{-1}$$

$$\phi(x) = \min_p [\theta(p)]^{-1}px.$$

It follows from (8) that $\theta(p)$ and $\phi(x)$ are both linearly homogeneous and satisfy (7).

2.4 *Compatibility*

The condition $H = H(R; x, p)$ for a utility relation R and a demand (x, p) to be *compatible* is the conjunction of conditions

(1) $$H' \equiv py \leqq px \Rightarrow xRy$$

$$H'' \equiv yRx \Rightarrow py \geqq px$$

signifying "maximum utility for the cost" and "minimum cost for the utility"; so H is the Pareto condition as applied to the competing objectives of gaining utility and saving money. But, with R semi-increasing, $H' \Rightarrow H''$, and with R continuous, $H'' \Rightarrow H'$. Thus, with the given limitation on R, H' and H'' are equivalent to each other and hence to H. A statement of H'' is

(2) $$\rho(p, x) \geqq px.$$

But by 2.2(5), since R is reflexive, this is equivalent to

(3) $$\rho(p, x) = px.$$

Then, if there is differentiability,

(4) $$\rho_p(p, x) = x,\ \rho_x(p, x) = \lambda p,$$

where, if there is homogeneity, $\lambda = 1$.

More generally, e compatibility, or *compatibility at a level of cost efficiency* e is the condition $H(R, e; x, p)$ given by

(5) $$\rho(p, x) \geqq epx.$$

Thus 0 compatibility is unconditional, 1 compatibility coincides with compatibility, and e compatibility implies e' compatibility for all $e' \leqq e$. With the *cost efficiency* $\bar{e} = \bar{e}(R; x, p)$ of (x, p) relative to R given by

(6) $$\bar{e} = \rho(p, x)/px,$$

e compatibility holds if and only if $e \leqq \bar{e}$.

From the demand (x, p) is derived the budget (x, u) with $ux = 1$, where $u = M^{-1}p$ is the associated exchange vector, $M = px$ being the expenditure. With this,

$$\bar{e} = \rho(u, x). \tag{7}$$

2.2(5) is equivalent to

$$\rho(u, x) \leqq 1. \tag{8}$$

The compatibility condition (3) is

$$\rho(u, x) = 1, \tag{9}$$

which, if there is differentiability, is equivalent to

$$\rho_u(u, x) = x, \tag{10}$$

and implies

$$\rho_x(u, x) = \lambda u \tag{11}$$

where $\lambda = \rho_x(u, x)x$, so that $\lambda = (u, x)$ if there is homogeneity, in which case (9) is equivalent to (11) with $\lambda = 1$,

$$\rho_x(u, x) = u. \tag{12}$$

Therefore, in this case, conditions (9), (10), and (12) are equivalent.

Consider a *demand configuration* D whose elements are demands $(x_t, p_t) : t = 1, \ldots, k$, and the derived *exchange configuration* E with elements (x_t, u_t), where $u_t = M_t^{-1}p_t$ and $M_t = p_t x_t$. Compatibility of a utility relation R with D is defined by simultaneous compatibility with each element of D and is equivalent to compatibility with E. Thus it is the condition $H(R)$ given by

$$\rho(u_t, x_t) = 1, \qquad t = 1, \ldots, k. \tag{13}$$

More generally, the condition $H(R, e)$ of e compatibility of R with D is

$$\rho(u_t, x_t) \geqq e, \qquad t = 1, \ldots, k. \tag{14}$$

If

$$\bar{e}(R) = \min \bar{e}_t(R), \tag{15}$$

where $\bar{e}_t(R) = \rho(r_t, x_t)$. Then

(16) $$H(R, e) \Leftrightarrow e \leqq \bar{e}(R).$$

Thus $H(R, 0)$ holds for all R unconditionally, and

(17) $$H(R) \Leftrightarrow H(R, 1) \Leftrightarrow \bar{e}(R) = 1 \Leftrightarrow \bar{e}_t(R) = 1.$$

2.5 *Consistency*

A demand configuration D or, equivalently, its derived exchange configuration E, is *consistent* if there exists a utility relation which is compatible with it, which is to say the condition H that there exists a utility relation R such that $H(R)$. More generally, e consistency, or consistency at the level of cost efficiency e, is defined by the existence of an e-compatible utility relation, that is, the condition $H(e)$ that there exists a utility relation R such that $H(R, e)$. Thus $H(0)$ holds unconditionally and $H(1) \Leftrightarrow H$; and $H(e) \Rightarrow H(e')$ for all $e' \leqq e$. The *critical cost efficiency* is defined by

(1) $$\bar{e} = \sup\, [e : H(e)],$$

so that $0 < \bar{e} \leqq 1$. Then

(2) $$H(e) \Rightarrow e \leqq \bar{e} \qquad e < \bar{e} \Rightarrow H(e)$$

and

(3) $$H(\bar{e}) \Leftrightarrow \bar{e} = 1 \Leftarrow H.$$

Let P denote any property for a utility function, such as homogeneity, or having a certain separation structure, or being on any special model. In particular, C can denote the classical property. Then let $H_P(R)$ be the condition that a utility relation R both have the property P and be compatible with D, and let $H_P(R, e)$ be the same with e compatibility instead. Then the condition H_P of P consistency is defined by the existence of R such that $H_P(R)$, and similarly for $H_P(e)$, or P consistency at the level of cost efficiency e. The P *critical cost efficiency* is defined by

(4) $$e_P = \sup\, [e : H_P(e)].$$

Thus, for any P and e,

(5) $$H_P(R, e) \Rightarrow H(R, e)$$

so that

(6) $$H_P(e) \Rightarrow H(e),$$

and hence $e_P \leqq \bar{e}$. The same things are defined in particular with $P = C$. Though obviously $H_C(R, e)$ is not implied by $H(R, e)$, it is nevertheless the case that

(7) $$H_C(e) \Leftrightarrow H(e),$$

and hence $e_C = \bar{e}$. In other words, the classical restriction does not affect consistency. In particular $H_C \Leftrightarrow H$, that is, consistency is equivalent to classical consistency.

In fact these results essentially are stronger than is apparent in this formulation. They are true when a utility relation is understood to be any order relation in Ω^n without any further restrictions whatsoever, specifically without dependence on assumptions that R be continuous and semi-increasing. But in that case the basic condition $H = H(R; x, p)$ is not equivalent to H', since now H' and H'' are independent, but must again be identified with the conjunction of H' and H''. The same is true with the modification which permits a partial cost efficiency $e < 1$.

Defining *cross-differences*

(8) $$D_{rs}^e = u_r x_s - e$$

and *e cyclical difference consistency*

(9) $$K(e) \equiv D_{rs}^e \leqq 0,\ D_{st}^e \leqq 0, \ldots,$$
$$D_{qr}^e \leqq 0 \Rightarrow D_{rs}^e = D_{st}^e = \cdots = D_{qr}^e = 0$$

it appears that

(10) $$H(e) \Leftrightarrow K(e),$$

and thus $K(e)$ provides a finite test of e consistency. It follows then that

(11) $$\bar{e} = \min_{r,s,\ldots,q} \max[D_{rs}, D_{st}, \ldots, D_{qr}].$$

But $K(e)$ is necessary and sufficient for the existence of $\lambda_r > 0, \phi_r > 0$ such that

(12) $$\lambda_r D_{rs}^e \geqq \phi_s - \phi_r.$$

Then with

$$\phi_t(x) = \phi_t + \lambda_t(u_t x - e) \tag{13}$$

$$\phi(x) = \min_t \phi_t(x)$$

it appears that $\phi(x)$ is a classical utility function, and that it is e-compatible with the demand configuration D. A consequence is (7), namely, that $\dot{e}$ consistency is equivalent to classical e consistency. The adjoint of $\phi(x)$ is

$$\psi(u) = \max [t : t \leqq \phi_t + \lambda_t(u_t x - e), \; ux \leqq 1]. \tag{14}$$

Numbers $\rho_{rs} = \rho(u_r, x_s)$ are determined as solutions of

$$\psi(\rho_{rs}{}^{-1} u_r) = \psi(u_s) \tag{15}$$

and equivalently as

$$\rho_{rs} = \min [u_r x : \phi_t + \lambda_t(u_t x - e) \geqq \phi_s]. \tag{16}$$

Then $1 \geqq \rho_{tt} \geqq e$, demonstrating e compatibility. In case $e = 1$, it appears that

$$\phi(x_t) = \phi_t = \psi(u_t) \tag{17}$$

and

$$\psi(\rho^{-1} u_t) - \psi(u_t) \leqq \lambda_t(\rho - 1). \tag{18}$$

With $\delta_{rs} = \rho_{rs} - 1$, it appears from (15), (16), and (17) that

$$\lambda_r \delta_{rs} \geqq \phi_s - \phi_r. \tag{19}$$

But since $\rho_{rs} \leqq u_r x_s$, also $\delta_{rs} \leqq D_{rs}$, λ_r and ϕ_r having been chosen only so that $D_{rs} = D_{rs}{}^1 = u_r x_s - 1$ satisfy (12).

Analogously for homogeneous e consistency, which can be denoted $\dot{H}(e)$, with *cross-ratios*

$$L_{rs}{}^e = u_r x_s / e, \tag{20}$$

and *e cyclical ratio consistency*

$$\dot{K}(e) \equiv L_{rs}{}^e L_{st}{}^e \dots L_{qr}{}^e \geqq 1 \tag{21}$$

which is necessary and sufficient for the existence of $\phi_r > 0$ such that

$$L_{rs}{}^e \geqq \phi_s / \phi_r. \tag{22}$$

It appears that

(23) $$H(e) \Leftrightarrow \dot{K}(e)$$

and, with ϕ_r as given,

(24) $$\phi(x) = \min_t \phi_t u_t x$$

is a linearly homogeneous classical utility function which is e-compatible with the given demands. A consequence is that homogeneous e consistency is equivalent to linearly homogeneous classical e consistency. Also it follows that the homogeneous critical cost efficiency is

(25) $$\dot{e} = \max\, [e : u_r x_s u_x x_t \ldots u_q x_r \geqq ee \ldots e].$$

The antithetic price function for $\phi(x)$ is

(26) $$\theta(u) = \min\, [ux : \phi_t u_t x \geqq 1],$$

so $\theta(u)\phi(x) \leqq ux$, but it appears that

(27) $$1 \geqq \theta(u_t)\phi(x_t) \geqq e,$$

as required for e compatibility. In case $e = 1$, then moreover $\phi(x_t) = \phi_t$, $\theta(u_t) = 1/\phi_t$.

3.0 PRICE INDEXES

3.1 *Theory of the Price Level*

The idea of the existence of a *level of prices* is made intelligible by assuming that utility cost can be factored into a product of price and quantity levels,

(1) $$\rho(p, x) = \theta(p)\phi(x),$$

which is equivalent to assuming that the utility relation R is homogeneous. Then for all p, x

(2) $$\theta(p)\phi(x) \leqq px,$$

and for utility cost efficiency of a demand (x, p),

(3) $$\theta(p)\phi(x) = px.$$

Hence if (x_0, p_0), (x_1, p_1) are a pair of demands compatible with R, say, corresponding to a *base* and a *current* observation,

(4) $$\theta(p_0)\phi(x_0) = p_0x_0, \qquad \theta(p_1)\phi(x_1) = p_1x_1$$

and

(5) $$\theta(p_0)\phi(x_1) \leqq p_0x_1, \qquad \theta(p_1)\phi(x_0) \leqq p_1x_0.$$

Then

(6) $$P_{10}X_{10} = M_{10}$$

where

(7) $$P_{10} = \theta(p_1)/\theta(p_0), \qquad X_{10} = \phi(x_1)/\phi(x_0)$$

and

(8) $$M_{10} = M_1/M_0,$$

where

(9) $$M_0 = p_0x_0, \qquad M_1 = p_1x_1.$$

Also

(10) $$p_1x_1/p_0x_1 \leqq P_{10} \leqq p_1x_0/p_0x_0$$

and

(11) $$p_1x_1/p_1x_0 \leqq X_{10} \leqq p_0x_1/p_0x_0.$$

From (6), $\phi(x_0) = \phi(x_1)$, that is, $X_{10} = 1$, if and only if $M_1 = P_{10}M_0$. This is the condition that an expenditure M_1 at prices p_1 be equivalent in purchasing power to an expenditure M_0 at prices p_0. All that must be specified to establish this purchasing power, or real value relation, is the *price index* P_{10}. Any current money M_1 can be *deflated* by the index to give the base equivalent $M_0 = M_1/P_{10}$.

In terms of derived budgets (x_0, u_0), (x_1, u_1) where

(12) $$u_0 = M_0^{-1}p_0, \qquad u_1 = M_1^{-1}p_1,$$

so that

(13) $$u_0x_0 = 1, \qquad u_1x_1 = 1$$

and with

(14) $$U_{10} = P_{10}M_0/M_1 = \theta(u_1)/\theta(u_0)$$

the foregoing relations are equivalent to

$$(15) \qquad \theta(u_0)\phi(x_0) = 1, \qquad \theta(u_1)\phi(x_1) = 1$$

$$(16) \qquad U_{10}X_{10} = 1$$

and

$$(17) \qquad 1/u_0x_1 \leqq U_{10} \leqq u_1x_0,$$

$$1/u_1x_0 \leqq X_{10} \leqq u_0x_1.$$

The condition for equivalent budgets is $U_{10} = 1$ or, equivalently, $X_{10} = 1$. But compatibility of a budget (x, u) with a homogeneous utility relation implies the compatibility of $(\rho x, \rho^{-1}u)$ for all $\rho > 0$. Hence the condition that $(\rho_0x_0, \rho_0^{-1}u_0)$, $(\rho_1x_1, \rho_1^{-1}u_1)$ be equivalent is that $\rho_1/\rho_0 = U_{10}$.

3.2 *Laspeyres and Paasche*

From 3.1(17), for homogeneous consistency of the given budgets, that is, their simultaneous compatibility with some homogeneous utility relation, it is apparently necessary that the Paasche index not exceed that of Laspeyres. Equivalently,

$$(1) \qquad u_0x_1u_1x_0 \geqq 1,$$

and in fact this is also sufficient. Also, if a homogeneous utility relation is constrained by compatibility with the budgets, then U_{10} is constrained to lie in the *Paasche-Laspeyres interval* defined by 3.1(17), which is nonempty by (1). In fact, the constraint set is identical with that interval. Without imposition of further constraints on utility, such as compatibility with further given budgets, or possession of special properties, there is no sharper specification of U_{10} than this. Various price index formulas single out various special points in the admissible set, which is nonempty subject to the homogeneous consistency condition (1). Thus the Paasche and Laspeyres formulas single out the extremes, and the Fisher formula singles out the geometric mean of these. But no principle is explicitly available here for discriminating between admissible points.

3.3 *Fisher*

Buscheguennce (1925) remarked that with the assumption of a homogeneous quadratic utility function the Fisher index is exact. In translation to present concepts, and with appropriate additional qualifications, given a pair of budgets (x_0, u_0), (x_1, u_1) where $x_0, x_1 > 0$, if R is compatible and homogeneous and has quadratic representation in a convex neighborhood containing x_0, x_1 then

$$(1) \qquad U_{10} = (u_1x_0/u_0x_1)^{1/2},$$

and reciprocally, and equivalently,

$$(2) \qquad X_{10} = (u_0x_1/u_1x_0)^{1/2}.$$

By *homogeneous quadratic consistency* of the pair of budgets can be meant the existence of such an R. Immediately, this is at least as restrictive as homogeneous consistency, which is equivalent to 3.2(1), and appearances suggest it is more restrictive. Therefore, there is some surprise that *for a pair of budgets, homogeneous quadratic consistency is equivalent to homogeneous consistency.* However, for more than two budgets, it is more restrictive.

Such a utility relation R corresponds to an antithetic pair of price and quantity functions of the form

$$(3) \qquad \theta(u) = (uBu')^{1/2}, \qquad \phi(x) = (x'Ax)^{1/2}$$

where $BA = 1$, in a region where they are defined and satisfy the functional inequality

$$(4) \qquad (uBu')^{1/2}(x'Ax)^{1/2} \leqq ux,$$

which is to say in the convex cone where $x'Ax$ is semi-increasing and quasi-concave, equality holding in equilibrium. Though such compatible R, if any, exist and are not unique, they all determine the unique value of U_{10} given by (1). But, as just remarked, a compatible homogeneous quadratic R exists if and only if a compatible homogeneous R exists. It follows that the Fisher index, where it is capable of interpretation at all, which is in the case of homogeneous consistency, is identifiable with the value of U_{10} determined with respect to a locally quadratic compatible homogeneous relation. Thus *the Fisher index cannot be divorced from the quadratic utility hypothesis.*

3.4 *Palgrave*

A demand (x, p) has a total expenditure

(1) $$M = \Sigma_i p_i x_i = px$$

which is a sum of individual expenditures

(2) $$M_i = p_i x_i$$

which represent a distribution of the total in shares

(3) $$\sigma_i = M_i/M = p_i x_i/px = u_i x_i$$

where $u = M^{-1}p$. The Laspeyres index is expressible as an arithmetic mean of price ratios between the base and current period with the expenditure shares as weights. For

(4) $$(U_{10})_u = u_1 x_0 = \Sigma_i u_{0i} x_{1i} = \Sigma_i u_{1i} x_{1i}(u_{0i}/u_{1i}) = \Sigma_i \sigma_{1i}(u_{0i}/u_{1i}).$$

The geometric mean which corresponds to this arithmetic mean, in which the same weights become exponents, is

(5) $$(U_{10}{}^p)_u = \Pi_i(u_{0i}/u_{1i})^{\sigma_{1i}}$$

This is Palgrave's formula, translated into present terms. By the general relation of an arithmetic mean to the corresponding geometric mean,

(6) $$(U_{10}{}^p)_u \leqq (U_1)_u.$$

Similarly the Paasche index $(U_{10})_l = 1/(U_{01})_u$ has associated with it the companion to the Palgrave formula

(7) $$(U_{10}{}^p)_l = \Pi_i(u_{0i}/u_{1i})^{\sigma_{0i}}$$

It is obtained by replacing the current shares by the base shares as exponents. Similarly

(8) $$(U_{10})_l \leqq (U_{10}{}^p)_l.$$

Just as the Fisher index cannot be divorced from the homogeneous quadratic utility function, so the Palgrave formula cannot be divorced from a quantity function of the extended Cobb-Douglas form.

(9) $$\phi(x) = \Pi_i(x_i)^{w_i}; \qquad w_i \geqq 0, \qquad \Sigma w_i = 1.$$

The antithetic price function is

$$\theta(u) = \Pi_i(u_i)^{\sigma_i}, \tag{10}$$

It is noticed that then, with $ux = 1$, $\sigma_i = u_i x_i$,

$$\theta(u)\phi(x) = \Pi_i(u_i x_i)^c = \Pi_i \sigma_i^c \leqq 1, \tag{11}$$

with equality if and only if $\sigma_i = w_i$ (as can be verified by the Kuhn-Tucker argument). The equilibrium conditions

$$\theta(u_0)\phi(x_0) = 1, \qquad \theta(u_1)\phi(x_1) = 1, \tag{12}$$

therefore require

$$\sigma_{0i} = w_i = \sigma_{1i}. \tag{13}$$

The consistency condition for this model of utility is therefore

$$\sigma_{0i} = \sigma_{1i}. \tag{14}$$

If this is satisfied, the companion pair of Palgrave indexes coincide, and, along with the Fisher index, provide just another point lying between the Laspeyres and Fisher indexes. However, the consistency conditions thus associated with Palgrave are more stringent than those associated with Fisher, which have been seen to be identical with the basic homogeneous consistency.

When Palgrave consistency is not satisfied, a Palgrave critical cost efficiency $e_P \leqq \dot{e} \leqq \bar{e} \leqq 1$ can always be determined, and, for any $e < e_P$, a Cobb-Douglas utility function can be constructed which is e-compatible with each of the given two budgets.

The Laspeyres, Paasche, Fisher, and Palgrave formulas appear to be the only traditional price index formulas involving base and current budget data which have a supporting utility theory.

3.5 *General Price Index Construction*

Ordinarily, the elements that are to enter into a construction of an index between a base (0) and current (1) location are regarded as data for the locations themselves. The conceptual basis for a price index is a homogeneous utility relation. Here it is understood that the purpose of budget data is to impose a constraint on the utility relation, by the compatibility requirement, and thereby to place a constraint on the admissible values of the index. There are two reasons why this framework is too simple. Any available budget data are pertinent by the same principle that the base and current data are pertinent. Therefore calculations should apply to a more general scheme of data, with two budgets as only a special case. This is particularly important and even essential when simultaneous comparisons are required between more than two locations. Then, with two or more budgets, there might not exist a homogeneous utility relation, or any other utility relation, with which they have simultaneous compatibility. That is, they might not be homogeneously consistent. In any case, exact consistency is too limiting a condition to insist on in practice, even if it is a basic theoretical requirement. Any budgets are homogeneously e-consistent, for some cost efficiency e, where $0 < e \leqq 1$; and with this limitation, homogeneous consistency is just the special case of homogeneous 1-consistency. Thus a simple way of accommodating inconsistency is to permit partial cost efficiency. (The method is stated in section 2.5.)

Let $(U_{rs}{}^e)_u$, $(U_{rs}{}^e)_l$ be upper and lower limits of the interval described by $U_{rs} = \theta(u_r)/\theta(u_s)$ when determined with respect to all homogeneous utility relations which are e-compatible with a finite set of budgets (x_t, u_t), $t = 1, \ldots, k$. Then

$$(1) \qquad e' \leqq e < \dot{e} \Rightarrow (U_{rs}{}^{e'})_l \leqq (U_{rs}{}^e)_l \leqq (U_{rs}{}^e)_u \leqq (U_{rs}{}^{e'})_u.$$

The intersection of all these intervals, for $e < \dot{e}$, is an interval with limits which coincide with $(U_{rs})_u = (U_{rs}{}^1)_u$, $(U_{rs})_l = (U_{rs}{}^1)_l$ if there is homogeneous consistency. Then in the special case where the available budgets are just the pair for $t = 0, 1$, these limits for U_{10} coincide with the Laspeyres and Paasche indexes,

$$(2) \qquad (U_{10})_l = 1/u_0x_1, \qquad (U_{10})_u = u_1x_0.$$

But more generally,

$$(3) \qquad (U_{rs})_u = \min_{ij\ldots k} u_rx_iu_ix_j \ldots u_kx_s, \qquad (U_{rs})_l = 1/(U_{rs})_u,$$

so

$$(4) \qquad (U_{rs})_u(U_{st})_u \geqq (U_{rt})_u,$$

and

(5) $$(U_{rr})_u \leqq 1$$

since $u_r x_r = 1$. A necessary and sufficient condition for homogeneous consistency is that

(6) $$(U_{rr})_u \geqq 1;$$

equivalently $(U_{rr})_u = 1$. In that case

(7) $$(U_{rs})_u (U_{sr})_u \geqq 1.$$

Equivalently,

(8) $$(U_{rs})_l \leqq (U_{rs})_u,$$

which generalizes the homogeneous consistency condition of 3.2. t requires that the generalized Laspeyres and Paasche indexes are coı. sistent as upper and lower limits, the one being at least the other. Since

$$(U_{rt})_u = \min_s (U_{rs})_u (U_{st})_u$$

it follows that (6) holds for all r if and only if (8) holds for all r, s. Let

$$U_u = \min_v (U_{rv})_u$$

$$= \min_{r,s,\ldots,q} u_r x_s u_s x_t \ldots u_q x_r$$

so $U_u \leqq 1$. A necessary and sufficient condition for homogeneous consistency is $U_u \geqq 1$; equivalently $U_u = 1$. With $U_u{}^e$ denoting U_u with each $u_r x_s$ replaced by $u_r x_s / e$, so that $(U^1)_u = U_u$, a necessary and sufficient condition for homogeneous e consistency is $U^e \geqq 1$. Since $U_u{}^e$ is a decreasing function of e, the homogeneous critical cost efficiency $\dot{e}$ is determined as the unique e such that $U_u{}^e = 1$.

3.6 *Extrinsic Estimation*

Consider k countries and m levels of income in each which are judged by extrinsic criteria, that is, not on the basis of value and cost analysis of demands, but so as to correspond in purchasing power, at respective prices. Thus let M_{ti} be the ith level of income in country t ($i = 1, \ldots, m$; $t = 1, \ldots, k$). If the prices in country t are p_t, $P_t = \theta(p_t)$ is their level, and X_{ti} is the utility of M_{ti} at those prices, then a cost efficiency level of at least e requires that

(1) $$M_{ti} \geqq P_t X_{ti} \geqq e M_{ti}.$$

But it is judged that, for all i, the X_{ti} are the same for all t, say, equal to X_i. Thus

(2) $$M_{ti} \geqq P_t X_i \geqq e M_{ti}.$$

The value system and the efficiency being undetermined, it is proposed to determine P_t and e satisfying (2) with e as large as possible.

For $e = 1$ to be admissible in (2), equivalently for

(3) $$M_{ti} = P_t X_i$$

to have a solution for P_t and X_i, it is necessary and sufficient that M_{si}/M_{ti} be the same for all i or, equivalently, that M_{ti}/M_{tj} be the same for all t. If this condition holds, then by choosing P_t in the ratio of M_{ti} for any i, and then determining X_i from (3) for any t, a solution of (3) is obtained and, hence, a solution of (2) with $e = 1$.

If this condition does not hold, then let

(4) $$P_{rs} = \min_i M_{ri}/M_{si}$$

(5) $$X_{ij} = \min_t M_{ti}/M_{tj}.$$

Then let $\bar{e}$ be the largest e such that

(6) $$P_{rs}P_{st} \ldots P_{qr} \geqq ee \ldots e$$

and equivalently

(7) $$X_{ij}X_{jk} \ldots X_{hi} \geqq ee \ldots e.$$

Then also $\bar{e}$ is the largest e such that

(8) $$P_{rs}/e \geqq P_r/P_s.$$

Then (8) has a solution for P_t and, equivalently,

(9) $$X_{ij}/e \geqq X_i/X_j$$

has a solution for X_i. Let P_t be any solution of (8) with $e = \bar{e}$. Then, with $e = \bar{e}$ and $P_t = \bar{P}_t$, (2) has a solution $X_i = \bar{X}_i$, necessarily a solution of (9) with $e = \bar{e}$.

Thus the largest possible e has been found such that (2) holds for some P_t, X_i; and such P_t, X_i have been found. Then $\bar{P}_{rs} = \bar{P}_r/\bar{P}_s$ are a set of price indexes between pairs of countries, which are consistent in that they satisfy the circularity test

$$\bar{P}_{rs}\bar{P}_{st}\bar{P}_{tr} = 1, \tag{10}$$

appropriate to a set of ratios, and which, with distance determined in the economic sense of cost efficiency, fit the constraints of the original data as closely as possible.

It should be noted that (1), with $P_{rs} = P_r/P_s$, implies

$$M_{ri}e_{ri} = P_{rs}M_{se}e_{se}, \qquad e \leqq e_{ri} \leqq 1 \tag{11}$$

for some e_{ti}. In other words, efficient parts $M_{ri}e_{ri}$ of the income M_{ri}, with efficiencies e_{ri} at least e, are determined to be of equivalent purchasing power by the price indexes P_{rs}. It could have been required to determine the largest $\bar{e}$ such that there exist $\bar{P}_{rs}$ and e_{ri} which satisfy (10) and (11), and this would have had the same result as the foregoing determination. Necessarily some $e_{ti} = 1$ and some $e_{ti} = \bar{e}$, and generally $w_{ti} = 1 - e_{ti}$ is an imputed inefficiency associated with income M_{ti} in country t.

3.7 *Rectification of Pair Comparisons*

For any price index formula P_{st} between two points in time, s and t, Fisher's "time reversal" test requires that $P_{ts}P_{st} = 1$, which requires in particular, what apparently is true for all the formulas discussed here, that $P_{tt} = 1$. Fisher defined the "time antithesis" to be $P_{st}^* = P_{ts}^{-1}$. Fisher "rectified" a formula by "crossing" it with its time antithesis so as to obtain an associated formula which satisfied the time reversal test. Then the time antithesis of the Laspeyres formula is the Paasche formula. By crossing these according to the geometric mean, Fisher's "ideal" index is obtained, which is therefore the rectification of the Laspeyres index, and similarly of the Paasche. It satisfies the time reversal test and for that reason he considered it ideal. Here a more general rectification procedure will be considered.

In fact, the important logic behind the time reversal test is that any index P_{rs} which is expressible as a ratio $P_{rs} = P_r/P_s$ must satisfy the test. Since a price index theoretically, at least here, arises as a ratio of price levels, meeting the reversal test is a significant requirement. But so just as well is the "circularity test," which he considered, but which none of the one or two hundred formulas he examined appeared to satisfy.

All familiar price index formulas satisfy what might be called the *identity test* $P_{tt} = 1$. The circularity test implies the equivalence of this to the reversal test, and its combination with either constitutes what should be called the *ratio test,* since it is the condition for the expression $P_{rs} = P_r/P_s$. The combination of the circularity and reversal tests is equivalent to the combination of the identity test with the *chain test*

$$P_{rs}P_{st} = P_{rt}. \tag{1}$$

Also the chain test implies the equivalence of the identity and reversal tests and in combination with either is equivalent to the ratio test. But an algebraic formula cannot satisfy the ratio test unless it immediately presents a ratio, and it cannot do this if as in most standard formulas, the data for different periods are not entered as separate factors, which could cancel in multiplication.

Another approach to "rectifying" a set of P_{rs} as closely as possible with respect to the ratio test is to reconcile them as closely as possible with a set of ratios P_r/P_s. Thus, with any given P_{rs}, and any e, 3.7(6) is necessary and sufficient for (8) to have a solution for P_t. The largest value $\bar{e}$ of e for 3.7(8) can be determined, and then a solution $\bar{P}_t$ of 3.7(8) with $e = \bar{e}$ can be found. With $\bar{P}_{rs} = P_r/P_s$

$$\bar{P}_{rs}/\bar{e} \leqq P_{rs} \leqq \bar{e}\bar{P}_{rs} \tag{2}$$

so that

$$P_{rs} = \bar{P}_{rs} \tag{3}$$

if and only if $\bar{e} = 1$, and generally $\bar{P}_{rs}$ is the best approximation to P_{rs} which satisfies the reversal and circularity tests. Here again approximation distance is in the economic sense of cost inefficiency, which is appropriate since an economic error is an inefficiency.

A difficulty which arises with multinational price comparisons by means of one of the standard price index formulas based on price-quantity data is that the cirularity test is not satisfied; so a chain of comparisons is not consistent with the direct comparison, that is

(4) $$P_{ri}P_{ij}\ldots P_{ks} \neq P_{rs}.$$

One way of resolving this difficulty is to combine the directly determined P_{rs} to determine $\bar{P}_{rs}$ as above. The latter do satisfy the circularity test, or, since $\bar{P}_{rr} = 1$, equivalently the chain test, and approximate the P_{rs} as closely as possible, in the manner described. It should be noticed that this difficulty is automatically avoided if the method of sections 2.5 and 3.6 is used, since those determinations each i. ive explicit or implicit reference to some particular utility function.

With just two periods, the ratio test reduces to the reversal test, so the process just considered can be seen as a generalization of Fisher's general rectification procedure. But now it will be seen more closely as a generalization.

The Fisher "ideal" index, arrived at as the rectification $\bar{P}_{01} = (P_{01}P_{10}^{-1})^{1/2}$ of the Laspeyres index P_{01}, by geometric crossing with its time antithesis P_{10}^{-1}, so as to obtain an index which satisfies the time reversal test, can also be arrived at as the best approximation, with efficiency distance, which satisfies time reversal. Thus consider

(5) $$\bar{P}_{01} = \bar{P}_{10}^{-1},$$

equivalently

(6) $$\bar{P}_{01} = \rho,\ \bar{P}_{10} = \rho^{-1},$$

such that

(7) $$P_{ij}/e \geqq \bar{P}_{ij},$$

equivalently

(8) $$\rho/e \geqq P_{01} \geqq e\rho,$$

(9) $$1/\rho e \geqq P_{10} \geqq e/\rho.$$

With P_{01}, P_{10} given, ρ is to be determined with e as large as possible. Equivalently

(10) $$1/e^2 \geqq P_{01}P_{10} \geqq e^2,$$

(11) $$\rho^2/e^2 \geqq P_{01}/P_{10} \geqq e^2\rho^2.$$

But with the largest e, and in fact any e, which satisfies (10), $\rho^2 = P_{01}/P_{10}$ automatically satisfies (11). The Fisher indexes $F_{01} = P_{01}/P_{10}$

and $F_{10} = F_{01}^{-1}$, among all numbers $\bar{P}_{01}$ and $\bar{P}_{10}$ such that $\bar{P}_{10} = \bar{P}_{01}^{-1}$, are closest to P_{01}, P_{10} in that they satisfy (8) and (9) with e as large as possible.

Since P_{ij} could here be given by any formula, this argument, which deals with a special case of the previous analysis, is mainly a comment on Fisher's procedure of rectifying a formula by geometric crossing with its time antithesis or even with any other formula.

4.0 THEORY OF MARGINAL PRICE INDEXES

4.1 *Marginal Price Indexes*

Let R be a utility relation for which the associated utilities cost function can be represented by

$$\rho(p, x) = \theta(p)\phi(x) + \mu(p). \tag{1}$$

By this property, R can be called a *linear cost utility*. With this classification, a homogeneous relation, which is characterized by the same utility property but with $\mu(p) = 0$, can be distinguished as a *homogeneous linear cost utility*. Thus here there is a particular generalization of homogeneity as applied to relations.

Since $\rho(p, x)$ is linearly homogeneous in p for all x, both $\theta(p)$, $\mu(p)$ in (1) must be linearly homogeneous, and $\phi(x)$ must be uniquely determined up to a linear transformation. Thus $\phi(x)$ is completely specified when its values are specified at two points which are not indifferent and then so are the other functions. A function associated with a linear cost relation in the same way that a linearly homogeneous function is associated with a homogeneous, or homogeneous linear, cost relation can be described as a *linear profile* function, for the reasons presented below. With this classification, a function which is linearly homogeneous appears as a *homogeneous linear profile* function.

From (1), for all p, x

$$\theta(p)\phi(x) + \mu(p) \leqq px \tag{2}$$

and for all p, equality holds for some x, and R is quasi-concave if and only if for all x equality holds for some p. Whenever a utility relation is considered, it will be because it is compatible with given demands. But compatibility is preserved when the relation is replaced by its normalization obtained by taking the adjoint of its adjoint. That relation is automatically quasi-concave, its superior sets being the convex closures of those of the original. It appears from this that there is no essential loss of generality, so far as present questions are concerned, and a simplification in exposition, in assuming all utility relations dealt with to be quasi-concave. This is a fortunate circumstance for standard theory, which seems always to assume indifference contours to be concave. But it does not mean they really are. It is just that, in the limited language of economic choice, it is impossible to communicate that they are not.

A further simplicity which follows from strict quasi concavity is that expansion loci are paths, with a unique consumption corresponding to every level of income. Sometimes it is as well to assume this, again for simplicity of exposition, and also because an arbitrarily small modification can replace concavity by strict concavity, so there is no significant difference when error is allowed.

Because, for any p, $\rho(p, x)$ is a utility function which represents R, so is $\phi(x)$. The adjoint is

$$\psi(u) = [1 - \mu(u)]/\theta(u), \tag{3}$$

and

$$\psi(M^{-1}p) = [M - \mu(p)]/\theta(p), \tag{4}$$

so that

$$\partial\psi(M^{-1}p)/\partial M = 1/\theta(p). \tag{5}$$

Thus $P = \theta(p)$ is the marginal price of utility $X = \psi(M^{-1}p)$ attained at elementary prices p with a level of expenditure M, and it is fixed when elementary prices are fixed.

The *profile,* for prices p, of a utility function $\phi(x)$ with adjoint $\psi(u)$, is $\psi(M^{-1}p)$. Here it appears that the profiles are linear. Such a property is preserved under linear transformations, but not more general ones. An equivalent characterization of linear cost utility relations is that they admit representation by a utility function with linear profiles. It is to be seen that still another equivalent characterization is that the expansion loci are linear.

To see this, assume, as would be permitted on grounds already stated, that $\phi(x)$ is quasi-concave. Then, by a general proposition, since its profiles also are concave, it is concave. Let y and z be two different demands at prices p which are compatible, that is yEp, zEp which is to say $\theta(p)\phi(y) + \mu(p) = py$; $\theta(p)\phi(z) + \mu(p) = pz$. Then, with $\beta + \gamma = 1$, it follows that

$$\theta(p)[\phi(y)\beta + \phi(z)\gamma] + \mu(p) = p(y\beta + z\gamma).$$

But then with $\beta, \gamma \geqq 0$, and $x = y\beta + z$, $\phi(x) \geqq \phi(y)\beta + \phi(z)\gamma$, since $\phi(x)$ is concave, and, as usual, $\theta(p)\phi(x) + \mu(p) \leqq px$. But this with the foregoing implies $\theta(p)\phi(x) + \mu(p) = px$, and shows that the demand of x at prices p is compatible, that is, xEp.

It has been shown that $y, z \, \varepsilon \, Ep \Rightarrow \langle y, z\rangle \subset Ep$, where $\langle y, z\rangle$ is the line segment joining y and z; that is, the expansion locus Ep is a convex set. But if it is a path, as it is if R is strictly quasi-concave, then it is a segment of a line. Since the line is in any case truncated within the commodity space, it cannot extend beyond a half-line. In fact, there may have to be a further interruption of a different nature, where the function becomes strictly quasi-convex, and where no demand is compatible, and this could leave at most a bounded segment. An intrinsic limitation of this kind is important in describing the range of incomes for which a comparison is valid.

It has been seen that if R is a quasi-concave utility order and (1) holds, then $\phi(x)$ has concave contours and profiles and hence is a concave function. Then it was deduced that the expansion sets Ep are convex, where xEp means $\rho(p, x) = 1$. Thus in particular, when the expansion sets are paths that cut every level of income (equivalently, every level of utility), in a single point, they must be straight lines to be convex.

Now a converse proposition will be shown. Suppose Ep are given as describing all levels of utility indicated by 0 and 1 and as convex. Then, between these levels R has the linear cost property (1).

For any u let $x_0(u)$, $x_1(u)$ denote any elements of the expansion set Eu in utility levels indicated by 0 and 1. Then, by hypothesis,

$$x_t(u) = x_0(u) + [x_1(u) - x_0(u)]t$$

where $0 \leqq t \leqq 1$, is also in Eu. Also, since $x_0(u)Rx_0(v)$ for all u, v, then

$$ux_0(u) = \min\,[ux : xRx_0(u)] \leqq ux_0(v).$$

Thus $ux_0(u) \leqq ux_0(v)$ and similarly, $ux_1(u) \leqq ux_1(v)$, for all u, v. It follows, multiplying these inequalities by $(t,\ 1 - t)$ and adding, that

$ux_t(u) \leqq ux_t(v)$ for all $0 \leqq t \leqq 1$. But $x_t(u) \varepsilon Eu$, for all u. It follows that, for all such t, $x_t(u)Rx_t(v)$ for all u and v. Thus, for all such t, $x_t(u)$ for all u describes an indifference surface. Hence, defining $\phi[x_t(u)] = t$, $\phi(x)$ is a utility function which represents R. Also, if $\phi(x) = t$, $yRx \cdot \Leftrightarrow \cdot ux \geqq ux_t(u)$ for all u. Hence

$$\rho(p, x) = \min [py : yRx] = \min [py : ux \geqq ux_t(u)] = px_t(p)$$
$$= px_0(p) + p[x_1(p) - x_0(p)]t = \theta(p)\phi(x) + \mu(p),$$

where

$$\theta(p) = p[x_1(p) - x_0(p)], \ \mu(0) = px_0(p),$$

as required.

The equilibrium relation E which holds between x, p in an R-compatible demand is given by

(6) $$xEp \equiv \theta(p)\phi(x) + \mu(p) = px.$$

Thus, with compatible demands (x_0, p_0) and (x_1, p_1) in a base and current period, or country, if

(7) $$\begin{aligned} M_0 &\doteq p_0x_0, & M_1 &= p_1x_1 \\ m_0 &= \mu(p_0), & m_1 &= \mu(p_1) \\ P_0 &= \theta(p_0), & P_1 &= \theta(p_1) \\ X_0 &= \phi(x_0), & X_1 &= \phi(x_1) \end{aligned}$$

then

(8) $$P_0X_0 + m_0 = M_0, \qquad P_1X_1 + m_1 = M_1.$$

But the m's and P's are determined by prices alone. Hence the condition $X_0 = X$ for *any* incomes M_0, M_1 to have the same purchasing power at prices p_0, p_1 is equivalent to

(9) $$M_1 - m_1 = P_{10}(M_0 - m_0)$$

where $P_{10} = P_1/P_0$ is the *marginal price index* between p_0 and p_1. Therefore, with any incomes M_0, M_1 constrained to purchasing power equivalence,

(10) $$\Delta M_1/\Delta M_0 = P_{10}.$$

This shows the definition of a marginal price index by its characteristic role as applied to income differentials which preserve equivalence, instead of to incomes themselves, about which nothing can be said without knowledge of the *original values* m_0, m_1. This contrasts with an ordinary price index P_{10} which gives

$$(11) \qquad M_1 = P_{10}M_0$$

as the relation between equivalent incomes, and hence the same relation between income differentials which preserve equivalence. Thus in the use of marginal price indexes there is a distinction between total incomes and income differentials which is effaced by ordinary price indexes.

It would seem that (9) is an appropriate relation for adjusting wages for price change as required by an escalator clause in a labor-management contract, to maintain economic equity, but (10) could be appropriate for adjustment of any other moneys which do not have the nature of base incomes, such as rents, allowances, and so forth.

4.2 *Method of Limits*

Continuing now, from 4.1(2),

$$(1) \qquad \begin{aligned} P_0X_1 + m_0 &\leqq p_0x_1 \\ P_1X_0 + m_1 &\leqq p_1x_0 \end{aligned}$$

and from (8),

$$(2) \qquad \begin{aligned} P_0X_0 + m_0 &= p_0x_0 \\ P_1X_1 + m_1 &= p_1x_1. \end{aligned}$$

Now from (11) and (12),

$$(3) \qquad (p_1x_1 - m_1)/(p_0x_1 - m_0) \leqq P_{10} \leqq (p_1x_0 - m_1)/(p_0x_0 - m_0).$$

Thus, for any given pair of demands (x_0, p_0) and (x_1, p_1), if the hypothesis of their compatibility with R, for any values for m_0 and m_1, is accepted, then the bounds for P_{10} shown in (3) are determined. In the particular case with $m_0 = m_1 = 0$ these bounds coincide with the Paasche and Laspeyres price indexes. This circumstance can be amplified further later.

Now for any incomes M_0 and M_1, not necessarily p_0x_0 and p_1x_1, to be equivalent in purchasing power at respective prices p_0, p_1 it is necessary, by 4.1(9) and (3), that they be in the relation T_{01} depending on m_0, m_1 given by

$$(4)\quad M_0T_{01}M_1 \equiv (p_1x_1 - m_1)/(p_0x_1 - m_0) \leqq (M_1 - m_1)/(M_0 - m_0) \\ \leqq (p_1x_0 - m_1)/(p_0x_0 - m_0).$$

Thus every M_0 corresponds to an interval of M_1 with upper and lower limits $(M_1)_u$, $(M_1)_l$ where

$$(5)\quad \begin{aligned}(M_1)_l - m_1 &= [(p_1x_1 - m_1)/(p_0x_1 - m_0)]\,(M_0 - m_0)\\ (M_1)_u - m_1 &= [(p_1x_0 - m_1)/(p_0x_0 - m_0)]\,(M_0 - m_0)\end{aligned}$$

which can define the *extreme equivalents* of M_0, and similarly with 0 and 1 interchanged. Then a particular value $\bar{M}_1$ between these limits is given by

$$(6)\quad (\bar{M}_1 - m_1)^2 = [(M_1)_l - m_1][(M_1)_u - m_1)],$$

which can define the *principal equivalent* of M_0. There may seem to be no reason for introducing this concept for a particular correspondent $\bar{M}_1$ of M_0. It is just a way of singling out a point in the interval M_0T_{01} of correspondents of M_0. However, M_0 is the principal correspondent of its principal correspondent: The one-to-one subcorrespondence $\bar{T}_{01}$ of the many-to-many correspondence T_{01} given by

$$(7)\quad M_0\bar{T}_{01}M_1 \equiv [(M_1 - m_1)/(M_0 - m_0)]^2 \\ = (p_1x_1 - m_1)(p_1x_0 - m_1)/(p_0x_1 - m_0)(p_0x_0 - m_0)$$

can define the *principal correspondence.* It holds between incomes M_0 and M_1 at prices p_0 and p_1 if and only if each is the principal correspondent of the other, in which case each is the principal correspondent of its principal correspondent. Note that if $m_0 = m_1 = 0$ then this is the correspondence associated with the Fisher price index.

The theory thus shows that the Laspeyres, Paasche, and Fisher indexes, understood in their role as price indexes, correspond to a special case, where the parameters m_0 and m_1 are zero.

A particular utility relation R which was presented earlier, has the property expressed by (1). With $\phi(x)$ fixed, by assigning values at two nonindifferent points, the other functions are fixed, and hence so are m_0 and m_1. For any M_0 there exists a unique M_1 such that

$$(8)\quad \psi(M_0^{-1}p_0) = \psi(M_1^{-1}p_1).$$

It has been shown that this together with x_0Ep_0, x_1Ep_1 for any x_0, x_1 implies $M_0T_{01}M_1$. Thus it would be exceptional that also $M_0\bar{T}_{01}M_1$. But the particular utility relation R makes (8) specific and also m_0, m_1. Usually in practice there is nothing available but fragmentary demand data, even when, as here, there are only two reference periods or countries.

4.3 *Expansion Lines and Critical Points*

If at each of the prices p_0, p_1 just one demand is available, x_0, x_1, then only a rather loose analysis can be developed, as has just been done, where the parameters m_0, m_1 are unspecified. Even this analysis can be taken further, but not here. Instead it will be supposed that a second demand is available at each of the prices, say, y_0, y_1. By implication then, since compatible linear cost utility is to be considered, for which expansion loci are line segments, or possibly half-lines, what is being considered is a pair of segments $K_0 = \langle x_0, y_0 \rangle$, $K_1 = \langle x_1, y_1 \rangle$ of demands associated with prices p_0, p_1.

Let L_0, L_1 denote the carrier lines of K_0, K_1. These are the lines joining the extremities. It can be shown that, if

$$(1) \qquad \begin{vmatrix} p_0(x_0 - y_0) & p_0(x_1 - y_1) \\ p_1(x_0 - y_0) & p_1(x_1 - y_1) \end{vmatrix} \neq 0,$$

(and if this is not so then it can be made so, by distributing the data slightly, or at least within bounds of its conspicuous inaccuracy), then there exists a unique pair of *critical points* c_0, c_1 on L_0, L_1 such that $p_0c_0 = p_1c_1$, $p_1c_0 = p_1c_1$. These need not be on K_0, K_1 nor even in the commodity space. They are distinguished as being indifferent with respect to every compatible utility relation. Then a pair of half-lines $L_0{}^*$, $L_1{}^*$ on L_0, L_1 with c_0, c_1 as vertexes are determined. The pair is selected according to the sign of the elements and the determinants of the foregoing nonsingular 2×2 matrix. Then the expansions (K_0, p_0), (K_1, p_1) are generally consistent, that is, compatible with any utility relation, regardless of properties, if and only if $K_0 \subset L_0{}^*$ or $K_1 \subset L_1{}^*$. Thus, should $L_0{}^*$, $L_1{}^*$ happen not to lie in the commodity space at all then certainly the expansions are inconsistent. However, for *local linear cost consistency,* equivalently compatibility with a utility function which has linear expansion loci *in a convex neighborhood containing* K_0, K_1, it is necessary and sufficient that $K_0 \subset L_0{}^*$ and $K_1 \subset L_1{}^*$.

With c_0, c_1 as the pair of critical points on the carrier lines L_0, L_1, let

x_0, x_1 now be any other pair of points $p_0(x_0 - c_0) > 0$, $p_1(x_1 - c_1) > 0$. Then for consistency it is necessary that also $p_0(x_1 - c_1) > 0$, $p_1(x_0 - c_0) > 0$. Then the *critical determinant*

(2)
$$\begin{vmatrix} p_0(x_0 - c_0) & p_0(x_1 - c_1) \\ p_1(x_0 - c_0) & p_1(x_1 - c_1) \end{vmatrix}$$

is nonzero by hypotheses (1). Then the *hyperbolic* and *elliptical* cases are distinguished by the sign, positive or negative, of (2). In the hyperbolic case, L_0^*, L_1^* correspond to $x_0 \;\varepsilon\; L_0$, $x_1 \;\varepsilon\; L_1$, where p_0x_0) p_0c_0, $p_1x_1 \geqq p_1c_1$. In this case, consistency of (K_0, p_0), (K_1, p_1) requires

(3)
$$\begin{aligned} x_0 \;\varepsilon\; K_0 &\Rightarrow p_0x_0 \geqq p_0c_0 \\ x_1 \;\varepsilon\; K_1 &\Rightarrow p_1x_1 \geqq p_1c_1 \end{aligned}$$

Inequalities are reversed for the elliptical case.

Let $F_0(M_0)$, $F_1(M_1)$ denote the points x_0, x_1 on L_0, L_1 with $p_0x_0 = M_0$, $p_1x_1 = M_1$.

With reference to the relation T_{01} given by 4.2(4), with the specification $m_0 = p_0c_0$, $m_1 = p_1c_1$ it can now be said that, at prices p_0, p_1, any incomes M_0, M_1 may be determined as of equivalent purchasing power with respect to some utility relation compatible with the expansions (K_0, p_0), (K_1, p_1), if and only if, first, these expansions are consistent; second, in the hyperbolic case,

(4)
$$\begin{aligned} M_0 &\geqq p_0c_0, & F_0(M_0) &\geqq 0 \\ M_1 &\geqq p_1c_1, & F_1(M_1) &\geqq 0, \end{aligned}$$

and correspondingly in the elliptical case; and finally, $M_0T_{01}M_1$.

With the appropriate qualifications about the range of M_0, M_1 it appears thus that the relation T_{01} determines, for any M_0, the best possible bounds, that is, the limits of M_1 that can be established as equivalent with respect to some compatible utility relation.

Two peculiarities may be noted. No restriction at all has been made for the utility relation in the foregoing, but now two restrictions will be considered. The first is that the utility relation be of the linear cost type, at least in a convex neighborhood containing K_0, K_1. However, even if this restriction is imposed on the utility relation just described, the description remains valid. This is remarkable only because on the face of the matter, it would seem that with this restriction the relation T should be contracted to a proper subrelation.

The other restriction to be considered is a stronger one. It requires the utility relation to be representable in a convex neighborhood containing K_0, K_1 by a general quadratic utility function. This implies qualification under the first restriction, since quadratic representation implies linear expansion. Again on the face of the matter, quadratic consistency is a stronger condition than linear expansion consistency. In regard to any number of expansions, it is. But it is surprising that for just a pair of expansions, it is equivalent. Then, under this common consistency requirement, it is natural to ask what is the subrelation, say T_{01}^*, of T_{01} corresponding to this further quadratic restriction. Certainly now it will be a proper subrelation, but since, if there are any, there are infinite compatible quadratics, it might be expected that T_{01}^* would not be one-to-one, but that, for every M_0, $M_0T_{01}^*$ would be a subinterval of M_0T_{01}, nonempty by consistency and with a variety of points because of the variety of compatible quadratics. However, it is established that $T_{01}^* = \bar{T}_{01}$, that is, the quadratically determined correspondence T_{01}^* coincides with the principle correspondence given by 4.2(7), and moreover this is one-to-one. Thus here there is a surprise opposite to the first. Introducing the values of m_0, m_1, that formula becomes

$$
\begin{aligned}
(5)\qquad M_0\bar{T}_{01}M_1 &\equiv [(M_1 - p_1c_1)/(M_0 - p_0c_0)]^2 \\
&= p_1(x_1 - c_1)p_1(x_0 - c_0)/p_0(x_1 - c_1)p_0(x_0 - c_0).
\end{aligned}
$$

It follows from the definition of the critical points c_0, c_1 on the carrier lines L_0, L_1 that c_0, c_1 in this formula could be replaced by any point c on the *critical transversal* to L_0, L_1 obtained by joining the L's, assuming L_0, L_1 are skew. But if L_0 and L_1 intersect in a point c, then both c_0 and c_1 coincide with c, and no such transversal is defined. It should be noted that if L_0, L_1 are skew, any compatible quadratic is singular, that is, its matrix of second derivatives, which is constant, is singular. In this case its expansion loci for p_0, p_1 do not lie in lines but in linear manifolds at least as large as the joins of c with L_0, L_1. For the expansion loci strictly to be lines, the quadratic must be regular, and in this case L_0 and L_1 must intersect.

If, in particular, the intersection is at the origin $c = 0$, then (5) becomes Fisher's formula. Then for consistency, the elliptical case is excluded entirely, since if $M_0 \leqq 0$ then $F_0(M_0) \geqq 0$ is impossible. Thus the consistency condition becomes simply $p_0x_0p_1x_1 \geqq p_0x_1p_1x_0$, corresponding to the remaining hyperbolic case. Since c_0 and c_1 appear as points where the gradient of any compatible quadratic must vanish, $c_0 = c_1 = 0$ corresponds to the case of a homogeneous quadratic. This reproduces the observation of Buescheguennce that the Fisher index is exact if a homogeneous utility function can be assumed to prevail. But, related to this, as a generalization, Wald has shown that if a pair of expansion lines are given, with associated prices, and it can be assumed that a general quadratic utility function prevails, then it is possible to determine a unique one-to-one correspondence between equivalent incomes at these prices. This is by his "new formula," which, because of Buescheguennce's proposition must be essentially a generalization of Fisher's formula. Consistency conditions were not treated and, hence, neither were the necessary restrictions on the range of those incomes for such comparison. But with the introduction of the concept of critical points certainly his formula must be identical with (13), which is transparently a generalization of Fisher. A generalization of Wald's formula appears in Afriat (1964).

This theory of marginal price indexes extends every feature of the theory of price indexes based on the traditional concept. Instead of a pair of demands (x_0, p_0), (x_1, p_1), which because of implicit homogeneity correspond in principle to a pair of linear expansions (x'_0, p_0), (x'_1, p_1), where x'_0, x'_1 are the rays through x_0, x_1, the data now consist of a general pair of linear expansions (K_0, p_0), (K_1, p_1), where K_0 and K_1 can arise from pairs of demands x_0, y_0 and x_1, y_1 not necessarily on the same ray. The Paasche-Laspeyres limits for a price index P_{10} become the limits given in (1) with $m_0 = p_0c_0$, $m_1 = p_1c_1$ for a marginal price index P_{10}. The index then has the role shown by 4.1(10), and m has the role shown by 4.1(9).

This theory of marginal price index construction, here restricted to data for two periods or countries, and dependent on consistency conditions, has a general extension for arbitrary data and with a relaxation of strict consistency to approximate consistency. But this development will be shown here only as it applies to the usual price indexes.

With all this, it still has to be claimed that marginal price indexes, as described here, are not yet general enough. They are not vulnerable to the objection made to price indexes, which was that the concept implies that the rich and poor have the same spending pattern. But they are vulnerable to the objection that the concept implies that rich and poor have the same *marginal pattern*. This is to say that an extra dollar given to a rich individual would be spent in identical fashion were it given to a poor one. This does not go so far as to say they enjoy all things in the same proportion and differ just in the scale corresponding to their different incomes, but it is a radical contradiction of reality nevertheless.

To escape this objection, a further method is possible, where the intervals of incomes to be compared can be dissected into consecutive corresponding subintervals, or steps, corresponding to different intervals of real income, where the foregoing scheme applies, but with different P_{10}, m_0, m_1 at each level. This corresponds to the concept of a utility relation determined by a finite set of indifference surfaces, each surface being the interface between consecutive intervals of real income. The interpolation between surfaces is by the unique linear cost utility relation they determine.

For arbitrary demand data, consistency of utility relations with such a form is not more restrictive than general consistency. Such a scheme for establishing equivalent real incomes, though it would not be put into operation by publication of a single number, as is the usual practice, would still have practical simplicity. It would establish corresponding income classes, and then different P_{10}, m_0, and m_1 for determining corresponding points in each pair of corresponding classes. Though a utility function conceptually underlies such information, there is no need to compute, let alone present, a particular one. In any case, such a scheme of information would present everything about such a utility function that would be relevant to the desired comparison.

The real-income classes correspond to any partition of the range. When there is just one class, the method is identical with the original marginal index method. Transition from one class to another can correspond to a significant shift of marginal pattern.

A more elaborate general analysis can apply to several periods or countries, and express approximation in terms of cost efficiencies. Any income in any period would have an imputed cost efficiency and an interval of corresponding incomes in every other period or country. Within each such interval of correspondents, a single point can be determined from the principal correspondence which is produced by the linear expansions across each real-income interval in the two periods or countries. This more elaborate method communicates information about underlying error and indeterminacy together with a one-to-one correspondence which represents a statistical resolution of both.

BIBLIOGRAPHY

Afriat, S. N. "Theory of Economic Index Numbers." Mimeographed. Cambridge, Engl., Department of Applied Economics, Cambridge University, May 1956.

———. "Preference Scales and Expenditure Systems." *Econometrica* 30 (1962): 305–323.

———. "A Formula for Ranging the Cost of Living." Abstract in R. L. Graves and P. Wolf, eds. *Recent Advances in Mathematical Programming: Proceedings of the Chicago Symposium, 1962.* New York, McGraw-Hill, 1962.

———. "The System of Inequalities $a_{rs} > X_r - X_s$." *Proceedings of the Cambridge Philosophical Society* 59 (1963): 125–133.

———. "An Identity Concerning the Relation Between the Paasche and Laspeyres Indices." *Metroeconomica* XV, I (1963): 38–46.

———. "On Bernoullian Utility for Goods and Money." *Metroeconomica* XV, I (1963), 38–46.

———. "The Construction of Utility Functions from Expenditure Data." *International Economic Review* 8, 1 (1967): 66–77.

———. "The Cost of Living Index." In M. Shubik, ed. *Studies in Mathematical Economics in Honor of Oskar Morgenstern.* Princeton, N.J., Princeton University Press, 1967, Chap. 23.

———. "The Construction of Cost Efficiencies and Approximate Utility Functions from Inconsistent Expenditure Data." Paper presented at the winter meeting of the Econometric Society, New York, 1969.

———. "The Method of Limits in the Theory of Index Numbers." *Metroeconomica* (1970).

Allen, R. G. D. "The Economic Theory of Index Numbers." *Economica,* New Series XVI, 63 (August 1949): 197–203.

Antonelli, G. B. *Sulla Teoria Matematica della Economia Pura* (1886). Reprinted in *Giornale degli Economisti* 10 (1951): 233–263.

Bowley, A. L. Review of *The Making of Index Numbers,* by Irving Fisher. *Economic Journal* 33 (1923) : 90–94.

——— "Notes on Index Numbers." *Economic Journal* (June 1928).

Buscheguennce. "Sur une classe des hypersurfaces. A propos de 'l'index idéal' de M. Irv. Fisher." *Recueil Mathematique* (Moscow) XXXII, 4 (1925).

Chase, Arnold E. "Concepts and Uses of Price Indices." Paper presented at the American Statistical Association meeting, August 1960.

Cournot, Augustine. *Researches into the Mathematical Principles of the Theory of Wealth* (1838). Translated by Nathaniel T. Bacon with an essay on Cournot and mathematical economics and a bibliography on mathematical economics by Irving Fisher (1924). Reprint: New York, Kelley, 1960.

De Finetti, Bruno. "Sulle stratificazioni convesse." *Ann. Mat. Pura Appt.* 4 (1949): 173–183.

Divisia, F. *Economique Rationelle.* Paris, 1928.

Dupuit, J. "De la mesure de l'utilité des travaux public" (1844). Reprinted in English translation as "On the Measurement of the Utility of Public Works," in *International Economic Papers,* No. 2. London, Macmillan, 1952.

Edgeworth, F. Y. "A Defense of Index Numbers." *Economic Journal* (1896): 132–142.

Fisher, Irving. *The Purchasing Power of Money.* New York, Macmillan, 1911.

———. *The Making of Index Numbers.* Boston, Houghton Mifflin, 1922.

———. "Professor Bowley on Index Numbers." *Economic Journal* 33 (1923): 246–251.

———. "A Statistical Method for Measuring Marginal Utility and Testing the Justice of a Progressive Income Tax." In *Economic Essays in Honor of John Bates Clark.* New York, 1927.

Fleetwood, William. *Chronicon Preciosum: or, an Account of English Money, The Price of Corn, and Other Commodities, for the last 600 Years—in a Letter to a Student in the University of Oxford.* London, 1707.

Foster, William T. Prefatory Note, to *The Making of Index Numbers* by Fisher (see above).

Frisch, Ragnar. "Annual Survey of General Economic Theory: The Problem of Index Numbers." *Econometrica* 4, 1 (1936): 1–39.

Georgescu-Roegen, N. "Choice and Revealed Preference." *Southern Economic Journal* 21 (1954): 119–130.

Gorman, W. M. "Separable Utility and Aggregation." *Econometrica* 27 (1959): 469–487.

———. "Additive Logarithmic Preferences: A Further Note." *Review of Economic Studies* 30 (1963): 56–62.

Haberler, Y. *Der Sinn der Indexzahlen.* Tubingen, 1924.

Hicks, J. R. *A Revision of Demand Theory.* Oxford, Clarendon Press, 1956.

Hotelling, H. "Demand Functions with Limited Budgets." *Econometrica* 3 (1935): 66–78.

Houthakker, H. S. "Revealed Preference and the Utility Function." *Economica,* N. S. 17 (1950): 159–174.

———. "La forme des courbes d'Engel." *Cahiers du Seminarie d'Econometrie* 2 (1953): 59–66.

———. "An International Comparison of Household Expenditure Patterns, Commemorating the Centenary of Engel's Law." *Econometrica* 25 (1957): 532–551.

———. "Some Problems in the International Comparison of Consumption Patterns." In *L'évaluation et le rôle des besoins de consommation dans les divers régimes économiques.* Paris, Centre National de la Recherche Scientifique, 1963.

International Labour Office. *A Contribution to the Study of International Comparisons of Costs of Living.* Studies and Reports, Series N, 17. Geneva, 1932.

Keynes, J. M. *A Treatise on Money,* Vol. I, *The Pure Theory of Money.* New York, Harcourt, Brace, 1930.

Klein, L. R., and H. Rubin. "A Constant Utility Index of the Cost of Living." *Review of Economic Studies* 15 (1947): 84–87.

Konus, A. A. "The Problem of the True Index of the Cost of Living." *Economic Bulletin of the Institute of Economic Conjecture* (Moscow), 1924).

Lange, O. "The Determinateness of the Utility Function." *Review of Economic Studies* 1 (1934): 218–224.

Laspeyres, E. "Die Berechnung einer mittleran Waarenpreissteigerung." *Jahrbücher für nationaloekonomie und Statistik* (Jena) XVI, 1871: 296–314.

Lerner, A. P. "A Note on the Theory of Price Index Numbers." *Review of Economic Studies* (1935): 50–56.

Little, I. M. D. *A Critique of Welfare Economics.* New York, Oxford University Press, 1957.

Liviatan, Nissan, and Don Patinkin. "On the Economic Theory of Price Indices." *Economic Development and Cultural Change* IX (1961): 501–536.

Mathur, P. N. "Approximate Determination of Indifference Surfaces from Family Budget Data." *International Economic Review* 5 (1964): 294–303.

Midgett, B. D. *Index Numbers.* New York, Wiley, 1951.

Morgenstern, Oskar. *On the Accuracy of Economic Observations.* Princeton, N.J., Princeton University Press, 1963.

National Bureau of Economic Research. *Problems in the International Comparison of Economic Accounts.* Studies in Income and Wealth, Vol. 20. Princeton University Press for NBER, 1957.

Paasche, H. "Über die Priesentwickelung der letzten Jahre, nach den Hamburger Börsennotierungen." *Jahrbücher für Nationaloekonomie und Statistik* (Jena) XXIII (1874): 168–178.

Palgrave, R. H. I. "Currency and Standard of Value in England, France and

India, and the Rates of Exchange between These Countries." *Memorandum Laid Before the Royal Commission on Depression of Trade and Industry,* 1886, Third Report, Appendix B, pp. 213–390.

Pareto, V. "Économie Mathématique." *Encyclopedie des sciences mathématiques,* 1911. Reprinted in English translation as "Mathematical Economics," in *International Economic Papers,* No. 5. London, Macmillan, 1955.

Prais, S. J. "Non-Linear Estimates of the Engle Curves." *Review of Economic Studies* 20 (1952–53): 87–104.

Prais, S. J., and H. S. Houthakker. *The Analysis of Family Budgets.* Cambridge, Engl., Cambridge University Press, 1955.

Rajoaja, V. "A Study in the Theory of Demand Functions and Price Indexes." *Commentationes physico-mathematicae, Societas Scientiarum Fennica* (Helsinki) 21 (1958): 1–96.

Report of the President's Committee on the Cost of Living. Office of Economic Stabilization. Washington, D.C., 1945.

Report of the Price Statistics Review Committee. *Government Price Statistics, Hearings.* Subcommittee on Economic Statistics of the Joint Economic Committee. Part I, pp. 5–99. 87th Cong., 1st sess., January 1961.

Rose, Hugh. "Consistency of Preference: the Two-Commodity Case." *Review of Economic Studies* 25 (1958): 124–125.

Roy, R. "La distribution du revenu entre les divers biens." *Econometrica* 15 (1947): 205–225.

Ruggles, Richard. "Price Indices and International Price Comparisons." In *Ten Economic Studies in the Tradition of Irving Fisher.* New York, Wiley, 1967.

Samuelson, P. A. "Evaluation of Real National Income." *Oxford Economic Papers* N. S. 2, 1 (1950): 1–29.

———. "Structure of a Minimum Equilibrium System." In R. W. Pfouts, ed., *Essays in Economics and Econometrics.* Chapel Hill, University of North Carolina Press, 1960.

Schumpeter, Joseph A. *History of Economic Analysis.* New York, Oxford University Press, 1954.

Slutsky, E. E. "Sulla teoria del biancio del consumatore" *Giornale degli Economisti* (1915). Reprinted as "On the theory of the budget of the consumer," translated by O. Ragusa, in G. J. Stigler and K. E. Boulding, eds. *Readings in Price Theory.* Chicago, Irwin, 1952.

Staehle, Hans. "A General Method for the Comparison of the Price of Living." *Rev. Econ. Papers,* New Ser., 2, 1 (1950): 1–29.

Stone, Richard. "Linear Expenditure Systems and Demand Analysis; an Application to the Pattern of British Demand." *Economic Journal,* 64 (1954): 511–524.

Stone, Richard, assisted by D. A. Rowe, W. J. Corlett, R. Hurstfield, and M. Potter. *The Measurement of Consumers' Expenditure and Behavior in the*

United Kingdom, 1920–1938, Vol. I. Cambridge, Engl., Cambridge University Press, 1966.

Stone, Richard, and D. A. Rowe. *Ibid.,* Vol. II.

Theil, H. "The Information Approach to Demand Analysis." *Econometrica* 33 (1963): 67–87.

Ulmer, M. J. *The Economic Theory of Cost of Living Index Numbers.* New York, Columbia University Press, 1949.

Ville, J. "Sur les conditions d'existence d'une orphelimite totale et d'un indice du niveau des prix." *Annales de l'Université de Lyon* (1946). Reprinted in English translation as "The Existence Conditions of a Total Utility Function," in *Review of Economic Studies* 19 (1951–52): 128–132.

Viner, J. "The Utility Concept in Value Theory and Its Critics." *Journal of Political Economy* 33 (1925): 369–387, 638–659.

Volterra, V. "L'economia matematica." Review of *Manuale di Economia Politica,* by V. Pareto. *Giornale degli Economisti* 32 (1906): 296–301.

Wald, A. "A New Formula for the Index of the Cost of Living." *Econometrica* 7, 4 (1939): 319–335.

———. "The Approximate Determination of Indifference Surfaces by Means of Engel Curves." *Econometrica* 8 (1940): 144–175.

———. "On a Relation Between Changes in Demand and Price Changes." *Ibid.* 20 (1952): 304–305.

Walsh, C. M. *The Measurement of General Exchange-Value.* New York, Macmillan, 1901.

Wold, H. O. A. "A synthesis of pure demand analysis." *Skandinavisk Aktuarietidskrift* 26 (1943): 85–118, 221–263; *ibid.* 27 (1944): 69–120.

Wright, Georg Henrik von. *The Logic of Preference.* Edinburgh, Scotland, Edinburgh University Press, 1963.

COMMENT

CARLOS F. DIÁZ-ALEJANDRO, Yale University

Afriat's main theses are set forth elegantly and forcefully. It makes no difference whether the distinctions be of time or place in the theory of index numbers (although he concentrates in his paper on time comparisons). Inherent in the price-level concept is the assumption that expansion paths are rays through the origin. The assumption of homogeneity, and the assumption that observed cost does not exceed minimum cost for value obtained, or "X-efficiency," are necessary if, in Fisher's terminology, we are to find a definite center of gravity of the shell fragments as they move in space.

Afriat is skeptical that we can meaningfully define "general purchasing power." Value, he tells us, is an attribute of a chooser, and his identity must be clear. Who is the chooser when we deal with public goods and "national wants"? Comparisons among countries whose wants are manifestly different highlight the problem, and Afriat doubts that comparisons of real national measures can be developed except where they refer directly to individuals. But even here, one can add, changes in taste by individuals threaten the basis for comparison. To take into account at least part of the plurality of purchasing power, he recommends the use of marginal price indexes.

The Afriat paper, rooted in utility theory, seems to say that the only purpose of index number construction is to measure *the* price level, general purchasing power, or welfare. Yet index numbers can be asked to perform other, often more modest, tasks. In those cases, index numbers may provide reasonably good answers, without much violence being done to the concept of what is being measured. Take, for example, the concept of productive capacity, used by Bergson in his paper included in this volume. But I suspect Afriat would point out that a clear definition of "general productive capacity" requires carefully spelled out assumptions, including explicit objective functions, which may not always be realistic.

A more modest task is to use index numbers to describe patterns of relative price structures in different countries, as a first step in analyzing the economic causes behind different patterns. For example, the relative prices of capital goods in Latin American countries and their changes through time can be described with such indexes. Unlike Fisher, we can say that the purpose of index numbers is not irrelevant to their construction. Indexes of relative prices in less developed countries (LDC's) can be a valuable tool in the analysis of development policies in those countries. They can reflect deviations from world market prices, due to commercial policies and other reasons, and indicate whether those deviations are becoming larger or smaller. One can test whether in fact the fastest-growing sectors in LDC's are those experiencing rising relative prices, thus yielding upward biases to national accounts measured at recent-year prices.

With a plentiful supply of computers, Afriat's plea for greater use of "marginal indexes" should be easy to respond to. Their greater use could cast light on patterns of income distribution, and on such matters as the impact of inflation on that distribution.

Because of its emphasis on very pure theory, the paper does not discuss several interesting issues to which more modest tools could be applied. One is Ruggles' redundancy problem: How many items should be gathered in the preparation of index numbers? Issues such as quality changes, new products, technical change, changes in taste, public goods and bads (for example, pollution), etc., are not explored in relation to index numbers.

We should be grateful to Afriat, however, for this useful and sophisticated reminder that one should be very careful when translating theory to empirical work, so as not to lose fidelity to the concept which one is supposed to be measuring.

MELVILLE J. ULMER, University of Maryland

Several points, central to the topic of this conference, are overlooked or in my judgment otherwise mistreated, in Afriat's paper. In the first place, he asserts that in principle the index number problem is the same whether the comparisons are over time or from place to place. This is a frequently repeated, and perhaps even an innocuous-sounding, statement, but I think that any resemblance it may have to the truth will tend to diminish rapidly the more we think about it. This is especially so, if "place-to-place" really refers to country-to-country comparisons, which I take to be the focus of attention here. Analytically as well as empirically, there is a distinct difference in the problems posed by temporal and locational real-income comparisons.

To clarify this difference, I should like to turn to the index number problem in its classic form, the one adhered to, in general, in Afriat's paper. The theory grew out of the problem of measuring a relative change in the cost of living from one period to the next in the same place, and ordinarily for people in a particular income class, such as urban workers. All the conditions of the problem, at least when the comparisons cover a short period of time, make it possible to adopt as reasonable assumptions: (a) constant tastes from one period to the next, and (b) a common set of commodities. Indeed, the basic facts of the problem dò not seem to be seriously violated by proceeding, in accordance with the technique of most analysts, to consider an individual representative of the income group, who is assumed to be enjoying a certain real-income level in the base period, say:

$$U_0 = U(q_{01}, q_{02}, \ldots, q_{0n}) \tag{1}$$

and a certain real-income level in the next period, say:

$$U_1 = U(q_{11}, q_{12}, \ldots, q_{1n}) \tag{2}$$

In these equations, the q's stand for the quantities of goods and services consumed, the first subscript indicating the time period and the second subscript indicating the commodity. The utility functions in periods 0 and 1 are, of course, identical, because of the assumption of constant tastes; but the particular levels of utility or preference reached in the two periods, U_0 and U_1, may be different. Any differences in the two utility levels naturally would flow from the differences in the q's, and these in turn would stem from two distinguishable factors: (a) differences in relative prices between the two periods, and (b) differences in money incomes.

Now in measuring price changes, if we use as weights the quantities of period 0, we of course have the Laspeyres index, and if we use the quantities of period 1, we have the Paasche index. It is true, as Afriat remarks, and as I pointed out more than twenty years ago,[1] that these two indexes do not provide the limits for the true index, or for anything else that is relevant. Indeed, strictly speaking, we may distinguish *two* true indexes. One would show the relative change in costs from one period to the next needed to maintain the plane of living actually enjoyed in period 0, and the other the corresponding relative change in costs for maintaining the plane of living actually experienced in period 1. In my own early study,[2] I showed how one could estimate the probable difference between the Laspeyres and the true index based on the given year's plane of living, and the difference

[1] Melville J. Ulmer, *The Economic Theory of Cost of Living Index Numbers*, New York, Columbia University Press, 1949 (reprinted, New York, AMS Press, 1968), pp. 38–39.

[2] *Ibid.*, pp. 49–60.

between the Paasche index and the true index based on the other year's plane of living, that is, the differences:

$$I_0 - I_L = d_0, \tag{3}$$

and

$$I_1 - I_P = d_1, \tag{4}$$

where I_L is the Laspeyres index, I_P is the Paasche index, and I_0 and I_1 are the corresponding true index numbers.

The expected differences, d_0 and d_1, turned out to be very small, probably less than 1 per cent, and since the Laspeyres and Paasche indexes were themselves very close to each other in the extensive period covered by my experiment, we may conclude that the two true indexes, at least in year-to-year comparisons, were virtually identical.

Now the main reason for relating this ancient tale is to refresh your memory concerning the fact that comparisons of real income over time are, from a theoretical point of view at least, relatively straightforward. In deflation, we have excellent justification for using the Laspeyres index, or something like it. And if we had reason to believe that changes in living costs were significantly different for different income groups, a matter that worries Afriat, we could at some additional expense compute separate indexes for some of the different income classes, and weight them appropriately when deflating consumer expenditures or personal incomes.

On the other hand, country-to-country comparisons are of quite a different order, and in particular raise questions that lie distinctly outside the theoretical framework I have just described—a set of special and difficult questions, incidentally, that Afriat notably neglects in his paper. First of all, in the international setting, and unlike comparisons over time, we are comparing the prices faced by *different* sets of individuals. Second, the different sets of individuals ordinarily have demonstrably different tastes, and are conditioned by different customs and institutions, and nearly always, also, consume and produce significantly different sets of commodities.

One empirical symptom of these differences is the enormous disparity between the Laspeyres and Paasche indexes in intercountry comparisons. Even for a relatively homogeneous group of countries such as the Latin American nations, as Ruggles has shown, these differences are huge. Whereas Laspeyres and Paasche indexes, in temporal comparisons, remain within 1 per cent of one another, even when the base periods are as much as eighteen years apart,[3] and always show the same trend in prices, in one of the experiments conducted by Ruggles the differences in the geographic comparisons averaged about 50 per cent, were sometimes more than 100 per cent, and often showed entirely different price trends.[4] For example, using an Argentine basket of goods, Ruggles found that prices were 15 per cent higher in Brazil than they were in Argentina. Using a Brazilian basket of goods, he found that prices were 15 per cent lower in Brazil than in Argentina. And this, incidentally, was one of the more modest disparities disclosed by his study.

The fact is that in comparing prices or real incomes in two or more countries, we have no justification for using any of the simplifying assumptions that appear to be appropriate in temporal comparisons. We cannot refer to a common utility scale, or to a single, common utility function. We cannot properly assume a common set of commodities, since not only are many goods and services physically different among countries, but often similar physical characteristics mask important functional differences. For example, the bicycle is still an important means of transportation in Holland, while in the United States it is primarily a play toy or a sporting good.

Consequently, in practice, those who compare real incomes among countries ordinarily do so from the standpoint of production or productivity rather than of utility or welfare, which is the focal point of Afriat's analysis. It may in fact be hopeless to try to attach quantitative welfare implications to differences in per capita real consumption or per capita personal incomes among countries. If there is any hope for such efforts, I think it must clearly involve recognizing the distinctive problems involved, which in turn means breaking away from what I have termed the classic theoretical framework. Thus, confronting frankly the existence of different utility functions and different commodities would stimulate the search for some connecting link that

[3] *Ibid.*, p. 55.

[4] Richard Ruggles, "Price Indexes and International Price Comparisons," in *Ten Economic Studies in the Tradition of Irving Fisher*, New York, Wiley, 1967, p. 186.

could conceivably relate them. For example, we might try, deliberately, to formulate equivalent budgets in two or more countries for given, selected planes of living: in other words, market baskets that in the judgment of informed investigators are approximately equivalent in terms of welfare, given the different tastes, habits, and customs of the countries involved. Stated another way, a panel of intelligent observers, all of whom were well acquainted with two or more of the countries involved, would provide the link necessary for relating utility scales internationally; and in terms of this common international scale they would designate equivalent combinations of goods and services. Pricing these market baskets in the respective countries would make it possible to compare per capita consumption or real personal incomes in a way that would illuminate what we usually mean by "differences in levels of living."

For the whole GNP, including investment goods, government expenditures, and exports and imports, the approach just described would not apply at all. More theoretical work, I think, needs to be done on the possibility of attaching welfare implications to international GNP comparisons, and I do not see that Afriat's paper gets into this at all. Meanwhile, we are left with the alternative of relating aggregate productions among countries, using some international value standard for weighting the individual commodities or commodity groups, after the manner, perhaps, of the pioneering work of Gilbert and Kravis. But even in this more modest framework of comparing physical production, and never mind welfare, serious problems arise in international comparisons that far surpass in magnitude, complexity, and number those encountered in studies over time, as anyone who tried them knows.

For example, I think a good case can be made for viewing what are called technological external diseconomies as *negative* outputs. Over short periods of time, say in the United States, we have no significant fluctuations in these, and we perhaps lose very little, if anything, by neglecting them. But among countries, we often have great differences. Thus, by placing electric and telephone cables underground, Great Britain adds appreciably to the total net value actually produced by its electric power and telephone industries. By placing our cables above ground, we subtract—and this is the external diseconomy

—from our apparent net output. What may appear as a higher cost of a similar service in one country may actually represent more net service. Such problems along with the others discussed above, I think, represent the truly critical issues in index number theory as it relates to international comparisons. The fact that they received little or no attention in Afriat's paper raises a serious question about the usefulness of the framework he has adopted.

IRVING B. KRAVIS, University of Pennsylvania

The theories of interspatial and intertemporal price comparisons are, as Professor Afriat indicates, identical. All that the pure theory covers are comparisons of the money incomes required to make a single individual at a given point in time and space indifferent between two structures of relative prices.[1] The assumption that a given individual has constant tastes over time is an empirical one which is not strictly true (tastes for a given individual change during his life cycle) and certainly has no theoretical justification. Thus, if we rely on rigorous theory, we are not justified in talking about differences in cost-of-living levels either between two times or two places except from the standpoint of a single individual at one of the times or places. This means that we cannot compare welfare between either two times or two places without leaving the confines of the theoretical model.

There is nothing in this that makes it any less warranted, in principle, to inquire about the income that would be required to make John Jones indifferent between the Chinese price structure and the U.S. 1969 price structure with a $10,000 income than it is to ask how much Jones would need at the 1968 U.S. price structure to make him indifferent between that and his 1969 opportunities. The differences between intertemporal and international comparisons in practice lie not in that the one is covered by the pure theory while the other is

[1] See F. Fisher and K. Shell, "Taste and Quality Change in the Pure Theory of the True Cost-of-Living Index," in J. H. Wolfe, ed., *Value, Capital, and Growth; Papers in Honor of Sir John Hicks,* Edinburgh, Edinburgh University Press, 1968.

not but rather in the extent of the differences in the patterns of expenditure and in operational problems of price sampling.

What economic statisticians do in fact is to construct index numbers that measure differences in prices between two situations on the assumption that not only the tastes of each situation but the quantities of each good purchased would remain the same if the prices of the opposite situation prevailed.

Usually, though not always, the observed differences in price structure and in expenditure patterns will be larger for two situations separated in space than for two situations separated only in time. As a result, the expenditure required to purchase the basket of goods of either situation at the prices of the other will greatly exaggerate that which would be required to leave an individual in either one of the situations indifferent between the two price structures.

The other major difference arises out of sampling problems. It is, on the whole, easier to choose a sample of items for which to compare prices over time than between places. The reason is that the correlations between price movements over time for different products and product variants within one country are easier to identify than are the correlations between price differences between countries for different commodities and subcommodities.

REPLY BY AFRIAT

The remarks of Diaz-Alejandro on essentials in my paper and also on an important absence are well taken. There are concepts and analytical techniques in it which are not close to the main interests of the conference and in any case cannot be treated briefly, so I must take up mostly the matter which is conspicuously absent, namely, the theory of production comparison. I will remark on some general questions about index numbers in responding to Ulmer's discussion.

In being asked to present a paper on comparison theory, there was a definite hint that I should attend to production. Unfortunately, I could think of nothing to say there that was essentially different from what might be said about consumption. Therefore, I took this to be an occasion for presenting a general position about index numbers, in the usual budget and utility framework.

In my approach to consumption analysis, which seems especially rewarding for index theory, there is analysis of a finite scheme of data by a finite method which does not involve special assumptions. This differs from the common method of associating the data with a special type of a function with parameters to be determined. The method applies just as well to production, especially joint production where production function technique is less workable. But this does not seem helpful for the production comparison question except possibly in the following fashion.

In an obvious sense, by turning everything around in my paper and making the appropriate verbal substitutions, the whole can be read as applying to production. All this depends on being able to entertain the concept of a capacity function, which determines the "capacity" necessary to produce a given output. This takes on the role of the utility function, upside down. The efficiency condition of minimum cost for utility gained becomes the condition of maximum profit for the capacity available. Everything goes parallel, only maxima and minima become minima and maxima and correspondingly all inequalities are reversed. The constructed ϕ's in my paper become capacity levels, and the reciprocals of the λ's multiplied by profits become marginal profitabilities of capacities. The approximation theory, which applies when the data reject the basic hypothesis and where exact efficiency is replaced by a certain level of efficiency, holds just as well with cost efficiency replaced by profit efficiency.

The outstanding question is whether or not the idea of a capacity function is acceptable. This is doubtful because productive capacity explicitly has many explicit dimensions which cannot be combined into one in a logical way. However, the question of productive capacity comparison seems to involve commitment to the idea that there is ultimately a single dimension which determines production possibilities. Therefore, if the question is to be pursued, it must be as if this were true. After hearing Mr. Usher's discussion of Mr. Bergson's paper, it seemed that this could be worth doing. My algebraical approach does in fact give a method for expressing several explicitly recognized capacities statistically as a single capacity. This is not presented in my paper and space cannot now be taken for it. The apparatus in my paper can be reinterpreted as for capacity comparison where no capacity variables are explicitly identified. The same dimension question inherently does not arise for utility. No doubt underneath utility there are many dimensions, but in the final act of valuation, as concerns choice under budgets, there is one that suffices.

Another question which could cause speculation is whether it is proper to think of the output of an economy as that which at the market prices gives maximum profit subject to the productive capacity limitation. This is especially true where there is decentralization. But some efficiency hypothesis must be made for an economic comparison, and this is clearly the only one which is available, in the particular framework of the question and the data. In an attempt to do something different, which I will come to later, welfare also is involved, and the efficiency hypothesis applies to welfare as limited by productive capacity. Market prices are less directly related to efficiency. Efficiency prices, which bear simultaneously on capacity and welfare, are defined in the hypothetical system, but are unknown. A further hypothesis is that market prices in the countries have only a tendency to be efficiency prices, and can be used, by taking their average, to estimate the efficiency prices. In other words, efficiency prices are taken to lie on the linear segment joining the market prices. This gives a method of capacity comparison in which the Paasche and Laspeyres indexes occur as limits, but in such a way that the puzzle about which should be greater or less, which presents itself in Usher's discussion, is entirely avoided.

Turning now to individual incomes rather than national ones, I note that Ulmer objected to an early remark in my paper, that the same elements are present in the question of real income comparison whether the reference be to different countries or different times.[1] Having in mind what I meant, which I believe is what is usually meant in this familiar observation, I still take it to be plainly true. This seems contrary to his suggestion that any resemblance it may have to the truth will tend to diminish rapidly, the more we think about it. It is likely therefore that what we are thinking about is

[1] The point of my remark was to suggest comparisons with a broader framework, where distinctions of time and place occur simultaneously.

different. Possibly it is that I am attending to the comparison question itself and he is attending to the further question of whether it is sensible to ask it. I have been unable to ask if he accepts this, in which case perhaps we could have some agreement, though I am unable to close either of our questions as firmly as he does.

To ask questions whether or not they are decidedly sensible is unavoidable, and is an old and no doubt worthy habit. I take it that the question of comparison of real incomes has some dilemmas about its significance, but my starting point has been that the question is in fact asked. I have taken a particular position about principles to be applied for answering it, given a specific scheme of data, and then been entirely concerned with the theory of computation which proceeds from that position. My limitation is that I have worked entirely within the framework of the question itself and the postulated scheme of the data. It could be that there is a different understanding of the question—possibly of greater relevance—which has been missed. It would be interesting to have a statement of it, preferably one which is entirely explicit. Such explicit statement calls for abstraction, and the only workable abstraction of which I am aware does not incorporate a distinction between comparisons which apply to different countries and different times.

Ulmer's remarks are stimulating as bringing basic questions into relief. My logical or methodological position must be different from his. Evidence of this is that, in my discussions, various formulas are derived in answer to formal questions which are posed about a scheme of data. Should two formulas, for different questions, turn out to be the same, they still have different theoretical contexts and correspondingly different meanings. The sometimes surprising experience that identical constructs appear in different contexts is common.

I have been unable to find any significance for the Paasche index outside the framework of the hypothesis that homogeneous utility prevails, together with efficiency. The data can reject that hypothesis, in other words fail in homogeneous consistency. The test turns out to be simply that the Paasche index not exceed the Laspeyres index. With that condition met, many homogeneous utility functions exist which are appropriate to the data under the efficiency hypothesis. To each corresponds a particular determination of the given cost index. The set of all such determinations describes an interval for which the Paasche formula determines the lower limit, and the Laspeyres formula the upper.

Should the homogeneity part of the hypothesis be dropped, a weaker test, for general consistency of the data, is appropriate. Then the range of determinations is wider, and the lower limit is given by a new formula, though the upper still by Laspeyres. In this more general context, the Paasche value ceases to be significant, and it need not even belong to the set of possible determinations. Here are propositions which are simple (though it is some work to prove them) and completely unambiguous. Whether or not they are useful for a particular application is another question entirely. That there is always the possibility of their routine application is undeniable. This is the same with linear regression analysis. It can always be done, whether or not it is worthwhile. I have in fact attempted to develop index number analysis as a form of routine statistical analysis, with various hypotheses concerning utility and efficiencies, and measures of significance, and a broader base for the data that can be used. Classical index formulas are recovered from more general formulas as corresponding to a case $k = 2$, and this general setting exposes their nature further. I believe that where index questions are asked, this kind of general method has scope for an answer, with the reservation that there is usually not one answer but a variety according to the hypothetical basis adopted.

This leads to a further question raised by Ulmer, one which rests on the manifest dissimilarities between countries and people. An answer, which is in my paper, is that in requesting an impartial general comparison, one is logically committed to viewing them statistically as basically similar, and to fashion a yardstick by ironing out their differences. It could be a yardstick of dubious value, but that reflects on the way the request is understood. I do agree with Ulmer that the intrinsic approach, that is, the approach based on demand data as reflecting utility, could be sterile where widely dissimilar countries are involved. More direct judgments from immediate experience have stronger bearing on the matter, as he suggests. Then statistical technique is needed for combining a variety of such judgments in a consistent way. This is what I have called extrinsic estimation, as distinct from direct utility and efficiency analysis of demand data, and my paper shows an effort in that direction. If there is a defense for the intrinsic method, it is that it is not involved in the hazard of direct judgments, but relies only on standard observations and the application to them of a statistical routine. With that it has a more scientific character. But its value for the intended use could be sacrificed in achieving this. No doubt it is better to be quite unscientific about an impossible question, and to be partly scientific about a question which is not altogether impossible—which is what the theory of extrinsic estimation attempts.

I must touch on one more item in Ulmer's discussion. He has not recognized the fault I found with the use of price indexes for establishing correspondence between equivalent individual real incomes. That fault cannot be mended by computing, as he suggests, different price indexes for different income classes. It could be mended by computing different marginal price indexes, together with one pair of correspondents in different income classes. If the income classes are arranged to correspond in real terms, as they would be in my method, then corresponding end points are such pairs of correspondents. My objection to conventional price indexes is that they have a theoretical meaning only in conjunction with the hypothesis that an individual will spend an increment of income on increments of goods in proportion to the totals he already has. That is, he will just move further along on the ray he is on. It is better to assume that the individual will move along a general straight line, not through the origin. This corresponds to the concept of a marginal index. If that is still not good enough, then the further scheme just described would be better. It allows for a shift in the direction of the line in making the transition from one real-income stratum to another, that is, piecewise linear approximation to expansion paths. It is clear, at least to me, that the concept of a marginal price index, which is a simple generalization of the concept which underlies the use of standard price indexes, is the proper practical instrument for establishing individual real-income comparisons, intertemporal or international, and by intrinsic or extrinsic estimation.

My general answer to Ulmer, or in agreement with him, is that the questions are not ended. Underneath index questions is a wider problem of developing statistical apparatus which can take better hold of that cardinal term of economic knowledge, the structure of wants. The classical approach is naive, but notice of its extraordinary persistence in economic thinking is hardly necessary to see it as basic. On its own, it corresponds to that stage of exercise in mechanics where there is nothing but point particles sliding on frictionless planes while held by light strings over inertialess pulleys, and so forth. Most likely it is necessary to have a thoroughly workable toy apparatus for such extravagant simplifications before there is the ground for supporting, and even operationally defining, further complexity. I have suffered the limitation of playing with that old toy, because I did not think it was working properly.

I have remarked on the absence of a treatment of production and productive capacity in my paper, and how something might have been done about it. Concerning welfare, if it is at all convincing, the story is identical, except that individuals become countries, consumption becomes production, and utility becomes welfare. However, following Bergson's paper and Usher's discussion of it, it occurred to me that something might be done which had a structure to it essentially different from that in pure consumption or pure production analysis. It is a theory of limits for the index of capacity comparison, where the upper limit is given by the maximum of the Paasche and Laspeyres indexes and the lower limit by the minimum. Thus, as is fortunate, it is inevitable that the formula for the upper limit is at least that for the lower; so they are consistent as upper and lower limits.

The basic efficiency hypothesis is welfare efficiency, or that production be such as to provide maximum welfare subject to capacity limitation. This, even with the homogeneity assumptions, is incapable of contradiction, given the data. This is in contrast to separate pure production or welfare analysis with prices having the usual efficiency role, expressing capacity-profit efficiency in one case and welfare-cost efficiency in the other. For the data to be consistent given the respective hypotheses, with the additional imposition of homogeneity, it is necessary that the Paasche be at least equal to the Laspeyres index in one hypothesis; and in the other, that it be at most equal. This comes from Usher's discussion. Thus usually one of the hypotheses of profit or cost efficiency must be rejected. There is an advantage in a theory of comparison which is not vulnerable to overthrow by rejection of its basic hypothesis, especially if the limits it provides are not worse than those from any other theory which holds up. This is the situation with welfare efficiency as against profit and cost efficiency. But to get the equivalent strength in regard to limits, the price data must enter, in the way already mentioned. Then finally, a theoretical interpretation of the equality of the Paasche and Laspeyres indexes can be shown. It appears as necessary and sufficient for the existence of universal homogeneous and convex technology and welfare in regard to which both countries can be presented with the three types of efficiency simultaneously, or more essentially, the two from which the third follows. The necessity is apparent again from Usher's discussion. All the hypotheses—about capacity, welfare, and various kinds of efficiencies—are of uncertain significance, and it is of no advantage to make any rigid commitments beyond those implicit in the comparison question itself. The best that can be attempted is a kind of analytical taxonomy, which gives a varied basis for interpretation of the data.

Bergson opened his paper with remarks, reaffirmed in subsequent discussion, which seemed a defense of his comparison work in the face of an unfriendly emphasis on exact concepts. It was argued that often many things have been done without a rigorous framework, but on intuitive grounds which might subsequently be proved sound. But perhaps just as well it might have been said that economics is rather different from other sciences, and the often ambiguous questions of economists are an original reality, and analysis had better give a tolerable account of them.