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Induced Technical Change and Patterns of International Trade

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1. INTRODUCTION

What largely distinguishes the modern from the classical theory of international trade is its exploitation of and dependence on the concept of a production function in its narrow conception as a relation between a constant stream of output on the one hand and constant streams or constant stocks of inputs on the other. More narrowly still, production functions are usually assumed to be concave and homogeneous of the first degree.

Differences observed in these input-output relations between countries at a given time, or within a country at different times, are termed technological. Within the neoclassical tradition (as opposed to that of Marx and Schumpeter), these differences have been generally regarded as exogenous to economic science. But empirical studies increasingly indicate that such differences "account for" a large fraction of both trade and growth. The growth studies are extremely well known and need not be mentioned here; among empirical investigations suggesting the importance of technical differences (and especially, technical lags) in the explanation of trade patterns, are the celebrated researches of Leontief [35, 36] as well as the studies of Kindleberger [32, 33], Kravis [34],

NOTE: This paper is dedicated to the memory of my late colleague Jacob Schmookler.

MacDougall [38], Hoffmeyer [24], Freeman [16, 17], Hufbauer [25], Brechling and Surrey [7], Vernon [64], and Gruber, Mehta, and Vernon [18]. Some of these studies also tend to suggest, as have empirical studies of economic growth, that "technology" is an economic activity responsive to economic forces, although not very much is known as yet concerning the precise form of such responsiveness. A highly suggestive, though admittedly speculative, study by Habbakuk [19] lays great emphasis on induced biases in the direction of technical change among countries.

A theorist could approach this problem in two ways. One is to "fix up" or supplement the conventional concept of a static production function by providing a recipe for changing it. Another is to abandon the static production function altogether and replace it with a dynamic function of a particular kind. Both procedures are scientifically respectable. If by postulating the existence of an extra planet an astronomer can reconcile Newtonian theory with observations, his procedure is justified even if the planet is not independently "observed"—provided, of course, that he does not have to keep inventing new planets to explain fresh sets of observations. On the other hand, the ill-fated concept of "ether," which the physicists were obliged eventually to abandon, illustrates the case in which it is better to do away with the old concepts altogether.

The two approaches just mentioned need not be mutually exclusive. Starting from a definite recipe for changing a static production function, one can define a dynamic one. Conversely, it is possible in some cases (cf. Arrow [3]) to define a concept similar to a static production function starting from a dynamic progress function. (A large part of the early controversies in capital theory dealing with the problem of "maintaining capital intact" may be interpreted as being addressed to this type of problem.) The important thing—whichever approach is used—is to be left with some structure and therefore some possibility of prediction.

In the present paper I shall adopt the first of the above approaches. This is not because I consider the concept of the static production function sacrosanct; rather, the approach seems to be a convenient one in the circumstances. A distinct advantage in retaining the static production function concept, at least for the present, is that the modern Heckscher-Ohlin-Lerner-Samuelson theory of international trade, with given or exogenously changing production functions, has been fairly well worked out. From the development that followed Hicks' Inaugural Lecture [23], on the part of Corden [9], Johnson [26, 27], Findlay and Grubert [15], Bhagwati [6], Takayama [58], Bardhan [5], Jones [28], and others, we have a catalog of possible outcomes according to whether technical progress is capital-saving or labor-saving, or is concentrated in export industries or import-competing industries. What is lacking as yet is a systematic theory leading to a presumption as to which among the great variety of possible "technical changes" can be expected to take place.

Three approaches to the "endogenization" of technical change have achieved prominence in the theoretical literature. One is that of Arrow [3], followed up by Levhari [37], Sheshinski [56], and others, which stresses experience or "learning by doing." A second is that of Hicks [22], Kennedy [30, 31], Samuelson [48, 49], and Drandakis and Phelps [10], with related developments by Fellner [13, 14], Amano [2], Kamien and Schwartz [29], and others; this approach stresses induced direction of technical change. A third is that of Uzawa [61] stressing induced intensity of technical change; this has been synthesized with the previous group of developments by Nordhaus [42] and Drandakis and Hu [11]. The present paper builds on and extends the second and third approaches.

A few remarks may be in order to justify this choice. In explaining "imitation lags" among countries, it is natural to appeal to concepts such as experience and "learning by doing," as has been done, for instance, by Hufbauer [25]; however, the mathematical content of Arrow's model is that labor productivity is an increasing function of cumulated gross investment, which for practical purposes can be identified with the stock of capital. The identification of this cumulated stock with "experience" is a possible but not necessary interpretation of the model [cf. 44], and a direct translation of these concepts to international trade—equating transfers of capital with transfer of experience—might be questionable or would at least require some reformulation in view of the problems of capital absorption. The intuitive concept of "experience" is as difficult to isolate as that of "technical change" itself—both tend to be defined as residuals—and in using the notion of "learning by doing" to explain "technical progress" one runs the risk of employing one ill-defined concept to explain another.[1]

Induced technical change

The approach I shall use is to adopt a neoclassical production function $F(AL, BK, CN)$ relating a constant flow of capacity output to constant stocks of labor, L, and capital, K, and constant flows of raw material inputs, N. These factors are to be considered as aggregates or surrogates; in particular, natural resources, N, should be thought of as a variable taking the place of raw materials of all kinds, so that an increase in it may represent recourse to further kinds of primary products. In other words, it replaces the actual extensive margin by a fictitious intensive margin. Inclusion of raw materials follows Meade [40] and certainly seems appropriate in dealing with international trade.

Technical change will be assumed to be of the factor-augmenting type, involving changes in the coefficients A, B, C; in the case of fixed technical coefficients, any technical change can be decomposed uniquely into factor-augmenting technical changes, so the factor-augmentation hypothesis seems reasonable as long as the elasticities of substitution between any pairs of inputs are small. It will be assumed for the most part that they are bounded below unity.[2] Following tradition, and so as not to compound problems, F will be assumed to be homogeneous of the first degree.[3]

[1] Arrow cites as an example [3, p. 156] the case of the Horndal iron works in Sweden which "had no new investment (and therefore presumably no significant change in its methods of production) for a period of fifteen years, yet productivity (output per man-hour) rose on the average close to 2 per cent per annum. We find again steadily increasing performance which can only [*sic*] be imputed to learning from experience." A number of alternative explanations might present themselves, however: (1) disembodied technical change due either to internal R&D or to knowledge acquired from trade journals, etc.; (2) improved quality of the labor force; (3) improved quality of raw material inputs; (4) errors of measurement. Moreover, in terms of the specific model, without new investment there would have been no learning from experience; one would presumably want to use cumulated output here rather than cumulated investment as an index of experience.

[2] It is well known [cf. 59] that the factor-augmenting coefficients cannot be identified if the elasticity of substitution is unitary. In the formal analysis to follow it can equally well be assumed that elasticities of substitution are bounded above unity, but the factor-augmentation hypothesis is difficult to justify in this case [cf. 1].

[3] Elsewhere [8] I have suggested a method for handling increasing returns to scale. For the difficulties that would be involved in introducing them here, see footnote 10 below.

The crucial additional assumption is that the rates of factor augmentation $\alpha(t) = \dot{A}(t)/A(t)$, $\beta(t) = \dot{B}(t)/B(t)$, $\gamma(t) = \dot{C}(t)/C(t)$, are related by a "Kennedy function" of the form $\alpha = \phi(\beta, \gamma; r)$ where ϕ is strictly concave, decreasing in β and γ and increasing in r, where r is the proportion of capacity output devoted to the process of factor-augmenting technical change. Actual output is then given by $(1 - r)F(AL, BK, CN)$. A degenerate limiting case of the above would be that in which $\alpha = \beta = \gamma =$ constant, independently of r, leading to Hicks-neutral technical change.

While r could be interpreted as the proportion of capacity output devoted to research and development (R&D), it need not be interpreted quite so narrowly, or equivalently R&D may be broadly interpreted as any activity leading to an increase in A, B, or C. In particular, it need not be limited to "inventions" and their developments. For example, it could include maintenance, repair, and adjustment of equipment, leading to better future performance, or to on-the-job training of personnel.[4] Neither of these necessarily involves production of new information in Arrow's sense [cf. 4, p. 616]. Even in the case of inventions, it has been stressed by Schmookler [52] and others that most inventions make use of relatively old basic knowledge: Schmookler has also argued [cf. 50, 51, 52] that invention is strongly subject to economic forces. Even if individual inventions are not predictable, inventive activity in general may have predictable consequences; this hypothesis lies at the basis of the Kennedy function or similar formulations.

The present model is, of course, open to a number of a priori objections. It can be argued that one cannot expect such a Kennedy function to remain invariant over time or that it is unreasonable to suppose that the alternative growth rates of factor augmentation coefficients should have a constant exponential form. If objection is made to *any* form of time invariance in the creation of technical change, this is tantamount to renouncing any attempt to explain the major components of economic growth, as well as one of the main bases for international trade. While

[4] The usual definition of a production function describes the maximum flow of output obtainable from given flows (or stocks, as the case may be) of inputs, in a given state of knowledge. As the illustrations just given indicate, an *ultimately* larger steady flow of output can in general be obtained from the given inputs by means of an initial investment, and a maximal flow of this sort (if it exists at all) would only be achieved with a zero rate of interest. Differences in production functions among countries could therefore be simply a reflection of differences in the rate of interest.

the Kennedy function may not provide the most suitable expression for what time invariance there is, it should be kept in mind that objections to it apply equally well to customary formulations in terms of exogenous exponential technical change which—as noted above—can be considered as degenerate limiting cases of the present model. It should also be kept in mind that the entities we are dealing with are aggregates, so that while particular methods of factor augmentation become exhausted, others take their places; e.g., silk gives way to rayon, and rayon to nylon. To subsume such complex processes in a single function is admittedly heroic, but perhaps no less so than to adopt the usual aggregative production function. Economists are very familiar with the latter concept, but it is sometimes forgotten that when the linear homogeneous production function was first introduced into English economic thought by Wicksteed, Edgeworth [12] greeted the concept with nothing less than derision.

A theory of induced technical change is not obliged to account for every technical change that takes place, nor to explain the entire sweep of history. If it can help explain a fair proportion of the rate and direction of technical progress over a relatively brief period, say ten to twenty years, this might be the best one could expect.

There are two limitations in the present model which should be pointed out. One is the representation of all technical change as disembodied; this is particularly limiting inasmuch as the international transmission of technical change often is effected by means of the export of new types of equipment. The other is the assumption that the *proportion* of capacity output devoted to R&D is the relevant variable, thus abstracting from the type of scale economies that result from the fact that a given *amount* of inventive activity could lead to improved techniques (higher factor augmentation coefficients) regardless of the scale of operations.[5]

When one introduces the possibility of dynamic variations in productive techniques, one has to introduce some kind of behavior assumptions governing their introduction. The formal model presented in the following section is set up as a social optimization problem; nevertheless I shall interpret the model as a descriptive one. To justify this, the follow-

[5] For alternative formulations of the treatment of investment in research, see Phelps [45].

ing interpretation will be suggested. Let us assume that all technical change is carried out by "firms," broadly interpreted to include governmental agencies involved in research activities, and that these entities choose the parameters of the appropriate Kennedy function so as to maximize the present value of their future profits given certain assumptions concerning the future. The simplest assumption to make is the usual one of static expectations: constant product and factor prices, constant factor proportions, and constant rates of factor augmentation. Under these conditions the problem reduces to maximizing the present value of output per head:

$$\int_0^\infty e^{-\delta t}(1 - r)\, A(0)e^{\phi(\beta,\gamma;r)t} f\left[\kappa(0)e^{[\beta-\phi(\beta,\gamma;r)]t},\ \nu(0)e^{[\gamma-\phi(\beta,\gamma;r)]t}\right] dt \tag{1.1}$$

where $f = F/AL$, $\kappa = BK/AL$; $\nu = CN/AL$, $L(0) = 1$, and $\delta = \rho - \lambda$ where ρ is the rate of discount and λ the rate of increase of the labor force employed. When (1.1) is maximized with respect to β and γ one obtains the conditions [6]

$$\begin{aligned} -\frac{\partial\phi}{\partial\beta} &= \frac{\kappa f_\kappa(\kappa, \nu)}{f(\kappa, \nu) - \kappa f_\kappa(\kappa, \nu) - \nu f_\nu(\kappa, \nu)}, \\ -\frac{\partial\phi}{\partial\gamma} &= \frac{\nu f_\nu(\kappa, \nu)}{f(\kappa, \nu) - \kappa f_\kappa(\kappa, \nu) - \nu f_\nu(\kappa, \nu)} \end{aligned} \tag{1.2}$$

as in Samuelson [48] and Drandakis and Phelps [10]. This states that the slopes of the Kennedy function should be equal to the imputed relative shares of capital and labor and of resources and labor, respectively. Likewise, maximization of (1.1) with respect to r yields, after some computations,

$$\frac{\partial\phi}{\partial r} = \frac{\delta - \phi(\beta, \gamma; r)}{1 - r} \cdot \frac{f(\kappa, \nu)}{f(\kappa, \nu) - \kappa f_\kappa(\kappa, \nu) - \nu f_\nu(\kappa, \nu)}. \tag{1.3}$$

The second factor on the right is the reciprocal of the imputed relative share of labor in total output.

The criteria (1.2) and (1.3) appear in (2.31) below and correspond to the optimality conditions in a situation of balanced growth. The more

[6] It is necessary to assume that δ is sufficiently large so that the integral converges; a sufficient condition for this is that $\delta > \phi\ (0, 0; 1)$ (see (2.18) below). It is further assumed that f is sufficiently regular to permit differentiation under the integral sign in (1.1).

general optimality conditions (2.21) and (2.22) given below could be interpreted as defining "rational expectations" on the part of firms. They might be considered, to that extent, as accounting for some of the expectational considerations stressed by Fellner [13, 14], though these considerations do not alter the basic long-run result.

If investment in technical change involves no production of new "knowledge," the above formulation presents no fundamental difficulties. Trouble occurs [cf. 4] when and to the extent that the research outlays of one firm result in knowledge useful to others. If the industry's R&D activities are completely monopolized, and if the benefits of research do not extend to other industries and cannot be absorbed by the corresponding industry in other countries, and if product and factor markets remain perfectly competitive—in these limited circumstances an optimal allocation of resources could be achieved. Although optimality is not to be expected in general, it is at least arguable that the quantitative distortions are not too great and that alternative and more realistic assumptions would not lead to radically different conclusions. For instance, the first innovator in the field must make allowance for being followed by imitators and may therefore fall short of the optimal r; on the other hand, imitators might crowd the field and collectively push beyond the optimal r. Since the error could go in either direction, the optimality assumption at least seems neutral, even if it is not very accurate.

Imports of primary products

One of the controversial topics of the past few decades has been that of the role and prospects of natural resources in international trade. The thesis set forth by Prebisch [47] and Singer [57] that there is a tendency towards a deterioration in the terms of trade of raw material producing countries has been challenged factually and has also given rise to theoretical discussion [cf. 65, 20, 21]. The analysis of Schultz [55] has the great virtue of focusing not on the terms of trade but rather on the ratio of expenditures on raw materials to gross national product; his figures show a clearly declining tendency since the early 1900's.

Nurkse [43] has described how industrial expansion in the nineteenth century was accompanied by a more than proportionate expansion in imports of raw materials, whereas in the twentieth century these imports have been much less than proportionate. The two explanations that received the greatest emphasis by Nurkse [43, p. 23] are low income elasticities of demand and technical progress in the use of natural materials as exemplified by the rise of synthetics industries. Nurkse recognized the existence of an "identification problem" in this analysis [43, p. 26]; this was also perceived by Schultz [55] who concluded that low income elasticities provided the correct explanation.[7] Schultz's conclusion was based on the proposition that consumer demand functions are empirically stable, whereas production functions are not; in his words [55, pp. 316–317]:

> The production function is . . . a venerable concept, based on received theory of long standing. It has not been a useful concept, however, in organizing data and gaining from them dependable insights about supply. . . . For a function to be useful it must either be fairly stable, or we must be able to predict how it will change. The stability of demand functions . . . is dependent upon what happens to tastes, while the stability of the supply function rests upon technology. Fortunately for demand analysis, tastes remain fairly stable. Technology, on the other hand, does not. Therefore . . . unless we can predict the changes in technology, estimates of production functions are comparatively useless in a logical positivistic sense.

Although it is not claimed that the model used here can predict changes in technology with any accuracy, it nevertheless supports Schultz's thesis in the weak sense that the declining share of raw materials expenditure in national income cannot be accounted for on the basis of the hypothesis of induced resource-augmenting technical progress alone. On the contrary, with consumer goods being represented in our model by a single aggregate commodity (implying unitary income elasticity), an ultimately constant share of raw material expenditure is predicted. The same conclusion is obtained if part of the primary product is assumed to be consumed directly and a constant proportion of income is devoted to it; on the other hand, a declining proportion of direct expenditures on primary products is shown to lead, under certain reasonable conditions, to a declining share of resource income to total income.

If foodstuffs are omitted from the list of raw materials, as well as forest products (other than pulpwood), Schultz's figures indicate only a slightly declining trend. Considering that certain services from natural resources, such as hydroelectric power, tourism, and unpriced or hard-

[7] Schultz's thesis is further elaborated in two other papers [53, 54].

to-price uses of land, water, and air for travel and communication, are omitted from the list, a case might be made for the proposition that except for foodstuffs (to which Engel's law applies) and rents imputed to them, the share of natural resources is relatively constant. This share is in any case small.[8]

The main result of the formal model in the next section can be summarized as follows. If the home country produces a single consumer good, using up labor, capital, and imported raw material, then an equilibrium growth path will be approached in which technical change is Harrod neutral as far as capital is concerned (that is, there will be no capital-augmenting technical change), whereas the rate of augmentation of natural resources will be determined by the formula $\gamma = (\lambda + \alpha)/(1 + \eta)$, where λ is the rate of growth of population, α the rate of labor-augmenting technical change, and η the elasticity of the foreign offer curve. Thus, if resources are in perfectly elastic supply ($\eta = \infty$), there will be no resource-augmenting technical improvement, whereas at the opposite extreme of perfectly inelastic supply ($\eta = 0$), resource-augmenting technical change will proceed at the rate $\lambda + \alpha$. The volume of primary products grows at the rate $(\lambda + \alpha)\eta/(1 + \eta)$, and the foreign terms of trade improve at the rate $(\lambda + \alpha)/(1 + \eta)$.[9]

These results must, of course, be interpreted with caution. The model does not distinguish the particular types of raw materials that enter into the production of synthetics from those they displace or supplement, nor does it distinguish whether they will be obtained from imported or domestic sources. If the foreign supply of natural rubber, cotton, wool, hides, etc., becomes inelastic while the domestic supply of petroleum, natural gas, pulpwood, etc., remains relatively elastic, and, if these latter items are technologically more suitable for resource-augmenting improvement, then a declining share of foreign primary products in national expenditure could be explained on this basis.

The question naturally arises as to how this type of analysis would extend to the standard Heckscher-Ohlin model of international trade, in which factor supplies in the several countries are given. It is well

[8] Vanek [62, 63] in his analysis of the resource content of traded products has made use of the a priori assumption of constancy of the relative share of resource income. He has not tested this hypothesis directly, however.

[9] These are asymptotic results. The speed with which the balanced growth solution is achieved depends on the magnitude of $\delta - \alpha$, and the solution can therefore be taken as approximating actual events only if the discount rate is not too small.

known that a uniform expansion in the home country would, in general, worsen the expanding country's terms of trade. This will lead to a relative decline in the price of the factor intensive in the export industry; for instance, if the country's exports are capital intensive, it will lead to a relative rise in wages. On the basis of the theory of induced technical change one would then expect that technical improvement should take place in the export industry, thus bringing down the wage-rental ratio and thereby restoring the relative share of labor and capital to its former level. On this basis, one would expect induced technical change to be "ultra-pro-trade biased" in Johnson's [26] terminology; this would also conform to the results of Gruber, Mehta, and Vernon [18] to the effect that R&D effort tends to be strongest in the export industries. The following definite results can be shown, however: If factor-augmenting technical change is obtainable by a Kennedy function in each industry, and if it is not transferable (via factor movement) to other industries, then balanced growth in rates of factor augmentation is possible only in the special case of "Cobb-Douglas" utility functions (unitary elasticity of substitution in consumption) and exactly matched expansion in the foreign offer function; in other words, balanced growth and constant relative shares are no longer to be expected. A detailed analysis of this extended model is postponed to another occasion.

2. *ON INDUCED TECHNICAL CHANGE IN THE FACE OF SCARCE NATURAL RESOURCES*

This section is devoted to analyzing a very simple model which may be conceived of as depicting in a crude fashion some of the long-run trade patterns between developed industrial nations on the one hand and underdeveloped suppliers of raw materials on the other.

Let the developed country produce a single consumer good which it exports in exchange for raw material inputs which are used together with labor and capital to produce the consumer good. Output of this consumer good is assumed to be given by

(2.1)
$$Y(t) = (1 - s(t) - r(t))F[A(t)L(t), B(t)K(t), C(t)N(t)]$$

where $L(t)$, $K(t)$, $N(t)$ are the quantities of labor, capital, and natural resources (raw material) used at time t, $A(t)$, $B(t)$, $C(t)$ are the respective factor augmenting coefficients of (disembodied) technical progress, and $s(t)$, $r(t)$ are the proportions of capacity output used for capital accumulation and for research and development, respectively [cf. 11]. Assuming capital to depreciate of the constant rate μ, net investment is then determined according to

$$(2.2) \quad \dot{K}(t) = s(t)F[A(t)L(t), B(t)K(t), C(t)N(t)] - \mu K(t), \quad (\mu \geqq 0).$$

Labor is assumed to grow at the constant rate λ:

$$(2.3) \qquad L(t) = L(0)e^{\lambda t}, \quad (\lambda \geqq 0).$$

Denoting

$$(2.4) \qquad \alpha(t) = \dot{A}(t)/A(t), \beta(t) = \dot{B}(t)/B(t), \gamma(t) = \dot{C}(t)/C(t),$$

I assume that technical progress is determined according to a Kennedy function of the form

$$(2.5) \qquad \alpha(t) = \phi[\beta(t), \gamma(t); r(t)]$$

where ϕ is strictly concave, decreasing in β and γ and increasing in r. Further assumptions concerning ϕ will be specified when needed below; see Kennedy [30], Samuelson [48, 49], Drandakis and Phelps [10], Nordhaus [42], Kamien and Schwartz [29], and Drandakis and Hu [11].

Domestic consumption (in the industrial country) of the consumer good is given by

$$(2.6) \qquad X(t) = Y(t) + Z(t)$$

where $Z(t)$ is the amount imported (if positive) or exported (if negative); in the present case $Z(t)$ will be negative. The imported good will be assumed to have a price of $p(t)$ relative to that of the exported good; thus, $p(t)$ is the foreign (underdeveloped) country's terms of trade. The home (developed) country faces the budget constraint

$$(2.7) \qquad p(t)N(t) + Z(t) = 0.$$

The foreign offer function will be of the general form

$$(2.8) \qquad N(t) = h[p(t), t]$$

of which we will consider below the special cases of perfectly elastic supply (constant p and unlimited supplies N) and perfectly inelastic supply (constant N).

Since the possibility of investment in technical progress in general precludes untrammeled perfect competition—requiring some form of monopoly control, licensing, or secrecy—our model will be cast in terms of an optimization problem. However, problems of commercial policy are beyond the scope of this study, and the terms of trade $p(t)$ will be assumed to be regarded as a parameter by optimizing firms and individuals. Thus, the offer function (2.8) will not figure as a constraint in the optimization problem, but will be substituted into the solution of the optimization problem which has (2.7) as the corresponding constraint. The problem is then to

$$\text{(2.9)} \qquad \text{Maximize } \int_0^\infty e^{-\rho t} X(t) dt$$

subject to the above constraints, other than (2.8).

I shall assume constant returns to scale,[10] so that

$$\text{(2.10)} \quad F[A(t)L(t), B(t)K(t), C(t)N(t)] = A(t)L(t)f\left[\frac{B(t)}{A(t)}k(t), \frac{C(t)}{A(t)}n(t)\right]$$

where

$$\text{(2.11)} \qquad k(t) = \frac{K(t)}{L(t)}, \; n(t) = \frac{N(t)}{L(t)}$$

and f (assumed increasing in both arguments and strictly concave) is defined by

$$\text{(2.12)} \qquad f\left[\frac{B(t)}{A(t)}k(t), \frac{C(t)}{A(t)}n(t)\right] = F\left[1, \frac{B(t)}{A(t)}k(t), \frac{C(t)}{A(t)}n(t)\right].$$

It will also be convenient to define the capital-labor and nature-labor ratios in terms of efficiency units:

$$\text{(2.13)} \qquad \kappa(t) = \frac{B(t)}{A(t)}k(t), \; \nu(t) = \frac{C(t)}{A(t)}n(t).$$

Finally, define

$$\text{(2.14)} \qquad x(t) = \frac{X(t)}{L(t)}, \; y(t) = \frac{Y(t)}{L(t)}, \; z(t) = \frac{Z(t)}{L(t)}.$$

[10] Introduction of scale economies in models such as these tends to give rise to problems of convergence of the integral (2.9). For instance, if r were to be replaced by Y in the argument of (2.5), representing a scale economy in the Kennedy function rather than the production function, one would obtain a divergent integral unless ϕ was bounded with respect to Y.

In view of (2.10) and (2.11), the investment equation (2.2) may be transformed to

$$\dot{k}(t) = s(t)A(t)f\left[\frac{B(t)}{A(t)}k(t), \frac{C(t)}{A(t)}n(t)\right] - (\mu + \lambda)k(t) \tag{2.15}$$

and the maximand (2.9) becomes

(2.16)

$$L(0)\int_0^\infty e^{-(\rho-\lambda)t}\left\{[1 - s(t) - r(t)]A(t)f\left[\frac{B(t)}{A(t)}k(t), \frac{C(t)}{A(t)}n(t)\right] - p(t)n(t)\right\}dt.$$

For convenience, denote

$$\delta = \rho - \lambda. \tag{2.17}$$

It will be assumed that

$$\delta > \phi(0, 0; 1), \tag{2.18}$$

and that the elasticity of substitution between any two of the factors is bounded below 1.

The appropriate Hamiltonian expression for this problem [cf. 46] is

$$H(q, a, b, c; k, A, B, C; r, s, \beta, \gamma, n; p) = (1 - r - s + qs) \tag{2.19}$$
$$Af\left(\frac{B}{A}k, \frac{C}{A}n\right) - pn - q(\mu + \lambda)k + aA\phi(\beta, \gamma; r) + bB\beta + cC\gamma,$$

where the time variable has been dropped for notational convenience. The state variables are k, A, B, C, the controls or instrument variables are r, s, β, γ, n, and the auxiliary variables q, a, b, c satisfy the differential equations

$$\begin{aligned}
\dot{q} &= \delta q - \frac{\partial H}{\partial k} = (\rho + \mu)q - (1 - r - s + qs)Bf_\kappa(\kappa, \nu) \\
\dot{a} &= \delta a - \frac{\partial H}{\partial A} = [\delta - \phi(\beta, \gamma; r)]a \\
&\qquad - (1 - r - s + qs)[f(\kappa, \nu) - \kappa f_\kappa(\kappa, \nu) - \nu f_\nu(\kappa, \nu)] \\
\dot{b} &= \delta b - \frac{\partial H}{\partial B} = (\delta - \beta)b - (1 - r - s + qs)kf_\kappa(\kappa, \nu) \\
\dot{c} &= \delta c - \frac{\partial H}{\partial C} = (\delta - \gamma)c - (1 - r - s + qs)nf_\nu(\kappa, \nu)
\end{aligned} \tag{2.20}$$

where f_κ, f_ν are the partial derivatives of f with respect to κ, ν, the latter being defined by (2.13). q may be interpreted as the imputed value of the activity of capital accumulation, expressed in relation to the value of the foregone production of current consumer goods. Likewise, a, b, c may be interpreted as imputed values, in terms of foregone current output of consumer goods, of the activities of investing in labor-augmenting, capital-augmenting, and resource-augmenting technical progress respectively.

In accordance with the Pontryagin maximum principle, the variables $s(t)$, $r(t)$, $\beta(t)$, $\gamma(t)$, $n(t)$ must be chosen so as to maximize H at each instant of time, t. Maximization with respect to $s(t)$ requires maximization of $1 - r(t) + [q(t) - 1]s(t)$ with respect to to $s(t)$, hence $s(t) = 1$, $[0, 1 - r]$, or 0 according as $q(t) >, =,$ or < 1. This is intuitively obvious; since the capital good and consumer good are physically identified (a unit of one can be transformed into a unit of the other in accordance with (2.2)), if the capital good has a higher imputed value than the consumer good then all output should be accumulated, and in the converse case none of it should be. The cases of interest will clearly be those in which $q(t) = 1$.

Maximization of H with respect to $r(t)$ requires maximization of $a(t)\phi[\beta(t), \gamma(t);\ r(t)] - r(t)f[\kappa(t), \nu(t)]$ with respect to $r(t)$. If $a(t)\partial\phi[\beta(t), \gamma(t);\ r(t)]/\partial r - f[\kappa(t), \nu(t)]$ is always positive, then $r(t) = 1$, or if always negative then $r(t) = 0$. We shall naturally be interested mainly in conditions leading to an interior maximum for which

$$(2.21)\qquad \frac{\partial\phi[\beta(t), \gamma(t); r(t)]}{\partial r} = \frac{f[\kappa(t), \nu(t)]}{a(t)},$$

and similarly in conditions under which H is maximized with respect to $\beta(t)$ and $\gamma(t)$ when

$$(2.22)\qquad \frac{\partial\phi[\beta(t), \gamma(t); r(t)]}{\partial\beta} = -\frac{b(t)B(t)}{a(t)A(t)};\ \frac{\partial\phi[\beta(t), \gamma(t); r(t)]}{\partial\gamma} = -\frac{c(t)C(t)}{a(t)A(t)}.$$

Finally, assuming f to be strictly concave with $f_\nu(\kappa, 0) = \infty$ and $f_\nu(\kappa, \infty) = 0$, maximization of H with respect to $n(t)$ entails

$$(2.23)\qquad [1 - r(t) - s(t) + q(t)s(t)]C(t)f_\nu[\kappa(t), \nu(t)] = p(t),$$

which simply states that the value of the marginal productivity of the raw material should be equated to its world price.

Following Drandakis and Hu [11] it is convenient to deal directly with the variables appearing in (2.22). Let these be denoted

$$u(t) = \frac{b(t)B(t)}{a(t)A(t)}, \; v(t) = \frac{c(t)C(t)}{a(t)A(t)}. \tag{2.24}$$

From (2.20), (2.4), (2.13) and (2.24) one verifies that

$$\begin{aligned} \dot{u}(t) &= [1 - r(t) - s(t) + q(t)s(t)]\frac{m[\kappa(t), \nu(t)]}{a(t)}\left\{u(t) - \frac{\kappa(t)f_\kappa[\kappa(t), \nu(t)]}{m[\kappa(t), \nu(t)]}\right\} \\ \dot{v}(t) &= [1 - r(t) - s(t) + q(t)s(t)]\frac{m[\kappa(t), \nu(t)]}{a(t)}\left\{v(t) - \frac{\nu(t)f_\nu[\kappa(t), \nu(t)]}{m[\kappa(t), \nu(t)]}\right\} \end{aligned} \tag{2.25}$$

where

$$m(\kappa, \nu) = f(\kappa, \nu) - \kappa f_\kappa(\kappa, \nu) - \nu f_\nu(\kappa, \nu), \tag{2.26}$$

this being $\partial F/\partial(AL)$, i.e., the marginal product of labor in terms of efficiency units.

It is also convenient to obtain the corresponding differential equations for $\kappa(t)$ and $\nu(t)$. From (2.13) and (2.15) these are

$$\begin{aligned} \dot{\kappa}(t) &= [\beta(t) - \phi[\beta(t), \gamma(t); r(t)] - \lambda - \mu]\kappa(t) + s(t)B(t)f[\kappa(t), \nu(t)] \\ \dot{\nu}(t) &= [\gamma(t) - \phi[\beta(t), \gamma(t); r(t)] - \lambda]\nu(t) + \frac{C(t)}{A(t)L(t)}\dot{N}(t). \end{aligned} \tag{2.27}$$

Let the offer function (2.8) be assumed to have the constant-elasticity static form

$$N(t) = \bar{N}^{1+\eta}p(t)^{\eta} \quad (0 \leqq \eta < \infty). \tag{2.28a}$$

For $\eta = 0$ this reduces to $N(t) = \bar{N}$. As $\eta \to \infty$ we obtain the limiting form

$$p(t) = 1/\bar{N}. \tag{2.28b}$$

If (2.28b) holds, $p(t)$ is constant in (2.23). If (2.28a) holds, the second equation of (2.27) becomes

$$\dot{\nu}(t) = [\eta\dot{\pi}(t)/\pi(t) + (1 + \eta)\gamma(t) - \phi[\beta(t), \gamma(t); r(t)] - \lambda]\nu(t) \tag{2.29}$$

where

$$\pi(t) = \frac{p(t)}{C(t)}; \tag{2.30}$$

$\pi(t)$ is the price of the raw material (the foreign terms of trade) expressed in efficiency units.

Balanced growth solution

In solving our system of differential equations, it is natural to look first for a singular solution. Suppose an "equilibrium" or balanced-growth solution exists with constant $u(t)$, $v(t)$, $\kappa(t)$, $\nu(t)$, $a(t) \neq 0$, $q(t)$, $r(t)$, $s(t)$. Denoting equilibrium values of variables by daggers, we obtain from (2.21), (2.22), (2.24), (2.25,), and the first two equations of (2.20), the system

$$(2.31)\qquad \begin{aligned} -\frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial\beta} &= \frac{\kappa^\dagger f_\kappa(\kappa^\dagger,\nu^\dagger)}{m(\kappa^\dagger,\nu^\dagger)} \\ -\frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial\gamma} &= \frac{\nu^\dagger f_\nu(\kappa^\dagger,\nu^\dagger)}{m(\kappa^\dagger,\nu^\dagger)} \\ \frac{1-r^\dagger}{\partial-\phi(0,\gamma^\dagger;r^\dagger)}\ \frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial r} &= \frac{f(\kappa^\dagger,\nu^\dagger)}{m(\kappa^\dagger,\nu^\dagger)}. \end{aligned}$$

If a solution to these equations exists, they may be combined in view of (2.26) into the single equation

$$(2.32)\qquad \frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial\beta} + \frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial\gamma} + \frac{1-r^\dagger}{\delta-\phi(0,\gamma^\dagger;r^\dagger)}\ \frac{\partial\phi(0,\gamma^\dagger;r^\dagger)}{\partial r} = 1,$$

which defines an implicit relation between γ and r.

Now from (2.23) and (2.30) we have $\dot{\pi}(t) = 0$ whence (2.29) yields

$$(2.33)\qquad \gamma^\dagger = \frac{\lambda+\phi(0,\gamma^\dagger;r^\dagger)}{1+\eta}$$

for $0 \leqq \eta < \infty$. This defines another implicit relation between γ and r. In the limiting case $\eta = \infty$ when (2.28b) holds, we have $\gamma^\dagger = 0$ from (2.23), and thus (2.33) remains valid upon substituting $\eta = \infty$. Together, (2.32) and (2.33) enable us to solve for $\gamma^\dagger$ and $r^\dagger$; some sufficient conditions for a unique solution will be specified below.

If the initial conditions are such that the system is in a state of balanced growth, then (2.33) and (2.23) imply that

$$(2.34)\qquad C(t) = C(0)e^{\frac{\lambda+\alpha}{1+\eta}t},\quad p(t) = p(0)e^{\frac{\lambda+\alpha}{1+\eta}t}$$

and consequently, from (2.28), imports of raw materials satisfy

$$N(t) = N(0) \exp \left\{ (\lambda + \alpha) \frac{\eta}{1 + \eta} t \right\}. \tag{2.35}$$

The term $p(t)n(t)$ of (2.16) therefore grows at the rate α and the integral converges in view of (2.18).

The above equations (2.34) and (2.35) are valid for $0 \leqq \eta \leqq \infty$. Thus, in the limiting case $\eta = \infty$ of unlimited supplies of natural resources, there will be no resource-augmenting technical progress, but instead the quantity of raw materials imported will grow at a rate $\lambda + \alpha$ equal to the rate of increase of the domestic labor force measured in efficiency units. At the opposite extreme $\eta = 0$ of fixed supplies of natural resources, resource-augmenting technical improvement will proceed at the rate $\lambda + \alpha$.

In either case, as well as in the cases in between, we have what may properly be called an economic rather than historical law of constant returns.

The remainder of this section will be devoted to providing a set of sufficient conditions for the existence of a unique solution to equations (2.31) and (2.33) in the variables $\gamma^\dagger$, $r^\dagger$, $\kappa^\dagger$, $\nu^\dagger$, enabling one to solve for the remaining variables; then we shall justify the attention paid to this equilibrium solution by sketching an argument to show that it is approached asymptotically by the optimal path.

Let the production function F be of the constant-elasticity-of-substitution type [cf. 60], with elasticity of substitution $\sigma < 1$, so that

$$f(\kappa, \nu) = (\mu_0 + \mu_1 \kappa^{1-1/\sigma} + \mu_2 \nu^{1-1/\sigma})^{\sigma/(\sigma-1)} \quad (\sigma < 1). \tag{2.36}$$

Then, denoting $\mu_1' = \mu_1/\mu_0$, $\mu_2' = \mu_2/\mu_0$, equations (2.31) reduce to

$$\begin{aligned} &-\frac{\partial\phi(0, \gamma^\dagger; r^\dagger)}{\partial\beta} = \mu_1' \kappa^{1-1/\sigma}; \quad -\frac{\partial\phi(0, \gamma^\dagger; r^\dagger)}{\partial\gamma} = \mu_2' \nu^{1-1/\sigma}; \\ &\frac{1 - r^\dagger}{\delta - \phi(0, \gamma^\dagger; r^\dagger)} \frac{\partial\phi(0, \gamma^\dagger; r^\dagger)}{\partial r} = 1 + \mu_1' \kappa^{1-1/\sigma} + \mu_2' \nu^{1-1/\sigma}. \end{aligned} \tag{2.37}$$

The terms $\mu_1' \kappa^{1-1/\sigma}$ and $\mu_2' \nu^{1-1/\sigma}$, which are respectively the ratio of capital outlays to labor outlays and of raw material outlays to labor outlays when factors receive the value of their marginal product, are monotone decreasing from ∞ to 0 as κ and ν respectively increase from

0 to ∞. Thus, if a solution $\gamma^\dagger$, $r^\dagger$ exists to (2.32) and (2.33), then a unique positive solution $\kappa^\dagger$, $\nu^\dagger$ exists to (2.37).

First, let us obtain sufficient conditions for the existence of a solution to (2.32) and (2.33). As the conditions are intricate, I shall not attempt to obtain general conditions, especially for uniqueness (which is not to be expected in general); it will suffice to display an example that meets our requirements, in order to ensure that the method of analysis is meaningful and the required assumptions plausible.

Denote

$$\alpha_1 = \alpha,\ \alpha_2 = \beta,\ \alpha_3 = \gamma \tag{2.38}$$

and let the function ϕ of (2.5) be defined implicitly by

$$\Phi(\alpha_1, \alpha_2, \alpha_3; r) = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij}\alpha_i\alpha_j - (\delta_1^2 - \delta_0^2)r^\epsilon - \delta_0^2 = 0 \tag{2.39}$$

for values of α_1, α_2, α_3 not all $\leqq 0$, where $c_{ij} > 0$ and the matrix $[c_{ij}]$ is symmetric positive definite, and where the following conditions also hold:

$$c_{11} \geqq 1;\ 0 < \epsilon < 1;\ \sqrt{c_{33}}\,\lambda < \delta_0 < \delta_1 < \delta < 1. \tag{2.40}$$

Then (2.32) and (2.33) have a solution satisfying (2.18), and such that $0 < r < 1$. This is shown by the following argument:

Define $\psi(r)$ implicitly by the identity

$$\Phi\left(\psi(r), 0, \frac{\lambda + \psi(r)}{1 + \eta}; r\right) \equiv_r 0. \tag{2.41}$$

Then the conditions $\delta_0 > \sqrt{c_{33}}\,\lambda$ and $c_{11} \geqq 1$ of (2.40) imply that

$$0 < \frac{\delta_0^2 - c_{33}\lambda^2}{c_{11} + c_{33} + 2c_{13} + 2\lambda(c_{13} + c_{33})} \leqq \psi(r) \leqq \delta_1, \tag{2.42}$$

for all $\eta \geqq 0$. Thus (2.18) is satisfied.

Defining $\chi(r)$ by

$$\chi(r) = \frac{1 - r}{\delta - \psi(r)} \frac{\partial\phi\left(0, \dfrac{\lambda + \psi(r)}{1 + \eta}; r\right)}{\partial r} \tag{2.43}$$

it follows from (2.42) that $0 < \chi(r) < \infty$ for $0 < r < 1$, $\chi(0) = \infty$, and $\chi(1) = 0$. On the other hand, (2.42) implies that the functions

$$(2.44)\qquad -\frac{\partial\phi\left(0, \dfrac{\lambda + \psi(r)}{1+\eta}; r\right)}{\partial\alpha_2} = \frac{\left(c_{21} + \dfrac{c_{23}}{1+\eta}\right)\psi(r) + c_{23}\lambda}{\left(c_{11} + \dfrac{c_{13}}{1+\eta}\right)\psi(r) + c_{13}\lambda}$$

$$(2.45)\qquad -\frac{\partial\phi\left(0, \dfrac{\lambda + \psi(r)}{1+\eta}; r\right)}{\partial\alpha_3} = \frac{\left(c_{31} + \dfrac{c_{33}}{1+\eta}\right)\psi(r) + c_{33}\lambda}{\left(c_{11} + \dfrac{c_{13}}{1+\eta}\right)\psi(r) + c_{13}\lambda}$$

are uniformly bounded above zero and below infinity throughout the interval $0 \leqq r \leqq 1$. The graph of

$$(2.46)\qquad 1 - \frac{\partial\phi\left(0, \dfrac{\lambda + \psi(r)}{1+\eta}; r\right)}{\partial\alpha_2} - \frac{\partial\phi\left(0, \dfrac{\lambda + \psi(r)}{1+\eta}; r\right)}{\partial\alpha_3}$$

must therefore cross that of $\chi(r)$ inside the open interval $0 < r < r_1 < 1$, where r_1 is the largest root of $\chi(r) = 1$. This proves that (2.32) and (2.33) have a solution with $0 < r < 1$.

The solution need not be unique. However, if the condition

$$(2.47)\qquad \delta_1 < \frac{\delta}{2} - \frac{c_{13}\lambda}{2(c_{11} + c_{13})}$$

is imposed, it may be verified that $\chi(r)$ is monotone decreasing (hence the root r_1 of $\chi(r) = 1$ is unique), and if $c_{11}c_{23} \geqq c_{21}c_{13}$ then (2.44) is monotone increasing. However, positive definiteness of the matrix $[c_{ij}]$ implies that $c_{11}c_{33} > c_{31}c_{13}$ hence (2.45) is monotone nonincreasing; nevertheless, if $c_{11}c_{33} - c_{13}^2$ or λ is sufficiently small (and certainly if $\lambda = 0$) the solution will be unique.

An example satisfying (2.40) and (2.47) is given by

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{9} \end{bmatrix}, \quad \begin{matrix} \lambda = .03,\ \delta = .10, \\ \delta_0 = .02,\ \delta_1 = .04. \end{matrix}$$

Returning to the solution of the equilibrium system, recall that $s(t)$ maximizes $1 - r(t) + [q(t) - 1]s(t)$ subject to

$$r(t) \geqq 0, \quad s(t) \geqq 0, \quad r(t) + s(t) \leqq 1, \tag{2.48}$$

hence $q(t) > 1$ implies $s(t) = 1 - r(t)$ and $q(t) < 1$ implies $s(t) = 0$.

To solve for $q^\dagger$, eliminate $B^\dagger$ from the equilibrium solutions of the first equations of (2.20) and (2.27) respectively to get

$$\frac{s^\dagger q^\dagger}{1 - r^\dagger - s^\dagger + s^\dagger q^\dagger} = \frac{\alpha^\dagger + \lambda + \mu}{\delta + \lambda + \mu} \; \frac{\kappa^\dagger f_\kappa(\kappa^\dagger, \nu^\dagger)}{f(\kappa^\dagger, \nu^\dagger)}. \tag{2.49}$$

If $q^\dagger > 1$ then $s^\dagger = 1 - r$ implying that the left side of (2.49) is $=1$ which is impossible, since the right side is <1. Conversely if $q^\dagger < 1$ then $s^\dagger = 0$ and the left side of (2.49) would vanish, which is also ruled out since the right side is positive. Thus, $q^\dagger = 1$ [cf. 61, 11]. The solution for $s^\dagger$ follows immediately from (2.49). $B^\dagger$ is then obtained from the first equation of (2.20), and $a^\dagger$ from the second.

$A(t)$, $b(t)$, $k(t)$ will all grow at the rate α, and $c(t)$, $n(t)$ at the rate $(\alpha\eta - \lambda)/(1 + \eta)$. The latter expression can be either positive or negative; if $\eta = 0$ it is equal to $-\lambda$, and as $\eta \to \infty$ it becomes $+\alpha$. Thus if raw materials are in fixed supply, they obviously decline in proportion to the growing labor force; but if they are in perfectly elastic supply they will actually grow in proportion to the labor force in order to keep up with its increasing efficiency.

Dynamic analysis

It remains to undertake the qualitative analysis of the system of differential equations (2.4), (2.15), and (2.20). These are eight equations in all, in the four state variables $A(t)$, $B(t)$, $C(t)$, $k(t)$ and four auxiliary variables $a(t)$, $b(t)$, $c(t)$, $q(t)$. Together they may be described as the system variables. The remaining variables that appear in those equations are the control variables $r(t)$, $s(t)$, $\beta(t)$, $\gamma(t)$, $n(t)$ which are functions of the eight system variables, as determined by solving (2.21), (2.22), (2.23), (2.28) simultaneously in conjunction with the criterion for determining $s(t)$. The latter problem is handled as in Uzawa [61] and Drandakis and Hu [11] by considering three phases: phase I in which $q(t) > 1$ hence $s(t) = 1$, phase II in which $q(t) = 1$ and $s(t) \in [0, 1]$, and phase III in which $q(t) < 1$ hence $s(t) = 0$.

In the following analysis, I shall limit attention to phase II.[11] If the economy is in phase II for a finite interval of time, then $\dot{\kappa}$ is determined not from (2.27) but with the help of the first equation of (2.20) which becomes

$$(2.50) \qquad (1 - r)\, Bf_\kappa(\kappa, \nu) = \rho + \mu = \delta + \lambda + \mu.$$

This states that the marginal net productivity of capital is equated to the rate of discount, ρ. Likewise, (2.23) reduces to

$$(2.51) \qquad (1 - r)\, Cf_\nu(\kappa, \nu) = p.$$

From (2.28) and the definition of ν in (2.13) and (2.11), we have

$$(2.52) \qquad \begin{aligned} \frac{\dot{p}}{p} &= \frac{1}{\eta}\left[\frac{\dot{\nu}}{\nu} - \gamma + \phi(\beta, \gamma; r) + \lambda\right] \text{ if } \eta > 0; \\ \frac{\dot{\nu}}{\nu} &= \gamma - \phi(\beta, \gamma; r) - \lambda \text{ if } \eta = 0. \end{aligned}$$

Differentiating (2.50) and (2.51) logarithmically, and substituting (2.52), we obtain for $\eta > 0$:

$$(2.53) \qquad \begin{aligned} &\frac{-\dot{r}}{1 - r} + \beta + \frac{f_{\kappa\kappa}(\kappa, \nu)}{f_\kappa(\kappa, \nu)}\dot{\kappa} + \frac{f_{\kappa\nu}(\kappa, \nu)}{f_\kappa(\kappa, \nu)}\dot{\nu} = 0 \\ &\frac{-\dot{r}}{1 - r} + \gamma + \frac{f_{\nu\kappa}(\kappa, \nu)}{f_\nu(\kappa, \nu)}\dot{\kappa} + \frac{f_{\nu\nu}(\kappa, \nu)}{f_\nu(\kappa, \nu)}\dot{\nu} = \frac{1}{\eta}\left[\frac{\dot{\nu}}{\nu} - \gamma + \phi(\beta, \gamma; r) + \lambda\right]. \end{aligned}$$

If $\eta = 0$, the second equation of (2.53) is replaced by the second equation of (2.52).

Since r is a function of κ, ν, u, v, a from (2.21) and (2.22), $\dot{r}$ will involve derivatives of these variables. Assuming $\partial^2\phi/\partial\beta\partial r = \partial^2\phi/\partial\gamma\partial r = 0$ as in (2.39), we obtain $\partial r/\partial u = \partial r/\partial v = 0$ and

$$(2.54) \qquad \frac{\partial r}{\partial \kappa} = \frac{f_\kappa(\kappa, \nu)}{a\dfrac{\partial^2\phi}{\partial r^2}} < 0, \quad \frac{\partial r}{\partial \nu} = \frac{f_\nu(\kappa, \nu)}{a\dfrac{\partial^2\phi}{\partial r^2}} < 0, \quad \frac{\partial r}{\partial a} = -\frac{f(\kappa, \nu)}{a^2\dfrac{\partial^2\phi}{\partial r^2}} > 0.$$

[11] All the analysis that follows is based on the presumption that the optimal path will be such as to reach the singular point with constant q. However, from the first equation of (2.20) one could have $s = 0$, $\dot{q}/q = \beta^\dagger$, leading to the possibility of a balanced growth solution in which investment in capital augmenting technical change takes the place of investment in capital accumulation. It is an interesting question, which I cannot go into here, as to what are the exact conditions for optimality of a path leading to one or the other of these points.

Thus,

$$\dot{r} = \frac{\partial r}{\partial \kappa}\dot{\kappa} + \frac{\partial r}{\partial \nu} + \dot{\nu}\frac{\partial r}{\partial a}\dot{a}. \tag{2.55}$$

Substituting this in (2.53), as well as the differential equation for $\dot{a}$, we obtain the desired system of differential equations. The first equation of (2.27) is then used together with (2.50) to solve for $s(t)$.

The procedure just outlined evidently involves some cumbersome substitutions. The dynamic properties also depend on the magnitude of $\partial^2\phi/\partial r^2$ in the denominators of (2.54). In what follows I shall therefore confine myself to detailed analysis of a limiting case in which $\partial^2\phi/\partial r^2 \to -\infty$ and r is fixed. This would correspond to the case in which δ_0 approaches δ_1 in (2.39).

Corresponding to (2.54) we have, from (2.21) and (2.22),

$$\begin{aligned} &\frac{\partial \beta}{\partial u} = -\frac{\partial^2\phi/\partial\gamma^2}{D} > 0; \quad \frac{\partial \beta}{\partial v} = \frac{\partial^2\phi/\partial\beta\partial\gamma}{D} \\ &\frac{\partial \gamma}{\partial u} = \frac{\partial^2\phi/\partial\gamma\partial\beta}{D}; \quad \frac{\partial \gamma}{\partial v} = -\frac{\partial^2\phi/\partial\beta^2}{D} > 0 \end{aligned} \tag{2.56}$$

where

$$D = \begin{vmatrix} \partial^2\phi/\partial\beta^2 & \partial^2\phi/\partial\beta\partial\gamma \\ \partial^2\phi/\partial\gamma\partial\beta & \partial^2\phi/\partial\gamma^2 \end{vmatrix} \tag{2.57}$$

and D as well as $\partial(\beta, \gamma)/\partial(u, v)$ is positive definite. The partial derivatives of β and γ with respect to κ, ν and a all vanish. The above holds with variable as well as fixed r, if $\partial^2\phi/\partial\beta\partial r = \partial^2\phi/\partial\gamma\partial r = 0$. From now on I consider the case of fixed r.

With r fixed we obtain from (2.53)

(2.58a)

$$\begin{bmatrix} \dot{\kappa} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\dfrac{f_{\kappa\kappa}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} & -\dfrac{f_{\kappa\nu}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} \\ -\dfrac{f_{\nu\kappa}(\kappa, \nu)}{f_\nu(\kappa, \nu)} & \dfrac{1}{\eta\nu} - \dfrac{f_{\nu\nu}(\kappa, \nu)}{f_\nu(\kappa, \nu)} \end{bmatrix}^{-1} \begin{bmatrix} \beta \\ \left(1 + \dfrac{1}{\eta}\right)\gamma - \dfrac{1}{\eta}[\phi(\beta, \gamma) + \lambda] \end{bmatrix},$$

an equation which remains valid when $\eta = \infty$. It may also be written

(2.58b)

$$\begin{bmatrix} \dot{\kappa} \\ \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\dfrac{f_{\kappa\kappa}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} & -\dfrac{f_{\kappa\nu}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} \\ -\dfrac{\eta f_{\nu\kappa}(\kappa, \nu)}{f_\nu(\kappa, \nu)} & \dfrac{1}{\nu} - \dfrac{\eta f_{\nu\nu}(\kappa, \nu)}{f_\nu(\kappa, \nu)} \end{bmatrix}^{-1} \begin{bmatrix} \beta \\ \\ (\eta + 1)\gamma - \phi(\beta, \gamma) - \lambda \end{bmatrix}$$

which is valid for $\eta = 0$.

We shall write the expressions on the right of (2.58) in the form

(2.59a)
$$\begin{aligned} \dot{\kappa} &= \mathcal{K}(\kappa, \nu, u, v, a) \\ \dot{\nu} &= \mathcal{N}(\kappa, \nu, u, v, a). \end{aligned}$$

To complete this system we have

(2.59b)
$$\begin{aligned} \dot{u} &= \mathcal{U}(\kappa, \nu, u, v, a) = (1 - r)\frac{m(\kappa, \nu)}{a}\left\{u - \frac{\kappa f_\kappa(\kappa, \nu)}{m(\kappa, \nu)}\right\} \\ \dot{v} &= \mathcal{V}(\kappa, \nu, u, v, a) = (1 - r)\frac{m(\kappa, \nu)}{a}\left\{v - \frac{\nu f_\nu(\kappa, \nu)}{m(\kappa, \nu)}\right\} \\ \dot{a} &= \mathcal{A}(\kappa, \nu, u, v, a) = [\delta - \phi(\beta, \gamma)]a - (1 - r)m(\kappa, \nu). \end{aligned}$$

Equations (2.59) form a self-contained system of five differential equations.

The Jacobian matrix $J = \partial(\mathcal{K}, \mathcal{N}, \mathcal{U}, \mathcal{V}, \mathcal{A})/\partial(\kappa, \nu, u, v, a)$ evaluated at the equilibrium point $(\kappa^\dagger, \nu^\dagger, u^\dagger, v^\dagger, a^\dagger)$ has a structure partially displayed as follows:

(2.60)
$$J = \begin{bmatrix} 0 & 0 & \partial\mathcal{K}^\dagger/\partial u & \partial\mathcal{K}^\dagger/\partial v & 0 \\ 0 & 0 & \partial\mathcal{N}^\dagger/\partial u & \partial\mathcal{N}^\dagger/\partial v & 0 \\ \partial\mathcal{U}^\dagger/\partial\kappa & \partial\mathcal{U}^\dagger/\partial\nu & \delta^* & 0 & 0 \\ \partial\mathcal{V}^\dagger/\partial\kappa & \partial\mathcal{V}^\dagger/\partial\nu & 0 & \delta^* & 0 \\ \partial\mathcal{A}^\dagger/\partial\kappa & \partial\mathcal{A}^\dagger/\partial\nu & \partial\mathcal{A}^\dagger/\partial u & \partial\mathcal{A}^\dagger/\partial v & \delta^* \end{bmatrix}$$

where

(2.61)
$$\delta^* = \delta - \phi(0, \gamma^\dagger) = (1 - r^\dagger)\frac{m(\kappa^\dagger, \nu^\dagger)}{a^\dagger}.$$

Denote

(2.62)
$$J_1 = \begin{bmatrix} \partial\mathcal{U}^\dagger/\partial\kappa & \partial\mathcal{U}^\dagger/\partial\nu \\ \\ \partial\mathcal{V}^\dagger/\partial\kappa & \partial\mathcal{V}^\dagger/\partial\nu \end{bmatrix}, \quad J_2 = \begin{bmatrix} \partial\mathcal{K}^\dagger/\partial u & \partial\mathcal{K}^\dagger/\partial v \\ \\ \partial\mathcal{N}^\dagger/\partial u & \partial\mathcal{N}^\dagger/\partial v \end{bmatrix}.$$

Then the characteristic polynomial of J is

$$(2.63) \qquad (\xi - \delta^*)\{[\xi(\xi - \delta^*)]^2 - \operatorname{tr}(J_1J_2)\,\xi(\xi - \delta^*) + \det(J_1J_2)\}.$$

Thus J has one positive characteristic root $\xi_1 = \delta^*$. Denoting

$$(2.64) \qquad \zeta = \xi(\xi - \delta^*),$$

the remaining four roots are found by first obtaining the roots ζ_1, ζ_2 of

$$(2.65) \qquad \zeta^2 - \operatorname{tr}(J_1J_2)\zeta + \det(J_1J_2) = 0$$

and then for each ζ_i solving the quadratic equation

$$(2.66) \qquad \xi^2 - \delta^*\xi - \zeta_i = 0 \qquad (i = 1, 2).$$

In order that there be a path satisfying (2.59) which approaches the point $(\kappa^\dagger, \nu^\dagger, u^\dagger, v^\dagger, a^\dagger)$ asymptotically, it is necessary and sufficient that at least one root ξ_i of (2.63) have a negative real part. If one of the roots of (2.65), say ζ_1, has a positive real part, then one of the two roots of (2.66), for $i = 1$, will have a negative real part. If $\det(J_1J_2) < 0$, (2.65) will have one positive and one negative root. If $\det(J_1J_2) > 0$, one of the roots of (2.65) will have a positive real part provided $\operatorname{tr}(J_1J_2) > 0$. A sufficient condition for the desired result is therefore that $\operatorname{tr}(J_1J_2) > 0$.

Let elasticities of substitution be constant as in (2.36). Then

$$(2.67) \qquad J_1 = \frac{1 - \sigma}{\sigma}\,\delta^*\begin{bmatrix} \mu_1'\kappa^{-1/\sigma} & 0 \\ 0 & \mu_2'\nu^{-1/\sigma} \end{bmatrix}.$$

From (2.62) it follows that

$$(2.68) \qquad \operatorname{tr}(J_1J_2) = \frac{1 - \sigma}{\sigma}\,\delta^*\left[\mu_1'\,\kappa^{-1/\sigma}\,\frac{\partial\mathcal{K}^\dagger}{\partial u} + \mu_2'\,\nu^{-1/\sigma}\,\frac{\partial\mathfrak{N}^\dagger}{\partial v}\right].$$

From (2.56), (2.57), (2.58) it may be seen that $\partial\mathcal{K}^\dagger/\partial u > 0$ and $\partial\mathfrak{N}^\dagger/\partial v > 0$ as long as

$$(2.69) \qquad \frac{\partial^2\phi(\beta, \gamma; r)}{\partial\beta\partial\gamma} > 0.$$

Now $\delta^* > 0$ in view of (2.18) and the existence of a solution to (2.32), (2.33), (2.37); therefore $a^\dagger > 0$ in (2.61). Together with (2.69), a sufficient condition that $\operatorname{tr}(J_1J_2) > 0$ is therefore that $\sigma < 1$ in (2.68).

Taking account of (2.58) as well as the strict concavity of f and ϕ, it may be verified that

$$|J_2| = \frac{1 + \eta - \partial\phi/\partial\gamma}{\eta\Delta} > 0 \tag{2.70}$$

where

$$\Delta = \begin{vmatrix} -\dfrac{f_{\kappa\kappa}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} & -\dfrac{f_{\kappa\nu}(\kappa, \nu)}{f_\kappa(\kappa, \nu)} \\ -\dfrac{f_{\nu\kappa}(\kappa, \nu)}{f_\nu(\kappa, \nu)} & \dfrac{1}{\eta\nu} - \dfrac{f_{\nu\nu}(\kappa, \nu)}{f_\nu(\kappa, \nu)} \end{vmatrix} > 0. \tag{2.71}$$

Consequently, det $(J_1J_2) = |J_1|\,|J_2|$ has the same sign as $|J_1|$.

The situation may be summarized as follows. If $\sigma > 1$ then det $(J_1J_2) < 0$ and (2.65) has one positive and one negative root. Consequently, (2.63) will have two positive, one negative, and two imaginary roots. If $\sigma < 1$ and if (2.69) is assumed, then det $(J_1J_2) > 0$ and tr $(J_1J_2) > 0$, which guarantees that (2.65) will have two roots with positive real parts. Then (2.63) will have either (a) three positive and two negative roots, or (b) one positive root, two complex roots with positive real parts, and two complex roots with negative real parts. In all of these cases, $(\kappa^\dagger, \nu^\dagger, u^\dagger, v^\dagger, a^\dagger)$ will be either a generalized saddle point or generalized focal point [cf. 41, pp. 187–88]. In the case $\sigma > 1$ there will be a path approaching the equilibrium point monotonically. In the case $\sigma < 1$, there will be either (a) a manifold of such paths, or (b) a path with a corkscrew-like approach to the equilibrium point.

The above local analysis does not, of course, prove the optimality of a path leading to the singular point. Such optimality appears very plausible on geometric grounds, but a formal proof will not be attempted here.

A generalization

The above analysis can be generalized in the following way. Suppose that part of the imported raw material is directly consumed and the rest used as input into the production of the export good. $X_i(t)$, $Y_i(t)$, $Z_i(t)$ will now denote consumption, net production, and import respectively of the ith good ($i = 1, 2$), the first being the export good (whence $Z_1 < 0$) and the second the raw material ($Z_2 > 0$, where

$-Y_2 = N = Z_2 - X_2$ is the amount used as input in the production of commodity 1). We denote

$$(2.72) \qquad x_i(t) = \frac{X_i(t)}{L(t)},\ y_i(t) = \frac{Y_i(t)}{L(t)},\ z_i(t) = \frac{Z_i(t)}{L(t)} \quad (i = 1, 2)$$

and $n(t) = N(t)/L(t) = -y_2(t)$ as in (2.14) and (2.11).

Assume that the utility function has the form

$$(2.73) \qquad \int_0^\infty e^{-\rho t}L(t)U[x_1(t), x_2(t)]dt = \int_0^\infty e^{-\delta t}U[x_1(t), x_2(t)]dt$$

where $L(0) = 1$, and suppose further that

$$(2.74) \qquad U(x_1, x_2) = x_1^{\theta_1}x_2^{\theta_2} \quad (\theta_i > 0,\ \theta_1 + \theta_2 = 1).$$

This implies unitary elasticity of substitution in consumption; more important, homogeneity of U implies a unitary income elasticity of demand, contrary to Engel's law.[12]

In place of (2.7), (2.28), the foreign offer function will be assumed to have the form

$$(2.75) \quad Z_2(t) = \bar{N}^{1+\eta}p(t)^{\eta}e^{\lambda^* t},\ Z_1(t) + p(t)Z_2(t) = 0 \quad (0 \leqq \eta < \infty)$$

with the limiting form $p(t) = 1/\bar{N}$ as $\eta \to \infty$. λ^* is a shift parameter representing the growth rate of the foreign country.

The four equations (2.20) become slightly modified: In each equation, the second term on the right is multiplied by $U_1 = \partial U/\partial x_1$. The extra condition $p = U_2/U_1$ is added, where $U_i = \partial U/\partial x_i$, and the remaining equations hold as before. Since $p = U_2/U_1 = (\theta_2/\theta_1)(x_1/x_2)$ from (2.74), it follows that

$$(2.76) \qquad \hat{U}_1 = -\theta_2\hat{p},$$

where the symbol ^ denotes logarithmic differentiation with respect to time, i.e., $\hat{p} = \dot{p}/p$, etc. From (2.75) we also obtain

$$(2.77) \qquad \hat{Z}_2 = \eta\hat{p} + \lambda^*.$$

If there is a singular solution in which Z_2, X_2, and $N = -Y_2$ grow at constant rates, and in which $\nu^\dagger = CN/AL$ is constant, then with constant $\alpha^\dagger$, $\beta^\dagger$, $\gamma^\dagger$ we have

[12] This is to be taken as a null hypothesis. The consequences of changing θ_i's are discussed below.

(2.78) $$\hat{X}_2 = \hat{Z}_2 = \hat{N} = \alpha^\dagger + \lambda - \gamma^\dagger.$$

If $r^\dagger$, $s^\dagger$, $q^\dagger$, $\kappa^\dagger$, $\nu^\dagger$ are constant, then (2.23) yields

(2.79) $$\hat{p} = \gamma^\dagger.$$

Thus from equations (2.77), (2.78), (2.79), we obtain

(2.80) $$\gamma^\dagger = \frac{\alpha^\dagger + \lambda - \lambda^*}{1 + \eta}.$$

From the first (modified) equation of (2.20) we obtain, for $\dot{q} = 0$,

(2.81) $$\hat{U}_1 = \beta^\dagger.$$

Together, (2.76), (2.79) and (2.81) yield

(2.82) $$\beta^\dagger = -\theta_2\gamma^\dagger.$$

Consequently, (2.32) and (2.33) are modified by replacing $\beta^\dagger = 0$ by (2.82). As long as $\gamma^\dagger > 0$ in (2.80), it follows from (2.82) that technical progress is strongly capital deepening, i.e., capital increases relative to output.

The dynamic properties of this singular solution may be briefly outlined. Define

(2.83) $$w(t) = U_1(t)/a(t)$$

where $U_1 = \partial U/\partial x_1$. Recalling that the right side of (2.21) is now to be multiplied by U_1, it may be written $w(t)f[\kappa(t), \nu(t)]$, and taking account of (2.24), it follows from (2.21) and (2.22) that β, γ, and r are functions of κ, ν, u, v, w.

In equation (2.52), λ is now replaced by $\lambda - \lambda^*$, as in the second equation of (2.53). The term on the right of the first equation of (2.53) becomes $(\theta_2/\eta)\,[\,\hat{\nu} - \gamma + \phi(\beta, \gamma; r) + \lambda - \lambda^*]$. As in the development following (2.53), only the case of constant r will be taken up; then (2.58a) is replaced by

(2.84)
$$\begin{bmatrix} \dot{\kappa} \\ \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\dfrac{f_{\kappa\kappa}(\kappa,\nu)}{f_\kappa(\kappa,\nu)} & \dfrac{\theta_2}{\eta\nu} - \dfrac{f_{\kappa\nu}(\kappa,\nu)}{f_\kappa(\kappa,\nu)} \\ -\dfrac{f_{\nu\kappa}(\kappa,\nu)}{f_\nu(\kappa,\nu)} & \dfrac{1}{\eta\nu} - \dfrac{f_{\nu\nu}(\kappa,\nu)}{f_\nu(\kappa,\nu)} \end{bmatrix}^{-1} \begin{bmatrix} \beta + \dfrac{\theta_2}{\eta} - \dfrac{\theta_2}{\eta}[\phi(\beta,\gamma) + \lambda - \lambda^*] \\ \left(1 + \dfrac{1}{\eta}\right)\gamma - \dfrac{1}{\eta}[\phi(\beta,\gamma) + \lambda - \lambda^*] \end{bmatrix}.$$

If, in addition to strict concavity of f, it is assumed that $f_{\kappa\nu} > 0$ (a condition satisfied by C.E.S. production functions), the matrix in (2.84) will have a positive determinant. Now in equations (2.59), the variable a in the arguments of the functions $\mathcal{K}$, $\mathfrak{N}$, $\mathfrak{U}$, $\mathfrak{V}$ is replaced by w, and in the first two equations of (2.59b), $m(\kappa, \nu)/a$ is replaced by $wm(\kappa, \nu)$. Finally, the third equation of (2.59b) is replaced by

$$\dot{w} = \mathcal{W}(\kappa, \nu, u, v, w) = (\hat{U}_1 - \hat{a})w \tag{2.85}$$

where

$$\hat{U}_1 - \hat{a} = -\frac{\theta_2}{\eta\nu}\mathfrak{N}(\kappa, \nu, u, v, w) + \frac{\theta_2}{\eta}[\gamma - \phi(\beta, \gamma) - \lambda + \lambda^*] - [\delta - \phi(\beta, \gamma)] + (1 - r)m(u, v)w. \tag{2.86}$$

The structure of the Jacobian matrix (2.60) remains unchanged. The expression for $|J_2|$ in (2.70) must now be multiplied by $1 - (\theta_2/\eta)\,\partial\phi/\partial\beta$, and Δ in (2.71) must be replaced by the determinant of the matrix of (2.84). The analysis goes through just as before.

Returning to the singular solution, it will now be shown for the case of fixed r that under certain conditions a decline in θ_2 (hence a decline in the proportion of consumers' income devoted to the primary product) will lead to a decline in the share $\nu^\dagger f_\nu(\kappa^\dagger, \nu^\dagger)/f(\kappa^\dagger, \nu^\dagger)$ of natural resources in the national income. If r is fixed, $\gamma^\dagger$ is determined from

$$\gamma^\dagger = \frac{\phi(-\theta_2\gamma^\dagger, \gamma^\dagger; r) + \lambda - \lambda^*}{1 + \eta} \tag{2.87}$$

(see equations (2.80), (2.82)). As long as $1 + \eta - ((\partial\phi/\partial\gamma) - \theta_2(\partial\phi/\partial\beta)) > 0$, $\gamma^\dagger$ will increase when θ_2 increases. Under these conditions, and assuming (2.69) to hold, we find that a fall in θ_2 leads to a decline in $-\partial\phi/\partial\gamma$ (hence a decline in the share of resources relative to labor's share) and to a rise in $-\partial\phi/\partial\beta$ (hence a rise in capital's share relative to labor's). Similarly a fall in θ_2 leads to a rise in the share of capital relative to resources. Consequently, as the proportion of income devoted to the primary product falls, so does the share of resources in national income, in accordance with Schultz's result.

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COMMENTS

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My comments will be addressed for the most part to questions raised by Professor Chipman's characteristically skillful employment of advanced mathematical techniques to deepen significantly our theoretical understanding of economic phenomena; at the end I will make a few remarks concerning the paper presented by Professor Jones at this session.

It may be useful to remark that Chipman's theory of optimal induced technological change is an example of the "dynamic comparative advantage," the subject of Professor Bruno's paper. Whenever there is capital accumulation relative to the growth of original factors, comparative advantage will change over time. But the full flavor of dynamic comparative advantage occurs when there is an element of irreversibility involved in the investment. Then indeed the justification for investment must involve the whole future course of effects on the marginal productivities. This effect appears in Bruno's paper in the rather simple sense that investment today in a given industry increases *its* absorptive capacity for future investment. In Chipman's model, the irreversible investment is the expenditure on research; accumulated research knowledge can neither be transferred to other industries nor undone to provide additional consumption. The justification for an irreversible investment is based on the sum of discounted marginal productivities over the future and not merely the current marginal productivity. (In this context it is misleading to state in equation (2.50)ff. that the product of capital is equated to the rate of discount; in fact the rate of discount is rated to the marginal net product of capital less an excise tax to pay for research expenditures.)

The Pontryagin principle is the natural mathematical tool for any type of model involving allocation of a current flow of resources among a number of activities when some of the activities are forms of capital accumulation. The principle can be given an economic interpretation: assign prices to stocks of different forms of capital to make them commensurate with current consumption. In the short run, resources are allocated so as to maximize income at these prices. The prices themselves must change over time in such a way as to make individuals just willing to hold the stocks, with income including capital gains from changing prices just equal to a rate of discount. Finally, the whole system—prices and quantities—has to tend to a stationary equilibrium. The Pontryagin principle is closely related to recursive optimization, which has been called dynamic programming, where the values of accumulation are imputed from the future. Professor Chipman has used the Pontryagin methods with great skill and economic tact.

The questions one might raise relate mainly to the economics of his model, though one technical problem will be mentioned below. In the first place, as he makes clear, his model is one of central planning; the entire benefit of research expenditures is internalized. Thus, there is no direct reason to argue that it is in any way descriptive; it may be true nevertheless that a perfectly optimizing model, such as Chipman's, may give an accurate qualitative, though not quantitative, picture of the workings of a more realistic model. One can easily imagine models, such as Professor Jones suggests in his paper, where research is carried out by firms who can appropriate part but not all of its benefits. The research done by other firms may be taken as parametric by any given firm, so that the complexities of oligopolistic interdependence may be avoided, and each firm is supposed to carry out an optimization of research over time. Such an analysis is probably only mildly more difficult than the one Professor Chipman has offered us.

In the second place, the assumptions about the relation between research and production function shifts are subject to serious question. These assumptions have become common in recent theoretical literature, and Professor Chipman has distinguished authority for them, but they are nevertheless both theoretically arbitrary and empirically unrealistic. In the Kennedy-Weizsäcker model, there is one kind of technological progress for each factor, namely, a rate of augmentation, and a stable function describing the trade-offs among these rates and some measure of research input. (Professor Chipman's measure, the *proportion* of total output devoted to research, seems especially arbitrary; it seems to imply that knowledge cannot be transmitted at all but must be re-created where needed.) Now there are two main problems in accepting this thesis: (1) is it reasonable to assume that rates of factor augmentation are stably related to research inputs? (2) is there any reason to assume a stable trade-off among rates of augmentation of different factors?

(1) The first point can be discussed most clearly in a one-factor model; suppose labor is the only factor. Then the model becomes, in Chipman's notation,

$$Y(t) = [1 - r(t)]\, A(t)L(t), \tag{1}$$

$$\dot{A}(t)/A(t) = \phi[r(t)]. \tag{2}$$

Equation (2) is supposed to be a specification of the knowledge-producing activity, where knowledge is measured by labor productivity. Clearly, it is reasonable to assume that the rate of production of new knowledge depends both on the resources devoted to it and on the amount of knowledge already accumulated. But why in the world should new knowledge display constant returns to accumulated knowledge at a fixed level of research input? By analogy with the usual economic analysis of more palpable branches of production one might argue for diminishing returns; the easier types of knowledge have been acquired first, and it becomes increasingly difficult (requires more resources) to make new discoveries. To put it another way, is it really credible that, if research inputs are held constant (even if measured proportionately rather than absolutely), then output will grow exponentially while material inputs remain constant. The laws of conservation of matter and energy alone seem to prohibit this.

(2) Analyzing technological progress into rates of factor augmentation is merely one possibility among many and has no very distinguished position either in terms of empirical evidence or analytic convenience. Professor Jones has dealt penetratingly, if briefly, with this point. But in addition the stability of the invention possibility frontier in any form has neither rationale nor empirical support. A gross reading of history suggests the bias is apt to vary from time to time due much more to changes in the state of knowledge than to changes in capital-labor ratios.

An analogy may help. Exploration, especially in the great days from

1492 to 1880 or so, was very analogous to research; resources were invested to produce knowledge about new places and resources. I suggest that which factors were in fact augmented depended primarily on what was in the explored countries and very little on factor ratios in the exploring countries. Columbus may have been impelled by a desire for spices, but it was the supply of corn which was increased.

I do not wish to decry the proposition that the magnitude of technological progress may be responsive to economic motivation, but I have grave doubts that its bias can be explained in any other than accidental and historical terms.

I conclude the discussion of Professor Chipman's paper with two less fundamental remarks. First, Professor Chipman's production function for research, which generalizes equation (2) above, is not a concave function of its arguments, since there are increasing returns to all factors (including both accumulated knowledge and current research expenditures). Just as in ordinary (finite-dimensional) maximization problems, it is then possible that the necessary conditions for a maximum may be satisfied by several policies, some of which indeed may not be even locally maximum. A study of this problem in a somewhat simpler but analogous model will be found in an unpublished dissertation by Larry Ruff.[1]

Second, Professor Chipman minimizes the importance of learning by doing as a cause of technological progress. But the notion of the "product cycle," so much emphasized at this conference, is certainly an example of learning by doing; it is assumed that the requirements for certain skill factors decline with experience in production.

Professor Jones has given, as expected, a beautifully lucid account of a two-factor economy. But with the growing emphasis on skill as a factor of production, we are at least in a three-factor world, and it is only sober sense to recognize that more factors should be distinguished and would be if we had the analytic tools and empirical evidence to do so. Now it is pretty clear that we are unlikely to be able to derive any theorems in the multifactor case comparable to those which hold in a two-factor world. We shall have to switch our emphasis in model-building from derivation of theorems to computation of implications.

[1] Ruff, "Optimal Growth and Technological Progress in a Cournot Economy," Institute for Mathematical Studies in the Social Sciences, Stanford University, 1968, Technical Report 11.

The algorithms by H. Scarf [2] for computation of equilibrium levels, prices, and quantities in a general competitive model, seem to be ideally suited for testing out the implications of alternative foreign trade models.

WASSILY LEONTIEF

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Four years ago I had an opportunity to discuss, in this very room, the then relatively new aggregative dynamic models with built in, endogenous technological change. I argued that despite or rather because of their formal elegance this type of theoretical approach could contribute very little to the understanding of real processes of economic growth. Much mathematical ingenuity has been invested since then in further elaboration of such models; Professor Chipman quotes and lists some twenty articles published on that special subject in the last three years. The formulation has become more elaborate and its mathematics higher. In the paper prepared by Chipman for this conference, he applies the full power of the perfected analytical tool to the solution of an important and well-defined problem: What effects can technological change have on imports of raw materials by advanced industrialized economies from the so-called less developed countries?

A study of Chipman's twenty typewritten pages of concise mathematical argument confirms me in my original contention: Elaborate aggregative growth models can contribute very little to the understanding of processes of economic growth, and they cannot provide a useful theoretical basis for systematic empirical analysis.

Taking for granted the internal consistency of Chipman's model, I propose to question the relevance of his conclusions and the validity of the factual assumption on which he builds his argument.

These conclusions are summarized on page 104; "If the home country produces a single consumer good, using up labor, capital, and imported

[2] H. Scarf, "On the Computation of Equilibrium Prices." W. Fellner *et al.* *Ten Economic Studies in the Tradition of Irving Fisher*, New York, 1967, pp. 207–30.

raw material, then an equilibrium growth path will be approached in which technical change is Harrod neutral as far as capital is concerned (that is, there will be no capital-augmenting technical change), whereas the rate of augmentation of natural resources will be determined by the formula $\gamma = (\lambda + \alpha)/(1 + \eta)$, where λ is the rate of growth of population, α the rate of labor-augmenting technical change, and η the elasticity of the foreign offer curve. Thus, if resources are in perfectly elastic supply ($\eta = \infty$), there will be no resource-augmenting technical improvement, whereas at the opposite extreme of perfectly inelastic supply ($\eta = 0$), resource-augmenting technical change will proceed at the rate $\lambda + \alpha$. The volume of primary products grows at the rate $(\lambda + \alpha)\eta/(1 + \eta)$, and the foreign terms of trade improve at the rate $(\lambda + \alpha)/(1 + \eta)$."

Addressing himself in the light of these conclusions to the "controversial topic of the past few decades"—the thesis set forth by Prebisch and Singer that there is a tendency toward a deterioration in the terms of trade of raw material producing countries—Chipman—on page 103—explains that:

"Although it is not claimed that the model used here can predict changes in technology with any accuracy, it nevertheless supports Schultz's thesis in the weak sense that the declining share of raw materials expenditure in national income cannot be accounted for on the basis of the hypothesis of induced resource-augmenting technical progress alone."

I submit that conclusions drawn from analysis of the properties of that particular model do not add to the understanding of the phenomena it is intended to explain, even if—heeding Chipman's advice—one interprets them with caution. They are wrong in the sense that they are derived from empirically unjustifiable assumptions. The fact that these assumptions conform strictly to the, by now well-standardized, theoretical construction code of the model-building industry lends additional justification to the following critical inquiry.

Trade statistics show that the amounts of raw silk, quebracho bark, or, say, vegetable dyes moving in international trade, attained their highest level many years ago and have been falling steadily since. They may eventually disappear from the list of internationally traded goods. Would it not be reasonable to ask whether the decline in the demand for these raw materials might not have been brought about by changes in some methods of production which, in their turn, could have been based on the acquisition of new technical knowledge brought about by investment in research?

Confronted with Chipman's elaborate refutation of so plausible an explanation, one naturally becomes suspicious of the factual assumptions on which he chooses to base his theoretical argument. The task of tracing these assumptions is complicated by the fact that instead of being presented all at once, additional assumptions are introduced "when needed," that is, in any intermediate stage of the argument at which the going gets too hard. His model contains an aggregative production function, a dynamic utility function and, in addition, the so-called Kennedy function—the *deus ex machina* of the modern pure theory of economic growth that permits the utility maximizing agent to control and, consequently, to predict in advance the future course of all technological change.

I propose to show that among all the structural assumptions built into Chipman's model the one pertaining to the effects of technological change on the shape of his new classical aggregative production function is solely responsible for the theoretical conclusion that I find difficult to accept.

Specifically, this is the assumption that all technological change is "factor-augmenting." The exclusion of all other kinds of structural change implies that the input-output relationships corresponding to all possible past, present, and future methods of producing any given good can be depicted on a single graph showing one set of conventionally shaped and properly numbered "isoquants." The effects of every admissible factor-augmenting innovation can be described simply by a change in the scales showing how many units of the particular factor are represented by each of the equal intervals marked off on that graph along the appropriate axis. With every "augmentation" of an input the number of physical units of the input contained in an inch or centimeter, as the case may be, is increased.

The elasticity of substitution between any two inputs can be said to remain unaffected by this particular kind of technical change in the sense that at any given point of the graph this elasticity remains the same, even after the numerical scales describing the position of that point in terms of input coordinates have been adjusted to reflect factor-augmenting technological change. For good measure Chipman limits the admissible shape of his production function still further by assuming that "the elasticity of substitution between any two factors is bounded below 1," which means that it is less than 1.

Under such conditions technological change cannot possibly eliminate any input from the production process that at any time has been part of the process. Only infinite augmentation in the magnitude of the efficiency coefficient of that factor, as compared to other factors, could conceivably drive the level of its input toward zero.

Following conventional usage and, I suppose, for the sake of casual realism, in addition to raw materials, Chipman's model includes the two primary inputs, labor and capital. Without affecting one way or another the nature or the substantive implication of his mathematical argument, its formulation can be simplified by dropping capital as a separate input. Thus Chipman's (2.1) can be reduced to the following relationships describing the balance between current consumption, production, exports of goods, and imports of raw materials in an economy that produces a single finished good from two inputs, labor and imported raw materials:

$$Y = F(AL, CN)(1 - r) - pN \tag{1}$$

where,

- Y is the net output.
- $F(\ )$ represents a homogenous production function.
- AL is the total labor input measured in "efficiency units"; L represents the number of workers and A, the corresponding technical efficiency (productivity) coefficient.
- CN is the total raw material input measured in "efficiency units"; N represents the amount of the raw material measured in natural units; C, the corresponding technical efficiency coefficient.
- r is the proportion of gross total output currently allocated to research and development, that in its turn brings about an increase in the magnitude of A or C or of both.
- pN represents the amount of the finished good exported in exchange for the imported N units of raw material; p is the price of that raw material expressed in units of the finished good which is being traded for it.

Because it is assumed to be homogeneous (constant returns to scale), the production function $F(\ )$ can be written in the following form:

$$(2) \qquad F(AL, CN) \equiv ALF(1, CN/AL) \equiv ALf(CN/AL)$$

If η represents the elasticity of the supply of raw material to the importing country,

$$(3) \qquad p = N^{\frac{1}{\eta}} \quad \text{and} \quad pN = N^{\frac{(\eta+1)}{\eta}}$$

Thus,

$$(4) \qquad Y = ALf(CN/AL)(1 - r) - N^{\frac{(\eta+1)}{\eta}}$$

The system is dynamic in the sense that each of the variables is considered from the outset to be a function of time. Following all other modern growth theorists Chipman centers his attention on a singular solution for the system better known as the state of the Golden Age, i.e., a state of even, uniform expansion in which the magnitude of each variable either grows (or contracts) exponentially or remains constant. In this particular instance the fraction r of total output allocated to generation of factor-augmenting technological change is assumed to remain constant while the labor force L is taken to be growing at an exogenously fixed annual rate, l.

Each of the other variables is permitted either to grow at some positive constant rate or not to change at all. Thus, under Golden Age conditions, the balance equation (4) acquires the following form,

$$(5) \qquad Y_0e^{vt} = A_0e^{at}L_0e^{lt}f(C_0e^{ct}N_0e^{nt}/A_0e^{at}L_0e^{lt})(1 - r) - N_0e^{\frac{\eta+1}{\eta}nt}$$

Moreover, the ratio of the total raw material to the total labor inputs (both measured in "efficiency units") and, consequently, the value of function $f(\ )$ is assumed to be constant too. The capital letters with the subscript o represent the magnitudes of the variables at the time $t = 0$; the small letters in the exponents represent their respective equilibrium (Golden Age) rates of growth. This is a departure from Chipman's own notation which, because of a large number of intermediate variables he has to handle, spills over much further into the Greek alphabet.

Thus, we have one equation with five unknowns: y, a, c, n, and r. Obviously, additional conditions are required for a unique determination of their values. The aforementioned Kennedy function—the assumed functional relationship between r, a, and c—is one of them. The rest are derived from the solution of a maximizing problem in which the values of the four variables are determined in such a way as to maximize, within the limitation imposed by the other conditions, the value of a dynamic social utility function. This does not enter into my reformulation of Chipman's argument at all. I do not have to follow this long path since Chipman's main conclusions concerning the effects of technological change on the demand for raw materials can be derived simply and directly from equation (5).

Consolidating some of the exponents we have,

$$(6) \quad Y_o e^{yt} = A_o L_o e^{(a+l)t} f(C_o N_o e^{(c+n)t} / A_o L_o e^{(a+l)t})(1 - r) - N_o e^{\frac{\eta+1}{\eta} nt}$$

The equality between the left-hand and the right-hand side of this equation can be maintained for all values of t only if the coefficients of t in the exponents appearing in its first, second, and third terms are equal, i.e., if

$$(7) \qquad y = a + l = \frac{\eta + 1}{\eta} n$$

The equality between the first and the middle terms means that the net output must increase at a rate equal to the sum of the rate of the factor-augmenting rise in the productivity of labor and the given rate of increase of the total labor force. The second equality describes a simple relationship between the last two magnitudes, the growth rate of the raw material input and the given elasticity of its supply.

The assumed invariance of the ratio of the two exponential expressions entered on the right-hand side of (5), under the function sign, implies,

$$(8) \qquad c + n = a + l$$

In case $\eta = \infty$, (7) is reduced to:

$$y = a + l = n$$

This is compatible with (8) only if $c = 0$, i.e., no resource augmenting technical change will affect the use of the imported raw materials if their supply happens to be perfectly elastic. In this case, (8) is reduced to:

$$n = a + l.$$

The import of raw material will grow at a rate equal to the growth rate of the labor force plus the augmentation rate of the efficiency of labor.

In case $\eta = 0$, i.e., if the supply of raw material happens to be fixed, the right-hand side of (7) can remain finite only if $n = 0$. According to (8), this implies:

$$c = a + l.$$

These are precisely the results anticipated by Chipman in the first part of his paper and stated concisely in his formulae (2.34) and (2.35). I reached them in a few brief steps using in my argument only his special assumption concerning the effect of technical change on the shape of the production function. After bringing into the argument the Kennedy function and a dynamic social utility function, each carrying with it a number of additional arbitrary assumptions, he obtains these results through an elaborate application of the powerful Pontryagin's "maximum principle."

Thus, we see that it is his special assumption concerning the "factor-augmenting" nature of all technical change that lies at the base of Chipman's refutation of the Prebisch-Singer thesis. In the light of incontrovertible empirical evidence—cited at the beginning of my remarks—I conclude that his theoretical argument simply disproves the validity of his basic factual assumption, an assumption that is incorporated—possibly for reasons of indisputable mathematical convenience—in most of the recently presented aggregative models of economic growth.

REPLY TO PROFESSOR LEONTIEF

JOHN S. CHIPMAN

If under hypothesis A, conclusion C follows if and only if assumption B holds, and if both B and C are empirically observed, I would say that we have an explanation of C. A is the hypothesis of induced factor-augmenting technical change, represented in terms of a Kennedy function, combined with other more traditional assumptions such as constant returns to scale and social behavior represented in terms of optimization. B is Engel's law, and C is the declining share of raw material producing countries in world income. Professor Leontief does not say whether he accepts C, but I shall assume that he does. He states that the assumption "pertaining to the effects of technological change on the . . . production function is solely responsible for the theoretical conclusion . . ." I am not sure whether he means by this to bring into question the validity of Engel's law. For the sake of the argument, however, I shall assume unitary income elasticity of demand and maintain that even under these conditions I do not believe that Professor Leontief has succeeded in establishing his point.

Professor Leontief states: "Trade statistics show that the amounts of raw silk, quebracho bark, [etc.] moving in international trade . . . have been falling steadily. . . . They may eventually disappear from the list of internationally traded goods." While I doubt that trade statistics can provide information about the future, my answer to the ensuing rhetorical question is simple: of course I attribute such a decline to technical change. Far from presenting an "elaborate refutation" of such an explanation, I accepted it as my starting point. Professor Leontief forgets that I stated that "natural resources N should be thought of as a variable taking the place of raw materials of all kinds, so that an increase in it may represent recourse to further kinds of primary products." In fact, I even cited silk myself in stating: "It should . . . be kept in mind that the entities we are dealing with are aggregates, so that while particular methods of factor augmentation become exhausted, others take their places; e.g., silk gives way to rayon, and rayon to nylon." The Prebisch-Singer thesis is not concerned with raw silk and quebracho bark, but with primary products *in general.* The very same trade statistics that show a fall in raw silk show a rise in petroleum and pulpwood—two of the most important ingredients of synthetic materials. But Professor Leontief would have us believe that technological change will allow the miracle fabrics of the future to be produced by labor alone—like the emperor's new clothes in Hans Andersen's fairy tale!

Even granting that we might be heading toward such a resource-free state, I am unable to see what the hypothesis of factor augmentation has to do with the issue. As I pointed out in my paper, under a Leontief fixed-coefficients technology $F(AL, CN) = \min(AL, CN)$, *any* technical change can be uniquely represented as factor augmenting. If my conclusions "are wrong in the sense that they are derived from empirically unjustifiable assumptions," the same must apply to the Leontief input-output system itself.[1] Professor Leontief's objection is that "only infinite augmentation in the magnitude of the efficiency coefficient of [a] factor, as compared to other factors, could conceivably drive the level of its input toward zero." Why should infinite augmentation be excluded? There is nothing in the factor augmentation hypothesis to rule out, say, $A = 1$ and $C = 1/(1 - t)$ if $t \leqq 1$, $C = \infty$ if $t > 1$. What *does* rule this out is the Kennedy function. But the Kennedy function does not rule out, say, $A = A_0 e^{\alpha t}$, $C = C_0{}^{\gamma t}$, $L = L_0 e^{\lambda t}$, with $\gamma > \alpha + \lambda$, entailing an ultimately infinite augmentation in the efficiency coefficient of natural resources as compared with labor. If $\eta > 0$ and $\dot{\nu}/\nu < \gamma - \alpha - \lambda$, this would lead to declining and ultimately negligible (if not actually zero) imports of raw materials, as well as of raw material prices. If $\eta = 0$ and $\dot{r} = 0$, one would have from (2.51)

$$\frac{\dot{p}}{p} = \gamma + \frac{\nu f''(\nu)}{f'(\nu)} = \gamma + (\gamma - \alpha - \lambda)\frac{\nu f''(\nu)}{f'(\nu)} = \frac{\gamma}{\sigma(\nu)}\left[\sigma(\nu) - \left(1 - \frac{\alpha + \lambda}{\gamma}\right)\right]\left[1 - \frac{\nu f'(\nu)}{f(\nu)}\right]$$

(where $f(\nu) = f(CN/AL) = F(1, CN/AL)$); hence as long as the elasticity of substitution $\sigma(\nu)$ were less than $1 - (\alpha + \lambda)/\gamma$, and $\nu f'(\nu)/f(\nu) \to 0$ as $\nu \to \infty$,[2] the terms of trade of the primary producing

[1] In his celebrated 1953 article cited in my paper [35], Professor Leontief himself used the criterion of factor augmentation to compare the technologies of different countries, in arriving at his conclusion that "in any combination with a given quantity of capital, one man year of American labor is equivalent to . . . three man years of foreign labor." Amano, in his 1967 article cited in my paper, showed that for C.E.S. production functions with $\sigma < 1$, essentially equivalent results would be obtained with a more general criterion of technological change.

[2] With C.E.S. functions $\nu f'(\nu)/f(\nu) \to 0$ if and only if $\sigma < 1$. With variable elasticity of substitution we must require that for some $\bar{\sigma} < 1$, $\sigma(\nu) \leqq \bar{\sigma}$ for all ν. This is *not* the same as assuming that $\sigma(\nu) < 1$ for all ν, as Professor Leontief claims; the boundedness away from unity can be shown to be indispensable.

countries would eventually deteriorate, even if natural resources were in fixed supply. Neither the factor augmentation hypothesis nor the form of the Kennedy function precludes these results; what does preclude them is the asymptotic condition $\gamma = (\alpha + \lambda)/(1 + \eta)$.

The crucial question is therefore whether the differential equations lead asymptotically to the balanced growth solution corresponding to Professor Leontief's equation (5), rather than to a solution such as the above with $\gamma > \alpha + \lambda$. As he points out, the result $\gamma = (\alpha + \lambda)/(1 + \eta)$ follows directly from (5); and this is how it was derived in my paper as well. The whole problem is to justify (5), and this is what requires the heavy mathematical apparatus that Professor Leontief so deplores.

CONCLUDING REMARKS

WASSILY LEONTIEF

In discussing Professor Chipman's paper I commented on two interrelated but nevertheless distinct aspects of his contribution.

First, I endeavored to demonstrate through an elementary mathematical argument that his theoretical conclusion concerning the relationship between the demand for imported raw material and technological changes in an aggregative neoclassical dynamic follow simply and directly from the following assumptions:

1. The input-output relationships within the system can be described by a homogeneous production function admitting only a factor-augmenting type of technological change.
2. The relevant relationship between technological change and demand for imported raw materials are those prevailing when the system finds itself in a state of steady uniform expansion, referred to at times as the Golden Age.

In Chipman's argument, the Kennedy function plays only the auxiliary role of excluding a priori the possibility of negative rates of factor-augmentation, i.e., of either a or c being less than zero. This, incidentally, limits critically the implications of Chipman's observations that "under a Leontief fixed coefficient technology . . . *any* technical change can be uniquely represented as factor augmenting." In describing technological change in terms of changes in the vector of input coefficients, I am prepared to take account of increases as well as of reductions in the magnitudes of individual coefficients. Chipman's Kennedy function, on the other hand, excludes the possibility of an increase in magnitude of any input coefficients under conditions prevailing in the Golden Age: each one of them must either fall or remain constant.

The dynamic aggregative utility function (permitting the application of Pontryagin's maximum principle that makes up the bulk of Chipman's paper) serves—so far as I am able to judge—the sole purpose of justifying the second of the aforementioned assumptions, that is, the reference to the Golden Age. Since the strict and narrow specification of the shape of that utility function seems to be dictated by requirements of mathematical convenience, rather than consideration of empirical relevance, this part of Chipman's argument can contribute very little to an explanation of actual relationships between technological change and the demand for raw materials.

The second point raised in my comments is that of the relationship of Chipman's theoretical constructs to observed facts. He doubts that "trade statistics cannot provide information about the future." What else could? Convenient a priori assumptions?

My reference to instances in which technological change has obviously led to spectacular reduction in the demand for a raw material—without appreciable increase in its supply price—was intended to justify a closer inquiry into the treatment of technological change within the framework of the neoclassical growth model. Since, at least in Chipman's formulation, this theory excludes the possibility of such phenomena, I conclude that it does not provide a viable conceptual framework for empirical inquiry.