# Appendix 5-A The Algebra of Rational Expections

The methodology of this appendix is similar to the methodology developed by Sargent and Wallace (1975).

We start from the system (5-1 to 3) and first solve Equation (5-3) for $r_t$

$$r_t = b^{-1}\phi m_t - b^{-1}p_t - b^{-1}y_t - b^{-1}u_{3t}$$

Substituting this last expression and (5-4) and (5-5) in (5-2) gives

$$\begin{aligned} y_t - y_{n,t} &= g + cb^{-1}\phi m_t - cb^{-1}p_t - cb^{-1}y_t - cb^{-1}u_{3t} \\ &\quad - cEp_{t+1} + cEp_t + u_{2t} \end{aligned}$$

Adding $(cb^{-1}y_{n,t} - cb^{-1}y_{n,t})$ in the above expression gives

$$\begin{aligned} y_t - y_{n,t} + cb^{-1}(y_t - y_{n,t}) &= g + cb^{-1}\phi m_t - cb^{-1}p_t \\ &\quad - cEp_{t+1} + cEp_t - cb^{-1}y_{n,t} + u_{2t} - cb^{-1}u_{3t} \end{aligned}$$

Then

$$\begin{aligned} y_t - y_{n,t} &= g/(1 + cb^{-1}) + (cb^{-1}/[1 + cb^{-1}])\phi m_t \\ &\quad - (cb^{-1}/[1 + cb^{-1}])p_t \\ &\quad - (c/[1 + cb^{-1}])Ep_{t+1} + (c/[1 + cb^{-1}])Ep_t \end{aligned} \tag{A5-1}$$

$$- (cb^{-1}/[1 + cb^{-1}])\, y_{n,t} + (1/[1 + cb^{-1}])u_{2,t}$$
$$- (cb^{-1}/[1 + cb^{-1}])u_{3t}$$

Equating this last expression to (5-1) we have

$$ap_t - aEp_t + ky_{c,t-1} + u_{1t} = g/(1 + cb^{-1}) + (cb^{-1}/[1 + cb^{-1}])\,\phi m_t$$
$$- (cb^{-1}/[1 + cb^{-1}])p_t - (c/[1 + cb^{-1}])Ep_{t+1} + (c/[1 + cb^{-1}]\,Ep_t$$
$$- (cb^{-1}/[1 + cb^{-1}])\, y_{n,t} + (1/[1 + cb^{-1}])u_{2t}$$
$$- (cb^{-1}/[1 + cb^{-1}])u_{3t}$$

Solving the last expression for $p_t$ the following expression is obtained

$$p_t = J_0Ep_t + J_1Ep_{t+1} + J_2\,(\phi m_t - y_{n,t}) + J_3y_{c,t-1} + J_4u_{1t} + J_5u_{2t} + J_6u_{3t} + J_7 \tag{A5-2}$$

where

$$J_0 = (a + c/[1 + cb^{-1}])/\theta$$
$$J_1 = (-c/[1 + cb^{-1}])/\theta$$
$$J_2 = (cb^{-1}/[1 + cb^{-1}])/\theta$$
$$J_3 = -k/\theta$$
$$J_4 = -1/\theta$$
$$J_5 = (1/[1 + cb^{-1}])/\theta$$
$$J_6 = -J_2$$
$$J_7 = (g/[1 + cb^{-1}])/\theta$$
$$\theta = a + cb^{-1}/(1 + cb^{-1})$$

Equation (A5-2) can be written more compactly as

$$p_t = J_0Ep_t + J_1Ep_{t+1} + N_t \tag{A5-3}$$

where

$$N_t = J_2(\phi m_t - y_{n,t}) + J_3y_{c,t-1} + J_7 + w_t$$

and

$$w_t = J_4u_{1t} + J_5u_{2t} + J_6u_{3t}$$

is a random variable normally distributed with zero mean.

Taking expectations in (A5-3) conditional on the information available as of the end of period $t - 1$ the following expressions are obtained:

$$Ep_t = J_0 Ep_t + J\,Ep_{t+1} + EN_t \tag{A5-4}$$

and

$$Ep_t = (J_1/[1 - J_0])\,Ep_{t+1} + (1/[1 - J_0])EN_t \tag{A5-5}$$

this last expression can be generalized to

$$Ep_{t+j} = (J_1/[1 - J_0])\,Ep_{t+j+1} + (1/[1 - J_0])\,EN_{t+j} \tag{A5-6}$$

Repeatedly substituting (A5-6) in (A5-5), we obtain,

$$Ep_t = (1/[1 - J_0]) \sum_{j=0}^{\infty} (J/[1 - J_0])^j EN_{t+j} + (J_1/[1 - J_0])^{n+1}\,Ep_{t+n+1} \tag{A5-7}$$

Now notice that $0 < J_1/(1 - J_0) = 1/1 - b^{-1}) < 1$, then it is assumed that $\lim_{n \approx \infty} (J_1/[1 - J_0])^{n+1} \approx 0$. Then the limit of (A5-7) for $n$ approaching infinity gives the following equation

$$Ep_t = (1/[1 - J_0]) \sum_{j=0}^{\infty} (J_1/[1 - J_0])^j / EN_{t+j} \tag{A5-8}$$

or, for period $t + 1$

$$Ep_{t+1} = (1/[1 - J_0]) \sum_{j=0}^{\infty} (J_1/[1 - J_0])^j EN_{t+j+1} \tag{A5-9}$$

From Equations (A5-8) and (A5-9) we notice that we should obtain some workable relationship for $EN_{t+j}$ in order to get an expression representing the formation of expected prices. We know from (A5-3) that

$$N_{t+j} = J_2\,(\phi m_{t+j} - y_{n,t+j}) + J_3 y_{c,t+j-1} + J_7 + w_{t+j} \tag{A5-10}$$

In order to apply the expectation operator above we need an assumption about the stochastic process followed by $y_{c,t}$. For simplicity I will assume that $y_{c,t}$ follows an autoregressive process

$$y_{c,t} = \alpha y_{c,t-1} + e_t$$

where $e_t$ is an independently distributed random term with zero mean. This assumption is not restrictive; any other process in the class of ARIMA processes can be assumed without loss of generality. It can be proved that the nature of the process for $y_{c,t}$ will be reflected in the autoregressive-moving average terms for $y_{c,t}$ in the transfer function for $p_t$ (see Appendix B for the derivation of the transfer function).

Applying the expectation operator in (A5–10) we obtain

$$EN_{t+j} = J_2 (E\phi m_{t+j} - y_{n,t+j}) + J_3 \alpha^j y_{c,t-1} + J_7 \qquad \text{(A5–11)}$$

Substituting (A5–11) in (A5–8)-(A5–9), and considering that $J_2/(1 - J_0) = 1/(1 - b)$ and that $J_1/(1 - J_0) = 1/1 - b^{-1})$ we have

$$Ep_t = (1/[1 - b]) \sum_{j=0}^{\infty} (1/[1 - b^{-1}])^j (E\phi m_{t+j} - y_{n,t+j}) \qquad \text{(A5–12)}$$

$$+ J_3/(1 - J_0) \sum_{j=0}^{\infty} (\alpha/[1 - b^{-1}])^j y_{c,t-1} + c_0$$

or

$$Ep_{t+1} = (1/[1 - b]) \sum_{j=0}^{\infty} (1/[1 - b^{-1}])^{j+1} (E\phi m_{t+j+1} \qquad \text{(A5–13)}$$

$$- y_{n,t+j+1}) + (J_3/[1 - J_0]) \sum_{j=0}^{\infty} (\alpha/[1 - b^{-1}])^{j+1} y_{c,t-1} + c_0$$

where

$$c_0 = (g/c[b^{-1} - 1]) \sum_{j=0}^{\infty} (1/[1 - b^{-1}])^j$$

It should be noticed that in Equations (A5-12) and (A5-13) the term $(\alpha/[1 - b^{-1}])$ is less than one so the coefficient of $y_{c,t-1}$ converges to a finite number.

Equations (A5-12) and (A5-13) correspond to Equations (5-6) and (5-7) of Section 2.