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DYNAMIC PROPERTIES OF A CONDENSED VERSION OF THE WHARTON MODEL

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1 INTRODUCTION

THE modern econometric model-building approach to the analysis of business-cycle phenomena is based, at least in part, on the Wicksell-Slutsky-Frisch-Kalecki hypothesis that business cycles can be adequately rationalized as the response of a stable dynamic system to random disturbances. According to this explanation of economic fluctuations, the lag structure incorporated into the econometric model provides the mechanism by which the disturbances are averaged to produce cyclical oscillations. The disturbances provide the energy essential in keeping the system from settling down to a steady-state equilibrium path. Although this view of economic fluctuations has not gone unchallenged, it remains a conceptually important element in the theory of business cycles.

Until recently, however, the extent to which the econometric models constructed over the past several decades are consistent with this theory of business cycles has not been rigorously considered. Moreover, the studies that have been concerned with this problem have, for the most part, relied on simulation techniques to generate experimental series that are then compared with observed series. In this paper, an analytical technique based on the spectral representation of a stochastic process is used to determine if one of these models, a condensed version of the Wharton Model, exhibits cyclical properties similar to those observed in aggregate economic time series.

In the next section, spectrum estimates based on series of postwar quarterly observations on several important macroeconomic variables are introduced to indicate the nature of business-cycle variations in these series. In Section 3, the spectrum-analytic approach to the determination of the dynamic properties of a linear econometric model is considered. A modified version of the Wharton Model, introduced in Section 4, is then analyzed, using the spectrum-analytic approach. The real-sector results are summarized in Section 5, and the complete system is examined in Section 6. The main results of the analysis are summarized in the concluding section.

2 THE NATURE OF BUSINESS CYCLES

A PRELIMINARY question that needs to be considered in connection with the use of spectrum-analytic techniques to study business-cycle phenomena is the extent to which business cycles manifest themselves in estimates of the power spectra of various aggregate economic time series. The presumption of business-cycle analysts is that economic variables exhibit sufficiently large and regular fluctuations to motivate a detailed study of this particular component. However, if spectrum estimates indicate that detrended economic variables are not serially correlated or, as Granger [8] suggests, that power decreases smoothly with frequency, then the spectrum estimates would not provide a meaningful basis for the isolation of business-cycle variations for special investigation. Several series are analyzed here in an attempt to shed some light on this issue and to provide a basis for comparison with the dynamic properties of the Wharton Model derived subsequently.

The spectrum densities of several of the endogenous variables in the Wharton Model have been estimated using quarterly observations for the period 1951–65. These variables include purchases of consumer durables, purchases of consumer nondurables, purchases of consumer services, investment in plant and equipment, nonfarm residential construction, inventory investment, and gross national product less government expenditure.[1] All variables are expressed in terms of constant (1958) dollars. Since these series are dominated by trends over the sample period, quarterly changes and deviations from a linear trend

[1] The reason for excluding government expenditure from gross national product is that government expenditure is an exogenous variable in the Wharton Model, whereas gross national product less government expenditure is endogenous to the system.

estimated by ordinary least squares were analyzed. This choice of detrending procedures is to a certain extent arbitrary. Indeed, relative rates of change or deviations from an exponential trend might be equally appropriate for this analysis. But given the relatively short sample period under investigation here, the differences between the arithmetic and logarithmic detrending methods is not very great. Moreover, the variance of the relative rate-of-change series tends to decrease over the sample period, especially after the 1960–61 recession. This phenomenon is particularly apparent in the analysis of growth rates based on data beyond the 1951–65 sample period employed in this study. This apparent nonstationariness militates against the use of the logarithmic forms of detrending.

The spectrum estimates obtained from the quarterly changes in the consumption and investment series are shown in Charts 2.1–2.6, and the estimates for the deviations from a linear trend are shown in Charts 2.7–2.12. The three curves plotted on a logarithmic scale correspond to the estimates obtained using a Parzen window, with the truncation points 10, 20, and 40.[2] These estimates of the spectrum density have been normalized so that the expected value of the estimate is 0.5 for a sequence of uncorrelated random variables. The estimates range from zero to one-half of a cycle per quarter in steps of $1/200$ of a cycle per quarter. The band of frequencies corresponding to the 40-month business cycle is centered on $15/200$ of a cycle per quarter (i.e., $200/15$ quarters per cycle $\times$ 3 months per quarter $=$ 40 months per cycle).

For ease of interpretation, two vertical lines corresponding to the 5-year and 2.5-year frequencies of oscillation are drawn on these

[2] The estimation procedure that was used is described in Parzen [17]. The form of the spectrum-estimator employed here is

$$s(\omega) = 0.5 \sum_{s=-m}^{m} \lambda(s)\hat{\rho}(s) \cos \omega s$$

where $\hat{\rho}(s)$ is an estimate of the autocorrelation function of the process defined by

$$\hat{\rho}(s) = \sum_{t=1}^{n-|s|} x(t + |s|)x(t) \Big/ \sum_{t=1}^{n} x(t)^2$$

where $x(t)(t = 1, 2, \ldots, n)$ is the sample sequence with its sample mean removed. The lag window, $\lambda(s)$, is defined by

$$\lambda(s) = \begin{cases} 1 - 6s^2(1 - |s|/m)/m^2 & 0 \leq |s| \leq m/2 \\ 2(1 - |s|)/m^3 & m/2 \leq s \leq m \\ 0 & |s| \geq m \end{cases}$$

CHART 2.1

Consumer Durables, Quarterly Changes
($\delta f = 1/200\ c/q$)

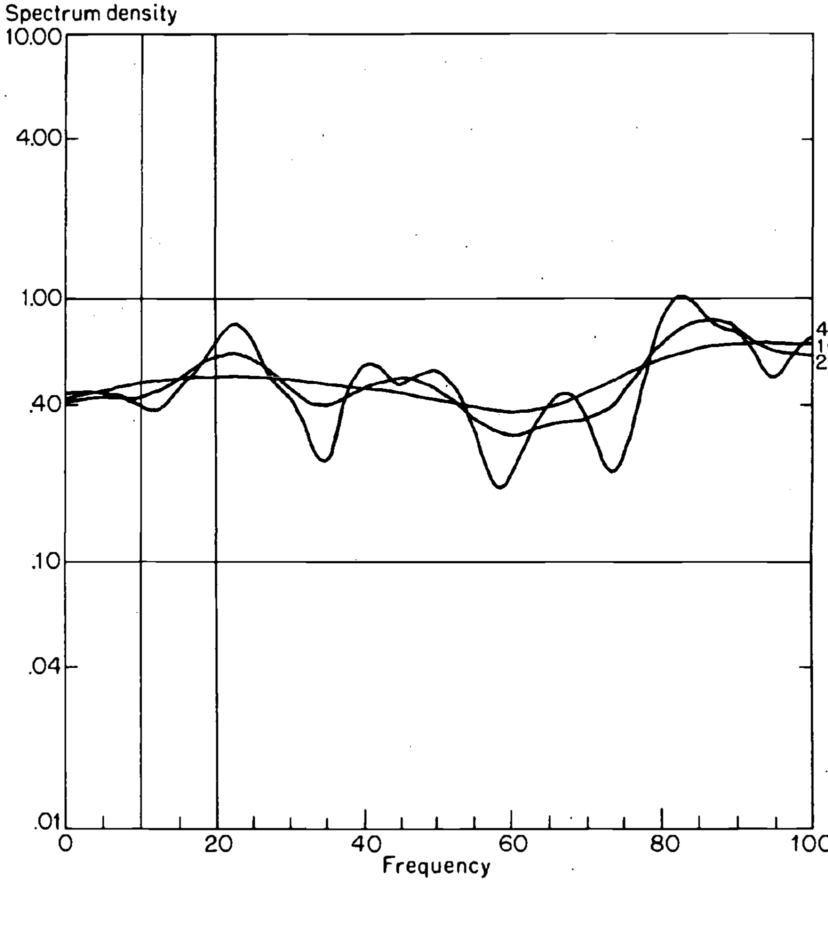


CHART 2.2

Consumer Nondurables, Quarterly Changes
($\delta f = 1/200\ c/q$)

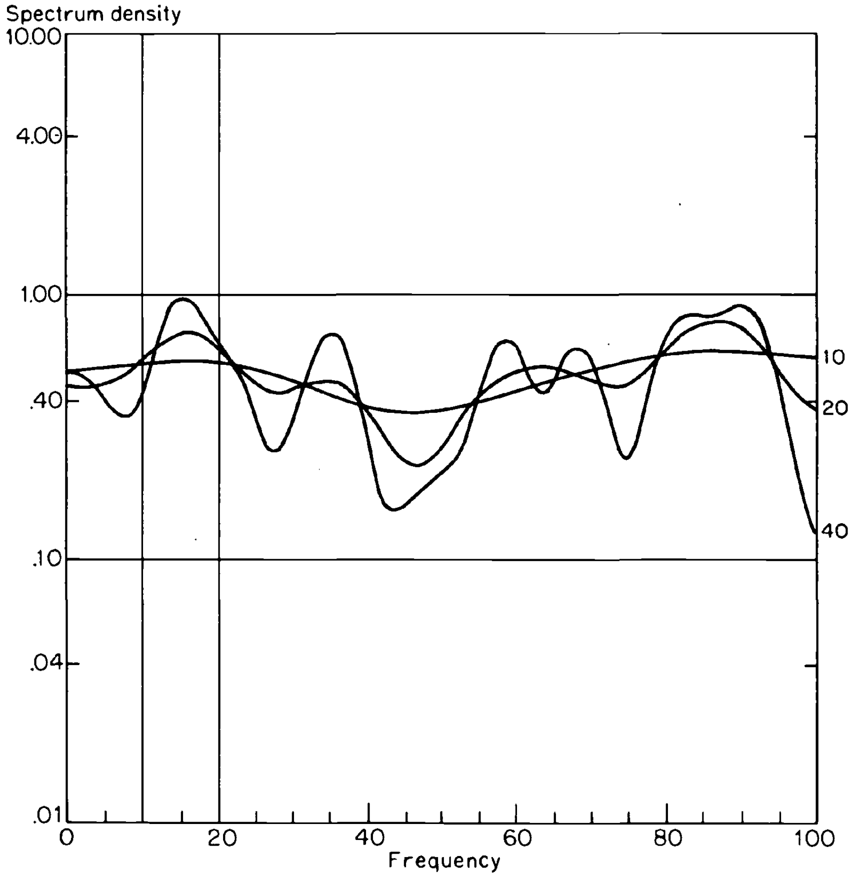


CHART 2.3

Consumer Services, Quarterly Changes
($\delta f = 1/200$ c/q)

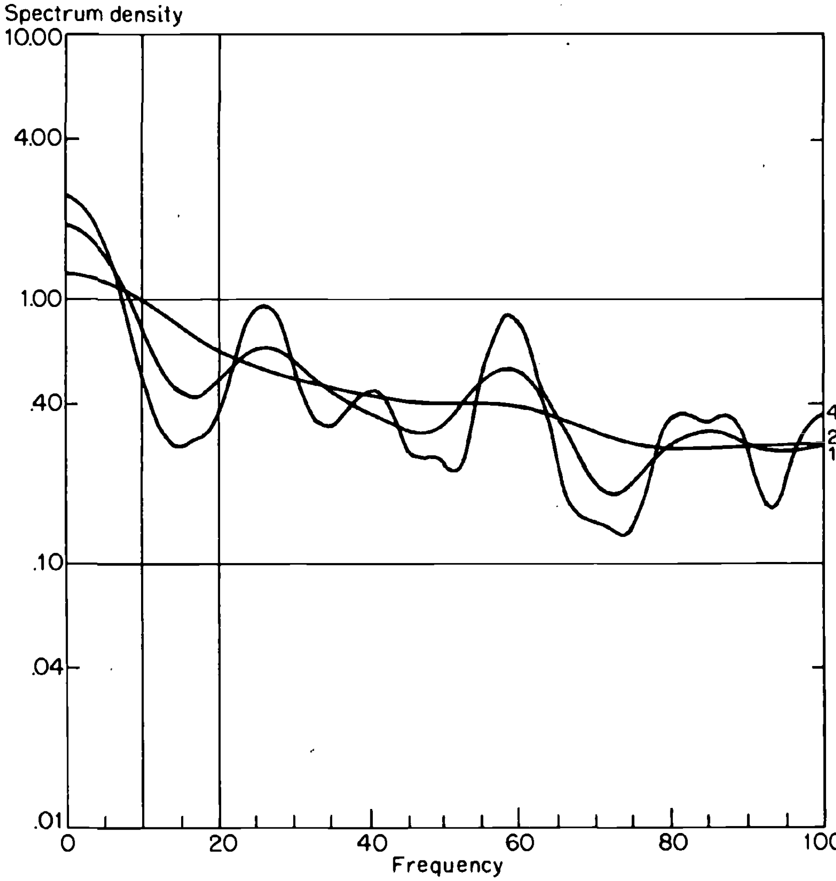


CHART 2.4

Plant and Equipment Investment
($\delta f = 1/200$ c/q)

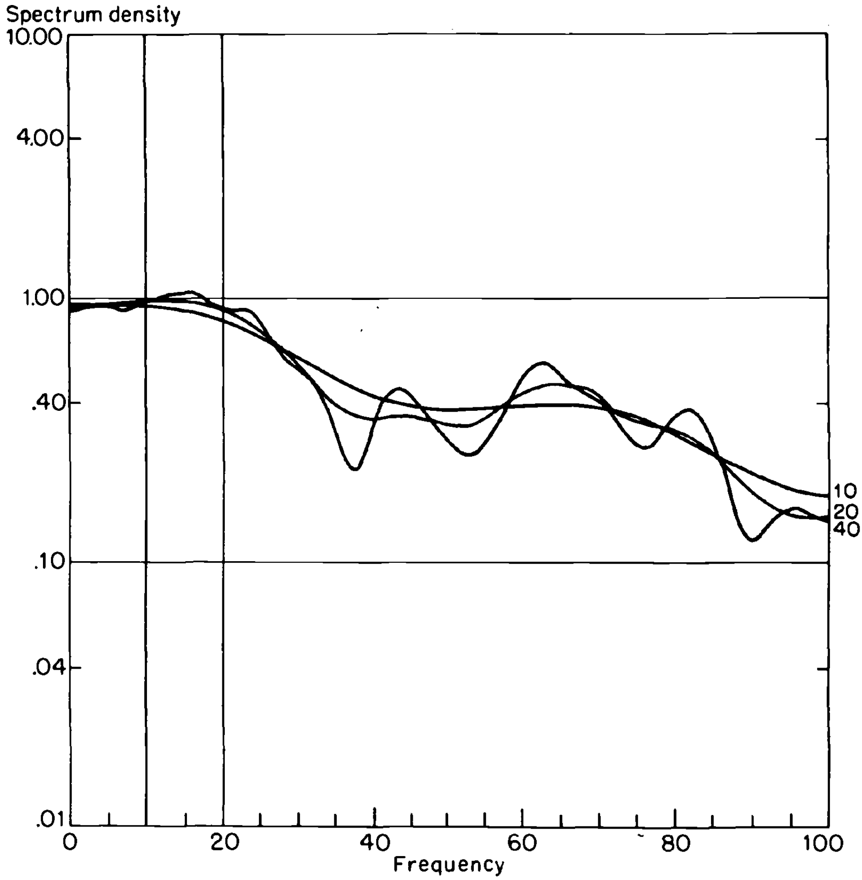


CHART 2.5

Residential Construction, Quarterly Changes
($\delta f = 1/200\ c/q$)

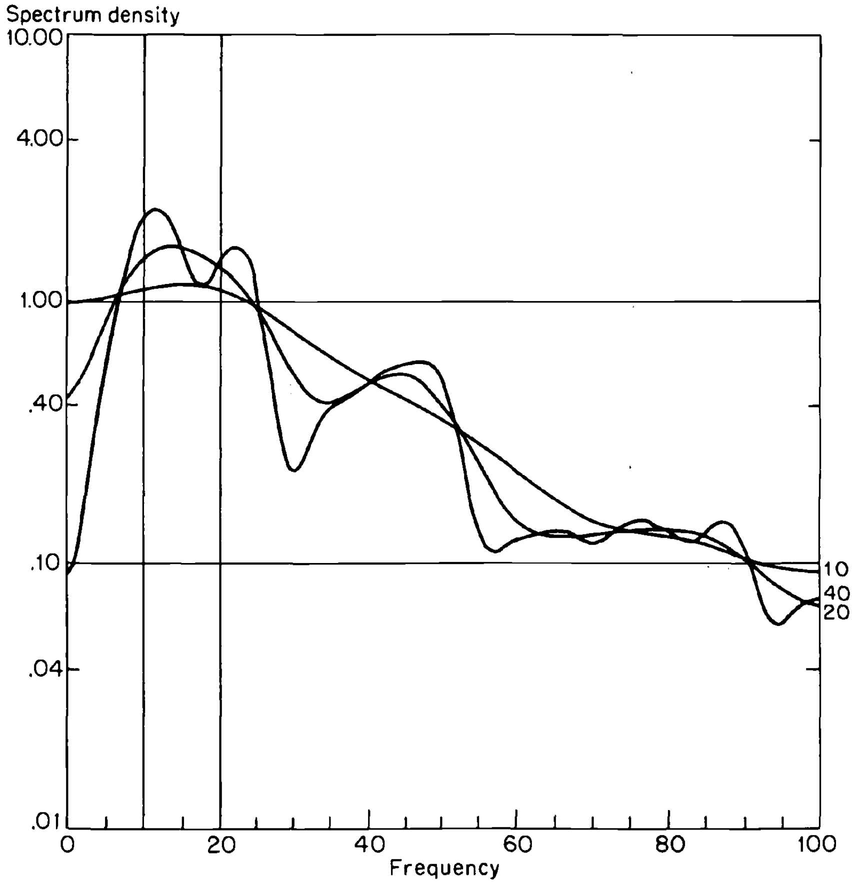


CHART 2.6

Inventory Investment
($\delta f = 1/200\ c/q$)

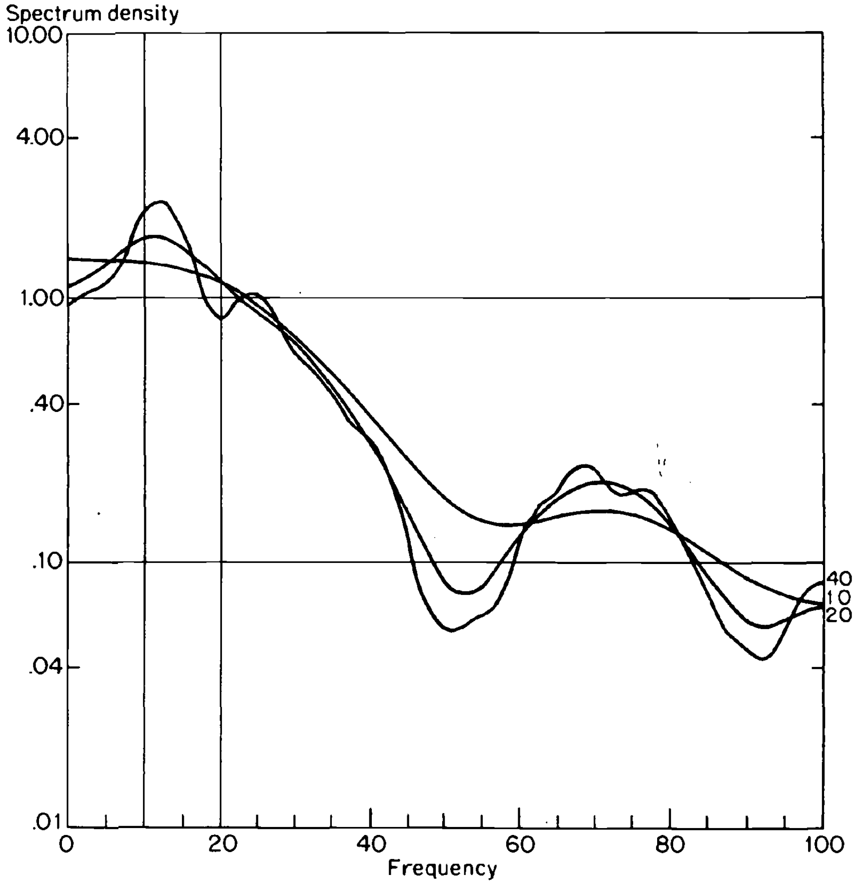


CHART 2.7

Consumer Durables, Linear Detrend
($\delta f = 1/200$ c/q)

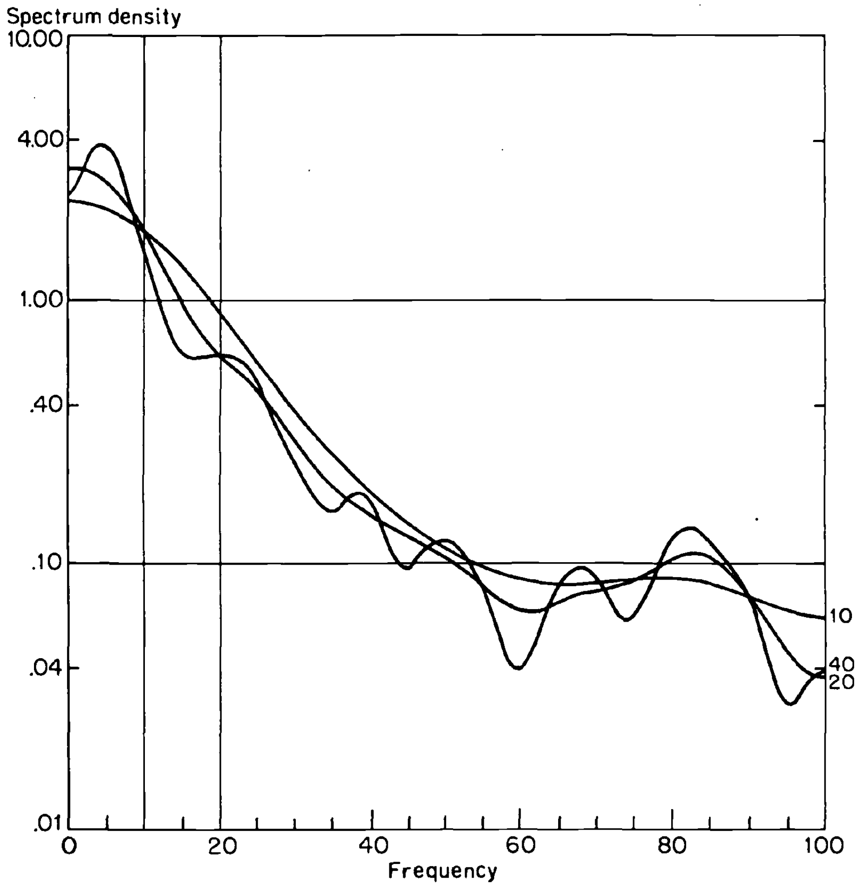


CHART 2.8

Consumer Nondurables, Linear Detrend
($\delta f = 1/200$ c/q)

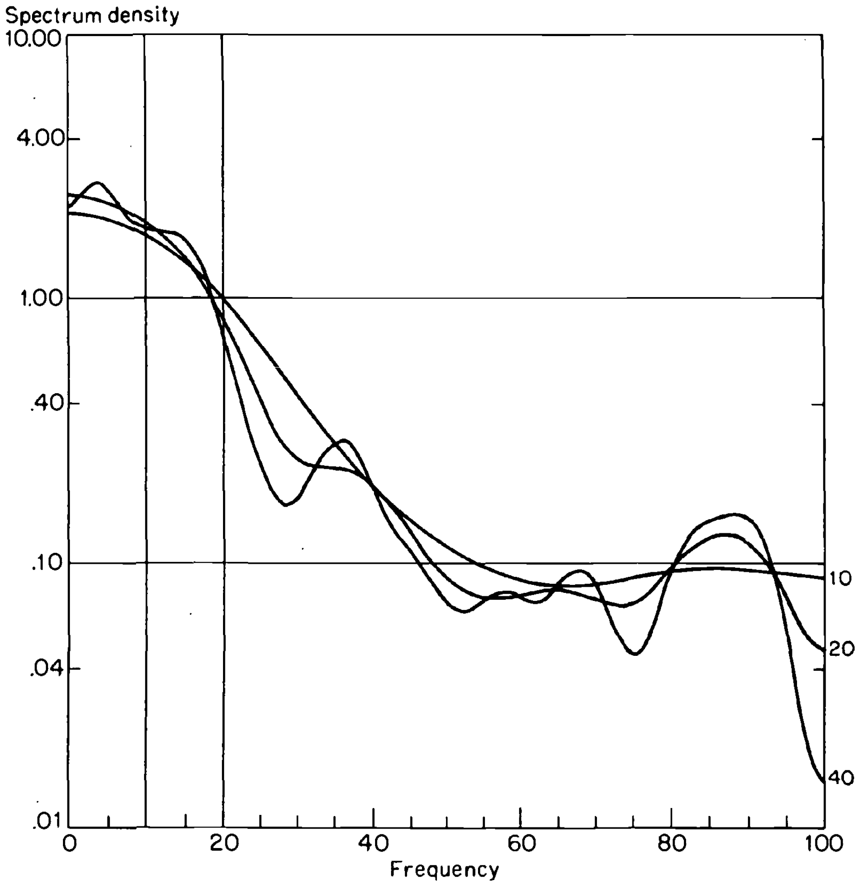


CHART 2.9

Consumer Services, Linear Detrend
(δf = 1/200 c/q)

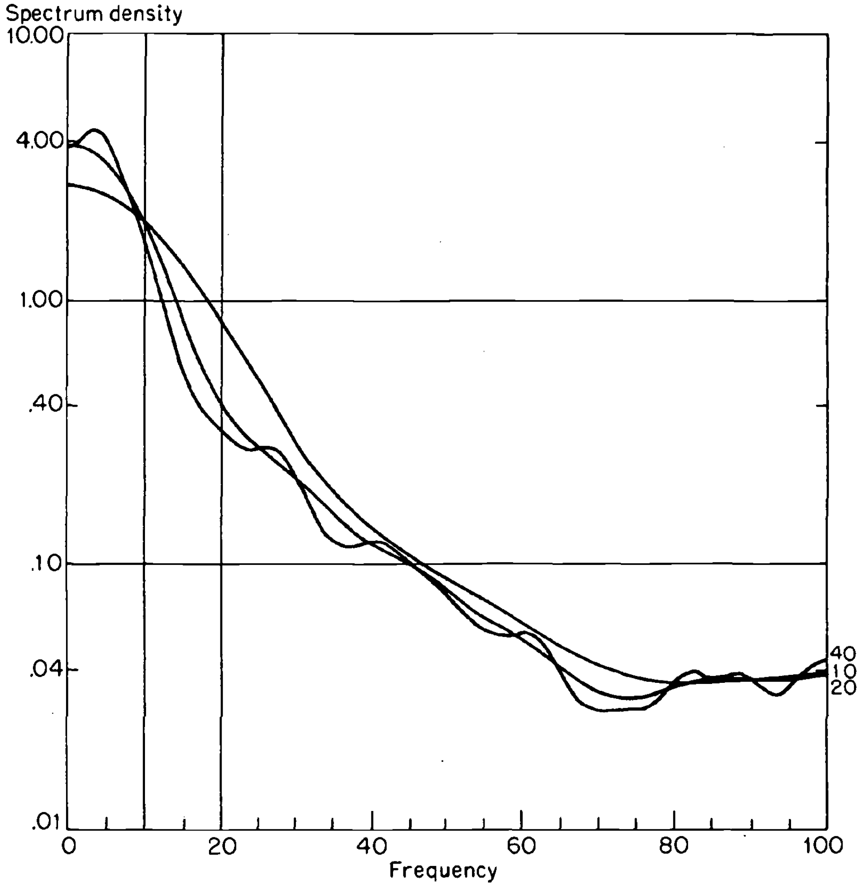


CHART 2.10

Plant and Equipment Investment, Linear Detrend
($\delta f = 1/200$ c/q)

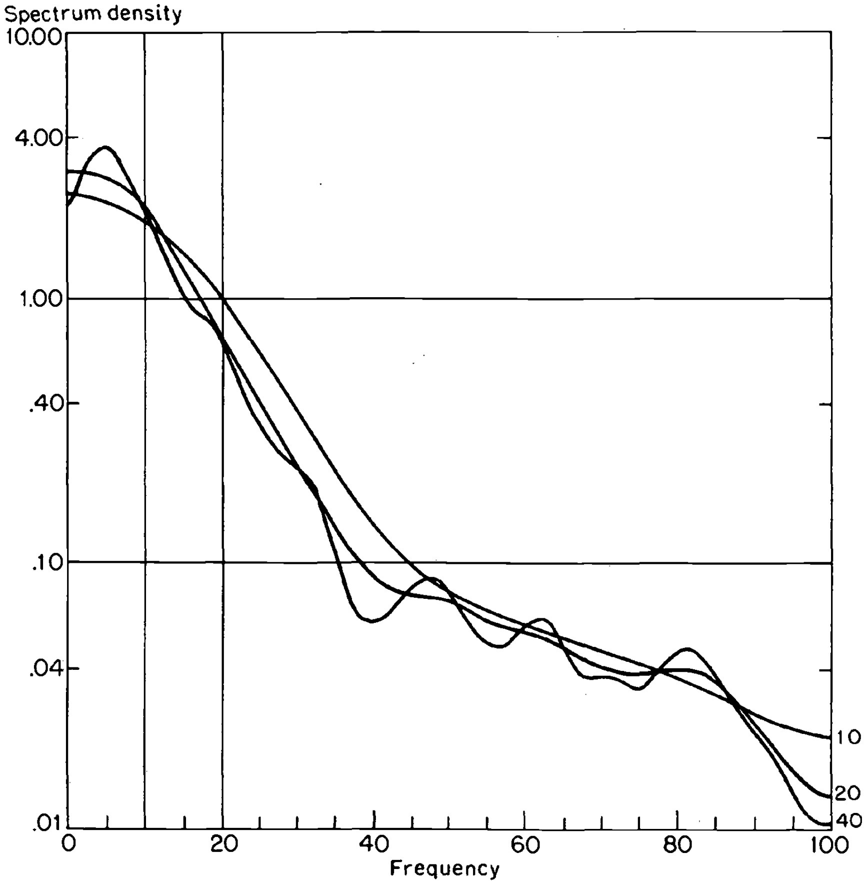


CHART 2.11

Residential Construction, Linear Detrend
($\delta f = 1/200\ c/q$)

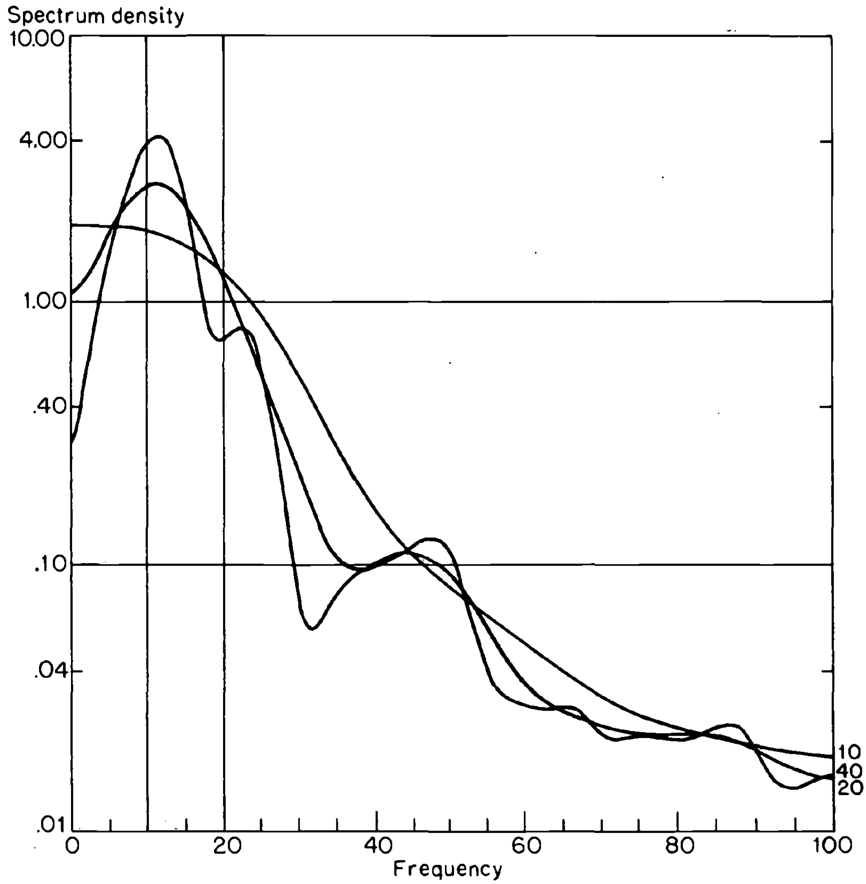
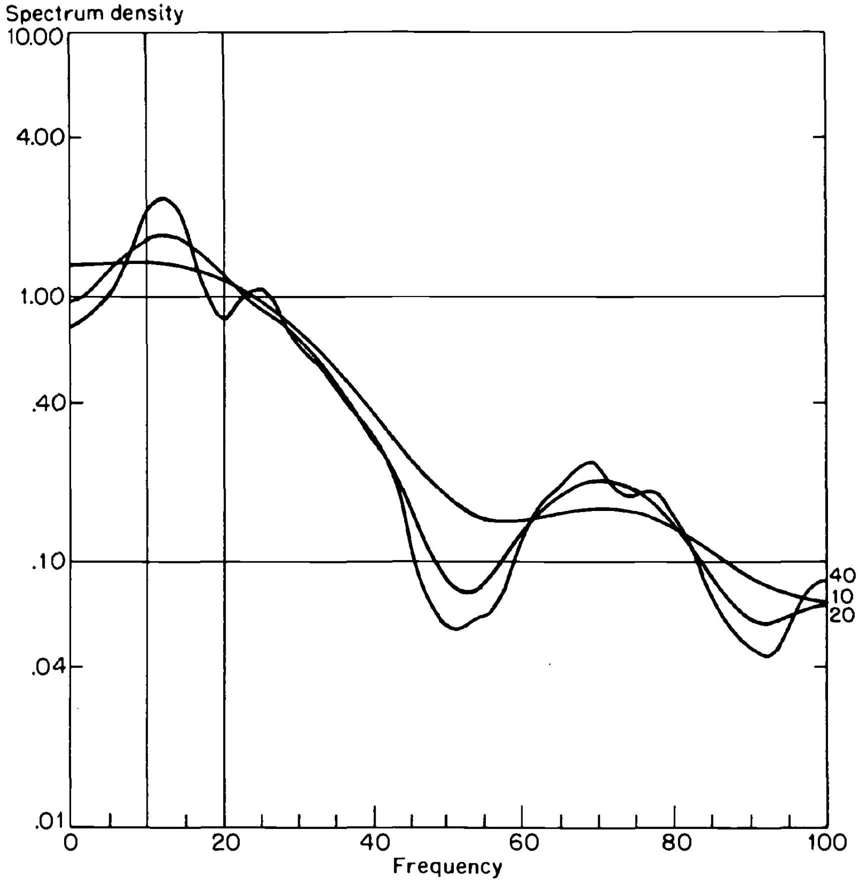


CHART 2.12

Inventory Investment, Linear Detrend
($\delta f = 1/200$ c/q)



diagrams. If the spectrum estimate exhibits a relative peak in this band of frequencies, this indicates that there is an important business-cycle component in the original series. The absence of a concentration of power in the band of frequencies corresponding to the business cycle means that the spectrum estimates do not provide a meaningful basis for the isolation and analysis of business-cycle variation in the original series. An examination of the spectrum estimates of the first-differenced series indicates that relative peaks emerge in the range of frequencies corresponding to 23 to 55 months per cycle. The rather pronounced peaks in the spectra of the investment series occur in the low-frequency end of this range (38–55 months per cycle), while the weaker peaks in the consumption series appear at higher frequencies. These relative concentrations of power, particularly in the residential construction and inventory investment series, indicate that there is a tendency for these series to oscillate with some degree of regularity.

Turning to the deviations from a linear trend, the spectrum estimates shown in Charts 2.7–2.12 indicate that, except at the very low and very high frequencies, the power decreases monotonically with frequency.[3] The two exceptions to this general pattern are residential

[3] The rather striking disparity between the first-differenced and linear-detrended consumption series may appear to be surprising, since the two transformations are often viewed as two ways to eliminate a trend from the data. However, the two transformations are expected to yield substantially different results. Consider, for example, the two series

$$\epsilon_t = x_t - x_{t-1}$$

$$\eta_t = x_t - a - bt$$

where ϵ_t is the differenced series and η_t is the linear detrended series. It is clear that

$$\epsilon_t = \eta_t - \eta_{t-1} + b$$

From this relationship, it follows that the power spectra of the two processes are related by

$$S_\epsilon(\omega) = 2(1 - \cos \omega)S_\eta(\omega)$$

This shows that if the deviations from trend, η_t, are serially uncorrelated so that $S_\eta(\omega)$ is constant, the spectrum of the first differences will increase monotonically with frequency. Conversely, if the first differences are uncorrelated, the deviations from trend are generated by a nonstationary process with a (pseudo-) spectrum that decreases monotonically with frequency. It will be noted that the spectrum estimates for the consumption series are consistent with these relationships. The reason for including the spectrum estimates of both the first-differenced and linear-detrended series is that it was not clear at the outset which of the two detrending procedures was particularly appropriate. In retrospect, the first-difference transformation appears to provide more interesting results from the point of view of business-cycle analysis.

construction and inventory investment, which both exhibit relative peaks at a frequency corresponding to a 50-month cycle. For these two investment series, the deviations from a linear trend, as well as the quarterly changes, tend to exhibit fairly regular business-cycle oscillations, judging from the spectrum estimates.

The spectrum estimates for the gross national product series are shown in Charts 2.13–2.16. The spectrum estimates of the first-differenced series shown in Chart 2.13 indicate that there are relative concentrations of power around 15.4 and 8.7 quarters per cycle. This is not surprising in view of the fact that *GNP* less government expenditure does not differ substantially from the sum of the consumption and investment series analyzed above. The spectrum estimates of the linear detrended *GNP* series shown in Chart 2.15 indicate a relative concentration of power around the frequency corresponding to a 50-month cycle. This again is not surprising in view of the fact that the detrended residential construction and inventory investment series exhibit concentrations of power around this same frequency. The low-frequency portions of the spectrum estimates of the two *GNP* series are shown on an expanded scale in Charts 2.14 and 2.16. In these two diagrams, the frequency axis ranges from zero to one-eighth of a cycle per quarter, so that these diagrams correspond to an enlargement of the first quarter of the diagrams to the left of them. These are included here to emphasize the fact that the business-cycle variation found in the components of gross national product, particularly in the investment series, is not transmitted to the aggregate in unattenuated form. Indeed, the aggregate exhibits much less cyclical variation than might be expected.

These estimates indicate, from a descriptive point of view, the reality of three- to five-year business cycles, particularly in the investment series. However, the statistical significance of these oscillations remains to be considered. The rather limited number of observations available creates several problems in connection with the construction of significance tests for spectrum estimates. Nevertheless, it may be useful to consider an approximate test of the significance of the variability that does emerge in the spectrum estimates.

Two tests of significance immediately suggest themselves. The first test is concerned with the null hypothesis that the spectrum esti-

CHART 2.13

Gross National Product, Quarterly Changes
($\delta f = 1/200\ c/q$)

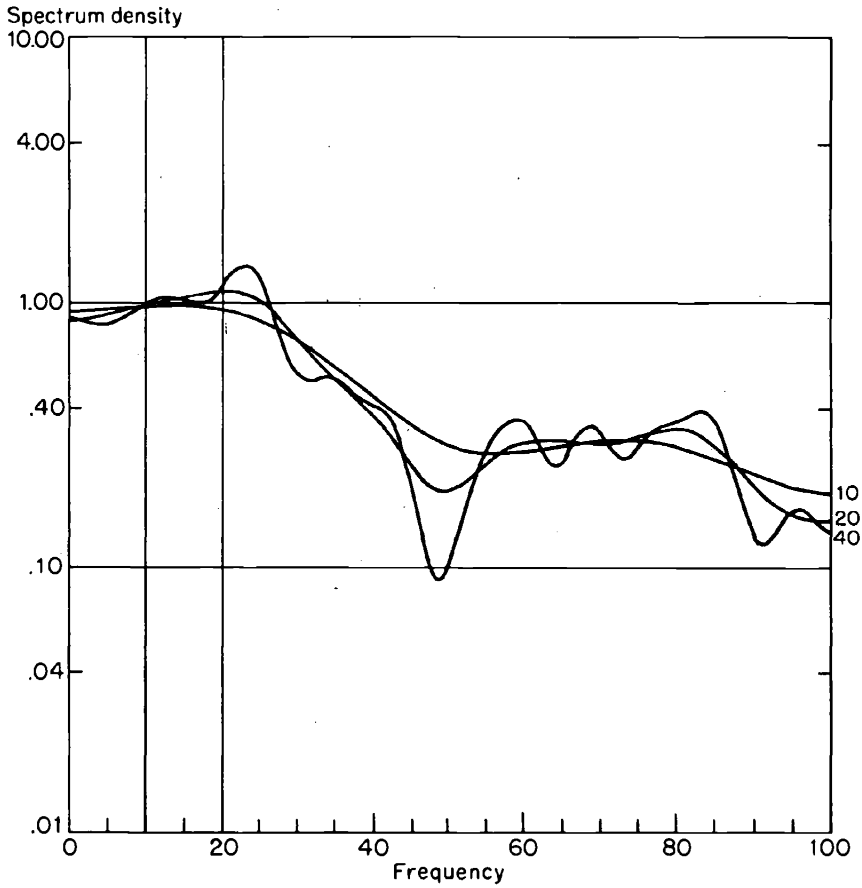


CHART 2.14

Gross National Product, Quarterly Changes
($\delta f = 1/800\ c/q$)

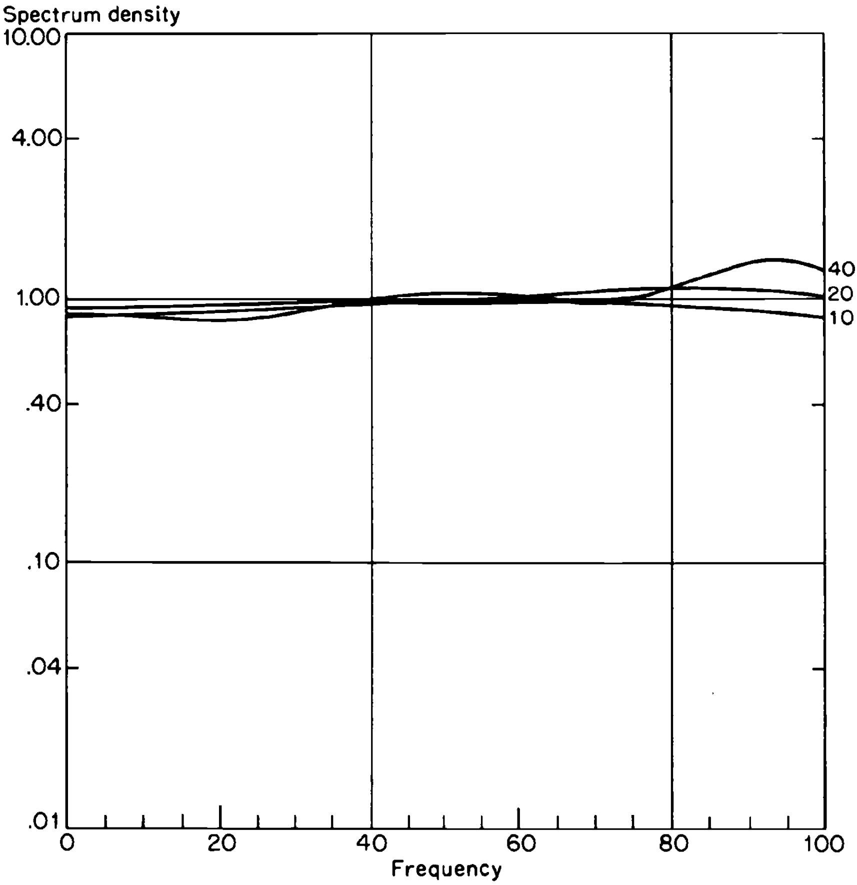


CHART 2.15

Gross National Product, Linear Detrend
($\delta f = 1/200$ c/q)

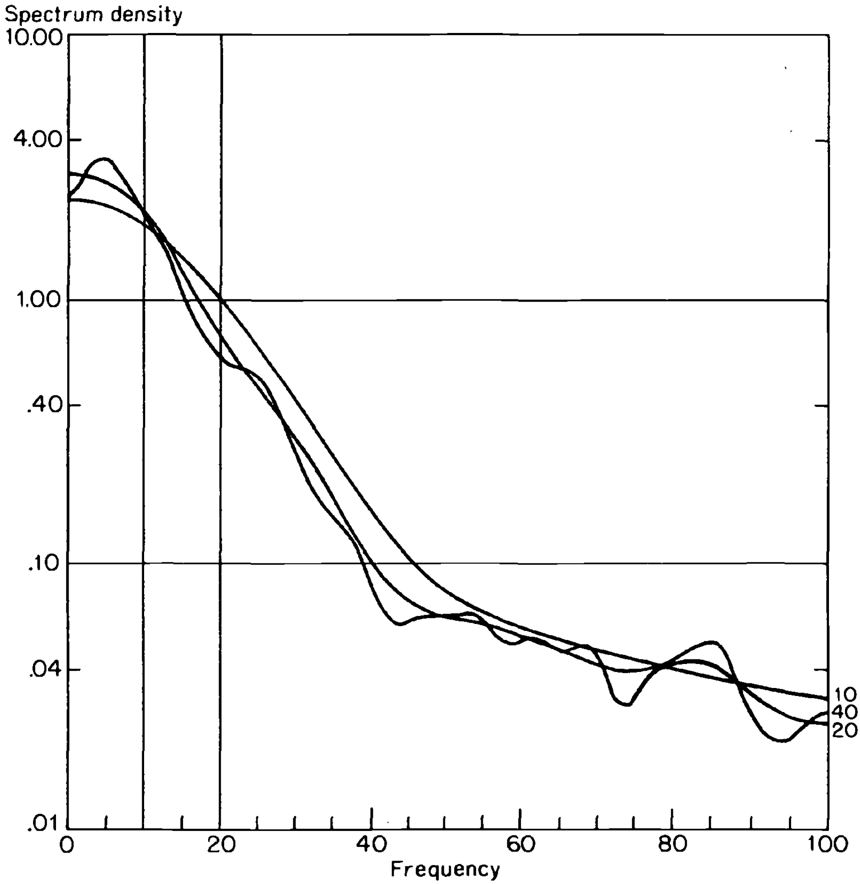
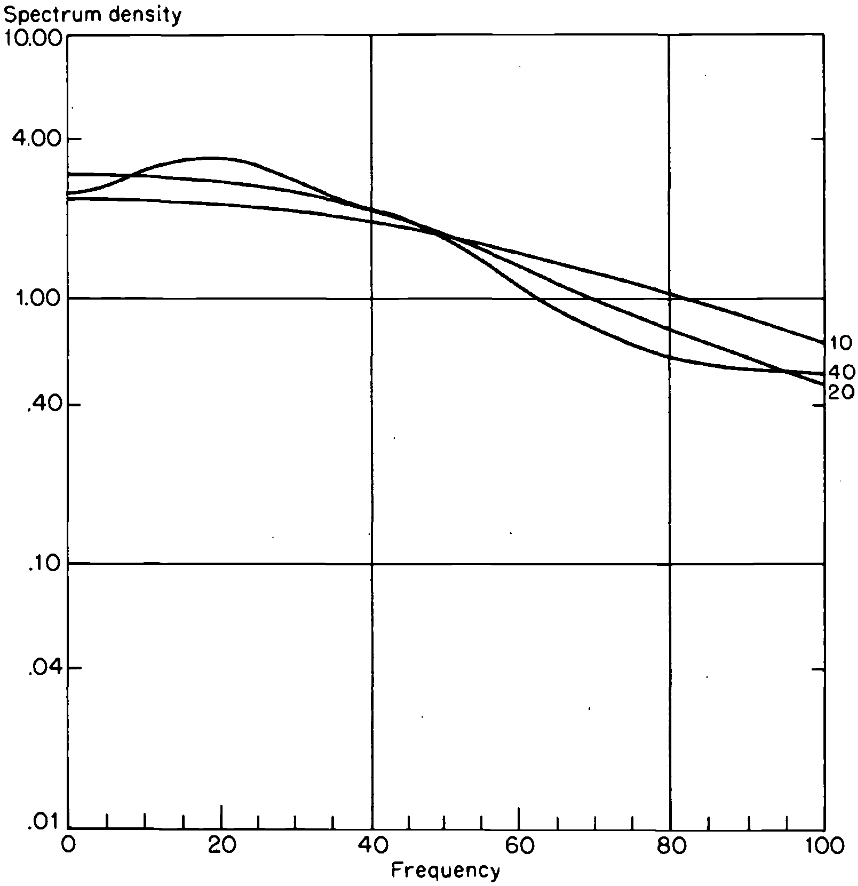


CHART 2.16

Gross National Product, Linear Detrend
($\delta f = 1/800\ c/q$)



mates are not significantly different from a constant; i.e., the underlying series are serially uncorrelated. On the assumption that the series are normally distributed, the spectrum estimates are proportional to a Chi-square variate.[4] A $100(1-2\alpha)$ per cent confidence interval for independent, normally distributed random variables can be determined from

$$Pr\ \{\chi^2_{1-\alpha}(l) \leqslant l\hat{S}(f)/S(f) \leqslant \chi^2_{\alpha}(l)\}$$

where for the Parzen window $l = 4n/m$ where n denotes the number of observations and m is the truncation point of the estimate. The upper bounds of the confidence band for the spectrum estimates shown in Charts 2.1–2.16 are as follows:

	$m = 10$	$m = 20$	$m = 40$
$\alpha = 0.05$	0.76	0.88	1.05
$\alpha = 0.025$	0.82	0.97	1.20

Provided an estimate exceeds the appropriate value in this table, the null hypothesis is rejected. An examination of the spectrum estimates shown in Charts 2.1–2.14 indicates that the only series for which the null hypothesis is not rejected at the 95 per cent level are the first differences of purchases of consumer durables and the first differences of consumer nondurables purchases. In all other cases, the spectrum densities deviate significantly from the estimates that would be expected if the series were serially uncorrelated.

A second and more important test within the context of business-cycle analysis is concerned with the significance of the relative peaks that emerge in the estimates of the spectrum densities. A convincing test of the significance of the peaks that emerge in the band of frequencies corresponding to the business cycle is particularly difficult to construct with the short series that are analyzed here.[5] The reason for this is that in order to resolve a peak in the spectrum, three independent estimates of the spectrum are needed: one estimate centered on the peak and one on either side of the peak. Since the bandwidth of the Parzen estimate is approximately $1/m$, where m is the truncation

[4] For a derivation of the Chi-square approximation, see Jenkins [15].

[5] The following discussion is based on Hatanaka and Howrey [11], to which the reader is referred for a more detailed account.

point of the estimate (i.e., an estimate centered on f is an average of the true spectrum over the interval $f \pm 1/m$), it follows that estimates centered on frequency points separated by at least $2/m$ are independent.[6] If f_1, f_2, and f_3 denote the center frequencies of three estimates of the spectrum where $0 \leqslant f_1 < f_2 < f_3 \leqslant 1/2$, it follows that these three estimates will be independent if, and only if, the following inequalities are satisfied:

$$f_1 + 1/m \leqslant f_2 - 1/m$$

$$f_2 + 1/m \leqslant f_3 - 1/m$$

With $f_2 = 1/14$ of a cycle per quarter, which corresponds to a 42-month cycle, it follows from the first of these inequalities that a truncation point $m \geqslant 28$ is required in order for the estimate centered on $f_1 = 0$ to be independent of the estimate centered on $f_2 = 1/14$.

Even this degree of resolution is not completely satisfactory, however, because the estimate centered on zero is subject to a downward bias.[7] This downward bias will tend to inflate the ratio of the peak estimate to the estimate centered on zero and will therefore distort the significance test. In order to circumvent this problem, at least partially, the estimate centered on f_1 should be independent of the zero frequency. This requires that the additional inequality

$$\frac{1}{m} \leqslant f_1$$

be satisfied. With $f_2 = 1/14$, a truncation point $m \geqslant 42$ is required in order for the estimate centered on f_2 to be independent of the estimate centered on $f_1 = 1/42 cpq$, which is, in turn, independent of the power at the origin.

These considerations suggest that a rough indication of the significance of the business-cycle peaks that emerge in estimates obtained with a truncation point $m = 40$ can be obtained by comparing the peak estimate centered on f_2 with the estimates centered on $f_2 \pm 1/20$. The ratios of these spectrum estimates will have an F-distribution, provided the original process is normal. The critical value at the 95 per cent level for the F-distribution with $4n/m = 6$ degrees of freedom in both

[6] This definition of the bandwidth of the spectrum estimate is due to Jenkins [15].
[7] For a discussion of this bias, see Hannan [10] and Hatanaka and Howrey [11].

the numerator and the denominator is 4.28. Provided the ratios exceed this value, the peaks will be considered significant.

An application of this test to the spectrum estimates shown in Charts 2.1–2.14 indicates that the only significant peak, according to this criterion, is the one that appears in the residential-construction series. However, this peak emerges in the vicinity of 16.7 to 18.2 quarters per cycle, so that the estimate centered on the comparison frequency of 100 to 200 quarters per cycle is biased downward. Therefore, this evidence of a significant business-cycle component may be somewhat biased.

Since the relative peaks that do emerge in the estimates of the power spectra of these relatively short series are not statistically significant according to the usual significance tests, these relative maxima do not provide strong evidence of the reality of business-cycle phenomena. Indeed, the purist might argue that these results confirm Granger's contention [8] that spectrum analysis indicates that economic time series do not exhibit any complicated dynamic patterns of statistical significance. However, several studies of considerably longer economic time series indicate more strikingly the relative importance of business-cycle variation. For example, the estimates obtained by Adelman [1] in connection with a study of the long-swing hypothesis indicate concentrations of power around the business-cycle frequency band. The relative contribution of major and minor cycles to variations of the rate of growth of various economic variables is even more apparent in the estimates described by Howrey [12]. These studies of the long-term development of the United States economy, together with the results described here, provide fairly clear motivation for the study of business-cycle phenomena. Therefore, it seems reasonable to use these spectrum estimates as bench marks for comparison with the spectra implied by the Wharton Model.

3 DYNAMIC PROPERTIES OF ECONOMETRIC MODELS

ONCE the parameters of an econometric model have been estimated, the dynamic properties of the resulting system of equations are often of considerable interest. These dynamic properties are frequently inferred from the deterministic system obtained by suppressing the disturbance terms from the model. However, Frisch [6] and Kalecki [16], among others, have argued that the disturbance terms are essential elements in the theory of business fluctuations. Moreover, Haavelmo [9], Fisher [5], and others have shown that the solutions obtained when the disturbance terms are neglected may show widely different patterns from the solution sequences that include the disturbances. Realism demands that econometric models be regarded as stochastic rather than deterministic systems to reflect the randomness of the behavior of the decision-makers which the model purports to describe.

Implicit in this theory of macrodynamics is the assumption that observed series are generated by a stochastic process which can be written in implicit functional form as

$$G[\tilde{y}(t), \tilde{x}(t), \tilde{\epsilon}(t)] = 0 \tag{3.1}$$

In this system, $G[\cdot]$ is a vector of functions with vector-valued arguments $\tilde{y}(t) = [y(t), y(t-1), \ldots]$ where $y(t)$ is a vector of endogenous variables at time t, $\tilde{x}(t) = [x(t), x(t-1), \ldots]$ where $x(t)$ is a vector of exogenous variables, and $\tilde{\epsilon}(t) = [\epsilon(t), \epsilon(t-1), \ldots]$ where $\epsilon(t)$ is a vector of random variables. This system can be written in the usual form of a linear econometric model as

$$By(t) + \Gamma z(t) = \epsilon(t) + \eta(t) \tag{3.2}$$

where B and Γ are coefficient matrices and $z(t)$ is a vector which includes both the exogenous and the lagged endogenous variables. The important point to note here is that even if the model were correctly specified, in which case $\eta(t)$ would be identically zero, the disturbance vector $\epsilon(t)$ would still impart a random element to the vector of endogenous variables.

In order to determine the dynamic properties of the system (3.2), it is convenient to rewrite the system as

$$B(L)y(t) = C(L)x(t) + u(t) \tag{3.3}$$

where $B(L)$ and $C(L)$ are matrices of polynomials in the lag operator L, and $u(t) = \epsilon(t) + \eta(t)$ is the vector of the sum of the two sources of error. Provided this system is stable, i.e., the roots of the determinantal polynomial $|B(\lambda)| = 0$ lie outside the unit circle, the solution for $y(t)$

will converge to

$$y(t) = B(L)^{-1}C(L)x(t) + B(L)^{-1}u(t) \tag{3.4}$$

where $B(L)^{-1}$ is the inverse of the matrix of linear operators $B(L)$. It is clear from this solution of the system that the variability observed in the vector of endogenous variables is the result of variations in the exogenous variables and variations in the disturbance series. If the econometric model is consistent with the notion that business cycles are generated in response to random shocks administered to the system, then the second term in the solution should exhibit a cyclical response path.

The response of the system to the disturbance process can be conveniently described in terms of the spectral representation of the stochastic process $B(L)^{-1}u(t)$. For purposes of estimation of the parameters of the system (3.3), it is generally assumed that $u(t)$ is generated by a covariance stationary stochastic process. If this condition is satisfied, then $u(t)$ possesses the spectral representation[8]

$$u(t) = \int_{-\pi}^{\pi} e^{i\omega t} dU(\omega) \tag{3.5}$$

where

$$U(\omega) = \frac{1}{2\pi}\{\omega u(0) - \sum_{t \neq 0} e^{-i\omega t}u(t)/it\} \tag{3.6}$$

The spectral representation of the stochastic response of the system, denoted by $\hat{y}(t)$, is obtained by substituting (3.5) into (3.4):

$$\hat{y}(t) = B(L)^{-1}\int_{-\pi}^{\pi} e^{i\omega t} dU(\omega) \tag{3.7}$$

An interchange of the order of the operations in (3.7) yields

$$\hat{y}(t) = \int_{-\pi}^{\pi} e^{i\omega t} T(\omega) dU(\omega) \tag{3.8}$$

where $T(\omega) = B(e^{-i\omega})^{-1}$. This shows that the spectral representation of $\hat{y}(t)$ is

$$d\hat{Y}(\omega) = T(\omega)dU(\omega) \tag{3.9}$$

[8] For a detailed discussion of spectral representations, the reader is referred to Yaglom [21].

Finally, the spectrum matrix of the endogenous variables of the system is given by

$$(3.10)\qquad \begin{aligned} F_{\hat{y}\hat{y}}(\omega) &= E[d\hat{Y}(\omega)d\hat{Y}^*(\omega)] \\ &= E[T(\omega)dU(\omega)dU^*(\omega)T^*(\omega)] \\ &= T(\omega)E[dU(\omega)dU^*(\omega)]T^*(\omega) \\ &= T(\omega)f_{uu}(\omega)T^*(\omega) \end{aligned}$$

In this final expression, $f_{uu}(\omega)$ denotes the spectrum matrix of the disturbance process $(= E[dU(\omega)dU^*(\omega)])$, T^* denotes the conjugate transpose of T, and E is the expectation operator.[9]

Up to this point it has been assumed that the system is stable. If this condition is not satisfied, the vector of endogenous variables does not converge to the expression given in (3.4) but rather is given by

$$(3.11)\qquad y(t) = P(t) + B(L)^{-1}C(L)x(t) + B(L)^{-1}u(t)$$

where $P(t)$ is a vector of functions of the form

$$(3.12)\qquad p_j(t) = k_{j1}\lambda_1^t + k_{j2}\lambda_2^t + \ldots + k_{jn}\lambda_n^t$$

where the λ_i are the characteristic roots of the system of equations and the k_{ji} are constants determined by the initial conditions.[10] The existence of roots greater than unity in absolute value thus imparts a trend (if $\lambda_i > 1$) or an explosive oscillation (if $|\lambda_i| > 1$ and λ_i is complex-valued) to the endogenous variables in the system. In this case, the spectrum matrix given in (3.10) still provides a useful description of the process as Chow and Levitan [4] and Quenouille [18] have shown.

For an analysis of the dynamic properties of the system, the power spectra given by the main diagonal elements of the spectrum matrix are particularly relevant. These power spectra characterize the response of the endogenous variables of the system to the disturbance process. Provided these power spectra exhibit relative concentrations of power at the frequencies corresponding to the business cycle, the model is consistent with the impulse-response theory of cyclical fluc-

[9] For an alternative derivation of the spectrum matrix in terms of the characteristic roots of the system, see Chow [2].

[10] For purposes of exposition, it is assumed that none of the roots are repeated. If there were repeated roots, one or more of the k_{ji} in equation (3.12) would be polynomials in t.

tuations. It should be emphasized that the absence of a relative concentration of power in the business-cycle frequency band of the spectra of the stochastic response does not imply that the system will not exhibit cyclical fluctuations. But if the model is correct, the absence of business-cycle power in the spectrum means that whatever business-cycle oscillations are observed in the endogenous variables are due not to the internal dynamics of the system but rather to corresponding oscillations in the exogenous variables or the disturbance process. In this case, the lag structure of the model by itself is simply not sufficiently complicated to explain the existence of business-cycle variations in the data.[11]

4 A CONDENSED VERSION OF THE WHARTON MODEL

THE macroeconometric model that is analyzed here is a forty-five equation version of the Wharton Econometric Forecasting Unit Model. The parameters of the model were estimated using two-stage least-squares techniques on quarterly data spanning the period 1948–65. For a detailed discussion of the specification and estimation of this system, the reader is referred to Rahman [19, Chapter 3]. For purposes of analysis, the nonlinear equations were linearized using the technique described by Goldberger [7]. The linear approximation that was used in the following analysis is given immediately below each of the original nonlinear equations in the system. An asterisk denotes an exogenous variable.

I. CONSUMPTION

$$(4.1)\quad Cd = 42.0689 + 0.1381Y - 41.7527Pd/Pc - 0.0630Kd_{-1} - 0.3037Un + 1.4236Cr^{*} - 2.2453Ds^{*} - 0.7017Cd_{-1}$$

[11] It should be emphasized that the spectrum matrix implied by a dynamic econometric model is used here exclusively for the purpose of describing the dynamic properties of the stochastic system. The use of the spectrum matrix implied by a system of equations to validate a model is considered in Howrey and Kelejian [14].

(4.1a) $$Cd = 0.1381Y - 0.0630Kd_{-1} - 0.3037Un - 0.7017Cd_{-1} - 41.7527Pd + 41.7527Pc$$

(4.2) $$Cn = 14.4115 + 0.2056Y + 0.1093 \sum_{i=1}^{4} Cn_{-i}$$

(4.3) $$Cs = -3.4493 + 0.0568Y + 0.2225 \sum_{i=1}^{4} Cs_{-i}$$

Cd	Purchases of consumer durables
Y	Personal disposable income
Pd	Implicit price deflator for consumer durables
Pc	Implicit price deflator for consumption
Kd	Stock of consumer durables
Un	Unemployment in per cent
Cr	Dummy variable for consumer credit conditions
Ds	Dummy variable for shortages in supply of automobiles
Cn	Purchases of consumer nondurables
Cs	Purchases of consumer services

II. INVESTMENT

(4.4) $$Ip = 0.2856 - 3.6972iL_{-1} - 0.0133Kp_{-1} + 0.1690X^{p}{}_{-1} + 0.004 \sum_{i=2}^{5} X^{p}{}_{-i}$$

(4.5) $$Ih = -45.0367 + 0.0450Y + 38.6202(Pr^{*}/Ph)_{-3} + 1.8152(iL - iS^{*})_{-3} + 0.2745 \sum_{i=1}^{5} MRGE^{*}{}_{-i}$$

(4.5a) $$Ih = 0.0450Y - 38.6202Ph_{-3} + 1.8152iL_{-3}$$

(4.6) $$\Delta Ii = -55.1446 + 0.2019S^{p} - 0.1992Ii_{-2} + 0.7100\Delta U_{-1} + 3.2084STR^{*}$$

Ip	Private investment in plant and equipment
iL	Average yield (per cent) on corporate bonds

Kp	Stock of investment in plant and equipment
X^p	*GNP* in the private sector
Ih	Private investment in nonfarm housing
Y	Personal disposable income
Pr	Price index for rent
Ph	Implicit price deflator for residential structures
iS	Average yield (per cent) on 4- to 6-month commercial paper
$MRGE$	Marriage rate per thousand population
Ii	Stock of business inventories
S^p	Sales in the private sector
U	Unfilled orders in manufacturing
STR	Dummy variable for steel strikes

III. FOREIGN TRADE

$$Fi = -4.4693 + 0.0683Y + 0.0650\Delta Ii - 2.3931Pi*/P + 0.0681 \sum_{i=1}^{4} Fi_{-i} \tag{4.7}$$

$$Fi = 0.0683Y + 0.0650\Delta Ii + 2.3931P + 0.0681 \sum_{i=1}^{4} Fi_{-i} \tag{4.7a}$$

$$Fe = -16.9099 + 12.7990Xwt^* + 15.8086Pwt^*/Pe + 0.1225 \sum_{i=1}^{4} Fe_{-i} \tag{4.8}$$

$$Fe = -15.8086Pe + 0.1225 \sum_{i=1}^{4} Fe_{-i} \tag{4.8a}$$

Fi	Imports
Y	Personal disposable income
Ii	Stock of business inventories
Pi	Implicit price deflator for imports
P	Implicit price deflator for *GNP*
Fe	Exports

Xwt	Index of world trade
Pwt	Price index of world trade
Pe	Implicit price deflator for exports

IV. PRODUCTION AND CAPACITY

(4.9)
$$\ln (X^p) = 2.3745 + 0.6551 \ln (N \times h) + 0.1479 \ln (Kp \times Cp) + 0.0048t$$

(4.9a)
$$0.0018X^p = 0.0109N + 0.6551h + 0.00014Kp + 1.0482Cp$$

(4.10)
$$\ln (X^p)^c = 2.3745 + 0.6551 \ln (N_L^p)^c + 0.1479 \ln (Kp) + 0.0048t$$

(4.10a)
$$0.0018X^p = 0.0109N + 0.6551Un + 0.00014Kp$$

X^p	GNP in the private sector
$(X^p)^c$	Full capacity private GNP
N	Number of employees in the private sector
h	Index of average weekly hours
Kp	Stock of investment in plants and equipment
$(N_L^p)^c$	Full employment private-sector labor force [$=(1 + UN)N$]
Cp	Wharton School index of capacity utilization [$=X^p/(X^p)^c$]
Un	Unemployment rate

V. HOURS AND WAGES

(4.11)
$$h = 0.0182 + 0.0002X^p + 0.0010\Delta X^p + 0.1995Cp$$

(4.12)
$$(w - w_{-4})/w_{-4} = 0.0391 - 0.007 \sum_{i=1}^{4} Un_{-i} + 0.3041(w_{-1} - w_{-5})/w_{-5} + 0.3315(Pc - Pc_{-4})/Pc_{-4}$$

(4.12a)
$$w = 0.3041w_{-1} + 1.0245w_{-4} - 0.3041w_{-5} - 0.0048 \sum_{i=1}^{4} Un_{-i} + 2.0454Pc - 2.0454Pc_{-1}$$

h	Index of average weekly hours
X^p	Private sector GNP
Cp	Wharton School index of capacity utilization
w	Wage rate in the private sector
Un	Unemployment (per cent)
Pc	Implicit price deflator for consumption

VI. TAX EQUATIONS

$$Tb = -3.45.5 + 0.0752NI + 0.3861t \tag{4.13}$$

$$Tr = 3.8250 + 0.9004Un + 0.4418t \tag{4.14}$$

$$Tp = -14.910 + 0.155(PI + SSI^* - Tr) \tag{4.15}$$

$$Tc = 4.2136 + 0.3589(Pcb - Iva) \tag{4.16}$$

Tb	Indirect business taxes and business transfer payments
NI	National income
Tr	Transfer payments
Un	Unemployment (per cent)
Tp	Personal tax and nontax payments
PI	Personal income
SSI	Personal contributions for social insurance
Tc	Corporate profits tax
Pcb	Corporate profits before taxes
Iva	Inventory valuation adjustment

VII. PRICE AND INTEREST RATE EQUATIONS

$$P = -0.1206 + 0.2908W/X^p + 0.1485Cp + 0.2114 \sum_{i=1}^{4} P_{-i} \tag{4.17}$$

$$P = 0.0005W - 0.0012X^p + 0.1485Cp + 0.2114 \sum_{i=1}^{4} P_{-i} \tag{4.17a}$$

$$Pw = 0.2183 + 0.5065P + 0.2916Cp \tag{4.18}$$

$$Pd = 0.2836 + 0.2011P + 0.5038Pw \tag{4.19}$$

$$Pn = 0.2313 + 0.4144P + 0.3565Pw \tag{4.20}$$

(4.21) $Ph = 0.0527 + 0.8314P + 0.0028Ip + 0.002Ih$

(4.22) $Pk = -0.2467 + 1.3783P - 0.0032Ip$

(4.23) $Pe = 0.0692 - 0.0672P + 1.0074Pw$

(4.24) $iS = 0.4411 + 0.9746id^* - 7.7878Fr^*$

(4.25) $iL = 0.2175 + 0.0972iS + 0.8779iL_{-1}$

P	Implicit price deflator for *GNP*
W	Wage bill in the private sector
X^p	Private sector *GNP*
Cp	Wharton School index of capacity utilization
Pw	Wholesale price index
Pd	Implicit price deflator for consumer durables
Pn	Implicit price deflator for consumer nondurables
Ph	Implicit price deflator for residential structures
Ip	Private investment in plant and equipment
Ih	Private investment in nonfarm housing
Pk	Implicit price deflator for nonresidential fixed business investment
Pe	Implicit price deflator for exports
iS	Average yield (per cent) on 4- to 6-month commercial paper
id	Discount rate (per cent)
Fr	Net free reserves as a fraction of total required reserves
iL	Average yield (per cent) on corporate bonds

VIII. NONWAGE INCOME

(4.26) $DIV = 0.1491 + 0.0723(Pca + D) + 0.1566 \sum_{i=1}^{4} DIV_{-i}$

(4.27) $PRI = 0.8182 + 0.1269\Delta(P \times X^p) + 0.2527 \sum_{i=1}^{4} PRI_{-i}$

(4.27a) $PRI = 0.1269\Delta X^p + 70.6833\Delta P + 0.2527 \sum_{i=1}^{4} PRI_{-i}$

DIV	Dividends
Pca	Corporate profits after taxes
D	Capital consumption allowances

PRI	Business and professional income plus rental income plus personal interest income
P	Implicit price deflator for GNP
X^p	Private sector GNP

IX. OTHER STOCHASTIC EQUATIONS

(4.28) $$IVA = -0.3906 - 17.7819\Delta Pw$$

(4.29) $$D = -7.4861 + 0.0595(Pk \times Kp)$$

(4.29a) $$D = 0.0595Kp + 61.9990Pk$$

(4.30)
$$\Delta U = -0.5260 + 0.1592\Delta S^p + 0.6497\Delta Gd^* + 4.2979Duw^*$$

IVA	Inventory valuation adjustment
Pw	Wholesale price index
D	Capital consumption allowances
Pk	Implicit price deflator for nonresidential fixed business investment
Kp	Stock of investment in plant and equipment
U	Unfilled orders in manufacturing
S^p	Private sector sales
Gd	Government purchases for national defense
Duw	Dummy variable for Korean War period

X. DEFINITIONS AND ACCOUNTING IDENTITIES

(4.31) $$X = Cd + Cn + Cs + Ip + Ih + \Delta Ii - Fi + Fe + G^*$$

(4.32) $$X^p = X - (wg^*/Pg^*)$$

(4.33) $$PX = Pd\,Cd + Pn\,Cn + Ps\,Cs + Pk\,Ip + Ph\,Ih + Pw\Delta Ii + Pe\,Fe - Pi^*\,Fi + Pg^*\,G^*$$

(4.33a) $$557.0P = 67.9Pd + 181.0Pn + 154.4Ps + 63.3Pk + 23.1Ph + 9.8Pw + 32.9Pe$$

$$PI = NI - Pcb + DIV + Tr + Igc^* - Soc^* \tag{4.34}$$

$$Y = (PI - Tp)/Pc \tag{4.35}$$

$$Y = IP - Tp - 476.5Pc \tag{4.35a}$$

$$NI = PX - Tb - D \tag{4.36}$$

$$Pcb = NI - W - Wg^* - PRI - F^* + Igc^* \tag{4.37}$$

$$W = w \times N \times h \tag{4.38}$$

$$W = 60.1w + 6.33N + 380.4h \tag{4.38a}$$

$$Un = 100(NL^* - N - Ng^* - Nf^* - Ns^*)/NL^* \tag{4.39}$$

$$Un = -1.3137N \tag{4.39a}$$

$$S^p = X^p - \Delta Ii \tag{4.40}$$

$$Pc = (PdCd + PnCn + PsCs)/(Cd + Cn + Cs) \tag{4.41}$$

$$Pc = 0.1684Pd + 0.4488Pn + 0.3828Ps \tag{4.41a}$$

$$Kd = \sum_{i=0}^{40} (0.924)^i Cd_{-i} \tag{4.42}$$

$$Kd = 0.929Kd_{-1} + Cd \tag{4.42a}$$

$$Kp = \sum_{i=0}^{60} (0.953)^i Ip_{-i} \tag{4.43}$$

$$Kp = 0.953Kp_{-1} + Ip \tag{4.43a}$$

$$Cp = X^p/(X^p)^c \tag{4.44}$$

$$Pca = Pcb - Tc \tag{4.45}$$

X	*GNP*
G	Government purchases of goods and services
Wg	Compensation of government employees
Pg	Implicit price deflator for government purchases
Igc	Interest paid by government consumers
Soc	Social Security contributions
F	Unincorporated farm income
NL	Civilian labor force

Ng	Number of government employees
Nf	Number of farm operators
Ns	Number of nonfarm self-employed persons

5 DYNAMIC PROPERTIES OF THE REAL SECTOR

IN VIEW of the complexity of this model, it may be useful to consider first a model of the real sector obtained by holding prices constant. This real-sector model indicates how the economy would be expected to behave if prices were not permitted to vary according to equations (4.17) to (4.23) but, instead, were fixed. The characteristic roots of this linear approximation to the real sector are shown in Table 5.1.[12] The first point that emerges from these calculations is that the linear approximation appears to be unstable, since the largest root is 1.0078. It is interesting to note that this root implies an annual rate of growth of some 3.2 per cent. However, in view of the sampling variability to which the parameter estimates, and hence the characteristic roots, are subject, it seems unlikely that the three largest real roots are significantly different from one another. But these calculations do suggest that the linear approximations are stochastically unstable in the sense that the variance of the endogenous variables increases with time. This is due, of course, to the accumulation of disturbance errors implied by the unit roots of the system of equations.

The complex roots give rise to transient oscillations with fairly short periods. The longest oscillation is a little more than eight quarters, but the modulus of this component is so small that it cannot be considered very important. The remaining complex roots have periods of less than five quarters. It appears, therefore, that the transient response of this model does not exhibit twelve- to fifteen-quarter business-cycle oscillations.

[12] The characteristic roots of the system were obtained through a two-step procedure. The model was first converted to a first-order system of the form

$$BY_t = CY_{t-1}$$

where Y_t is the vector of current endogenous variables augmented by the lagged endogenous variables necessary to effect the first-order conversion. The characteristic roots of the matrix $E = B^{-1}C$ were then computed. A standard computer program (Share No. 3006.01 by J. E. Van Ness) was used in the characteristic-root calculation.

TABLE 5.1

Characteristic Roots of the Real Sector of the Condensed Wharton Model

Real Part	Imaginary Part	Modulus	Period
1.0078			
1.0042			
1.0000			
0.9534			
0.9379			
0.8944			
0.8779			
0.8351			
0.7717			
0.7372			
0.6228			
0.4922			
0.2967			
0.2020			
−0.0000			
−0.3373			
−0.4588			
−0.5204			
−0.5251			
−0.5504			
−0.5883			
−0.6083			
−0.8861			
−1.0047			
0.0026	±0.0026	0.0037	8.14
0.0710	±0.1742	0.1882	5.31
0.0001	±0.0001	0.0001	4.17
0.0017	±1.0056	1.0056	4.00
−0.0000	±0.0001	0.0001	4.00
−0.0499	±0.4793	0.4819	3.75
−0.0679	±0.6219	0.6255	3.74
−0.0606	±0.5531	0.5564	3.74
−0.0599	±0.5450	0.5483	3.74
−0.0645	±0.5814	0.5850	3.74
−0.0026	±0.0027	0.0037	2.68
−0.1338	±0.0988	0.1663	2.51

The stochastic response of this model is similarly devoid of business-cycle dynamics. The spectrum matrix of the endogenous variables of the real-sector model was computed using equation (3.10), introduced above. Since interest centers on how the system responds to uncorrelated shocks, the spectrum matrix of the residual process was obtained from

$$\hat{f}_{\mu\mu}(\omega) = \frac{1}{2\pi} \hat{\sum}_{uu} \tag{5.1}$$

where $\hat{\Sigma}$ is an estimate of the contemporaneous covariance matrix of the disturbance vector u. The power spectrum of gross national product implied by the real-sector model is shown in Chart 5.1.[13] This power spectrum, which is representative of the spectra of the consumption and investment variables in the model, indicates that power decreases with frequency, except for the relative concentrations of power at one-fourth and one-half of a cycle per quarter. The spectrum of quarterly changes in *GNP* shown in Chart 5.2 increases with frequency, which indicates that short-run variations in quarterly changes in *GNP* are more important than long-run changes.[14] Once again there is a relative concentration of power at one-fourth of a cycle per quarter.

The rather unexpected behavior of the power spectra at the frequencies corresponding to the annual and semi-annual variation in *GNP* can be explained by an examination of the wage equation in the model. Suppose that the four-quarter percentage changes in this wage equation are replaced by four-quarter absolute changes in the wage rate. This would yield a wage equation with the auto-regressive structure

$$w_t = w_{t-4} + 0.3041(w_{t-1} - w_{t-5}) + \epsilon_t \tag{5.2}$$

[13] On this and the following diagrams, the spectrum is evaluated at $\omega_j = 2\pi j/40 (j = 1, 2, \ldots, 20)$ unless otherwise indicated. The reason that the power at the frequency $\omega = 0$ is not included is that, due to the fact that one of the roots of the determinantal equation is unity, the matrix $B(e^{-i\omega})$ is singular when $\omega = 0$. Since $B(e^{-i\omega})$ must be inverted to obtain the transfer matrix $T(\omega)$ defined in equation (3.8), the transfer matrix, and hence the spectrum, is not defined at $\omega = 0$.

[14] The spectrum of quarterly changes was obtained by multiplying the original spectrum by the gain of the first-difference transformation. This yields $f_{\delta y}(\omega) = |1 - e^{-i\omega}|^2 f_y(\omega)$ where $f_y(\omega)$ is the spectrum of the original variable and $f_{\delta y}(\omega)$ is the spectrum of the first-differenced variable.

CHART 5.1

Gross National Product, Real-Sector Model

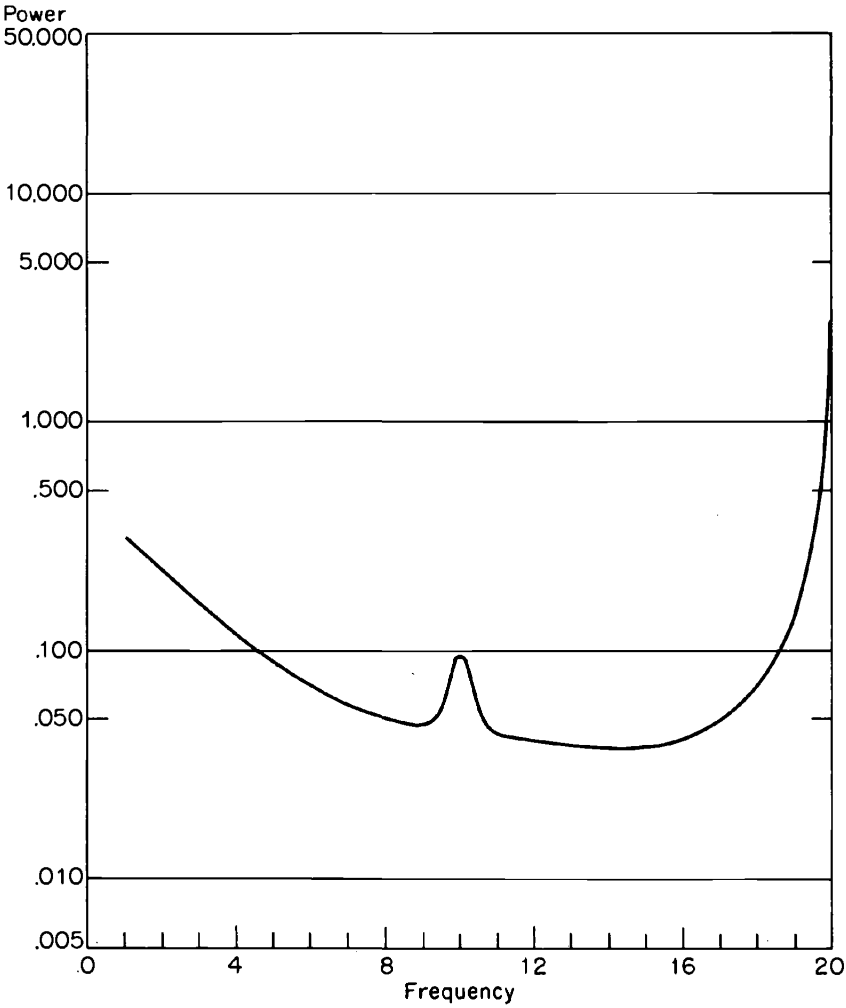
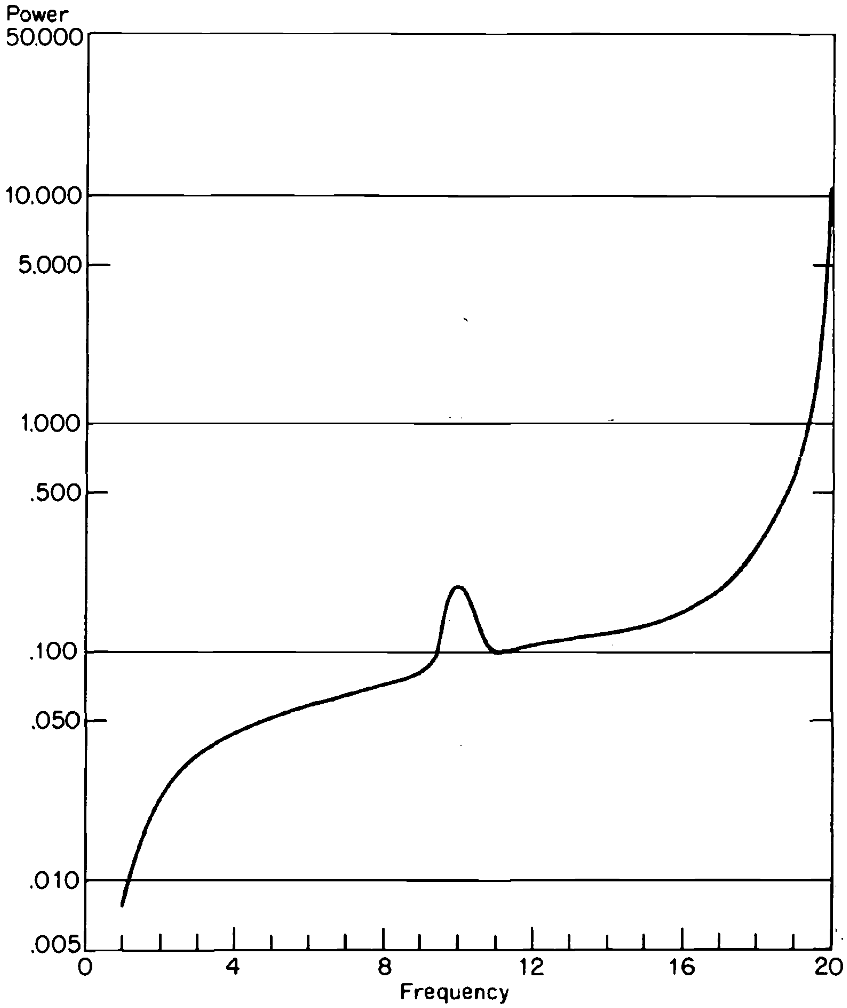


CHART 5.2

Changes in Gross National Product, Real-Sector Model



If the disturbance process ϵ_t is serially uncorrelated with variance σ_ϵ^2, the spectrum of the wage rate is

$$f_w(\omega) = |T(\omega)|^2\sigma_\epsilon^2/2\pi \tag{5.3}$$

where the frequency response function, $T(\omega)$, is

$$T(\omega) = [(1 - 0.3041e^{-i\omega})(1 - e^{-i4\omega})]^{-1} \tag{5.4}$$

Since the factor $(1 - e^{-i4\omega})$ exhibits singularities at $\omega = 0$, $\pi/2$, and π, the power spectrum of the wage rate will exhibit relative concentrations of power at each of these frequencies. These concentrations of power affect the power spectra of the other endogenous variables in the system. This indicates that the wage equation imparts a strong seasonal pattern to the endogenous variables of the model.

In order to determine the dynamic properties of the system in the absence of the somewhat peculiar behavior of the wage rate, the wage equation was deleted from the system. The power spectra of gross national product and changes in *GNP* are shown in Charts 5.3 and 5.4 for the resulting modified real-sector model. The absence of a relative peak at one-fourth of a cycle per quarter in these spectra confirms the suspicion that the form of the wage equation is responsible for the relative peaks in the real-sector model.

The basic point that emerges from this analysis of these models is that neither version of the real-sector model is consistent with an impulse-response explanation of business cycles. The lag structure of these models does not impart the sort of smoothing that is required for the model to respond cyclically to random disturbances. Any cyclical behavior that this model might exhibit is therefore due to serial correlation in the disturbance process, or to business-cycle variations in the exogenous variables.

6 DYNAMIC PROPERTIES OF THE COMPLETE MODEL

THIS section is devoted to an analysis of a linear approximation to the complete model. Due to the peculiar behavior of the wage variables, as described above, the wage equation is not included in the model. This

CHART 5.3

Gross National Product, Modified Real-Sector Model

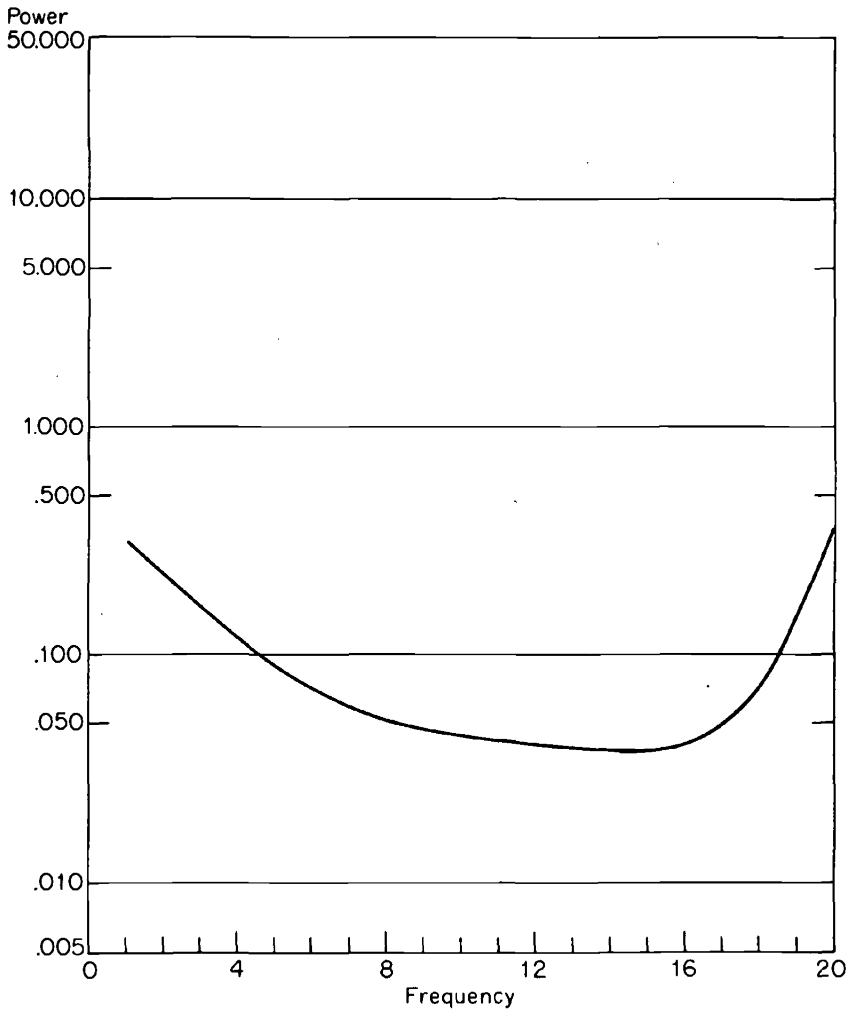
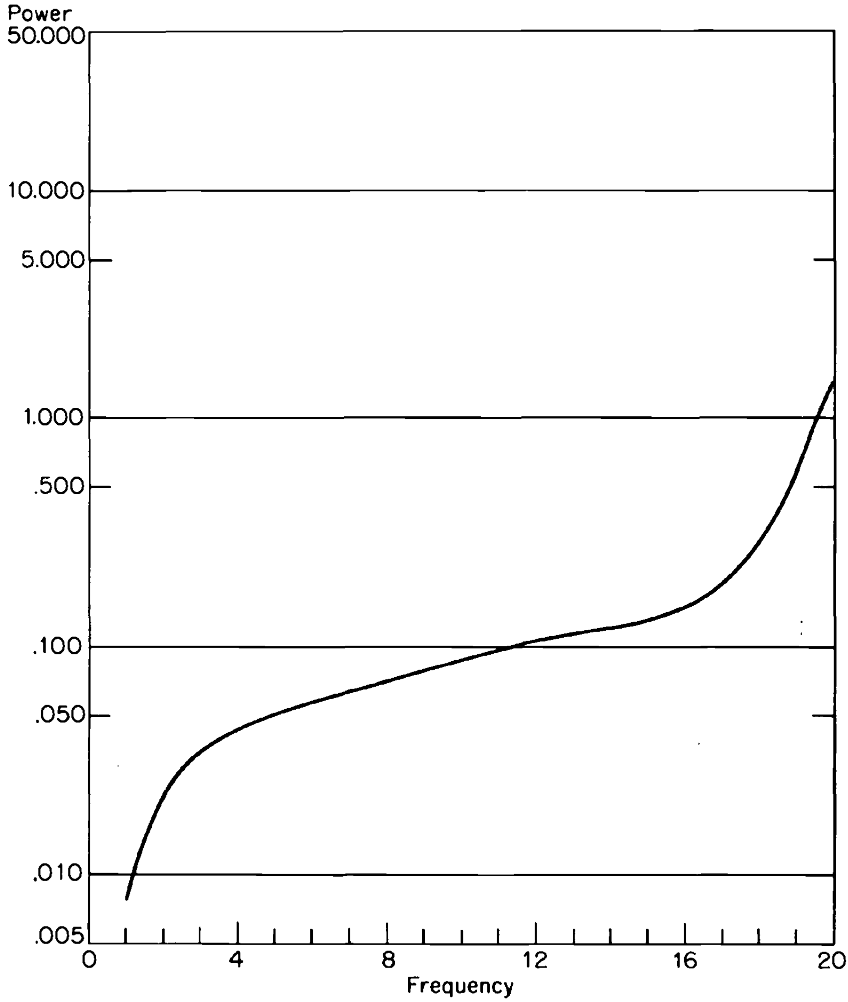


CHART 5.4

Changes in Gross National Product, Modified Real-Sector Model



leaves a system of twenty-nine stochastic equations with endogenous variables lagging as many as four quarters.

The characteristic roots of this system are shown in Table 6.1. The real roots of the determinantal equations range from −0.850 to 1.005. The root slightly greater than one implies an annual rate of growth of some 1.9 per cent. Once again there is a cluster of complex roots with periods of about four quarters. This, of course, is not too surprising in view of the repeated use of a four-quarter lag in the moving averages of the endogenous variables as predetermined variables in the structural equations of the model. As was found for the real-sector model, this model is unstable, and the transient response does not exhibit twelve- to fifteen-quarter business-cycle oscillations.

The power spectrum of gross national product implied by the complete system is shown in Chart 6.1, and the spectra of the three investment series—plant and equipment, residential construction, and inventory investment—are shown in Charts 6.2 to 6.4. These spectra were computed on the assumption that the disturbance process is serially uncorrelated. Hence, the spectrum matrix of residuals used in equation (3.1) was computed using equation (5.1), with an estimate of the contemporaneous covariance matrix corresponding to the complete model. The spectrum of *GNP* implied by the model indicates that the variations in gross national product are dominated by very long and very short oscillations. Once again, business-cycle variations are absent from the model. The power spectra of the first differences of these variables are shown in Charts 6.5 to 6.8. A comparison of the spectra of the investment series, implied by the model with the spectrum estimates in Section 2, indicates quite clearly that the lag structure of the model does not explain the dynamic behavior of these series. This means that the response of this system of equations to random disturbances does not provide an adequate explanation of business cycles.

TABLE 6.1

Characteristic Roots of the Modified Condensed Wharton Model

Real Part	Imaginary Part	Modulus	Period
1.0047			
1.0000			
0.9619			
0.9448			
0.8779			
0.8651			
0.8375			
0.7663			
0.7404			
0.6217			
0.4918			
0.1887			
0.0011			
0.0001			
0.0000			
−0.0013			
−0.3291			
−0.4583			
−0.5504			
−0.6086			
−0.8500			
0.0808	±0.1356	0.1578	6.08
−0.0001	±0.0016	0.0200	3.99
−0.0012	±0.0200	0.0200	3.85
−0.0621	±0.9117	0.9138	3.83
−0.0517	±0.5527	0.5551	3.78
−0.0498	±0.4788	0.4814	3.75
−0.0607	±0.5497	0.5530	3.74
−0.0672	±0.5780	0.5819	3.73
−0.0843	±0.6141	0.6199	3.68
−0.0903	±0.6566	0.6628	3.68
−0.1429	±0.1485	0.2061	2.69
−0.5858	±0.0104	0.5859	2.01
−0.5221	±0.0032	0.5221	2.00

CHART 6.1

Gross National Product, Condensed Wharton Model
($\delta f = 1/40$ c/q)

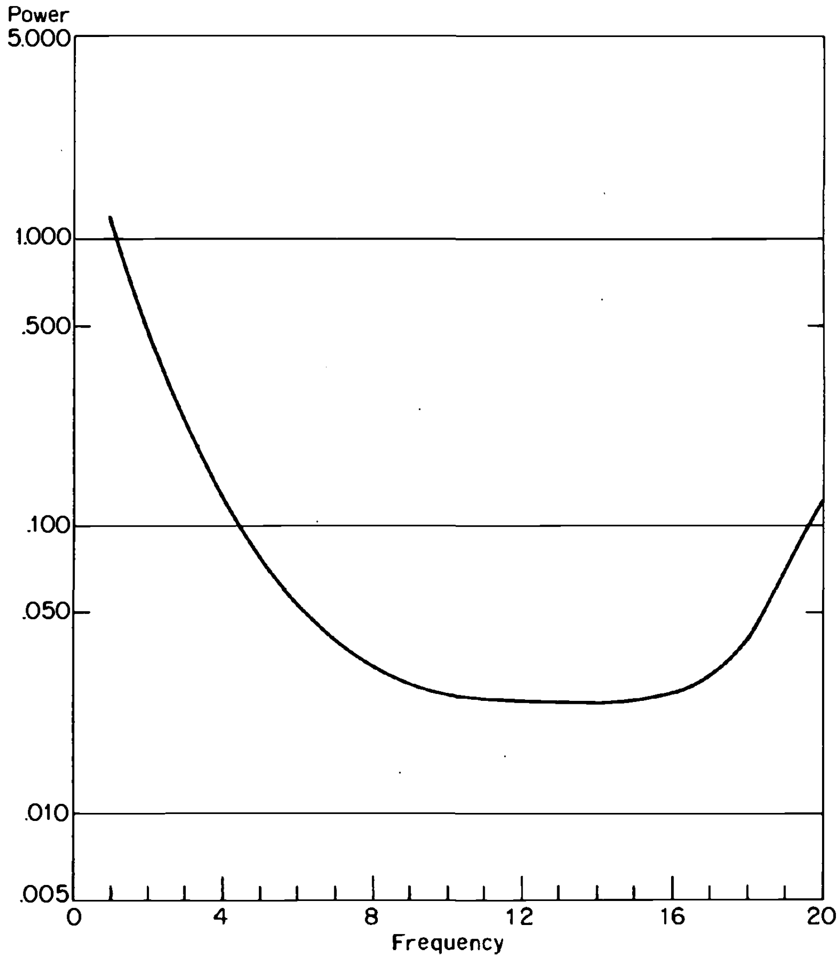


CHART 6.2

Plant and Equipment Investment, Condensed Wharton Model
($\delta f = 1/40$ c/q)

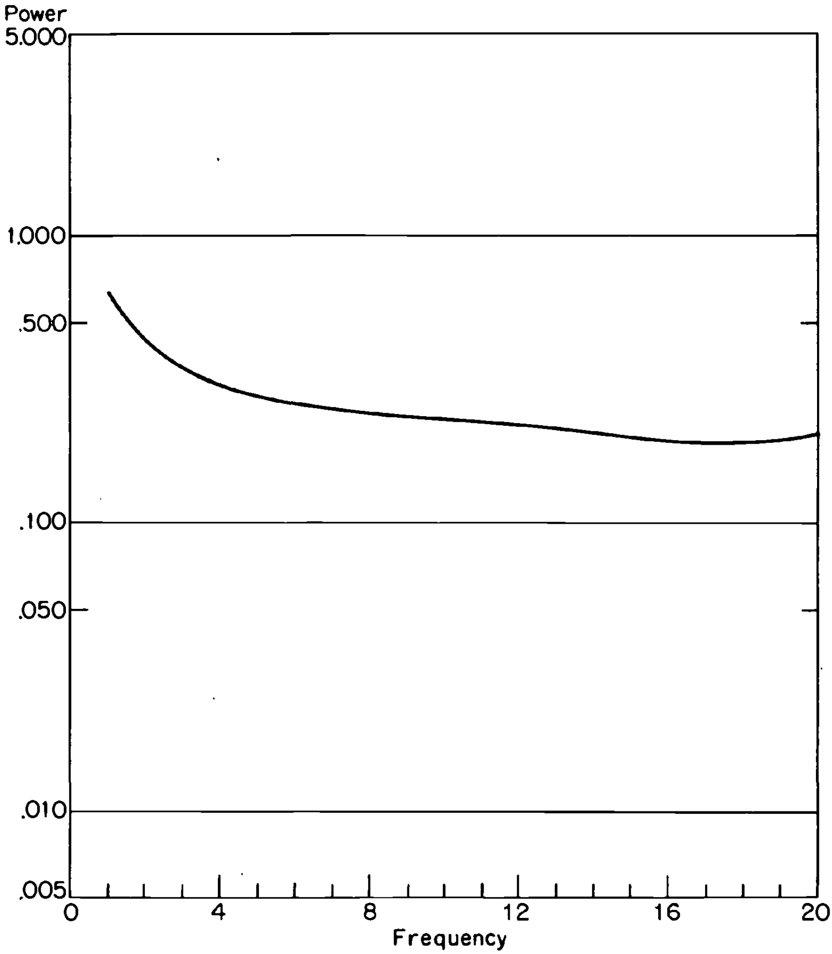


CHART 6.3

Residential Construction, Condensed Wharton Model
($\delta f = 1/40$ c/q)

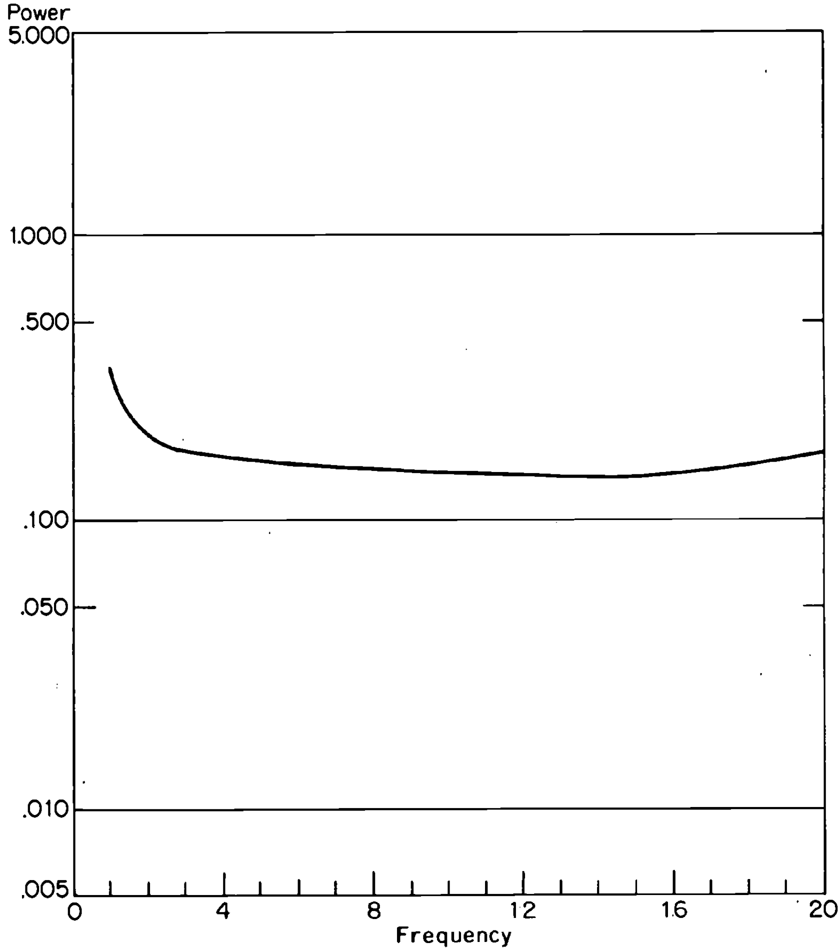


CHART 6.4

Inventory Investment, Condensed Wharton Model
($\delta f = 1/40$ c/q)

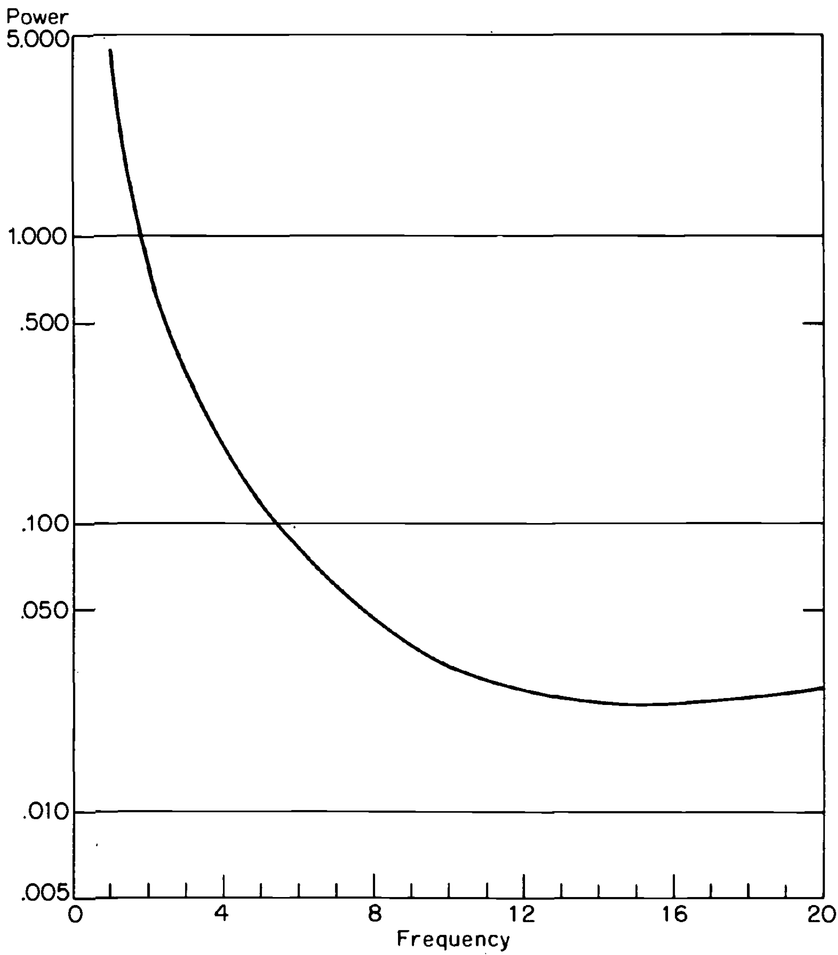


CHART 6.5

Gross National Product, Quarterly Changes, Condensed Wharton Model
($\delta f = 1/40\ c/q$)

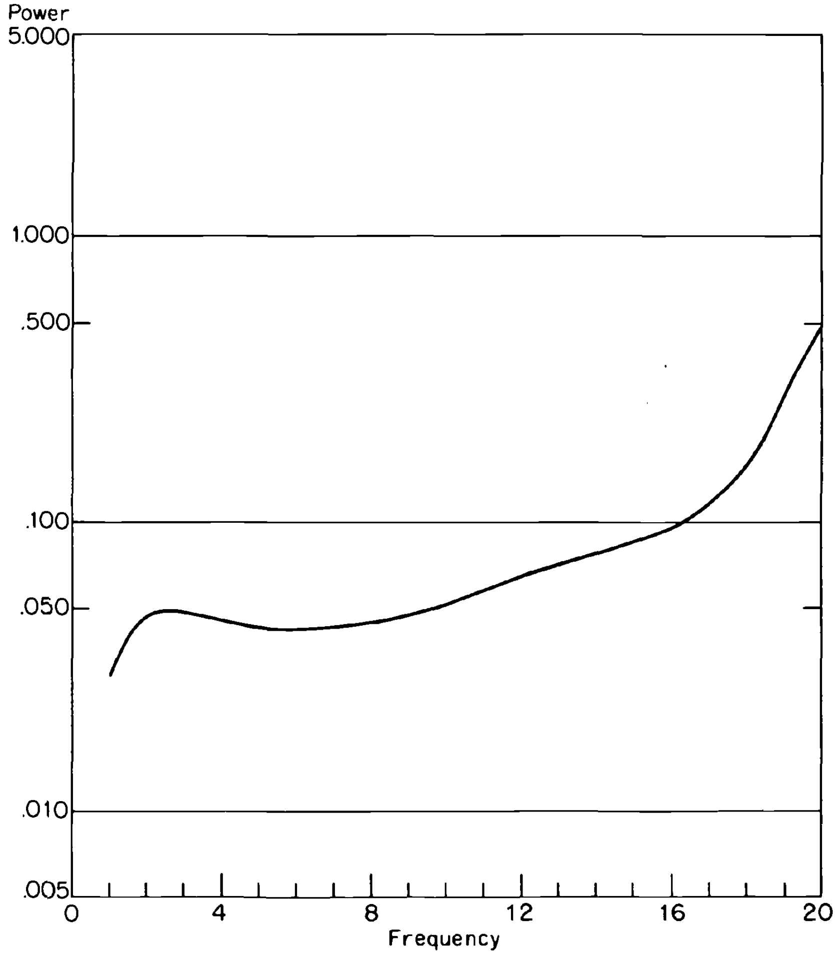


CHART 6.6

Plant and Equipment Investment, Quarterly Changes, Condensed Wharton Model
($\delta f = 1/40$ c/q)

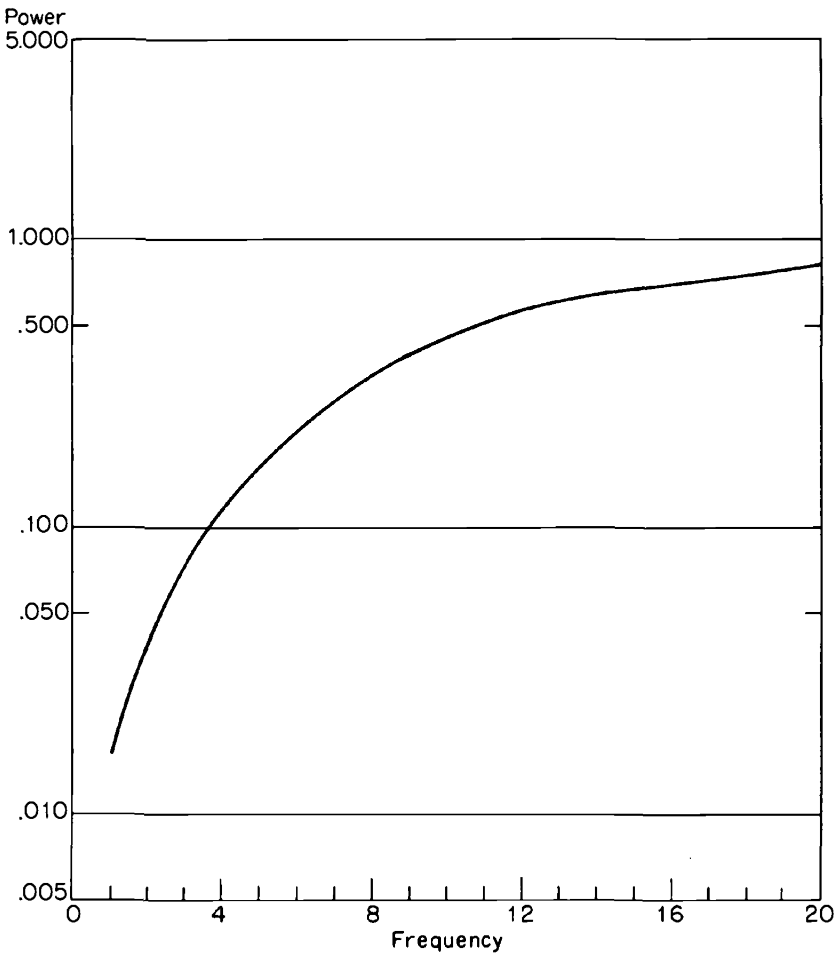


CHART 6.7

Residential Construction, Quarterly Changes, Condensed Wharton Model
($\delta f = 1/40$ c/q)

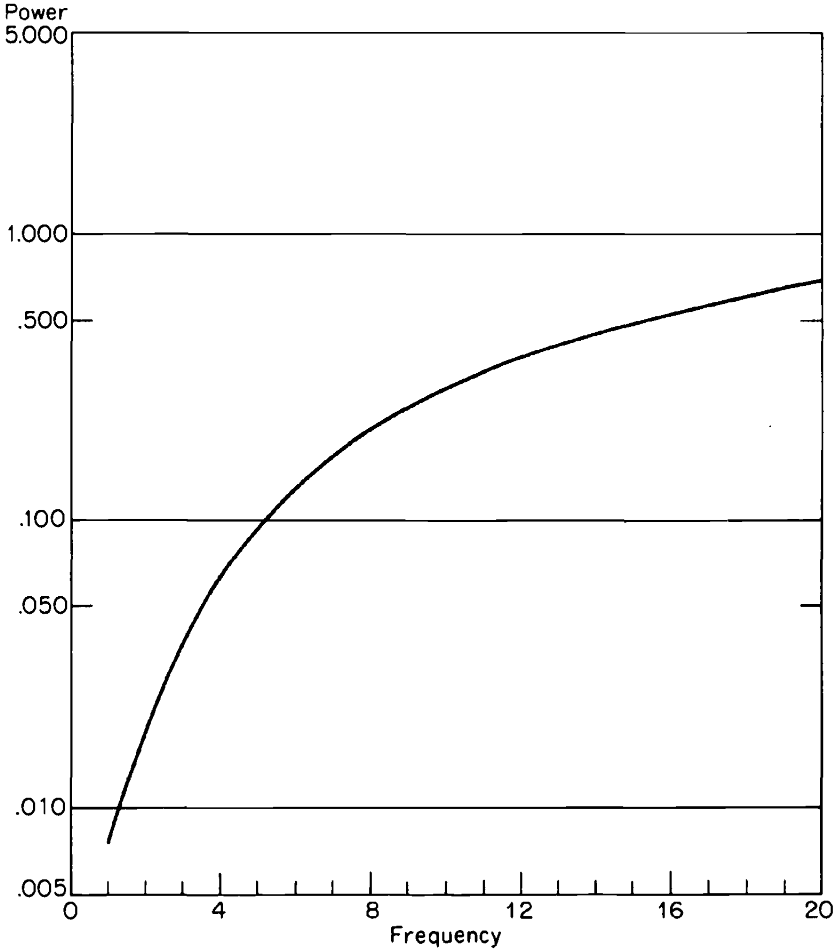
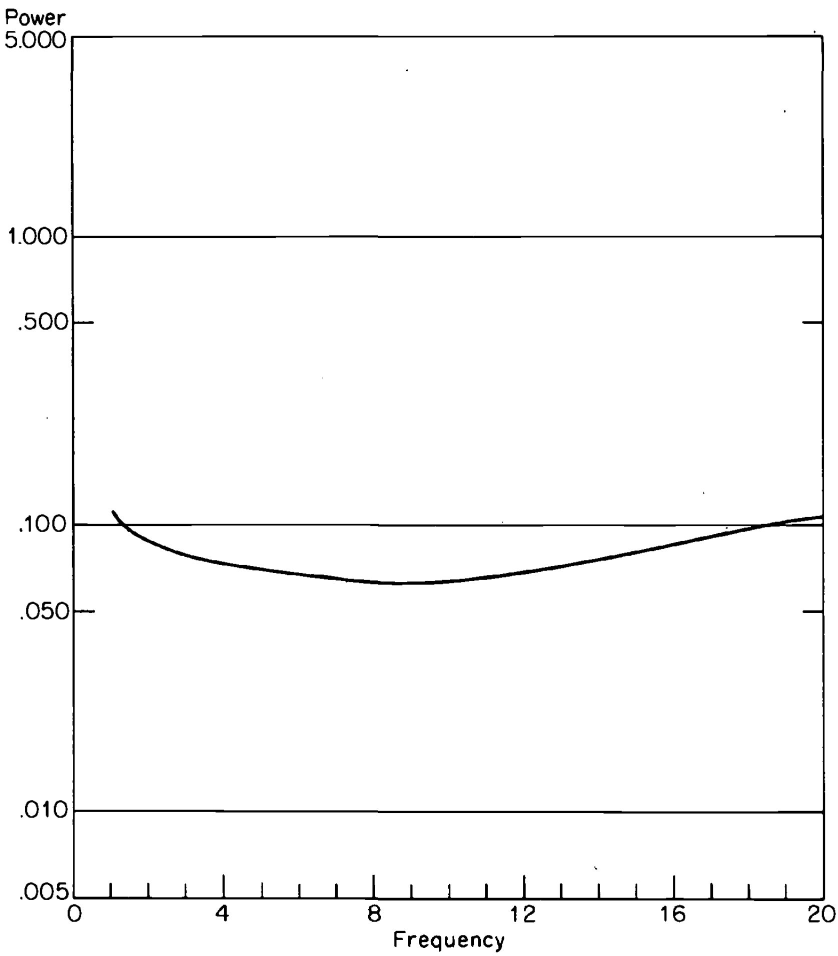


CHART 6.8

Inventory Investment, Quarterly Changes, Condensed Wharton Model
($\delta f = 1/40$ c/q)



7 ADDENDUM: THE CHARACTERISTIC ROOTS OF THE MODEL

DURING the course of this investigation of the dynamic properties of a condensed version of the Wharton econometric model, an attempt was made to calculate the characteristic roots of a linear approximation to the model. While this was not the major purpose of the study, it was felt that an examination of the roots of the determinantal polynomial of the system might yield additional insights into the dynamic behavior of the endogenous variables of the model, as characterized by their implied power spectra. Although some of the difficulties associated with the extraction of roots of large systems of equations were recognized at the outset, the magnitude of this problem was, perhaps, not fully appreciated. This root-extraction problem may be illustrated by the divergent results obtained by Kei Mori, using the same input data but a different root-extraction algorithm.[15] The real roots obtained using the alternative computational methods are shown in Table 7.1 and the complex roots are shown in Table 7.2. The major differences between these two sets of roots can be summarized as follows:

(*a*) Howrey finds 17 nonzero real roots whereas Mori finds only 15. There is very little difference between the Mori roots and fifteen of the Howrey roots. However, the Howrey routine finds two more real roots, 0.962 and 0.865, than the Mori routine.

(*b*) The Howrey routine finds 13 pairs of complex roots, whereas the Mori calculations reveal 14 pairs of complex roots. Twelve of these pairs of roots are in close agreement, but the remaining roots are quite divergent. The Mori routine finds $.8190 \pm 0.1934i$ and $0.8249 \pm 0.0884i$, whereas the Howrey routine yields $1.1344 \pm 0.0743i$. The Mori roots are particularly interesting because they have periods of 27 and 59 quarters (6.75 and 14.75 years, respectively), with damping factors of

[15] The author is indebted to Kei Mori for permission to quote his as yet unpublished results, and for his helpful comments on this section of the paper. It should be noted that the results described in this section are preliminary in the sense that further research on the calculation of characteristic roots is being conducted by Mr. Mori.

> 0.84 and 0.83, respectively. The Howrey root is also of some interest, since it indicates that the system is unstable, with a rather rapid speed of divergence.

In view of these differences, a resolution of the "characteristic-root problem" seems desirable.

Two types of tests have been carried out in an attempt to shed some light on the divergent results which have been obtained from the alternative root-extraction routines. The first involves a simple check on the characteristic roots. This calculation turns out not to be sufficiently sensitive to discriminate among the disparate results. The second test involves a comparison of the dynamic properties of two-difference equations constructed from the alternative sets of characteristic roots. A comparison of the transfer functions associated with

TABLE 7.1

Real Roots of a Condensed Version of the Wharton Model

Mori		Howrey	
Root (1)	Determinant (2)	Root (3)	Determinant (4)
1.0044	−0.5725E-12	1.0047	−0.1303E-13
1.0000	0.1251E-14	1.0000	0.3753E-14
		0.9619	−0.1205E-14
0.9452	−0.3220E-13	0.9448	0.5568E-15
0.8779	0.0	0.8779	−0.1388E-16
		0.8651	0.2200E-17
0.8307	−0.1052E-13	0.8375	−0.7183E-17
0.7725	−0.3105E-14	0.7663	0.1679E-17
0.7428	0.6026E-15	0.7404	−0.8744E-18
0.6249	−0.1415E-15	0.6217	0.1205E-18
0.4746	−0.2429E-17	0.4918	−0.7536E-21
0.1823	0.2313E-28	0.1887	0.9871E-33
−0.3559	0.2316E-21	−0.3291	−0.2126E-26
−0.4601	−0.1930E-20	−0.4583	0.1193E-23
−0.5503	−0.2836E-21	−0.5504	−0.1795E-22
−0.6084	0.7123E-18	−0.6086	0.9367E-20
−0.8112	−0.3885E-06	−0.8500	−0.6025E-09

TABLE 7.2

Complex Roots of a Condensed Version of the Wharton Model

Mori				Howrey			
Root		Determinant		Root		Determinant	
Real (1)	Imaginary (2)	Real (3)	Imaginary (4)	Real (5)	Imaginary (6)	Real (7)	Imaginary (8)
0.818989	0.193381	−0.4480E-09	0.1362E-08				
0.824922	0.088362	−0.3237E-11	−0.3044E-11				
−0.085540	0.653141	−0.1047E-13	−0.9649E-14	−0.090339	0.656622	−0.1819E-16	−0.4022E-17
−0.081975	0.641751	−0.6380E-14	−0.7157E-15	−0.084341	0.614124	0.6145E-19	0.8727E-19
−0.061888	0.612939	0.8352E-17	−0.1114E-16	−0.062122	0.611678	0.9758E-20	0.5476E-20
−0.066189	0.577634	0.4655E-18	0.1661E-18	−0.067228	0.578015	0.7277E-21	−0.4279E-21
−0.049586	0.479951	−0.2751E-19	−0.5123E-19	−0.049768	0.478793	0.1398E-22	0.5938E-22
−0.054793	0.551901	0.7478E-19	−0.1123E-18	−0.051741	0.552709	0.2389E-21	0.4991E-22
−0.061157	0.551579	0.1050E-18	0.4583E-19	−0.060927	0.549703	−0.4893E-20	0.1298E-19
−0.586495	0.009598	0.1652E-19	−0.2246E-18	−0.585848	0.010441	0.8960E-21	0.3808E-21
−0.523214	0.002567	−0.6601E-21	0.7965E-21	−0.522081	0.003232	0.3104E-24	−0.1886E-23
−0.131558	0.134792	−0.8103E-28	−0.6311E-28	−0.142884	0.148467	−0.1953E-31	0.3872E-32
0.065069	0.131656	0.2986E-30	−0.1098E-30	0.080766	0.135630	0.3058E-34	−0.1255E-33
−0.001349	0.019952	0.2896E-52	−0.6744E-55	−0.001234	0.019957	−0.9976E-53	−0.1065E-51
				1.134415	0.074333	−0.1532E-06	−0.1445E-07

the difference equations indicates that it is extremely difficult to distinguish between the two equations. This suggests that from the point of view of stochastic systems analysis, it may not be particularly important which of the two sets of roots corresponds to the truth.

7.1 EVALUATION OF THE DETERMINANTAL POLYNOMIAL

As a first step in the resolution of this issue, an independent calculation of the value of the determinantal polynomial was attempted. In particular, the complete system of equations, involving lags of up to order five, was converted to a first-order system of the form

$$BY_t = CY_{t-1} \tag{7.1}$$

or

$$Y_t = EY_{t-1} \tag{7.1a}$$

where $E = B^{-1}C$. If λ is a characteristic root of E, then it should satisfy the condition

$$|E - \lambda I| = 0 \tag{7.2}$$

or

$$|C - \lambda B| = 0 \tag{7.2a}$$

The determinant of $[C - \lambda B]$ was evaluated for each of the characteristic roots, with the results shown in Tables 7.1 and 7.2. These results can be summarized as follows:

(*a*) The real roots .962 and .865 are not blatantly spurious, since the determinant corresponding to these roots is zero to at least fourteen places. The value of the determinantal polynomial is closer to zero for these two roots than for several of the other roots on which the algorithms agree—notably, the roots close to unity.

(*b*) This check casts suspicion on the negative real roots -0.81 (Mori) and -0.85 (Howrey). In neither case is the determinantal polynomial as close to zero as might have been expected.

(*c*) The complex roots $0.82 \pm 0.19i$ and $0.82 \pm 0.09i$ obtained by

Mori both pass this root check. However, the unstable root $1.13 \pm 0.07i$ appears to be spurious on the basis of the determinantal polynomial test.

These results contribute to some extent to a resolution of the problem. The basic difference that remains is that the Howrey routine yields two real roots, 0.962 and 0.865, whereas the Mori routine produces two complex roots, $0.819 \pm 0.193i$ and $0.825 \pm 0.088i$. Unfortunately, the determinantal polynomial calculations offer virtually no help in the resolution of these differences. Apparently, a more delicate approach is necessary.

7.2 TRANSFER FUNCTION ANALYSIS

The inability of the determinantal polynomial calculations to discriminate between the two sets of characteristic roots means that an alternative approach to the problem is necessary. One possibility is to consider the extent to which the two sets of results diverge from one another in terms of their implications about dynamic behavior. If the differences between the dynamic properties of the two sets of roots are negligible, then from a pragmatic point of view, there would appear to be little need to explore the matter further.

The specification of a comprehensive, quantitative measure of the dynamic properties of a large set of roots is not an easy matter. However, within the context of stochastic systems analysis, it seems quite natural to compare the alternative stochastic difference equations

$$\Delta_M(L)y(t) = \epsilon(t) \tag{7.3}$$

and

$$\Delta_H(L)y(t) = \epsilon(t) \tag{7.4}$$

where the zeros of $\Delta_M(\lambda)$ correspond to the characteristic roots obtained by Mori, and the zeros of $\Delta_H(\lambda)$ correspond to the roots obtained by Howrey. The power spectra of these two stochastic processes are given by

$$f_j(\omega) = G_j(\omega)f_\epsilon(\omega) \qquad (j = M,H) \tag{7.5}$$

where

$$G_j(\omega) = |\Delta_j(e^{-i\omega})|^{-2} \qquad (j = M,H) \tag{7.6}$$

and $f_\epsilon(\omega)$ is the power spectrum of $\epsilon(t)$. This indicates that the functions $G_j(\omega)$ provide a useful basis for a comparison of the dynamic properties of the two systems.

The natural logarithm of $G_j(\omega)$ for these two equations is shown in Table 7.3. In addition, the logarithm of the gain (squared) corresponding to the first differences of $y(t)$ generated by (7.3) and (7.4) is shown in this table. An examination of the results for the low-frequency end of the spectrum indicates that for both of these equations, the ratio

TABLE 7.3

Dynamic Properties of the Two Sets of Roots

	Mori		Howrey	
	Log of Gain (squared)		Log of Gain (squared)	
Quarters (1)	Level (2)	First Difference (3)	Level (4)	First Difference (5)
200.0	35.51	28.59	32.62	25.70
100.0	32.85	27.31	28.51	22.97
$66.\overline{6}$	31.09	26.37	25.49	20.76
50.0	29.67	25.52	23.03	18.89
40.0	28.38	24.67	20.95	17.25
$33.\overline{3}$	27.13	23.79	19.15	15.81
28.6	25.87	22.83	17.57	14.53
25.0	24.58	21.81	16.17	13.41
22.2	23.26	20.73	14.94	12.41
20.0	21.94	19.62	13.84	11.52
18.2	20.66	18.52	12.87	10.74
$16.\overline{6}$	19.43	17.47	12.01	10.05
15.4	18.28	16.47	11.25	9.45
14.3	17.21	15.55	10.58	8.92
$13.\overline{3}$	16.24	14.72	10.00	8.47
12.5	15.36	13.96	9.49	8.09
11.8	14.56	13.28	9.06	7.78
11.1	13.85	12.69	8.69	7.53
10.5	13.23	12.17	8.39	7.33
10.0	12.68	11.71	8.15	7.19

of output variance to input variance is a smoothly decreasing function of frequency. Precisely the same result is apparent for the first differences. Thus, despite the individual differences between the two sets of roots, the over-all dynamic properties of the two sets of roots under consideration are quite similar with respect to both levels and first differences. In an attempt to confirm these results, the resolution was increased by a factor of four so that instead of computing the gain at twenty points ranging from 200.0 to 10.0 quarters per cycle, as in Table 7.3, the gain was calculated at eighty points ranging from 800.0 to 10.0 quarters per cycle. Once again, no local maxima were found in the gain functions.

These results may appear to be somewhat surprising at first glance, since they indicate that even though the Mori root set includes complex roots with only modest damping factors, these complex roots do not appear to impart discernible cyclical properties to the solution. The fact of the matter is that the cyclical response path associated with the complex roots is swamped by the contribution of the positive real roots. The way in which this can occur is demonstrated by the following example. Consider a third-order difference equation with a real root, λ_1, and a pair of complex roots, λ_2 and λ_3:

$$(1 - \lambda_1 L)(1 - \lambda_2 L)(1 - \lambda_3 L)y(t) = \epsilon(t) \tag{7.7}$$

The gain (squared) is the product of the three factors; i.e.,

$$G(\omega) = G_1(\omega) \times G_2(\omega) \times G_3(\omega) \tag{7.8}$$

where

$$G_j(\omega) = |1 - \lambda_j e^{-i\omega}|^{-2} \tag{7.9}$$

Even though $G_2(\omega) \times G_3(\omega)$ exhibits a relative maximum at ω_0, it is still possible for $G(\omega)$ to decrease smoothly. This is particularly likely if ω_0 is close to zero, and the real root is close to unity, in which case $G_1(\omega)$ decreases rapidly as ω increases from zero to ω_0.

The transfer function analysis of this section suggests that from the point of view of stochastic systems analysis, it may not be particularly important to discriminate between the Mori and Howrey sets of roots, since both sets of roots imply similar response patterns. A more general conclusion of this analysis is that it is not an easy matter to infer the over-all response of a stochastic system from an examination of the individual roots of the system. It is equally important—if not more important—to determine the combined influence of the entire set of roots. Finally, it might be noted that the results obtained here are consistent with the earlier calculations of the dynamic response path of the Wharton Model and, hence, tend to increase confidence in the numerical methods used in this study.

8 CONCLUDING REMARKS

THIS paper has been devoted to an examination of the Wicksell-Slutsky-Frisch-Kalecki proposition that business cycles can be rationalized as the response of a stable dynamic model to random disturbances. Spectrum-analytic techniques were used to determine the dynamic properties of the impulse-response mechanism implicit in a condensed version of the Wharton Model. It was found that the power spectra implied by the model demonstrate that this model does not exhibit the twelve- to fifteen-quarter oscillations in response to random disturbances that are found in the original series. This means that the model under discussion implies that the source of business cycles is to be found in oscillations in the exogenous variables, or in the disturbance terms, and is not due to the dynamic structure of the system.

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DISCUSSION

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This paper is reminiscent of Irma and Frank Adelman's study of the Klein-Goldberger Model to determine whether it had the same cyclical characteristics as the American economy when both are measured by the National Bureau turning-point analysis.

Howrey compares the power spectra of certain variables from a forty-five equation Wharton Model with the corresponding empirical series from the American economy.

Carrying through this analysis on both the mathematical and computational levels for a model of this complexity is an impressive achievement in itself. The author does not mention how he factored the characteristic equation of the model to obtain its 56 roots. He may have some computer programs that would be of interest to other researchers. I will not be so unkind as to raise the question of the numerical accuracy of these difficult calculations. Not being a specialist in spectral analysis, I will not undertake to evaluate the statistical adequacy of the tests for cyclical peaks and so on, but, rather, will concentrate on the economic issues involved in the study.

NOTE: This review is the responsibility of the author and does not necessarily reflect the views of the Urban Institute or the Department of Labor, whose support is gratefully acknowledged.

Howrey starts off by distinguishing between various classes of business-cycle theory, deciding on a study of the Wharton Model to determine whether it would generate realistic cyclical behavior when subjected to random disturbances. He clearly recognizes that even though a model did *not* have a cyclical response, it could display cycles as the result of fluctuations of exogenous variables. Indeed, his findings either support the conclusion that the model does not have a cyclical response, or raise a question about the accuracy of the model as a description of the American economy. However, certain qualifications must be added to this conclusion. These will be touched on later.

Before examining the details of Mr. Howrey's analysis, we might briefly examine alternative approaches that could be used to answer the question under consideration for a linear system. Nonlinear analyses are much more complex. First, I already have mentioned the National Bureau's turning-point analysis. (Its stress on peaks and troughs implies nonlinear dynamics, although this is seldom made explicit.[1] Second, a model can be solved for a sequence of time periods until it reaches *equilibrium*, and may then be disturbed by introducing an exogenous impulse, allowing one to observe the transient response for cyclical fluctuations as it returns to equilibrium. Third, the model can be subjected to a sinusoidal disturbance, and the model solved for a sequence of periods until the system reaches a steady-state sinusoidal fluctuation. This process is repeated for a pattern of frequencies in order to determine those frequencies at which peaks in the system re-

[1] Linear systems tend to fluctuate in sinusoidal patterns. Measurement of the amplitude and timing of a sinusoidal fluctuation in the presence of noise would be best done as follows. We get a good measure of amplitude by averaging the fluctuations in the region of the peak, averaging in the region of the trough, and measuring the distance between these two averages. The most accurate timing measure of a sinusoidal fluctuation is the point at which the fluctuation crosses its middle level.

The NBER turning-point analysis is quite suitable for various nonlinear systems in which some critical event reverses the fluctuation. However, this analysis, when applied to noisy sinusoids, will tend to exaggerate the amplitude, because it is measured as the distance between random peaks and random troughs. The timing similarly is subject to random variation, depending on when the random peak occurred on top of a smooth-topped sinusoidal "peak."

Economic systems appear not to be linear, but many model estimates reveal only slight nonlinearities so the sinusoidal hypothesis may not be too far from the mark. Consequently, use of the NBER turning-point analysis should be made with considerable caution, taking account of its tendency to overestimate amplitude, and its tendency to be uncertain in its measurement of lead-lag relations.

sponse occur. Fourth, the real and complex roots of the characteristic equation may be determined computationally by factoring a high-order polynomial. Fifth, the power spectrum may be calculated for the system when it is excited by a matrix of random disturbances with known covariance and autocorrelation. Sixth, since fluctuations of exogenous variables that are not treated as random may account for part of the fluctuations of the system, the previous method can be modified to incorporate the additional contribution to the power spectrum.

The NBER method is the most suitable for nonlinear systems. The transient response is the easiest to perform on a model, but it will be influenced somewhat by the particular point at which the system is disturbed. Unfortunately, it is difficult to perform the same experiment on the national economy, so direct comparisons are not possible. The sinusoidal response is a little more trouble to calculate than the transient but the other points mentioned there apply equally well. Knowing the characteristic roots of the system tells a great deal, particularly if any of them are unstable, but how much each of these roots will be excited, and how they interact, can only be determined for particular sets of initial conditions and disturbances. In short, it is difficult to judge the system response from the roots. Finally, the power spectrum gives a clear picture of the amplitude of the system's response at various frequencies, when it is subjected to a realistic matrix of disturbance. Since the American economy is continually being bombarded by comparable disturbances, measurements performed on empirical time-series can yield spectra that are directly comparable to that of the model. However, nonrandom exogenous disturbances must be added to complete the picture before the business-cycle comparison is complete.

Of these alternatives, Howrey has used methods four and five. However, he did not include the autocorrelation of his disturbances, so the conclusion must be treated with caution. Since the random residuals are unobserved—but nonetheless real—variables having their own characteristic dynamic structure, the significance of including them but suppressing their autocorrelation is questionable. Perhaps one gains insight by dissecting a process to get at its separate dynamic components, but conclusions must be carefully stated. Since his work seems to suggest the importance of exogenous nonrandom disturbances, perhaps we can look forward to a later paper using the sixth approach.

In his examination of six representative variables from the American economy, Howrey finds evidence of cyclical peaks, but he concludes that only one, residential construction, is statistically significant. He examines the spectra of quarterly differences and deviations from *linear* trend. One would have thought that exponential growth would have been more suitable both for the economy and for linear dynamic models.

Howrey fails to explain to the reader that quite different spectra should be expected for the first differences and for the deviations from trend. If $x = A \sin ft$, then $\dot{x} = (Af) \cos ft$. Since the amplitude of x tends to rise with frequency f, this tends to offset partially the normal tendency for dynamic systems to be less responsive to high-frequency disturbance components.[2]

Casual examination of his empirical spectra suggests dual cyclical peaks—in inventory investment, and in residential construction. His analysis of quarterly changes in *GNP* seems to indicate peaks at 2.2 and 3.8 years, which seem suggestive of inventory, and construction and capital-goods, cycles respectively.

Turning now to the Wharton Model, which is nonlinear, it was first necessary to make a linear approximation, because the spectral analysis explicitly assumes linearity. Holding prices constant, the characteristic roots and the spectra of the *real* economy are examined. A growth rate of 62 per cent per quarter suggests that this system is hardly to be taken seriously—at least, for its static properties.

The wage equation contains a four-quarter difference which introduces a peak in the spectrum of the system corresponding to annual fluctuations. To cure this, the wage equation was simply deleted. But it seems rather strange that a model which was intended to be in *real* terms would not even contain an equation for money wages. These runs on the "real sector," both with and without the wage equation, are highly questionable.

The "complete" model has a more reasonable growth rate, but the wage equation is still omitted, with the result that the important wage-price interaction is lost. No peaks are evident in the spectra, and

[2] The argument for discrete time is not quite this simple, but the above serves to make the point. Note also that differencing will tend to shift the frequencies of cyclical peaks, and to shift the timing of the fluctuation by roughly 90 degrees.

hence the conclusion is reached that the dynamics of the model in responding to random disturbances do not generate business cycles. The reader should understand clearly that autocorrelated random disturbances will produce a different spectrum, and possibly a different conclusion about the business-cycle characteristics of the model.

A small point of questionable noncomparability that would tend to favor the opposite outcome in the comparisons between the model and the world, is the difference in data time-spans for model estimation and spectra calculations – 1948–65 and 1951–65, respectively.

The conclusion that the model's dynamics do not generate business cycles must be qualified in two ways. First, the original model was nonlinear, and errors introduced in the linear approximation may accumulate to produce serious differences in its dynamic behavior. Second, a wage equation should be introduced. It would be a simple matter to run a transient response on the original nonlinear model (first method above).

Perhaps the most significant finding of the Howrey paper is that a one-year difference in a Phillips' wage-change equation introduces a peak in the system spectra that seems counter to the empirical evidence of the American economy. This equation should be changed in the Wharton Model.

Knowing that the model failed to produce twelve- to sixteen-quarter oscillations, one might use hindsight to ask whether we could reasonably expect to detect dynamic lags of that duration reliably from relations estimated to predict quarterly changes. I would have thought that pressing a quarterly model to make forecasts eight quarters into the future was about the limit of any reasonable chance of success.

We know that most models tend to under-forecast change for reasons that are not fully clear. Similarly, most models tend to under-predict fluctuation amplitudes. We recognize that errors in the observation of variables bias the regression coefficients toward zero. For forecasting, this bias is quite suitable, because reduced weight should be put on noisy variables. However, when we are discussing the degree of dynamic stability of a system, we need unbiased estimates of the system parameters.

Since there is a great deal of circular causality in an economy, its degree of stability will be strongly influenced by the extent to which disturbances are amplified or attenuated as they travel around the many loops in a model economy. For example, one causal loop is given by the income, expenditures, production, income sequence. Regression estimates can be made of the functions, but if there were errors in observing the independent variables, the regression coefficients would be biased toward zero, and the response of a disturbance traveling around causal loops would be decreased. The clear consequence is that we would overestimate the stability of the system. This phenomenon should lead us to treat with caution Howrey's conclusion that there is no evidence in the estimated model of business-cycle peaking. We might experiment with the sensitivity of the model's stability to estimation biases. All parameters could be increased by, say, 20 per cent, and stability observed.

In closing, I would like to commend Howrey's efforts at validating the Wharton Model against empirical data in a new way. We must do *much* more work in testing our models.

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I believe that I should begin by emphasizing that I am not an econometrician. My interest in being here derives from my interest, as a statistician, in the general methodology of fitting models to time-series data. I have a great stake in believing that the art of econometric-model-building and the general theory of time-series model-making would benefit from closer interaction. Therefore, my reaction to Howrey's paper is a very favorable one, since I regard it as a valuable contribution to the study of the interrelationship between—

(1) the qualitative ideas (such as the notion of a business cycle) which econometricians are trying to quantitatively model;

(2) current econometric models; and

(3) modern time-series analysis (especially systems theory, adaptive prediction theory, and spectral analysis).

NOTE: These comments were made at the Conference except for the next-to-last paragraph, which is new.

I believe Howrey's perceptive work helps bring to our attention the following important questions—

(1) How can empirical spectral-analysis of observed economic time-series quantify the notion of business-cycle oscillations?

(2) Given business-cycle oscillations in endogenous variables, can one explain them by partitioning them among causes, such as the following:

> (*a*) the internal dynamics of the equivalent linearized "*system,*" representing the dependence of endogenous variables on exogenous variables;
>
> (*b*) business-cycle *oscillations in exogenous* variables;
>
> (*c*) serial correlation in the *disturbance* process.

(3) Given a current econometric model, as typified by a condensed version of the Wharton School Model, does it support the Wicksell-Frisch-Kalecki (WFK) proposition (see Section 7): "business cycles can be rationalized as the response of a stable dynamic system to random disturbances."

I believe that the conclusions stated by Howrey in his Section 7 are basically supported by the numerical calculations reported in the paper. However, as Howrey indicated in his oral presentation, these calculations need to be carefully checked, and may then lead to different conclusions.

In my comments, I would like to discuss some broad methodological issues which I believe should guide further research on the questions I have listed above.

We must always carefully formulate what we mean by the spectrum of a time series. In order that the qualitative notion of a business cycle can be quantitatively captured by the notion of spectrum (and more precisely, by the notion of "spectrum of a stationary time-series"), we must be aware that the spectrum we compute from observed time-series is very much dependent on the transformations we first apply to our data (such as first-differencing or linear detrending). I should note that Howrey, also, points out this fact in his footnote 3.

To illustrate the alternatives available to us in defining the notion of spectrum, let us consider an endogenous variable $Y(t)$, exogenous variable $X(t)$, and noise $N(t)$ satisfying

$$Y(t) = X(t) + N(t)$$

A possible model for $X(t)$ (which includes $X(t) = a + bt$, a straight line) is

$$\Delta^2 X(t) \equiv X(t) - 2X(t-1) + X(t-2) = \epsilon(t)$$

where $\epsilon(t)$ is a zero mean stationary time-series. The model for $X(t)$ can be written

$$X(t) = a + bt + \Sigma\Sigma\epsilon(t)$$

where the "double sum over epsilon" process is a highly nonstationary one. If we subtract from $Y(t)$ a linear trend, the residual series still has the sample spectral shape of the "double sum over epsilon" process. This is the shape which Granger has called the typical spectral shape of an economic time-series; the spectral density is decreasing steadily from a maximum at zero frequency.

From the foregoing considerations, I infer that of the two methods of trend elimination considered by Howrey—first differencing and linear detrending—*differencing gives the "right" answer.* In our simple model, it leads to the spectrum of the series

$$\Delta^2 Y(t) = \epsilon(t) + \Delta^2 N(t)$$

If $N(t)$ is white noise, $\Delta^2 N(t)$ is a second-order moving average with a high-frequency spectrum. Therefore, any low-frequency features in the spectrum of $\Delta^2 Y(t)$ must be those of $\epsilon(t)$. In our simple model for $Y(t)$, business-cycle oscillations in the differenced series must be ascribed to business-cycle oscillations in the spectrum not of $X(t)$, but in the spectrum of $\epsilon(t) = \Delta^2 X(t)$.

Turning now to the multiple-time-series case, let us use Howrey's notation in his equation (3.3):

$$(*) \qquad B(L)y(t) = C(L)x(t) + u(t)$$

This is the general form of a typical econometric model, and the question is to study how the spectrum of $y(t)$ is partitioned among the following five parameters of the model: $B(L)$, $C(L)$, the spectrum of $u(t)$, the spectrum of $x(t)$, and the cross-spectrum of $u(t)$ and $x(t)$.

Howrey computes the roots of $B(L)$ for two simplified versions of the Wharton Model, and studies the spectrum of $B(L)^{-1}u(t)$, under the assumption that $u(t)$ is a vector white-noise series. He finds that the spectrum of $B(L)^{-1}u(t)$ not only does not exhibit business-cycle oscillations, but is also basically flat.

On examining the table of "complex roots of a condensed version of the Wharton Model" distributed at the meeting, I am not so sure that this conclusion would hold for revised numerical calculations using the two new complex roots found—(.819, .193) and (.825, .088).

In any event, I believe that it would be interesting to see further empirical work along the lines of Howrey's paper, not only computing the roots of $B(L)$ for other models, but also examining all five of the parameters of the basic representation (*).