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An Empirical Model of United States Economic Growth: An Exploratory Study in Applied Capital Theory

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Introduction

RECENT publications in economics, both books and journals, are full of contributions to our knowledge of the process of economic growth.[1] At the same time, anyone who attempts to read through this literature cannot avoid a sense of frustration because of the conspicuous absence either of relevant data or of an adequate theoretical framework, depending upon whether he is attempting to test a theory of his own, or trying to organize a body of empirical information into a coherent system. Upon reflection, the conclusion seems unavoidable that, so far, communication between those whose primary interest is the construction of a theory of the process of economic growth and those who are concerned primarily with the organization of empirical knowledge about the process of economic growth has been poor at best.

We are often reminded that the Keynesian Revolution in economics had the profound impact that it did on the thinking of economists because the theoretical contribution of Keynes coincided with the then new availability of empirical information on aggregative income

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I also wish to express my gratitude to Professor James Tobin of Yale University, who generously permitted me to see his unpublished manuscript on monetary theory.

[1] See the References at the end of this paper for a list of more recent contributions on this subject. The list is not meant to be exhaustive.

and output that resulted from the painstaking efforts of the group of economists in and around the NBER. It was good fortune for economists and for those who use the findings of economists as the basis of their policy decisions that the concepts used to organize the income and output data and the concepts in terms of which Keynes formulated his theory were sufficiently similar for these two bodies of knowledge to complement each other.

While some of the more important contributions to the current theoretical discussion of the problem of economic growth were made a number of years ago [18] [36] [54],[2] the attempt to estimate parameters in growth models started only in very recent years. Similarly, systematic organization of the data dealing with growth, particularly those relating to accumulation of capital, is very new, as evidenced by the yet uncompleted NBER project on capital formation. It may still be too early to expect these two quite separate enterprises to be well coordinated. However, if we are to increase our knowledge of the process of United States economic growth, such a coordination must eventually be made; and this essay is an attempt to make a modest contribution in this direction.

The model discussed in the first section is a very simple one-commodity model which is a slight variation of the models often used in classroom exercises, and it is presented here briefly to provide motivation for the direction of generalizations undertaken in the subsequent sections. The second section of the paper is addressed mainly to two questions: first, which of the simple, convenient properties of the single-commodity model can be preserved when the model is generalized to contain many commodities, particularly if the production of each commodity is subject to its own distinct rate of technological change? Second, is it possible to define a concept of aggregate capital for which data are likely to be available, and for which a useful interpretation can be given in terms of a less aggregated model? The third and last section of this paper deals with a special case of the model presented in the second section. In this special case only two goods, a consumption good and a capital good, are distinguished. The consequences of the introduction of government and a monetary system into our model are then investigated, and some preliminary empirical findings are presented. In view of the complexity of the model and the inadequacies of the data, these

[2] The number in brackets refers to the similarly numbered item in the References at the end of this paper.

empirical findings should not be regarded as anything more than rough consistency checks for the model and an illustration of the use to which the model can be put if more reliable data become available.

I. A Model with a Single Good

Models of growth in which there is only one good have been studied by a number of authors, particularly by Harrod [25] [26], Domar [11], Solow [42], Tobin [46], and Ando and Modigliani [1] [2]. The purpose of reviewing them in this section is to provide a point of reference for later models to be considered in this paper and to facilitate the interpretation of these simple models in terms of a less aggregated model. Solow has shown that the extreme instability of the Harrod-Domar model of growth is due to the fixed coefficient production function that they implicitly assume. Thus, one of the crucial questions that must be faced at the outset is the choice of the production function. In this paper, I shall adopt, in contrast to Harrod and Domar, the Cobb-Douglas production function.

The aggregate production function is at best what Samuelson calls the "Clark-Ramsey Parable" [38], and the capital stock which goes into the aggregate production function cannot represent concrete, physical capital. In the next section of this paper, it will be shown that it is best interpreted as representing the value of existing capital, heterogeneous as it is, valued at reproduction cost, adjusted for changes in the general price level.[3] Even if the capital coefficients are absolutely fixed for production of each specific commodity, the composition of commodities produced varies over time, and there is no reason to believe that the relation between the aggregate value of capital and the value of output must remain constant over time—i.e., that the capital-output ratio should remain fixed. It seems more reasonable to represent this relationship in a somewhat more flexible form. In addition, in growth models we are dealing with a very long-run, broad pattern of economic relationships, ignoring the short-run adjustment problems. In the short run, there may be severe limitations on the way existing physical capital can be combined with labor to produce output. Given enough time, however, the composition and structure of capital can be changed, and the value of capital available per unit of labor can be shifted

[3] More exactly, it should be the market value of the existing capital, but the market value would be the same in the equilibrium as defined in this paper.

much more easily. These considerations suggest that a production function with a fixed capital coefficient is not an appropriate representation of reality. This, however, is not a sufficient reason to adopt the Cobb-Douglas production function, and my reason for doing so is largely the ease with which it can be handled analytically.[4]

I shall work with a production function of the form

$$Y_t = Xe^{gt}E_t^{1-\beta}K_t^{\beta} \tag{1.1}$$

where Y_t = the rate of output per year at time t

E_t = the rate of employment per year at time t (in man-hours)

K_t = the stock of productive capital employed at time t, measured in terms of output

g = the rate of technological change, assumed to be exogenous and constant over time

β = constant, relative share of income accruing to capital

X = a scale factor

e = the base of a natural logarithm

I assume either that the depreciation of capital takes place in the declining balance form at a fixed rate, or that capital, once built, never depreciates but has a given probability of becoming unusable, independent of its past history or age. These two interpretations lead to the identical mathematical formulation; so the reader may adopt either of them. In addition, technological change reduces the value of existing capital by making more efficient capital available. For example, suppose that capitals I and II have been produced last year and this year, respectively, at identical costs, but that because of the changes in technology, when combined with the same amount of labor, I produces only 0.8 times as much output as II. In this case an original unit of I is treated as 0.8 unit. Since technological change is assumed to be occurring at a constant rate, this consideration increases the rate of depreciation described above, and no further complication in our analysis is needed. With this interpretation of the meaning of K, the demand for capital and employment can be written as marginal conditions as follows:

[4] Recently, Arrow, Chenery, Minhas, and Solow have suggested a somewhat more general form of homogeneous production function of the first order, which includes both the fixed coefficient case and the Cobb-Douglas case as special cases, and yet is relatively easy to work with [4]. A substantial part of the results reported in this paper appears to be sustained even if the Cobb-Douglas function is replaced by this more general function, though the difficulties of estimation will be increased enormously.

$$r_t + \delta = \frac{\partial Y_t}{\partial K_t} = \beta \frac{Y_t}{K_t} \tag{1.2}$$

$$w_t = \frac{\partial Y_t}{\partial E_t} = (1 - \beta) \frac{Y_t}{E_t} \tag{1.3}$$

where w_t = wage rate in terms of output
r_t = rental rate on capital per year, in terms of output
δ = the rate of depreciation of capital, including the rate of obsolescence, as defined above

The supply of labor is assumed to be given exogenously, and takes the form:

$$L_t = L_0 e^{nt} \tag{1.4}$$

The market equilibrium condition in the labor market is then:

$$E_t = L_t \tag{1.5}$$

where L_t = labor supply at time t
L_0 = a constant
n = a constant, representing the rate of increase of labor supply

The supply of capital is given by the savings function. It is here that I will depart from Solow [42] and follow the formulation proposed by Ando and Modigliani [1] [2]. Most writers in the past, including Harrod [25], Domar [11], Solow [42], and Tobin [47], have assumed a constant saving-income ratio in their growth models. This is not as unrealistic an assumption as it may sound at first, since work by Goldsmith [20] [21] shows that the saving-income ratio for the United States does appear to be reasonably stable in the long run. Nevertheless, I will adopt the consumption function given below:

$$C_t = \alpha_1 w_t E_t + \alpha_2 A_t \tag{1.6}$$

where C_t = the rate of consumption per year
A_t = the value of net worth held by consumers, in terms of output
α_1 and α_2 are parameters, and assumed to be constant over time

The rationale and the empirical evidence for this consumption function are reported elsewhere [3]. In the context of the present paper, equation (1.6) has the advantage that (1) it is consistent with the stable saving-income ratio in the long run; (2) it provides a more explicit behavioral hypothesis about the consumers' holding of wealth; (3) in the short run, it makes the saving-income ratio move with income; (4) the market equilibrium condition for capital stock can be stated (as in equation 1.8 below), rather than the market equilibrium condition for the increment of capital stock—in my opinion, the former condition is more appropriate as a part of a growth model than the latter.

In this simple model, in which government activities and the existence of money and other financial assets, as well as land, are ignored, the rate of change of A is equal to the rate of output less the rate of depreciation (defined broadly to include the rate of obsolescence as described above) less the rate of consumption. Thus,

$$\dot{A}_t = Y_t - \delta A_t - C_t \tag{1.7}$$

Again in this simple model, net worth of consumers consists entirely of ownership of the capital good; so the supply of capital is equal to the net worth of consumers. Hence, the market equilibrium condition for capital is given by

$$K_t = A_t \tag{1.8}$$

As in any general equilibrium system, one of the market equilibrium conditions (in this case, that for output) is redundant. There are eight equations (1.1) through (1.8), in eight unknowns, Y_t, A_t, K_t, L_t, E_t, C_t, w_t, and r_t.

To analyze the behavior over time of this system, let us note that Euler's theorem applied to equation (1.1) gives

$$Y_t = w_t E_t + (r_t + \delta) K_t \tag{1.9}$$

Appropriate substitutions then result in:

$$\begin{aligned} \dot{A}_t &= w_t E_t + (r_t + \delta) K_t - \delta A_t - \alpha_1 w_t E_t - \alpha_2 A_t \\ &= (1 - \alpha_1) w_t E_t + (r_t - \alpha_2) A_t \end{aligned}$$

$$\frac{\dot{A}_t}{A_t} = k_t + (1 - \alpha_1)\left(\frac{w_t E_t}{A_t} - \frac{k_t + \alpha_2 - r_t}{1 - \alpha_1}\right) \tag{1.10}$$

where k_t is any number. In particular, we are free to define

$$k_t = \frac{\dot{K}_t}{K_t}$$

We are specifically interested in the existence of a growth path of this system on which k_t and r_t remain constant over time. If such a path exists and is unique, we shall define it as the *equilibrium* path of growth for this system. Equation (1.1) can be rewritten as

$$\frac{\dot{Y}_t}{Y_t} = g + \beta\frac{\dot{K}_t}{K_t} + (1 - \beta)\frac{\dot{E}_t}{E_t} \tag{1.11a}$$

Equations (1.2), (1.9), and (1.10) indicate that, if k_t and r_t were to remain constant over time, so must the ratio of w_tE_t to A_t, and furthermore the rates of growth of Y_t, w_tE_t, and A_t must all be the same. With this consideration, inserting the definition of k_t into equation (1.11a) yields

$$k_t = g + \beta k_t + (1 - \beta)n = \frac{g}{1 - \beta} + n \tag{1.11}$$

Equations (1.10) and (1.11) imply that if the rate of growth of the labor force is given by n and the rates of growth of A and of Y are both given by k defined by (1.11), the rate of rental on capital, r_t, will remain constant over time, thus defining an equilibrium path of growth. It can be shown that such a growth path in fact exists and that it is stable. To show this, let us define a new variable $a_t \equiv w_tE_t/A_t$. Substitution of this definition and that of k_t into equation (1.10) gives

$$a_t \equiv \frac{w_tE_t}{A_t} = \frac{k_t + \alpha_2 - r_t}{1 - \alpha_1} \tag{1.12a}$$

In addition, equations (1.2) and (1.3) and the definition of a imply

$$r_t + \delta = \frac{\beta}{1 - \beta}a_t \tag{1.12b}$$

Equation (1.12) can be solved to yield:

$$r^* = \frac{\beta(k + \alpha_2) - \delta(1 - \beta)(1 - \alpha_1)}{1 - \alpha_1(1 - \beta)} \tag{1.13}$$

$$a^* = (1 - \beta)\frac{k + \alpha_2 + \delta}{1 - \alpha_1(1 - \beta)} \tag{1.14}$$

To show that the path defined above is stable, suppose that, by some accident,

$$\frac{w_tE_t}{K_t} = a^* + \Delta a,\ \Delta a > 0.$$

Then, through equation (1.2),

$$r_t = r^* + \Delta r = \frac{\beta}{1-\beta}(a^* + \Delta a) - \delta$$
$$= \left(\frac{\beta}{1-\beta}a^* - \delta\right) + \frac{\beta}{1-\beta}\Delta a$$
$$= r^* + \frac{\beta}{1-\beta}\Delta a$$

Substitution of these values into equation (1.10) yields

$$\text{(1.15)} \qquad \frac{\dot{A}_t}{A_t} = k + \Delta a\left[(1-\alpha_1) + \frac{\beta}{1-\beta}\right]$$

Substituting this result into (1.11a) finally gives

$$\text{(1.16)} \qquad \frac{\dot{Y}_t}{Y_t} = k + \beta\Delta a\left[(1-\alpha_1) + \frac{\beta}{1-\beta}\right]$$

Comparison of (1.15) and (1.16) shows that if a is greater than a^*, i.e., if the ratio of labor income to capital is greater than its equilibrium value, capital is growing at a faster rate than output. But because of equation (1.3), the rate of growth of labor income is identical to the rate of growth of output. Hence, the equilibrium path of growth defined above is stable.[5]

The model analyzed above is obviously too simple, and contains a number of drastic assumptions that are not tenable if it is to be capable of helping us interpret long-run data for the United States. At this point it would be useful to make some observation on the less tenable assumptions underlying the above model in order to determine the directions in which it must be generalized. As in any work of this kind, the adequacy of a model depends on its purpose. As was indicated earlier, the main purpose of this paper is to explain the data relating to the accumulation of capital stock in the United States economy and its relation to the growth of capacity to produce output in the long run, neglecting short-run cyclical fluctuations. It is quite feasible that the short-run fluctuations may have serious effects on the long-run trend of the economy, making it necessary to analyze the short-run fluctuations and the long-run trends of the

[5] It must be added here that the above analysis is valid only to the extent that the parameters in the consumption function, α_1 and α_2, are invariant under changes in r. However, it can be easily shown that, if α_2 is a function of r while α_1 is independent of r (as is likely to be the case), the above analysis is completely unaffected provided that the absolute value of the first derivative of α_2 with respect to r is less than unity.

economy simultaneously; but such an undertaking must be deferred to a future paper.[6]

In the above model, aggregate capital, K, was not clearly defined, and was taken to mean the sum of the value of capital stock, without a justification. This, in turn, was equated to the value of consumers' net worth. The meaning of aggregate capital, and of the aggregate production function that has aggregate capital as one of its arguments, has been the subject of sharp controversy in recent years. We have learned from these controversies that this question can be discussed fruitfully only in the context of a specific model. Accordingly, in the next section, we shall define a model involving many capital goods that are distinct from one another, and endeavor to exhibit a set of assumptions under which the concepts of aggregate capital and aggregate production functions are meaningful. Furthermore, consumers' net worth in reality includes, in addition to the value of reproducible physical capital, the debt of the United States government (a part of money and United States securities), non-reproducible wealth (land), consumer durables, and a few other items. Problems arising from these discrepancies, and the roles of government and of the monetary system will be discussed in the third section of this paper.

Before turning to the task of generalizing the model, let us pause briefly and consider whether the simple model presented above possesses any resemblance to reality in terms of the available data.

I assume the following numerical values for the basic parameters of the system:

$$\alpha_1 = .65 \qquad g = .017$$
$$\alpha_2 = .07 \qquad \beta = .35$$
$$n = .006 \qquad \delta = .04$$

The values for α_1 and α_2 are taken from a study by Ando and Modigliani reported elsewhere [3], and these values, after rounding, appear to be reasonably stable over the period 1910–59, excluding war years. The value of n is the average rate of increase of man-hours per year according to Kendrick [28]. This figure appears to be surprisingly low, particularly since the average rate of increase of the labor force, according to *Historical Statistics* [50], is close to 0.017. However, I believe that the figure given by Kendrick is about as accurate as any that could be found. The figure for g is the aver-

[6] Some preliminary analysis dealing with this problem has been reported elsewhere [1] [2].

age of the rate of technological change reported by Solow [45]. The value of β is a problem, since there are a variety of estimates reported in a number of sources. I take the figure of 0.35, a round figure lifted from Solow in the work cited above, to make it consistent with the value of g. These figures imply, through equation (1.11), that the rate of growth of output should be roughly 3.2 per cent per year. The average rate of growth of net national product as implied by the data reported by Goldsmith [21] for 1896–1950 is roughly 3.4 per cent per year, a figure somewhat larger than the one mentioned above. The cause of this discrepancy may be that the periods to which each of the parameters refer do not match exactly. I shall use the figure 3.2 per cent for k for this illustration.

I need to guess at the value of one more parameter, the rate of depreciation. Goldsmith uses various rates of depreciation for different types of capital goods in his *Study of Savings*, and a rough computation to get a weighted average of these rates suggests that the over-all rate is about 4 to 5 per cent per year [21]. For the computation in this section, I accept the figure of 4 per cent for the rate of depreciation. Then, equation (1.13) implies that the rate of return on capital should be roughly 4.5 per cent per year, and equation (1.14) implies that the ratio of labor income to the value of consumer net worth should be approximately 0.16.

The table below shows historical values of ratios of property income to consumer's net worth and of labor income to consumer's net worth, averaged over the periods indicated. It should be clear that the comparison of these figures with the theoretical expectations of r^* and a^* suggested above cannot be more than an encouraging indication that our analysis should be generalized and refined, since there is a great deal of discrepancy at this stage of our analysis between the theoretical concepts used in the model and the definitions forced on us by the empirical data.[7]

[7] Since the model in this section does not allow for taxes and government expenditures, a decision must be made as to whether to use income after or before taxes. The figures reported here are labor income *after* taxes, but the ratio is of nonlabor income *before* taxes to the value of net worth. The choice may appear inconsistent, and the justification must await the results of the later model, involving taxes explicitly. As the model does not allow explicitly for the existence of intangible assets such as securities of the United States government and money and for nonreproducible tangible assets (land), we are also faced with the choice between total net worth of consumers and net worth of consumers less intangible assets and land. The figures in the following table are computed using total net worth. Here again, discussion of the possible bias due to this choice must be postponed until a later section, where I shall deal with a model that explicitly allows for the existence of intangible assets and land.

	$\frac{Y - wE}{A}$	$\frac{wE}{A}$
1910–17	.054	.16
1921–29	.062	.16
1935–41	.034	.16
1947–58	.058	.17

Even so, the correspondence between the figures reported in the preceding table and the theoretical expectations implied by the figures for the parameters given earlier is reasonably good. The fairly low value of the ratio of property income to net worth for 1935–41 is undoubtedly the consequence of the depressed state of the economy and partial unemployment of capital stock, and suggests that if we are to do serious statistical work using this type of model, we must adjust for underutilization of capital and labor.

On the other hand, for all other periods the actual rate of return is much higher than that implied by the values of parameters through equation (1.15). This is partly because the model does not allow for the existence of uncertainty and of monopoly power, which in the real world tend to make the rate of return somewhat higher. In addition, the actual figures reported, as mentioned above, represent the rate of return before taxes, and the effects of taxes on these rates must be explicitly treated before a meaningful comparison can be made.

In addition, there are reasons to believe that the values of the basic parameters, particularly those for g and n, have changed somewhat during the first half of the twentieth century; and if so, we must allow for these changes in our calculations.

We conclude this section with the acknowledgment that the model presented here is totally inadequate and must be generalized and refined substantially in order to explain the long-run data of the United States economy even approximately, and with the cautious hope that when generalized and refined, this type of model may yield reasonably satisfactory results.

II. A Model with One Consumption Good and Many Capital Goods

In this section, the consequences of recognizing heterogeneity of capital goods will be analyzed in detail, and the conditions examined under which the concept of aggregate capital can be meaningfully defined.[8]

II. A. PROPERTIES OF THE MODEL

The system to be analyzed is given by equations (2.1)–(2.11) below. In all cases, $i = 1, 2, \ldots, J$ and $j = 1, 2, \ldots, J - 1$.

$$O^i = X_i e^{g_i t} \prod_{j=1}^{J-1} K^{ij\beta_{ij}} E^{i(1-\beta_i)}, \tag{2.1}$$

where
$$\beta_i = \sum_{j=1}^{J-1} \beta_{ij}$$

$$p^j(r^j + \delta_j) = \beta_{ij} \frac{O^i}{K^{ij}} p^i \tag{2.2}$$

$$p^J w = (1 - \beta_i) \frac{O^i}{E^i} p^i \tag{2.3}$$

$$L_0 e^{nt} = \sum_{i=1}^{J} E^i \equiv E \tag{2.4}$$

$$K^j = \sum_{i=1}^{J} K^{ij} \tag{2.5}$$

$$C = \alpha_1 w E + \alpha_2 \frac{A}{p^J} \tag{2.6}$$

$$C = O^J \tag{2.7}$$

$$O^j = \dot{K}^j + \delta_j K^j \tag{2.8}$$

$$A = \sum_{j=1}^{J-1} p^j K^j \tag{2.9}$$

$$r = r^j + \frac{\dot{p}^j}{p^j} \tag{2.10}$$

$$p^J = 1 \tag{2.11}$$

The convention adopted is that indexes $1, 2, \ldots, J - 1$ represent capital goods, and the index J denotes the consumption good. The time subscript is omitted except where any possibility of confusion exists. In the definitions below, $i = 1, 2, \ldots, J$ and $j = 1, 2, \ldots, J - 1$:

[8] It will be assumed that there exists a single homogeneous consumption good. It is possible to generalize the model to cover the case of a number of heterogeneous consumption goods and still keep most of the conclusions unaltered, provided that price and income elasticities of demand for all consumption goods are unity. Cf. the results given by Dhrymes [10].

O^i = the rate of output of ith good, in physical units
E^i = man-hours employed in production of the ith good
K^{ij} = quantity of jth capital employed in production of the ith good, in physical units
K^j = the existing stock of the jth capital good
r^j = the rate of rental accruing to the jth capital good, in terms of the jth capital good
p^i = price of the jth good
w = wage rate, in terms of the consumption good
C = the rate of consumption, in terms of the consumption good
A = net worth of consumers
r = the rate of interest

Equations (2.1) through (2.11) constitute a system of $J(J + 4) + 2$ equations in the same number of variables. With i and j having the ranges noted above, the parameters of the system are:

δ_j = the rate of depreciation of the jth capital good
n = the rate of growth of labor, measured in man-hours
g_i = the rate of technological change in the production of the ith good; assumed to be exogenous in the system
β_{ij} = parameters of production functions
α_1, α_2 = parameters of the consumption function
X_i = scale factors in the production functions
L_0 = scale factor in the labor supply

Equation (2.11) states that the consumption good will be used as the nummeraire in this section.

Equations (2.1) are production functions. As in the case of (1.1), it is assumed that technological change is neutral and constant over time, though the rate (g_i) is *different* for different industries. Unlike the case of (1.1), however, K^{ij} represents the number of machines of the jth type employed in the ith industry. Because of this, one may seriously question the appropriateness of the Cobb-Douglas production function, with its unitary elasticity of substitution. At best, this form of production function can be justified only as an approximate description of the relationship between output and inputs in the long run when everything is optimally adjusted. Equations (2.2) and (2.3) are marginal conditions, and they define the demand conditions for labor and capital stock; (2.4) and (2.5) are market equilibrium conditions for labor and capital stock, respectively. Equation (2.6) is the consumption function, which has already been discussed in Section I, while equation (2.7) is the market equilibrium condition for the consumption good. Equation (2.8) gives the market equilibrium conditions for the capital goods, and states that the supply of each capital good must be equal to the rate of net increase of demand for the stock of that capital good plus the depreciation of the stock. Equation (2.9) defines net worth of consumers. Equation (2.10) represents the well-known proposition that, for the markets for many capital goods to be simultaneously in equilibrium, it is necessary that the rate of rental of each capital good measured in terms of itself plus the rate of change of its price must be equal to the rate of interest.[9]

The condition equivalent to equation (1.7) in the foregoing section I can be derived from the above system, and is exhibited below:

(2.12)

$$\dot{A} = P^J wE + \sum_{j=1}^{J-1} p^j(r^j + \delta_j)K^j + \sum_{j=1}^{J-1} P^j \dot{K}^j - \sum_{j=1}^{J-1} \delta_j \dot{P}^j K^j - P^J C$$

It will be convenient to define here the following notations:

$$a \equiv \frac{P^J wE}{A}$$

$$\pi_j \equiv \frac{\dot{P}^j}{P^j} \qquad j = 1, 2, \ldots, J-1$$

$$k_j \equiv \frac{\dot{K}^j}{K^j} \qquad j = 1, 2, \ldots, J-1$$

$$\phi_j \equiv \frac{P^j O^j}{P^J O^J} \qquad j = 1, 2, \ldots, J$$

Note that $\phi_J = 1$ by definition.

The behavior of the system defined by equations (2.1) through (2.11) can be analyzed in a number of ways. The analysis presented below concentrates on the situation in which all k_j's are constant over time. It will be shown that a growth path on which all k_j's are constant exists, and that such a growth path (which will be called the "equilibrium growth path" in this paper) exhibits a number of distinctive characteristics.

The reason for the special attention given to this path is that, regardless of the initial conditions, the system will eventually approach the equilibrium path of growth, given enough time.[10] Thus,

[9] For a discussion of this condition, see Samuelson [37] [39].

[10] The stability for the general case has not been rigorously proved, but a sufficient number of special cases has been proved; so we feel the conjecture in the text is a reasonably safe one.

if the basic structure of the United States economy bears any resemblance to the system specified by equations (2.1)–(2.11), it may be expected to fluctuate around the equilibrium path, and the long-run data for the United States economy should be roughly consistent with the general characteristics of the equilibrium path, showing minor deviations at all times and major deviations at some times. Therefore, in order to see whether or not the long-run data for the United States confirm in broad outline the predictions of the model proposed in this paper, it is only necessary to exhibit the characteristics of the equilibrium path of growth—a task much easier than that of setting down the properties of the model in general. It must be emphasized that the data should not be expected to conform to the properties of the equilibrium path in detail; but the long-run data are very rough in any case, making it impossible to judge predictions of the model in detail. However, our decision to compare the data against the equilibrium path does imply that we should exclude from our consideration observations for years of the Great Depression and war, and that we should make adjustments in the data for cyclical underemployment of labor and resources.

If k_j's are to remain constant over time, the equations (2.8) imply[11]

$$\frac{\dot{K}^j}{K^j} = \frac{\dot{O}^j}{O^j} = k_j;\ j = 1, 2, \ldots, J - 1 \tag{2.13}$$

Our strategy is to suppose that there exists a set of values of k_j, π_j, ϕ_j, r^j, r, and a which is consistent with the system (2.1)–(2.11) and which can remain constant over time, and then attempt to exhibit such a set of values of these variables in terms of the parameters of the system (2.1)–(2.11). If we can so express them, then we shall have proved the existence of the equilibrium path on which these variables remain constant.

From equation (2.2), for any given j and a pair of indexes i and i', we may write

$$\frac{\beta_{ij}K^{i'j}}{\beta_{i'j}K^{ij}} = \frac{P^{i'}O^{i'}}{P^iO^i} = \frac{\phi_{i'}}{\phi_i}$$

$$\frac{K^j}{K^{ij}} = \sum_{i'=1}^{J} \frac{K^{i'j}}{K^{ij}} = \sum_{i'=1}^{J} \frac{\beta_{i'j}\phi_{i'}}{\beta_{ij}\phi_i}$$

[11] Write (2.8) in the form

$$\frac{O^j}{K^j} = \frac{\dot{K}^j}{K^j} + \delta_j$$

Since δ_j is a constant, $\dot{K}^j/K^j$ is constant if, and only if, O^j/K^j is constant for all t, implying (2.13).

Hence, the relative distributions of capital stocks among industries are given by

$$\text{(2.14a)} \qquad \frac{K^{ij}}{K^j} = \frac{\beta_{ij}\phi_i}{\sum_{i=1}^{J} \beta_{ij}\phi_i}; \quad \begin{array}{l} i = 1, 2, \ldots, J \\ j = 1, 2, \ldots, J-1 \end{array}$$

Similarly, the relative distribution of labor among industries is given by

$$\text{(2.14b)} \qquad \frac{E^i}{E} = \frac{(1-\beta_i)\phi_i}{\sum_{i=1}^{J} (1-\beta_i)\phi_i}; \; i = 1, 2, \ldots, J$$

Equations (2.14) indicate that if the ϕ's remain constant over time, the distribution of factors of production over industries must also remain constant. These constancies, through production functions (2.1) and equation (2.13), imply:

$$\text{(2.15)} \qquad k_i = g_i + \sum_{j=1}^{J-1} \beta_{ij}k_j + (1-\beta_i)n; \; i = 1, 2, \ldots, J-1$$

Equation (2.15) can be solved to give values of k_j's, $j = 1, 2, \ldots, J-1$, in terms of the g's, β's, and n.

The constancy of the ϕ_j's, through their definition and equations (2.13), implies that

$$\text{(2.16)} \qquad \frac{\dot{O}^J}{O^J} = \pi_j + k_j; \; j = 1, 2, \ldots, J-1$$

In addition, from the production function for O^J, we have

$$\text{(2.17)} \qquad \frac{\dot{O}^J}{O^J} = g_J + \sum_{j=1}^{J-1} \beta_{Jj}k_j + (1-\beta_J)n$$

When the values of the k_j's obtained from (2.15) are first substituted into (2.17) and the value of $\dot{O}^J/O^J$ so obtained is then substituted into (2.16), the equilibrium values of the π_j's, $j = 1, 2, \ldots, J-1$, may be obtained. Thus, the equilibrium values of the k_j's and the π_j's can be expressed in terms of n, the β's, and the g's.

There remains the problem of expressing the equilibrium values of ϕ_j, r^j and a in terms of parameters.

Returning to equations (2.2), using (2.8) and (2.14), the following relations can easily be derived:

$$(2.18)\qquad r^j + \delta_j = \beta_{jj}\frac{\frac{O^j}{K^j}}{\frac{K^{jj}}{K^j}}$$

$$= \beta_{jj}(k_j + \delta_j)\frac{\sum_{i=1}^{J}\beta_{ij}\phi_i}{\beta_{jj}\phi_j};\ j = 1, 2, \ldots, J-1$$

Equations (2.10) and (2.18) yield

$$(2.19)\qquad \pi_1 + \beta_{11}(k_1 + \delta_1)\frac{\sum_{i=1}^{J}\beta_{i1}\phi_i}{\beta_{11}\phi_1} - \delta_1$$

$$= \pi_j + \beta_{jj}(k_j + \delta_j)\frac{\sum_{i=1}^{J}\beta_{ij}\phi_i}{\beta_{jj}\phi_j} - \delta_j;\ j = 2, 3, \ldots, J-1$$

From equations (2.3) and (2.4), it can be seen that

$$(2.20)\qquad p^J wE = \sum_{i=1}^{J} p^i(1 - \beta_i)O^i$$

Hence, the definition of a can be rewritten as

$$(2.21)\qquad a = \frac{\sum_{i=1}^{J} p^i(1-\beta_i)O^i}{\sum_{j=1}^{J-1} p^j K^j} = \frac{\sum_{i=1}^{J}(1-\beta_i)\frac{p^iO^i}{p^JO^J}}{\sum_{j=1}^{J-1}\frac{p^jO^j}{p^JO^J}\frac{K^j}{O^j}} = \frac{\sum_{i=1}^{J}(1-\beta_i)\phi_i}{\sum_{j}^{J-1}\frac{\phi_j}{k_j+\delta_j}}$$

Let us define k_J by

$$(2.22)\qquad k_J \equiv \frac{\dot{O}^J}{O^J} = \pi_j + k_j;\ j = 1, 2, \ldots, J-1$$

Note that, for any j, on the equilibrium path,

$$\frac{\frac{d(p^jK^j)}{dt}}{p^jK^j} = \pi_j + k_j = k_J$$

Hence, it is clear, from the definition of A, that $\dot{A}/A = k_J$.

In view of the definitions of A and r given by (2.9) and (2.10), equation (2.12), after substitution into it of (2.6), may be rewritten as

$$(2.12a)\qquad \dot{A} = p^J wE + rA - \alpha_1 p^J wE - \alpha_2 A$$

When both sides of the above equation are divided through by A, and the resulting terms on the right-hand side are rearranged by the addition and subtraction of k_J, it becomes

$$\frac{\dot{A}}{A} = k_J + (1 - \alpha_1)\left(a - \frac{k_J + \alpha_2 - r}{1 - \alpha_1}\right) \tag{2.23}$$

It is clear from equation (2.23) that A grows at the rate k_J if, and only if, the relation

$$a = \frac{k_J + \alpha_2 - r}{1 - \alpha_1} \tag{2.24}$$

holds.

We now have equations (2.18), (2.19), (2.21), and (2.24), giving $2J - 1$ conditions to determine $2J - 1$ quantities, $(J - 1)\phi_j$'s, $(J - 1)r^{j}$'s, and a. Since equations (2.15), (2.16), and (2.17) have already determined k_j's and π_j's, it has now been established that there exists a growth path on which the rates of change of prices, π_j, the rates of growth of capital stock, k_j, the relative shares of value of each output to the value of output of the consumption good, ϕ_j, the own rental rates on each capital good, r^j, and the ratio of labor income to the value of consumers' net worth, a, are all constant.

It is easy to show that on the equilibrium path of growth having the properties described above, the over-all share of gross income (defined not to include capital gains) going to labor, denoted by $1 - \beta$, is given by

$$1 - \beta \equiv \frac{P^J wE}{\sum_{j=1}^{J} P^j O^j} = \frac{\sum_{j=1}^{J} \phi_j(1 - \beta_j)}{\Sigma\phi_j} \tag{2.25}$$

The ratio of saving to income, denoted by s, is given by

$$s \equiv \frac{\dot{A}}{\sum_{j=1}^{J} P^j O^j} = \frac{\dot{A}}{A}\,\frac{(1 - \beta)}{\frac{P^J wE}{A}} = k_J\,\frac{(1 - \beta)}{a} \tag{2.26}$$

Equation (2.26) states the familiar proposition that the saving-income ratio is the rate of growth of output measured in terms of the consumption good times the asset-income ratio. However, it should be noted that the definition of the saving-income ratio given above is somewhat unusual in that saving is defined to *include* capital gains, while income is defined to *exclude* capital gains.[12]

While, in principle, π_j, k_j, ϕ_j, r^j, and a can all be solved for in terms of parameters of the system, to do so in fact is quite tedious because of the nonlinearity of some of the equations involved. However, it is instructive to write down the expressions for the equilibrium values of these magnitudes for the case in which $J = 2$, i.e., there is only one consumption good and one capital good. Denoting equilibrium values by starred symbols, it can be shown that:

$$k_1^* = \frac{1}{1 - \beta_{11}} g_1 + n, \tag{2.27}$$

i.e., the rate of growth of the capital stock and of the capital-good production.

$$k_J^* = g_J + \beta_{J1} \frac{1}{1 - \beta_{11}} g_1 + n, \tag{2.28}$$

i.e., the rate of growth of the consumption-good production.

$$\pi_1^* = g_J - \frac{1 - \beta_{J1}}{1 - \beta_{11}} g_1, \tag{2.29}$$

i.e., the rate of change of price of the capital good relative to that of the consumption good.

$$\phi_1^* = \frac{(k_1^* + \delta_1)[1 - \alpha_1(1 - \beta_{J1})]}{(1 - \beta_{11})(k_1^* + \delta_1) + \alpha_2}, \tag{2.30}$$

i.e., the ratio of the value of the capital-good output to that of the consumption-good output.

$$r^{J*} = \frac{\beta_{J1}\alpha_2 + k_1^*[\beta_{11}(1 - \alpha_1) + \alpha_1\beta_{J1}] - \delta_1(1 - \alpha_1)(1 - \beta_{11})}{1 - \alpha_1(1 - \beta_{J1})}, \tag{2.31}$$

i.e., the rate of rental of the capital stock in terms of itself.

$$a^* = \frac{(k_1^* + \delta)(1 - \beta_{11}) + \alpha_2(1 - \beta_{J1})}{1 - \alpha_1(1 - \beta_{J1})}, \tag{2.32}$$

i.e., the ratio of labor income to the value of consumers' net worth.

The value of k_1^* given by equation (2.27) is very similar to the value

[12] The saving-income ratio in which income is also defined to *include* capital gains, but to *exclude* depreciation, is given by

$$s' \equiv \frac{\dot{A}}{p^J wE + rA} = \frac{k_J}{a + r}$$

for the rate of growth for the single-good model given by equation (1.11). In particular, this expression is independent of any characteristic of the production function for the consumption good. This is because the consumption good does not contribute to the production of the capital good, and the distribution of factors between the two industries remains constant along the equilibrium growth path. The rate of growth of the consumption-good output, given by equation (2.28), indicates that this rate is the sum of the rates of growth of the labor force and technological change in consumption-good production, supplemented by the modified rate of technological change in capital-good production. The factor modifying the rate of technological change in capital-good production takes account of the importance of capital in the production of the consumption good, and the effectiveness of technological improvement in capital good production in the production of the capital good. The rate of change of relative prices, given by equation (2.29), is the difference between the rates of technological change in the consumption and the capital-good-producing industries, modified by the intensities with which capital is used in both industries. The faster the technological improvement in the consumption-good industry and the slower the technological improvement in the capital-good industry, the greater the rate of increase in the price of the capital good relative to the consumption good. This tendency will be stronger the more labor-intensive the production of the consumption good, and the more capital-intensive the production of the capital good.

Since the distribution of factors between the two sectors is constant on the equilibrium path of growth, and the rates of technological change in the two sectors are not the same, the rates of growth of output in physical terms are not the same in the two sectors. However, the system as a whole will generate changes in relative prices such that they insure that the *value* of output will grow at the same rate in all sectors. The expressions (2.29) and (2.30) represent these properties of the model.

Finally, equations (2.31) and (2.32) express the equilibrium rate of rental of capital in terms of itself, and the equilibrium ratio of labor income to the value of consumers' net worth both measured in terms of the consumption good.[13]

[13] It may be noted that if there are no consumption requirements, i.e., $\alpha_1 = \alpha_2 = 0$, and there is no nonproducible factor of production, i.e., $\beta_i \equiv 0$ for all i, then the system represented by equations (2.1) through (2.11) reduces to a special case of the von

In abstract theory, the presence of many heterogeneous capital goods will merely make the expressions (2.27) through (2.32) more complex. In principle, it is possible to write down these expressions, and investigate their characteristics for any finite number of capital goods. However, in order to utilize them in empirical studies, the situation is not so simple. We should like to estimate the parameters of the system, g's, β's, α's, and n, and inquire, for suitably selected periods with reasonably full employment without abnormal shocks such as wars, whether or not observed values of k^{**}'s, π^{**}'s, ϕ^{**}'s, r^{**}'s, and a^* are in fact reasonably close to those given by substituting into equations (2.27) through (2.32) empirically observed values of the parameters. Data needed for such an inquiry are available for very aggregated sectors at best, and even then must be used with extreme caution. I have put together a set of data which may enable us to work with the two-sector version of the above model. But a serious question arises in the empirical interpretation of the aggregated variables in the above model, particularly in the meaning of aggregate capital. It is well known that various indexes of aggregate capital (for instance, on the one hand, the market value of aggregate capital deflated by a single price index such as an implicit GNP deflator; and on the other, the so called "real capital stock" in most of the National Bureau publications, constructed by deflating various segments of capital stock by their own price indexes and then summing) move substantially differently over time. We must make sure, therefore, that the empirical definition of aggregate capital most consistent with our model will be adopted. We shall turn to this question in the next section.

II. B. AN INTERPRETATION OF THE CONCEPTS OF AGGREGATE CAPITAL AND OUTPUT

We require our definition of aggregate capital and output to be such that the relation between them reflects accurately an aspect of the system defined by equations (2.1) through (2.11). Furthermore, we wish the relation between such aggregate concepts to be in the form of the Cobb-Douglas production function. In addition, when such a relation is treated as though it is in fact a production function,

Neumann model of growth, and the "equilibrium path of growth" defined above is a von Neumann ray. One indication of this is seen in the fact that, under such conditions, r^{J*} given by (2.31) reduces to k_1^*, a well-known property of the von Neumann ray. On the other hand, under these conditions, a number of other concepts introduced in this section must be modified to remain well defined.

the results should be capable of meaningful interpretation—the exponents of labor and capital should represent the shares of income accruing to labor and capital, and the rate of technological change measured by Solow's method of residuals should be a weighted average of the rates of technological change in the disaggregated functions.

Since, in the system defined by equations (2.1) through (2.11), technological changes are occurring in different industries at different speeds, the relative size of the stocks of different capital goods will be constantly changing. In such a situation it is difficult to define a concept of the aggregate capital stock in terms of physical units of individual capital goods. However, we note that the values of the stocks of all capital goods, p^jK^j's, are growing at the same rate if the system is moving along the equilibrium path of growth. In view of this, it is tempting to ask whether the value of capital can be used as the measure of capital to be introduced into the production function.

Let us, then, consider the following function for any i, where $i = 1, 2, \ldots, J$.

$$O^{i*} = X_i^* e^{g_i t} E^{i(1-\beta_i)} \prod_{j=1}^{J-1} \left(\frac{p^j}{p^k} K^{ij}\right)^{\beta_{ij}} \tag{2.33}$$

where p^k is some price index. For the moment, all we shall require of $\dot{p}^k$ is that $\pi_k \equiv p^k$ be constant over time if the system is on the equilibrium path of growth. Equation (2.33) is not a production function. It is considered here simply in order to provide a possible clue to the interpretation of aggregate capital. (2.33) may be rewritten as

$$O^{i*} = X_i^* e^{g_i t} E^{i(1-\beta_i)} \prod_{j=1}^{J-1} K^{ij\beta_{ij}} \prod_{j=1}^{J-1} \left(\frac{p^j}{p^k}\right)^{\beta_{ij}} \tag{2.33a}$$

Now, from the analysis in the preceding section, it is known that $\dot{p}^j/p^j = \pi_j$ is constant on the equilibrium path of growth, and hence, $\pi_j - \pi_k$ is constant also. Therefore, on the equilibrium path of growth, the movements of O^{i*} defined by (2.33) and of O^i given by the proper production function will be strictly parallel if[14]

$$\sum_{j=1}^{J-1} \beta_{ij}(\pi_j - \pi_k) = 0 \tag{2.34}$$

[14] A sufficient, but certainly not necessary, condition for (2.34) to hold is that technological changes proceed in all capital goods industries at the same speed.

Let us consider p^k to be defined by the above equation (2.34), i.e.,

$$\text{(2.34a)} \qquad \pi_k = \frac{\sum_{j=1}^{J-1} \beta_{ij}\pi_j}{\sum_{j=1}^{J-1} \beta_{ij}}; \; \frac{\dot{p}^k}{p^k} = \pi_k$$

On the equilibrium path of growth, it is also known that $(p^j/p^k)K^{ij}$ is growing at the rate $k_J - \pi_k$ for all j. Let us write

$$\text{(2.35)} \qquad \frac{p^j}{p_k} K^{ij} \equiv K_o^{ij} e^{k^* t}; \; k^* = k_J - \pi_k$$

where K_o^{ij} is some constant, and substitute this expression into (2.33).

$$\text{(2.36)} \qquad O^{i*} = X_i^* e \quad E^{i(1-\beta_i)} e^{k^* t \sum_{j=1}^{J-1} \beta_{ij}} \prod_{j=1}^{J-1} K_o^{ij\,\beta_{ij}}$$

Since $\prod_{j=1}^{J-1} K_o^{ij\beta_{ij}}$ is a constant over time, it may be subsumed under X_i^*; and in terms of time series data, the share of capital will turn out to be approximately $\sum_{j=1}^{J-1} \beta_{ij} = \beta_i$ if (1) the system does not deviate too far from the equilibrium path of growth; (2) O^i is used as the measure of output; and (3) some index which moves parallel over time to $(p^j/p^k)K^{ij}$ is used as the measure of capital. An index satisfying condition (3) is given by

$$\text{(2.37)} \qquad K^* = \sum_{j=1}^{J-1} \frac{p^j}{p^k} K^{ij}$$

Thus, one proposition has been established which is important in giving us some guide to the treatment of aggregative data: if some price index, p^k, can be found which satisfies the condition (2.34), then the use of an output index, O^i, and the capital index, K^*, will enable us to estimate g_i reasonably well. The exponent of K^* may be interpreted as the share of capital.

In order to see if it is possible to define and interpret the marginal product of K^*, let us for a moment suppose that an index p^k satisfying (2.34) can be found and that O^{i*} in equation (2.33) is replaced by O^i. Does the partial derivative of O^i with respect to K^* have any interpretation? To obtain some clue to the answer to this question, note first that

$$\frac{\partial O^i}{\partial\left(\frac{p^j}{p^k}K^{ij}\right)} = \frac{\partial O^i}{\partial K^*}\frac{\partial K^*}{\partial\left(\frac{p^j}{p^k}K^{ij}\right)} \tag{2.38}$$

Equation (2.37) implies that $\partial K^* \Big/ \partial\left(\frac{p^j}{p^k}K^{ij}\right) = 1$, and, by direct differentiation of (2.33)

$$\frac{\partial O^i}{\partial\left(\frac{p^j}{p^k}K^{ij}\right)} = \beta_{ij}\frac{O^i}{K^{ij}}\frac{p^k}{p^j} \tag{2.39}$$

Not only does the expression on the right-hand side of equation (2.39) have a very ambiguous meaning at best because the units are wrong, but this equation fails to define $\partial O^i/\partial K^*$ in (2.38), since (2.39) says that the left-hand side of equation (2.38) has different values depending on j.

The right-hand side of equation (2.39) has a well-defined meaning only when $p^k = p^i$. In that case it is the marginal product of K^{ij} in terms of K^{ij} in the production of O^i. Suppose, then, that $p^k = p^i$, and consider a new concept O^{i**} defined by

$$O^{i**} \equiv O^i + \sum_{j=1}^{J-1}\left(\frac{\dot{p}^j}{p^i} - \frac{p^j}{p^i}\delta_j\right)K^{ij} \tag{2.40}$$

Then

$$\begin{aligned} \frac{\partial O^{i**}}{\partial\left(\frac{p^j}{p^i}K^{ij}\right)} &= \beta_{ij}\frac{O^i}{K^{ij}}\frac{p^i}{p^j} + \frac{\dot{p}^j}{p^j} - \delta_j \\ &= r^j + \delta_j + \pi_j - \delta_j \end{aligned} \tag{2.41}$$

Hence, through an equation similar to (2.38),

$$\frac{\partial O^{i**}}{\partial K^*} = r^j + \pi_j = r \tag{2.42}$$

The above analysis suggests that, if O^{i**} (output less depreciation of capital used in production of O^i in terms of O^i plus capital gains or losses on capital goods used in production of O^i) is used to measure output instead of O^i, then the marginal product of K^* in the production of O^{i**} is equal to the rental on any capital good in terms of itself plus capital gain, $r^j + \pi_j$, the value of which is independent of j.

Equation (2.42) is valid only if p^i is proportional to p^k defined by equation (2.34a). This is equivalent to saying that the marginal product of K^* is well defined only if the rate of technological change in the production of the ith good is equal to some appropriately weighted average of the rates of technological changes in the production of capital goods. This is a reasonable, but not very helpful, result, since such an accidental equality cannot be counted on. However, there is one consolation. Let equation (2.2) be rewritten in the form

$$p^j K^{ij}(r^j + \delta_j) = \beta_{ij} O^i p^i \tag{2.2a}$$

Adding $p^j K^{ij}(\pi_j - \delta_j)$ to both sides gives:

$$p^j K^{ij}(r^j + \pi_j) = \beta_{ij} O^i p^i + p^j K^{ij}(\pi_j - \delta_j) \tag{2.2b}$$

The summation of this expression over j, where $j = 1, \ldots, J - 1$, on both sides, noting that $r^j + \pi_j = r$ for all j, results in:

$$r \sum_{j=1}^{J-1} p^i K^{ij} = \beta_i O^i p^i + \sum_{j=1}^{J-1} p^j K^{ij}(\pi_j - \delta_j)$$

and hence

$$r = \frac{\beta_i O^i p^i + \sum_{j=1}^{J-1} p^j K^{ij}(\pi_j - \delta_j)}{\sum_{j=1}^{J-1} p^j K^{ij}}; \; i = 1, 2, \ldots, J \tag{2.43}$$

Equation (2.43) says that if the share of capital in the value of total gross output in any industry is adjusted for the total capital gains and losses and for the depreciation on all capital goods used in the industry, and is divided by the value of all capital goods used in the industry, the result should be the same in all industries, and it should be equal to the rate of interest. Furthermore, it is clear from the foregoing analysis that equation (2.43) holds even when the data are aggregated over any number of industries. Thus, in spite of the difficulties of defining the precise meaning of $\partial O^J / \partial K^*$, there is a practical way of estimating the value of r from readily available aggregate data.[15]

We conclude, then, that the measure of aggregate capital stock should be the value of capital deflated by some price index satisfying

[15] At the risk of laboring an obvious point, I shall restate the implication of equation (2.43). Only when the rental rate is redefined to include the rate of capital gain can it be computed through equation (2.43), using aggregative data. This is reasonable in a model such as the one in this paper in which complete certainty and perfect knowledge are assumed, since, under these assumptions, individuals must be indifferent between the gain from the rental paid to them and the gain from the increase in value of the capital good they own.

the condition (2.34) as closely as possible. There remains the question of aggregating output, to which I shall now turn.

Suppose that the production function (2.1) is replaced by

$$O^i = X^{i*} e^{g_i t} E^{i(1-\beta_i)} K^{*i\beta_i}; \; i = 1, 2, \ldots, J \tag{2.44}$$

where K^{*i} is defined in the analogous manner to K^* in equations (2.37) and (2.34). If it is permissible to assume that the price indexes for defining K^{*i} for all i are sufficiently similar so that the movements of K^{*i} over time are also similar (provided that the system is on the equilibrium path of growth), then equation (2.44) would imply the following approximate equalities on the equilibrium path of growth:

$$k_i = g_i + (1 - \beta_i)n + \beta_i k^* = k_J - \pi_i = k^* + \pi_k - \pi_i \tag{2.45}$$

Let us define the "total output," O^k, of the capital goods industries and the "aggregate" capital, K^k, and "aggregate" labor force, E^k, employed in the capital goods industries by

$$O^k = \sum_{j=1}^{J-1} \frac{p^j O^j}{p_k}; \; K^k = \sum_{i=1}^{J-1} \frac{p^j K^{ij}}{p^k}; \; E^k = \sum_{j=1}^{J-1} E^j \tag{2.46}$$

and consider the "aggregate" production function of capital goods

$$O^k \simeq X_k e^{g_k t} K^{k\beta_k} E^{k(1-\beta_k)} \tag{2.47}$$

where β_k is defined, in a manner analogous to equation (2.25), by the aggregate share of gross income originating in capital goods industries going to capital, i.e.,

$$\beta_k = \frac{\sum_{i=1}^{J-1} \phi_i \beta_i}{\sum_{i=1}^{J-1} \phi_i} \tag{2.48}$$

Since assumptions leading up to equation (2.46) are very severe, equation (2.47) cannot be expected to hold exactly in reality. To remind ourselves of this fact, we have written (2.47) as an approximate equality rather than as an equality. However, to the extent that the underlying system defined by equations (2.1) through (2.11) is moving along the equilibrium path of growth and that equation (2.47) holds, it will exhibit most of the characteristics of a production function; and the usual marginal conditions discussed earlier in this section will apply to equation (2.47). However, one other concept must be clarified, namely g_k.

As long as the system defined by equations (2.1) through (2.11) is moving along the equilibrium path of growth, equation (2.47) implies that

$$k^* = g_k + \beta_k k^* + (1 - \beta_k)n \tag{2.49}$$

Substituting into the above expression definition (2.48) and equation (2.45) and rearranging terms, g_k can be expressed as

$$g_k = \frac{\sum_{i=1}^{J-1} \phi_i g_i}{\Sigma \phi_i} + \frac{\sum_{i=1}^{J-1} \phi_i(\pi_i - \pi_k)}{\Sigma \phi_i} \tag{2.50}$$

In some sense, it might be argued that the aggregate measure of the rate of technological change should be given by

$$g_k^* = \frac{\sum_{i=1}^{J-1} \phi_i g_i}{\Sigma \phi_i} \tag{2.51}$$

If the residual method of Solow is applied to equation (2.47), using the aggregate measures of output and capital given by (2.46), the resulting estimate of the rate of technological change is g_k, given by (2.50), rather than g_k^*, given by (2.51). To this extent, this method may be said to give a biased measure of the rate of technological change. Fortunately, however, the last term on the right-hand side of equation (2.50) is likely to be very close to zero, since π_k itself is constructed as a weighted average of π_i's, given by equation (2.34a).

In this section we have outlined the implications of the model involving many heterogeneous capital goods on aggregate magnitudes for which data are likely to be available. It has been found that, because the rates of technological changes are different in different industries, it is not possible to define meaningful aggregate concepts except in terms of market values. However, so long as the underlying microsystem is moving along the equilibrium path of growth, market values of output and capital stock may be used to define aggregate output and aggregate capital, deflating them by some appropriate general price index. We may utilize these data to measure the rate of interest, the average rate of technological change, and other magnitudes, and to interpret the foregoing in terms of the characteristics of the underlying microsystem. In the next section, we shall take advantage of these findings and attempt to interpret the data for the United States economy. In so doing it is necessary to remember that these theoretical propositions refer to the behavior of the system on the equilibrium path of growth, while the data are generated by the economy with all its cyclical fluctuations and the shocks from two world wars. Careful choice of periods and adjustments in data will, therefore, be necessary.

III. Introduction of Government and a Monetary System

III. A. MODIFICATIONS OF THE MODEL

My remaining task is to modify the system analyzed in Section II to allow for the roles of government and of the monetary system. I will then attempt to make the best possible guesses at the order of magnitudes of the parameters of the system, and to check the internal consistency of the model.

It is clearly necessary to make a compromise about the level of aggregation for this purpose, and I shall consider the situation in which only two commodities, one capital good and one consumption good, are distinguished. The capital good, then, takes on the meaning of aggregative capital discussed in Section II.B. In my empirical work it will be identified with the value of capital deflated by a single capital goods price index.

The system I shall use in this section is defined by modifying that given by equations (2.1)–(2.11) in Section II. Equations (2.1) through (2.5) can be carried over without any change, and will be renamed for the purpose of discussion in this section as (3.1) through (3.5). The running subscripts and superscripts should be

$$j = k;\ i = k,\ c;$$

representing the capital good (k) and the consumption good (c).

Equations (2.6) through (2.10) are modified as follows:

$$C = \alpha_1(1 - \tau_1)wE + \alpha_2 \frac{A}{p^c} \tag{3.6}$$

$$O^c = C + Q^c \tag{3.7}$$

$$O^k = \dot{K} + Q^k + \delta_k \tag{3.8}$$

$$A = p^k K + D \tag{3.9}$$

$$r = r^k + \frac{\dot{p}^k}{p^k} \tag{3.10}$$

where Q_t^c = the rate of government purchase of the consumption good at time t

Q_t^k = the rate of government purchase of the capital good at time t

τ_1, τ_2 = the average rates of taxes on labor income and on property income, respectively

D_t = the outstanding debt of the government held by the public

All these quantities are assumed to be exogenous to the system. In particular, D is not an independent decision variable for government, and is given by

$$\dot{D}_t = p_t^c Q_t^c + p_t^k Q_t^k + r_t D_t - \tau_1 p^c w_t E_t - \tau_2 r_t A_t \tag{3.11}$$

It is convenient to define a new concept, "disposable income" of consumers, by

$$Y \equiv (1 - \tau_1) wE + (1 - \tau_2) rA \tag{3.12}$$

It should be noted that Y is different from the usual national income definition by the inclusion of capital gains.

Equation (2.11) will be replaced by a description of the monetary sector, which is given by

$$M^d = M^d(A, Y, r) = p^c M^d\left(\frac{A}{p^c}, \frac{Y}{p^c}, r\right) \tag{3.13}$$

$$M^d = M^s \tag{3.14}$$

where M^d = demand for money

M^s = supply of money, assumed to be given exogenously to the system

All together, equations (3.1) through (3.14) constitute a modified version of (2.1) through (2.11), with $i = c, k; j = k$.

The nature of the supply of money, M^s, must be spelled out. In this paper, it is assumed that the banking system issues money in exchange for individuals' indebtedness to it or for government debt. Under this assumption, the possession of money by one individual is precisely offset against someone else's debt. When all individual balance sheets are aggregated, money is completely canceled out, and the total outstanding government debt appears as an item of assets for individuals. The volume of money is then controlled, for instance, by the reserve requirement. This is the opposite of the other extreme assumption, made by Tobin among others, that all money is paper money issued by the government [4]. In reality, of course, there exist both kinds of money in the economy. I adopt the assumption stated above partly because others have used the opposite assumption and it may be interesting to compare these results with theirs, but mostly because in the United States, the volume of money that is not offset by individual indebtedness or by bank holdings of government securities is insignificantly small. It may be noted here that it is only under this assumption that the definition of D given by equation (3.11) is strictly correct.

Also, under this assumption we can derive the budget equation for consumers, corresponding to equation (2.12):

$$\dot{A} = Y - p^c C \tag{3.15}$$

In order for this model to be complete, a demand function for government debt by the public and a market equilibrium condition for government debt must be specified. The demand for government debt by the public is ultimately a result of the portfolio selection behavior of individuals. Suppose that the demand function for government debt by the public is given by

$$D_t^d = D^d(A_t, r_t, r_t^{k*}, \sigma) \tag{3.16}$$

where σ is a parameter representing factors (other than the rates of return) differentiating the government debt and physical capital, and the market equilibrium condition for it by

$$D_t^d = D_t \tag{3.17}$$

Its supply is given by equation (3.11). Equation (3.14) must now be abandoned; and r^{k*}, not necessarily equal to r, is defined by

$$r_t^{k*} \equiv r_t^k + \frac{\dot{p}_t^k}{p_t^k} \tag{3.18}$$

The possibility of the nonzero difference between r, the rate of interest on the government debt, and r^{k*}, the rate of return on capital including capital gains, arises because of the possible difference in risk associated with holdings of physical capital and of government debt. In equation (3.16) σ represents this difference in risk. Thus, equation (3.15) must be modified to make A a function of r, r^{k*}, and σ. We will then have an expanded system, consisting of equations (3.1) through (3.10), (3.12), (3.13), (3.14), (3.16), (3.17), and the new variable, D^d.

But this creates rather than solves a difficulty. I have abstracted from uncertainty throughout the analysis in this paper. In order to accommodate the problems arising from uncertainty, the real part of the system has to be modified rather substantially, introducing, among other things, market imperfections and gradual rather than instantaneous adjustment processes. Yet, in order to define the demand function for debt within the framework of this system, it is necessary to introduce the uncertainty of the return on capital. To put it another way, if there is no uncertainty associated with return on capital from the point of view of individuals choosing their portfolios, ownership of physical capital and ownership of government debt are indistinguishable from each other provided that equation (3.10) holds; the system reduces to equations (3.1) through (3.14). But in such a system, there is no explicit mechanism by which savings of individuals are divided into purchases of government debt and of physical capital. Though the problems arising from uncertainty are among the most fundamental in the analysis of economic growth and fluctuations, they cannot be dealt with in this paper. I shall, instead, adopt the following simple convention: the government, by some appropriate procedure, always succeeds in selling all the debt issues which it wishes to sell, resulting in the condition summarized by equation (3.10). As a result, the value of physical capital demanded in the economy at any time is the total net worth generated through equation (3.15), less the value of total outstanding government debt.[16]

Let us adopt the following notations,

$$q_c \equiv \frac{Q^c}{O^c};\ q_k \equiv \frac{Q^k}{O^k};\ d \equiv \frac{D}{p^k K}$$

and the definitions of a, π, k, and ϕ as in Section II:

$$\frac{p^c(1-\tau_1)wE}{A} \equiv a;\ \frac{\dot{p}^k}{p^k} \equiv \pi_k;\ \frac{\dot{p}^c}{p^c} \equiv \pi_c;\ \pi \equiv \pi_k - \pi_c$$

$$\frac{\dot{K}}{K} \equiv k;\ \frac{p^k O^k}{p^c O^c} \equiv \phi$$

Suppose that q_k, q_c, τ_1 and τ_2, and d are constant over time.[17] Does there exist an equilibrium path of growth on which a, π, ϕ, r^k, and k are constant? And, if there is such a path, does it exhibit similar characteristics to those described for the model discussed in Section II?

[16] This situation may be relaxed slightly by assuming that the measure of uncertainty associated with holding the capital good is constant over time, and expressed by the difference between r_t and r_t^{k*}, replacing equation (3.10) by $r = r^k + (\dot{p}^k/p^k) - \rho$, where ρ is positive and constant.

[17] These are not strictly independent decision variables, as discussed below.

Examination of the analysis developed in Section II and of the modifications introduced in this section shows that the existence of the equilibrium path of growth is not disturbed by the modification introduced in this section, and that a number of the properties of the system on the equilibrium path remain unchanged. In particular, the distribution of factors among industries, given by equation (2.14), is unaffected, as are the equilibrium values of k and π, as exemplified by expressions (2.27), (2.28), and (2.29).

On the other hand, the equilibrium values of ϕ, r^k, and a are slightly altered, and the equations corresponding to (2.18) and (2.24) are given by the following:

$$\text{(3.19)} \qquad \phi = \frac{(1-\tau_1)(k+\delta)(1-\beta_c)}{a(1-d)(1-q_k)-(1-\beta_k)(1-\tau_1)(k+\delta)}$$

$$\text{(3.20)} \qquad r^k+\delta = \beta_k \frac{O^k}{K^k} = \frac{k+\delta}{1-q_k}\,\frac{\beta_c+\beta_k\phi}{\phi}$$

$$\text{(3.21)} \qquad a = \frac{\alpha_2+k+\pi_1-(1-\tau_2)r}{1-\alpha_1} = \frac{\alpha_2+k+\tau_2\pi-(1-\tau_2)r^k}{(1-\alpha_1)}$$

The final equilibrium values of r^k and a are then given by

$$\text{(3.22)} \quad r^{k*} = (\alpha_2+\tau_2\pi)\frac{\beta_c(1-q_k)(1-d)}{(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)}$$
$$+k\frac{\beta_k(1-\alpha_1)(1-\tau_1)+\beta_c[\alpha_1(1-\tau_1)-(1-\tau_1)+(1-q_k)(1-d)]}{(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)}$$
$$-\delta\frac{(1-\tau_1)(1-\alpha_1)[(1-\beta_k)-q_k(1-\beta_c)]}{(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)}$$

(3.23)

$$a^* = (\alpha_2+\tau_2\pi)\frac{(1-\tau_1)[(1-\alpha_1)(1-\beta_c)+q_k\beta_c(1-d)]}{(1-\alpha_1)[(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)]}$$
$$+k\frac{(1-\tau_1)(1-\alpha_1)[1-\beta_k(1-\tau_2)]}{(1-\alpha_1)[(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)]}$$
$$+k\frac{\beta_c[q_1(1-d)(1-\tau_2)-\tau_2(1-\alpha_1)(1-\tau_1)]}{(1-\alpha_1)[(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)]}$$
$$+\delta\frac{(1-\tau_1)(1-\tau_2)[1-\beta_k-q_k(1-\beta_c)]}{(1-\beta_c)(1-\tau_1)(1-\alpha_1)+\beta_c(1-d)(1-\tau_2)}$$

Substitution of these values into (3.19) will then give us the equilibrium solution for ϕ.

It is clear that, when $q_k = \tau_1 = \tau_2 = d = 0$, equations (3.22) and (3.23) reduce to equations (2.31) and (2.32).

From these equations, one may conjecture about the influence of changes in fiscal policy, i.e., changes in values of q_k, q_c, τ_1, and τ_2, on the values of r, a, and ϕ in the long run, and the productivity of labor at any given point in time. However, it would be more illuminating to discuss these important questions after approximate numerical values for the parameters are given. Before proceeding to the problem of empirical estimates of parameters, however, two remaining problems must be clarified. First, the government has one more policy instrument at its disposal, namely, the supply of money; and we must investigate what consequences, if any, would result within the framework of this model when the supply of money is changed. Second, D is a consequence of past decisions on expenditures and taxes on the part of the government, and it appears that the government cannot vary the value of d arbitrarily.

In order for the system to be maintained on the equilibrium path of growth suggested above, the proportion of government debt in consumer net worth must be kept constant. This is because, on such a path, both A and $p^k K$ must grow at the rate $k + \pi$, and hence, it must be that $\dot{D} = k + \pi$, also. Substituting this in equation (3.11) and using the definition of d given above, we have

$$d = \frac{\dfrac{k+\delta}{1-q_k}\left(\dfrac{q_c}{\phi} + q_k\right) - \dfrac{\tau_1}{1-\tau_1}a - \tau_2 r}{\pi + k - r + \dfrac{\tau_1}{1-\tau_1}a + \tau_2 r} \tag{3.24}$$

Equation (3.24) defines the ratio of government debt to the value of total consumer net worth which must be maintained in order for the system to move along the equilibrium path of growth, given values of q_k, q_c, τ_1, and τ_2. Thus, equations (3.22) and (3.23) do not give the complete equilibrium solution for a and r, as they appear to do. In order to obtain the equilibrium solution, equations (3.19), (3.20), (3.21), and (3.24) must be taken together and solved simultaneously for a, r, ϕ, and d. This is not too difficult, but the resulting solutions for these variables are very long expressions; and it would be more useful to deal with this problem when actual numerical values of β's and α's and of k and π have been obtained.

The ratio of government debt to the value of total net worth of consumers can also be changed through monetary policies and through the resulting changes in the prices of the consumption good. Let us suppose that the demand function for money, equation (3.13), takes the special form given by

$$M^d = m_1(r)(O^c p^c + O^k p^k) + m_2(r)A \tag{3.25}$$

where the first term on the right-hand side of this equation represents the transaction demand for money, the second term represents the asset demand for money, and m_1 and m_2 are both decreasing functions of r.[18]

It is convenient to define

$$m(r) = \frac{M^d}{p^k K} \tag{3.26}$$

On the equilibrium path of growth, it must be the case that

$$\begin{aligned} m(r) &= m_1 \frac{O^k p^k}{K p^k} \left(\frac{O^c p^c}{O^k p^k} + 1 \right) + (1 + d) m_2 \\ &= m_1 \frac{1 - q_k}{k + \delta} \left(\frac{1}{\phi} + 1 \right) + (1 + d) m_2 \end{aligned}$$

(3.27)

$$\begin{aligned} m(r) = m_1(r) \frac{1 - q_k}{k + \delta} &\left[\frac{a(1 - d)(1 - q_k) - (1 - \beta_k)(1 - \tau_1)(k + \delta)}{(1 - \tau_1)(k + \delta)(1 - \beta_c)} + 1 \right] \\ &+ (1 + d) m_2(r) \end{aligned}$$

Note that, since $m_1(r)$ and $m_2(r)$ as well as a are all decreasing functions of r, the partial derivative of m with respect to r must be negative.

In the standard Keynesian analysis, when the money supply is changed, the level of prices is supposed to change little or at best very slowly. Consequently, the change in the supply of money is largely reflected in the credit conditions as represented by the rate of interest in the money market. In this model it is assumed that the level of prices as well as the rate of interest adjust instantaneously whenever the supply of money is changed, so as to satisfy equations (3.1) through (3.14). Because the major concern of this paper is the problem of long-run growth, the consideration that it takes time for prices and other variables to adjust to new conditions is neglected, except for the adjustment through savings as described by equation (3.15).

Now suppose that the system has been moving along the equi-

[18] Under the assumption of perfect certainty, it would be foolish for anyone to hold money as a part of his portfolio; consequently, m_2 really should be identically zero.

librium path defined by equations (3.19), (3.22), (3.23), (2.27), and (2.29), with the monetary authority allowing banks to increase the supply of money at the rate $k + \pi$. Suppose further that at some point in time the monetary authority permits the banks to increase the supply of money by some amount over and above the regular increase of $(k + \pi)M$. In order to increase the supply of money, banks must offer loans at a rate of interest slightly below the rate that was prevailing previously. Individuals (firms) then find it profitable to borrow money from banks, and attempt to purchase the capital good, thereby bidding up its price. This in turn induces the producers of capital goods to produce more capital goods and to demand more factors of production, raising the price of the capital good even further, along with the wage rate, and leading to an increase in the level of money income and the money value of net worth of consumers. This in turn increases the demand for the consumption good, raising its price. This process will go on until the prices of both the capital good and the consumption good are sufficiently bid up so that the equations (3.1) and (3.14) are again satisfied. However, the resulting equilibrium is not the same as the one prevailing before the increase in the supply of money took place. When the level of prices is rising, the value of a part of consumers' net worth, p^kK, rises, but not the remaining part, D.

How this will affect the resulting equilibrium will depend on how the increase in the money supply is brought about. In the United States, this is ordinarily done through the purchase of government securities by the central bank, enabling commercial banks to acquire additional indebtedness by the public. In this case, we have the classical "Pigou-Patinkin" effects, generating more savings.[19]

How this process ends in the long run depends upon what the government does with q's and τ's. If these parameters are maintained at the same values as those that had prevailed before the increase in the supply of money took place, then the proportion of debt in the net worth of consumers will gradually be increased, and the system will return to the original equilibrium path of growth after some passage of time. If, on the other hand, the values of these parameters are changed, and in particular, if they are changed in

[19] I have reported elsewhere [3] that the marginal propensity to consume net worth, α_2, is roughly 0.05. The value of total government debt outstanding plus gold stock is between, say, \$300 billion and \$400 billion. Therefore, 10 per cent changes in the price level will result in some \$1.5 to \$2 billion changes in consumption, a negligible change compared to other repercussions of such a large change in the price level.

such a way that the final equilibrium value of d is less than that which prevailed before the increase in the supply of money, the capital-output ratio in both industries will be increased, the equilibrium value of the return on capital will be less than before, and the system will move to a new equilibrium path of growth which is characterized by a higher output in both industries than would have been the case if such a change had not taken place.

III. B. SOME PRELIMINARY EMPIRICAL RESULTS

The data needed for the estimation of the parameters in the model described in Section III.A are roughly as follows:

1. Output of the consumption-good industry and of the capital-good industry, gross of depreciation.
2. Price indexes for output of the capital-good industry and of the consumption-good industry.
3. The value of capital used in both industries.
4. Depreciation of capital in both industries.
5. Man-hours employed in both industries.
6. The share of capital in both industries, or, equivalently, labor income in both industries.
7. Some method of adjusting the value of capital in both industries for underutilization of capital.
8. Data needed to estimate parameters of the consumption function; more specifically, consumption, labor income after taxes, and the value in net worth of consumers.
9. Data relating to government activities; specifically, expenditure by government, taxes on labor and nonlabor income, and the value of debt.

The sources and derivation of the data are described and discussed in moderate detail in a separate appendix, and may be obtained at cost from the author on request.

Difficulties arise mainly from four sources. The first is simply lack of information that goes back to the beginning of the twentieth century; but this is to be expected. The second is the inconsistency among the data taken from various sources. For instance, personal income in current dollars reported by Creamer [8] moves from $29.4 billion in 1909, through $69.2 billion in 1920, to $83.4 billion in 1929; while the same item, apparently having the same definition, reported by Goldsmith [21] moves from $26.9 billion in 1909, through $75.8

billion in 1920, to $85.1 billion in 1929. Since no single source can provide all the data necessary for a work of the kind attempted in this paper, the combined use of a number of different sources containing such discrepancies as that described above is at present a necessity; and this is a very serious problem indeed when working with an internally consistent model such as the one here presented. It also casts serious doubts on the meaning of the empirical results obtained.

Third, several important implications of the model presented in Section III.A. are stated in terms of current values. For instance, according to the model, the ratio of the value of labor income to the value of net worth of consumers should remain constant over time, but it has nothing to say about the ratio of "real" labor income to "real" net worth as ordinarily defined (where each component of these aggregates is deflated by its own price and then summed). However, there appears to be an increasing tendency to report "real" series alone, without showing the current-value series. The most notable example of this tendency is the work of Kendrick [28], in which very few time series are given in current prices.

Fourth, although the division between the capital-good-producing sector and the consumption-good-producing sector is a very convenient and appealing one from the theoretical point of view, in reality industries are not neatly classified in this manner; and it is necessary to make a number of rather arbitrary decisions in allocating various production activities to the two sectors. In this paper, I have largely followed the method suggested by Eckaus and Lefeber [16] in their recent paper, with some modifications. This method of allocation is described in the separate appendix referred to earlier. In retrospect, I feel that this method in its general outline is as good as any that we can devise, but a number of substantial improvements can be made, given time and patience.

Before turning to the examination of actual numerical results, it may be helpful to recall the type of questions often raised in the theoretical and empirical literature on the process of economic growth. It is often suggested that such magnitudes as the relative share of income between labor and capital, the ratio of capital to output, and the long-run saving-income ratio are extremely stable over time. Whether or not these ratios are in fact constant depends very much on the precise definitions of the variables in terms of which the ratios are computed. In my theoretical discussion, a framework has been developed that specifies which of these ratios should be expected to be stable over time. Among them, the constancies of the relative income shares are the least interesting; since I assume the Cobb-Douglas production function for all industries, the constancy of the over-all relative income share follows directly from the constancy of ϕ, the ratio of value of output of the capital-good industry to that of the consumption-good industry. That some of the other ratios should be expected to be constant on the equilibrium path of growth is somewhat more surprising. Any one of the basic equations of the model, taken by itself, does not imply such constancies; but all of them taken together apparently do. Furthermore, the model specifies the values of these ratios in terms of the values of the basic parameters of the system. It is interesting, therefore, to see, first, whether some of these ratios are in fact historically constant, and second, if they are, whether their observed values are reasonably close to those predicted by the values of the parameters of my model.

Second, there has been much discussion recently on the measurement of technological change, beginning with an important contribution of Solow [45]. This discussion, as far as I am aware, has been directed toward the measurement of technological change for the economy as a whole. We are all aware, however, of the conceptual difficulty involved in defining aggregate capital and output. In Section II.B, one possible interpretation was formulated for the link between the concepts of aggregate capital and output and disaggregated capital goods and outputs. It turns out, however, that my solution calls for the use of the value of capital and value of output, each deflated by a price index satisfying certain conditions, and not "real" output and "real" capital as usually defined. It is interesting, therefore, to compare the results which I obtain in this paper with, for instance, those of Solow, who used "real" capital and "real" output.

The parameters of the system for two periods, 1900–28 and 1951–58, are reported in Table 1. The choice of periods is, by necessity, arbitrary. The Great Depression years of the 1930's and the World War II years are certainly not relevant to a model such as the one under consideration here, and should be excluded. The period 1948–58 may be used instead of 1951–58, but I have been persuaded that the years 1948–50 should be excluded, partly because of the effects of the Korean War on the United States economy and partly because these were still years in which the effects of the depression

TABLE 1

AVERAGE VALUES OF THE BASIC PARAMETERS FOR SELECTED PERIODS, 1900–57

Periods Between		n	τ_1	τ_2	q_c	d	δ	β_c	β_k	g_c	g_k	Aggregate g
1900–01	1927–28	.014	0	0	.04	.05	.04	.43	.25[a] (.15)	.013	.013[b] (.014)	.013
1951–52	1956–57	.003	.13[c] (.09)	.20[c] (.32)	.14	.30	.04	.37	.25	.017	.012	.015

NOTE: The data needed to compute these values are available in a separate appendix, which may be obtained at cost on request to the author. The figures reported are the averages of annual values, except that the figures for 1900–28 are averages excluding 1918, 1919, and 1920. For figures on α_1 and α_2, see table in Section I and Ando and Modigliani [3].

[a] From all indications, the value of output in manufacturing in this paper is substantially underestimated relative to all other figures in the period 1900–28. As a consequence, the relative share of income of capital and, hence, β_k, appears to be substantially understated if the value of output in manufacturing is taken at face value, resulting in the figure reported in parentheses. However, if a rough adjustment is made for the underestimation of the value of manufacturing output, then β_k becomes 0.25. This is the figure used throughout this paper.

[b] The figure 0.013 is the result under the assumption that β_k is 0.25. If β_k is assumed to be 0.15, then g_k would be 0.014, reported here in parentheses.

[c] The figure is arrived at by assuming that one-half of corporate income tax is a tax on property income, while the other half is a tax on labor income, on the basis that wages would be somewhat higher if the corporate income tax did not exist. The figure in parentheses will result if all corporate income tax is considered a tax on property income.

and the Second World War were being worked out, particularly the rapid accumulation of the real assets which had been depleted.

The footnotes to Table 1 give detailed comments on the nature of the individual estimates. The following features of the estimates may be noted here.

1. Even given the roughness of the data, there appears to be very little doubt that β_c is greater than β_k. This, in turn, implies that, provided the return on capital and the wage rate is in fact the same for both industries, as required by the model, the consumption-good industry is more capital-intensive than the capital-good industry. This appears to be a surprising result at first, but when it is realized that the output of the consumption-good industry includes the service of residential structures, the result is perfectly understandable.

2. It appears that, for the earlier period, the rate of technological change in the capital-good industry was greater than that for the consumption-good industry, but this relation has been reversed for the later period.

3. The ratio of government debt to the value of physical capital, d, is reported to be 0.05 for the period 1900–28. This figure, as indicated in the footnote, is the average for the entire period except for the First World War years. However, it starts at 0.04 in 1900, remaining stable until the First World War years, when it rises to 0.13, and then declines to 0.08 in 1929. This gradual change in d might cause serious discrepancies among the observed values of r, a, and other ratios and those implied by the parameter values in Table 1.

Of the five ratios for which the model gives predictions (a, r, k, π, and ϕ), k and π really do not provide a test of the consistency of the model, since data used to measure these are directly used to obtain the estimate of g_c and g_k. Of the remaining three, ϕ is most sensitive to the short-run cyclical fluctuations of economic activity, and probably less meaningful for testing the consistency of a long-run growth model of the kind under consideration here. Because of these considerations, in Table 2, the actual values of a and r^k and the

TABLE 2

COMPARISON OF OBSERVED VALUES OF a^* AND r^{k*} WITH THOSE IMPLIED BY THE MODEL AND TABLE 1, 1900–57

Periods Between		a^*		r^{k*}	
		Actual	Implied	Actual	Implied
1900–01	1927–28	.152	.151	.062[a] (.066)	.047
1951–52	1956–57	.173	.196[b] (.206)	.054	.040[b] (.039)

SOURCE: See Note, Table 1. The implied values are computed through equations 3.40 and 3.41, using values of the parameters given in Table 1. When the actual value of aggregate k deviates from that implied by Table 1, however, the actual value of k is used.

[a] The figure reported here is the average of annual values excluding 1918, 1919, and 1920. If these years are included, the figure reported in parentheses is obtained.

[b] This figure is calculated by using the values of τ_1 and τ_2 given in Table 1. If the figures of τ_1 and τ_2 given in parentheses in Table 1 are used, then the implied values of a^* and r^{k*} reported in parentheses will result.

corresponding values implied by the parameters reported in Table 1 through the use of the model of Section III.A are given.

The actual value of a^* is remarkably stable over time. For the period 1900–28, its highest value is 0.160, in 1920, and its lowest value is 0.133, in 1915, without its showing any visible trend. Furthermore, these extreme values are rather isolated phenomena, and if the highest three and the lowest three are ignored, its highest value is 0.155 and its lowest, 0.137. The figure reported in Table 1 is the average for the entire period. That the implied value of a^* for the corresponding period is so close to the actual value would appear to be very good, but this conclusion is not entirely warranted. The reason is that, while the actual value of a^* has been computed as the ratio of total labor income in the economy against the actual total net worth of consumers, which includes nonreproducible tangible assets (namely, the value of land), the model through which the implied value of a^* has been computed does not allow for the existence of land in the economy. Therefore, the actual value of a^* should be somewhat smaller than the implied value. It should be concluded, therefore, that for the period 1900–28, the implied value of a^* is somewhat too small relative to the actual value of a^*.

For 1951–57, the value of a^* is again very stable, but it exhibits a slight tendency to fall somewhat over this period, namely, from 0.179, in 1951, to 0.171, in 1957. The accumulation of net worth during the World War II years was extremely small, and the ratio a reached the peak of 0.23 in 1944. It recovered quickly, falling to roughly 0.18 in 1947, but further recovery appears to be very slow. Thus, the decline of a mentioned above may be attributed to the desire of consumers to accumulate up to the normal level those assets which were depleted during the World War II years. The value reported in Table 2 is again the average value for the period. The implied value of a^* for this period reported in Table 2 is 0.196, a figure considerably larger than the actual value of a^*. But, as suggested earlier, due to the treatment of land in the model, the implied value of a^* should be somewhat larger than the actual value. It is difficult to say how much larger, but it may be noted that the value of private land for this period is roughly 15 per cent of the total net worth of consumers, suggesting that the maximum adjustment would be to reduce the implied value of 15 per cent. This would make the implied value of a^* for this period slightly less than 0.17. In any case, the result of the comparison of the implied value of a^* with its actual value may be considered reasonably satisfactory.

The actual value of r^k for both periods fluctuates somewhat more than that of a. In particular, the value of r^k for the World War I years is extremely high. The figure reported for 1900–28 in Table 2 is the average excluding the World War I years, but the figure including the World War I years is shown in parentheses. Except for this fact, both periods can be discussed together here, since the details for both periods are very similar, i.e., the actual figures are much larger than the implied figures, though both actual and implied figures become smaller over the years.

The discrepancy between the actual and implied values of r^{k*} is quite large. Since the model from which the implied values of r^{k*} have been computed assumes perfect competition and perfect certainty, while in the real world uncertainties and various degrees of monopoly powers are important facts of life, it is to be expected that the actual rate of return on capital will be somewhat larger than the equilibrium values implied by the model. However, since we cannot measure quantitatively the effects of monopoly power and uncertainty in this context, it is impossible to determine whether the discrepancy displayed in Table 2 is too large or too small. This is one of the unsolved problems in this paper.

Before concluding this section on empirical results, I wish to report that experiments have been performed to compare the results of Solow [45] in the measurement of the rate of technological change and the conclusions reached through equation (2.50) in Section II.B. The comparison is not exact, since the data used by Solow are not identical with those used here. Solow's result is that, for the period 1910–49, the average rate of technological change was roughly 0.017 per year. According to my method, using the value of output of the consumption good and the value of capital deflated by a single price index, the rate of technological change for the similar period, after the adjustment factor shown in equation (2.50) has been allowed for, is also roughly 0.017. This finding appears to indicate that, for this period, at least, Solow's conclusion would not be altered if the method suggested by the model analyzed in this paper were substituted for his. On the other hand, for any subperiod that may reasonably be chosen for comparison, the results appear to be radically different. It is difficult to conclude anything definite from these subperiods, however, since the results for shorter periods must be affected by cyclical movements in the economy. Furthermore, the shorter the period, the more likely it is that the weakness of the data will reduce the reliability of the estimate.

Concluding Remarks

I have attempted, in this rather lengthy paper, to specify a model of United States economic growth which conforms to some of the more recent theories of economic growth, but which is, at the same time, concrete enough to act as a framework for interpreting the existing data for the long-run trend of the United States economy. It turned out that, on the one hand, the model I have developed requires some data that are not readily available, while on the other, the available data from different sources are inconsistent, presenting a number of almost insurmountable difficulties in the empirical work for this paper. Under the circumstances, within the limits of the time and resources at my disposal, I have tried to accomplish two objectives. First, to investigate the properties of the model I have specified, paying special attention to the problems of interpreting aggregative concepts and data in terms of detailed, disaggregated models; and second, to do my best to make approximations using the existing data, and to suggest the order of magnitudes of the crucial parameters in this model.

There are a number of constants that are often discussed in the literature, such as the relative share of income, the ratio of the value of capital to the value of output, and others. These ratios are useful concepts empirically, but the theoretical explanations of the constancy of these ratios have not been, in my opinion, very satisfactory. I have, therefore, specified a model which does not assume these ratios to be explicitly constant by adopting (1) the Cobb-Douglas production function, (2) different rates of technological progress in different industries, and (3) a consumption function which does not require the saving income ratio to be constant at all times; and I have inquired whether such a model is capable of implying the constancy of these ratios at least on the equilibrium path of growth. The answer to this question turned out to be affirmative. In addition, the model discussed here gives explicit solutions for these ratios as well as other ratios in terms of (1) the parameters of the production functions and the consumption function, and (2) exogenous factors such as the rate of growth of the labor force, the rates of technological change, and governmental policy variables. As a result, the model enables us to make specific numerical predictions for these ratios when its parameters are estimated and the exogenous factors are known.

Because of the difficulties in adjusting the existing body of data to information that is relevant for this model, the empirical results presented in this paper cannot be taken too seriously. However, the indications are that if it is possible to improve the approximations made here, the model presented should be moderately useful for interpreting data concerning the long-run trend of the United States economy. The improvements required of the existing body of data as well as a new set of data needed for this purpose are relatively minor, and I have little doubt that this will be possible in the very near future.

In addition to the quality of empirical approximation, a number of obvious improvements can be made in this model in order to make it more realistic and useful. The list of such future improvements surely includes: a more careful treatment of the role of government, particularly the services provided by government for private economic activities; the separation of producer durables, inventories, and structures in the concept of capital; a more detailed consideration of the monetary system; and the introduction of elements of uncertainty. Furthermore, it would be interesting to inquire how the transition of the system from one equilibrium path of growth to another would take place, following, for example, a change by the government in one of its policy variables.

In reality, the United States economy contains a number of characteristics that cause it to deviate significantly from the smooth path of equilibrium growth defined by this model. In fact, these deviations are so significant that during the period between 1930 and the late 1950s, the larger part of macroeconomic analysis was devoted almost exclusively to the study of the causes and effects of these deviations. On the other hand, as the list of references at the end of this paper indicates, fairly strong interest in the underlying economic structures has recently been shown, with less emphasis placed on the deviations discussed above. In principle, these two phenomena—the growth of the economy due to the growth of the labor force, of technological innovations, and of the accumulation of capital; and cyclical fluctuations of the economy—cannot be studied separately, since they are interrelated parts of the economic process over time.

However, given the current state of economic theory and empirical knowledge, it is extremely difficult to begin with a model which takes into account simultaneously all the complex factors affecting the growth of the economy and those affecting the cyclical fluctuations, estimates all parameters, and then analyzes the properties of the system. Under the circumstances, there seem to me to be two possible strategies in dealing with the problems of understanding the workings of the economy, involving both growth and fluctuations. The first is to start with a model which lays strong emphasis on the factors causing cyclical fluctuations in the economy, and inquire whether such a model is capable of generating a steady growth path if all the exogenous factors are kept constant in some sense, and if it is, then what the properties of the growth path so generated are. The work of Duesenberry [15] is an excellent example of this approach. The second strategy is to construct a model in which the economy functions without any friction, such as the one discussed in this paper, and inquire whether such a model is capable of generating a reasonable growth path that conforms to the characteristics of the real economy when the data are adjusted for cyclical fluctuations. Then, if it does, a number of frictions can be introduced into such a model —for example, wage rigidities, nonzero time required for adjustments in markets and in relations involving stocks and flows, imperfect knowledge—and study the consequences. That such a procedure would lead to a model involving both growth and fluctuations in the case of a single-good model has been reported elsewhere [1] [2].

I think there is no inherent reason for either one of these two approaches to be superior to the other. I therefore feel that both approaches should be tried, since the problem to which both of these approaches are directed—namely, the workings of the economy, involving both fluctuations and growth—is too important to leave any promising lead unfollowed. There are some definite advantages in starting from the growth model: the growth model itself may be required to be consistent with the classical general equilibrium model; and somewhat greater insight may be gained into the contributions of each specific friction to the cyclical fluctuations, since they may be introduced separately or sequentially. This paper is meant to be a modest beginning to such an inquiry.

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COMMENT

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As Ando has suggested, his models are very much in the spirit of Solow's "neoclassical" model.[1] Indeed it might be said that his models are modifications of the Solow model. As theoretical models, they seem to be eminently satisfactory. Ando spells out his assumptions rather carefully, and his models are internally consistent. If one wishes to quarrel with them, one must do so on the basis of the assumptions.

If the purpose of a model is theoretical conjecture, it is difficult even to disagree with its underlying assumptions. Surely it is legitimate and useful to postulate special conditions on the economic environment and to inquire as to the logical consequences of these conditions. So long as we feel free to use the laboratory control of special assumptions we may obtain interesting results. This is true even though the special environmental conditions assumed do not exist and have never existed; after all, it has often been observed that the real world is a special case.

[1] Robert M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, February 1956, pp. 65–94.

However, Ando is not interested in purely theoretical conjecture. He speaks of applied capital theory, and he proposes empirical testing of the models. The objections that I wish to raise center on the models' suitability for empirical testing.

In each of the models, output or product depends on three variables: technological change, employment, and capital. The first two are determined by time series, that is, they are functions of time alone. (Strictly speaking, it is the supply of labor that is determined by time, but since employment and supply of labor are identically equal in the model, this does not matter.) In the case of technological change, the use of a time series explanation may be the most satisfactory means of handling the problem, but I do not think that it is suitable in the case of employment. Granted that we are discussing very long-run models, an assumption that employment depends only on time precludes the possibility of secular unemployment. Furthermore, it also precludes the possibility of examining causes and results of secular unemployment.

The treatment of capital is also unsatisfactory from an empirical standpoint. The difficulty lies in the passive role assigned to investment. Savings are determined by the consumption function, and the demand for investment is determined by the marginal productivity equation of perfect competition theory. Presumably each model adjusts, through the production function and the marginal productivity equation, so that savings and investment are always equal. In view of the importance often accorded investment in discussions of economic growth, the absence of a realistic explanation of the determinants of investment seems odd. Further, in view of the importance of capital as the only endogenously determined variable in the output function, it seems doubly odd that investment should assume such a passive role.

Clearly, Ando's postulates and assumptions lead to an equilibrium model of growth. Indeed, his model is of an ambulatory equilibrium type; it moves in equilibrium. Now, there is a strong precedent for this kind of equilibrium model, varying all the way from the most primitive model of the *General Theory* to, say, the Solow model alluded to above. It seems to me that such models are legitimate for theoretical purposes, but I have strong doubts that they can achieve the empirical goals toward which Ando aims.

The quest for a stable equilibrium motion leads to certain peculiar results, a few of which can be cited quickly. For example, the ratios of rate of change of output to output, of rate of change of capital to capital, of rate of change of employment to employment, and of rate of change of technological change to technological change are all constants. In the cases of labor and technology, the constants arise from the specific nature of the two time series assumed to generate these variables. It may be observed that this constancy arises fundamentally from assumption rather than from factual study. This is a questionable procedure in a model intended for empirical testing.

A further special assumption of the models is that the ratios of rate of change of output to output and of rate of change of capital to capital are equal to each other and are constant. This is a characteristic of all of the models, and it leads to peculiar results. In the simplest model we find that $k = \dot{Y}_t/Y_t = \dot{K}_t/K_t$, or $\dot{Y}_t = (Y_t/K_t)\dot{K}_t$. Thus, the rate of change of output depends on only one time derivative, that of capital. The time derivatives of labor and technological change do not matter. Fundamentally, this is a consequence of the insistence upon a stable moving equilibrium.

A feature of the models that commands immediate attention is the total absence of any lagged variables. We may note that the absence of lags is simply unrealistic. The adjustment of macroeconomic variables is not instantaneous. Econometric models are replete with lags, and they frequently prove to be important. In addition, the absence of lags forces a reliance on instantaneous time derivatives. Again, it would seem open to question that such a usage is suitable for empirical testing.

As the model stands it is undoubtedly dynamic, but it is dynamic in a peculiar way. The time path of the variables depends not only on their past values but is also conditioned by the mere passing of time. This is true because of the time series that generate technological change and employment. The passage of time feeds new values of these variables into the system instantaneously and continuously, and the system adjusts instantaneously to the new values. Thus, it is not possible for the system to remain in an equilibrium in which all variables are constant; the passage of time alone prevents this.

The model is supported by an assumption that there are no economic fluctuations. If the author were saying, "I am examining a different world in which economic fluctuations do not exist, to see what equilibrium growth conditions are like in such a world," we could not object at all. Instead, he is saying, "I am examining the real world, and I am assuming there are no economic fluctuations."

This is not sound. To compare the real world with his model, he must adjust the real world data for the effects of economic fluctuations. (I do not know how this should be done.) To rely on an analogue, we may ignore seasonal variations in theoretical discussion, but when we make use of actual data, we adjust for seasonal variations.

Of course, Ando is aware of the difficulties that arise from his models, and he believes that they can be overcome in the future by refining the models and by obtaining more satisfactory data. He may be right about this, but I am doubtful. In the empirical parts of the paper, when the observed values are not compatible with the model, he has nothing of a precise nature to suggest as a means of reconciliation. As a consequence, the empirical work presented may be suggestive, but it is far from conclusive.

The paper has welcome additions to the theory of economic growth models; in this connection the work dealing with problems of aggregation and the effects of money and of government policy may be noted. But these are theoretical advantages. In its present form the paper has little that appears to be promising for empirical research.

REPLY by Albert Ando

Pfouts suggests in effect that the model I have presented has no relation whatever to the United States economy. My general answer to this criticism is contained in my paper, particularly in the concluding section.

I feel, however, that four specific points Pfouts makes should be answered in order to avoid further misunderstanding.

First, Pfouts says that investment in my model is completely *passive* because investment is determined by the marginal conditions. The meaning of the word *passive* in this context must be that I do not consider explicitly the time necessary for adjustment of capital stock by producers. I am concerned with long-run growth characteristics of the United States economy in my paper, and I consider the omission of these adjustment problems to be a justifiable approximation. Whether or not this is so is an empirical question, and must be treated as such. On the other hand, I have shown elsewhere that when frictions are explicitly introduced into this form of demand for capital, a stock-flow adjustment model of the Goodwin-Chenery type will result. In fact, for the case of a single-good model, I have already reported a study of the consequences of the introduction of such frictions, and I think this is a convenient and useful procedure to follow.[1]

Second, Pfouts seems to miss the important point that, because the *supply* of labor is given as a function of time while the *demand* for labor is given by the marginal conditions (the distinction Pfouts dismisses in parentheses), I have an option to study the causes of secular unemployment in this type of model, although I did not choose to do so. This fact is not my discovery, but was suggested by John R. Hicks in his now-famous paper, "Mr. Keynes and the 'Classics,' a Suggested Interpretation,"[2] and has been discussed by numerous authors since.

Third, Pfouts finds it "simply unrealistic" that my model does not contain any lagged variables. This statement must be based on his failure to realize that my model contains very complex lag structures implicitly, just as it contains both flow and stock variables. For instance, if Pfouts attempted to write down the consumption function utilized in my model totally in terms of flow variables, he would have found an equation involving infinite series of lagged variables.

Fourth, Pfouts seems to think that I have assumed various constants, such as the ratio of the *value* of the capital goods output to the *value* of the consumption goods output (some of the ratios he mentions are not constant, or do not play any significant role in my analysis, but this is unimportant). On the contrary, there is nothing in my model as originally set down that suggests these constants in a very obvious manner, except in the trivial sense that if these constants can be derived as implications of my model, the hypotheses specifying my model must have contained these implications in the first place.

Pfouts contends that my model is unrealistic because these constants emerge. This is a strange accusation indeed, because they have often been empirically observed and have presented themselves as puzzles awaiting *explanation* by economic theorists. I consider it a merit of my analysis that it begins with a rather flexible model containing any number of sectors, and manages to explain these aggregative constants, such as the relative shares of income and the ratio of the value of capital to the value of output.

I need not comment on his point that I must adjust data as in the

[1] See references [1] and [2] in my text.
[2] *Econometrica*, 1937, pp. 147–159.

case of seasonal fluctuations, since rather careful adjustments are in fact made, as explained in my text.

To sum up, I have attempted in my paper to study those periods of the United States economy in which reasonably full employment prevailed, and to inquire what kind of underlying structure may be responsible for generating certain uniformities that characterize these periods, leaving to a future study the causes which tend to make the economy deviate from such a position. If Pfouts chooses to tell me that this is not a proper procedure, then we differ in our opinions about research strategy. If, however, he does grant me the freedom to pursue such a strategy, then his criticisms appear to be based wholly on misunderstandings of my model, or on misinterpretation of the data, or both, since my model does explain a number of uniformities that have been observed in the data for the United States economy.