APPENDIX A

A Geometrical Model for the Analysis of Consumer Investment Decisions

Introduction

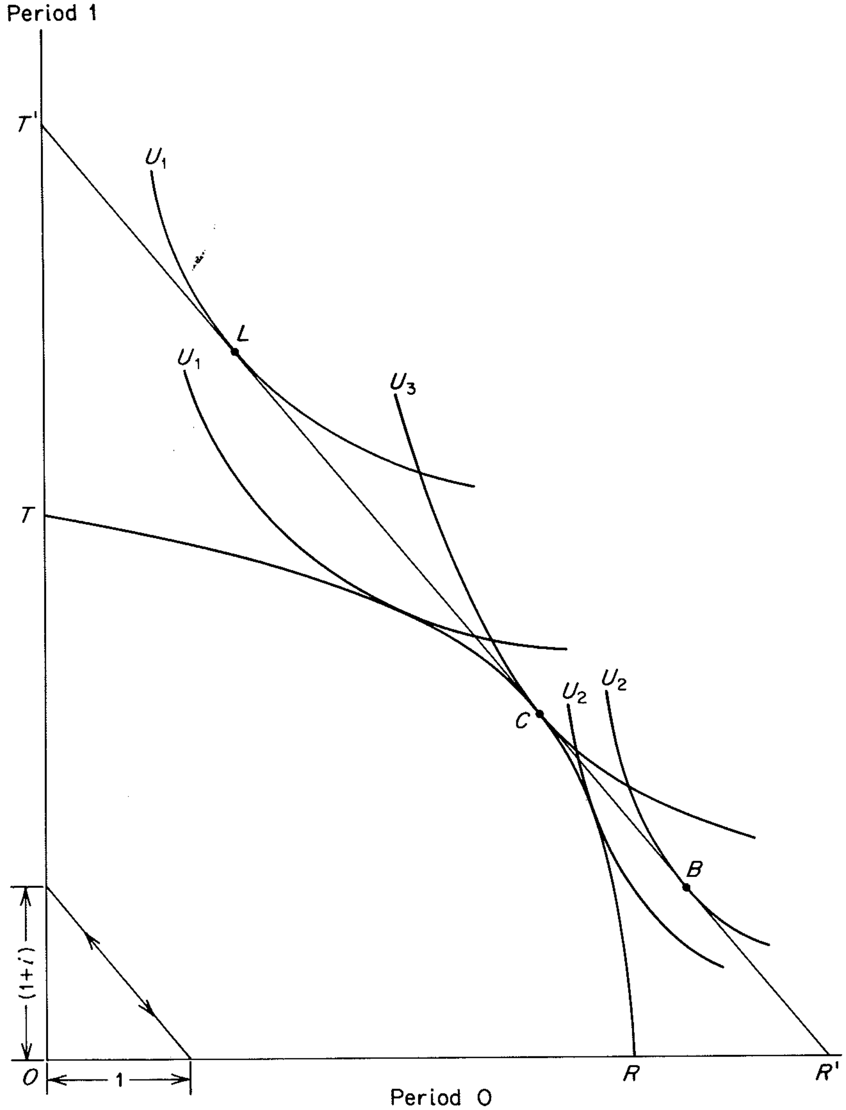
Fisher's theory relates the rate of interest to investment opportunities and time preference.[1] The problem is to allocate resources between two periods, 0 and 1, so that the present value of an individual's consumption, i.e., real income, is maximized. If K_0 and K_1 are consumption in periods 0 and 1, respectively, the objective is to maximize $K_0 + K_1/(1+i)$, where i is the market interest rate at which any amount of resources may be loaned or borrowed. Fisher is concerned with the analysis of optimal choice, given perfect foresight and perfect capital markets, since the assumption that any amount can be borrowed or loaned at the market interest rate is reasonable only under those conditions.

The design of Figure 2, following Fisher, shows an individual in command of *OR* current resources. The market interest rate is shown by the slope of the line in the lower left hand corner minus unity, i.e., by $(1+i)/1-1$. Arrows are drawn in both directions to indicate that both borrowing and lending take place at that rate. The individual has a series of independent market investment opportunities given by *RCT;* by giving up or investing resources during period 0, he can transform period

[1] Fisher, *Theory of Interest.*

FIGURE 2

Optimal Consumer Choice with Perfect Capital Markets



0 resources into period 1 consumption at the rate indicated by the slope of the investment opportunity function.

If neither borrowing nor lending were possible, the individual would have to choose his optimum consumption pattern from the locus of points on *RCT*. Given a utility function relating consumption in periods 0 and 1 (the usual utility indifference map) investment would be carried to the point where the marginal rate of return was equal to the marginal rate of time preference. If U_3 was part of the utility function, the optimum point would clearly be *C*. If U_2 was part of the utility function, on the other hand, less investment would be carried on, and if U_1 was, more investment would lead to the optimal solution. The optimal investment pattern cannot be located without reference to the structure of preferences for current versus future consumption.

This conclusion no longer follows when borrowing and lending are permitted. The range of choice is now widened, since it will pay to move along the investment opportunity curve until it becomes tangent to the market interest rate. At this point further investment cannot be worthwhile regardless of the individual's time preferences, since from point *C*—the tangency of *RCT* with *R′T′*—the individual can move along the line *R′CT′* by either borrowing or lending at the market rate. Following Hirschleifer's notation, let us call *R′CT′* the market opportunities function. This represents the best set of opportunities that can exist, given the individual's investment function and the market interest rate.[2]

The distribution of consumption between periods 0 and 1 depends on the individual's utility function. An individual with U_1 would choose to lend along *R′CT′* until he reached *L*, where his marginal rate of time preference is equal to the interest rate. An individual with U_2 would choose to borrow along *R′CT′*, winding up at *B* with the same marginal conditions. An individual with U_3 would neither borrow nor lend, since his marginal rate of time preference happens to be the same as the interest rate at precisely the point where the marginal yield from investment is equal to both.

Several points are worth emphasizing. First, individual time prefer-

[2] As Hirschleifer points out ("Optimum Investment Decision"), *R′CT′* is also a line of constant present value since it represents varying combinations of K_0 and $K_1/(1+i)$, all of which are equal. $0R'$ is obviously equal to $0T'/(1+i)$, given the construction, and i is positive since $0T' > 0R'$.

ence determines how much present versus future consumption is optimal, but does not influence the optimum investment pattern if unlimited borrowing or lending at a constant rate is permitted. Second, in equilibrium it must be true that the marginal yield from investment, the interest rate (marginal borrowing or lending cost), and the marginal rate of time preference are all the same.

Imperfect Capital Markets

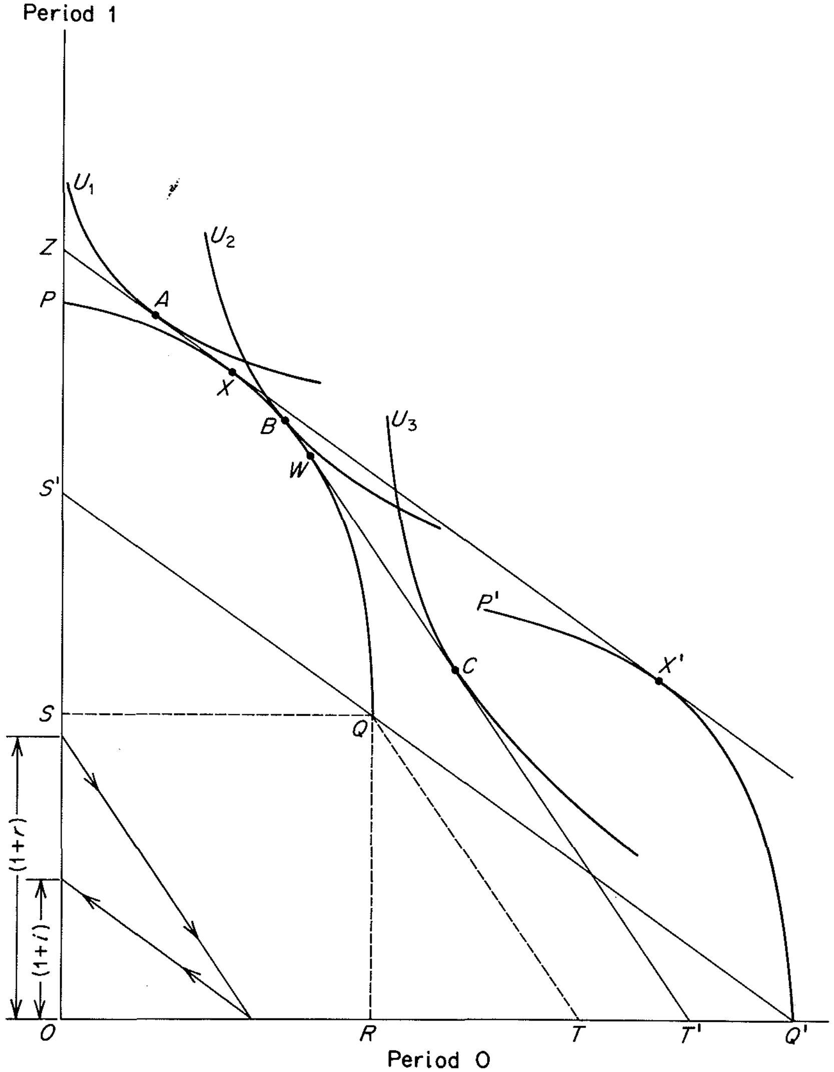
As was noted earlier, this analysis essentially relates to choice under conditions of perfect certainty and perfect capital markets. If both conditions hold, we can think of an individual with an income stream of K_0 and K_1 as having command over $K_0 + K_1/(1+i)$ resources, since he can borrow K_1 at the market rate and use it in any way he chooses. Let us drop both these assumptions and think of individuals as having some amount of current and expected income, and being able to borrow only at higher rates than those at which they can lend. Figure 3 shows the curves for an individual beginning at point Q with OR current income and OS expected income. He can lend at the rate indicated by the relatively flat slope in the lower left hand corner minus unity, i.e., at $[(1+i)/1-1]$, or i. Borrowing is assumed to be more costly, $(1+r)/1-1$, or r.[3] The investment opportunity function is QWP.

We can now proceed to construct a market opportunity curve. The individual has current income equal to OR. He can borrow at the rate r, along the line QT. Alternatively, he could invest along QWP and borrow by moving to the right along any line parallel to QT. It is clear that the best set of market opportunities involves, first, moving along QWP to W, where the marginal yield from investment is equal to the marginal cost of borrowing, then borrowing along the line WT'. Thus WT' is one segment of the market opportunities line. He can also continue upwards along QWP to point X, when the marginal yield from investment falls to the lending rate i; at this point, it is better to lend than to invest. Hence, the best set of market opportunities is the line $T'WXZ$.

[3] The lending rate shown in Figure 3 is actually negative, since $1+i$ is less than 1. This is irrelevant to the analysis, and only presents a sharp visual contrast between lending and borrowing rates.

FIGURE 3

Optimal Consumer Choice with Imperfect Capital Markets: Unlimited Borrowing and Lending, $r>i$



Depending on the preference function, the consumer will end by either borrowing (point C for the consumer with a preference function including U_3), lending (point A on U_1), or doing neither (point B on U_2). It is always true that the marginal yield from investment is equal to the marginal rate of time preference, but it may or may not be true that these two rates are equal to the market lending or borrowing rate. As might be supposed intuitively, an individual who borrows will equate the borrowing rate with the other two; one who lends will equate the lending rate; and one who does neither will have an equilibrium marginal investment yield and time preference somewhere between the borrowing and lending rates.[4] The "appropriate" rate for discounting future consumption, therefore, depends on whether the individual is a lender, borrower, or a Shakespearean.

The calculation of present value will give the correct solution, that the present value of $K_0 + K_1/(1+i)$ is a maximum at the point where utility is maximized, only if the equilibrium marginal rate of time preference (or the marginal investment yield) is used as the discount rate. This rate varies among individuals, being equal to the lending rate for the consumer with U_1, the borrowing rate for the consumer with U_3, and somewhere between the two for consumers with U_2. There is no unique discount rate that will yield the correct solution for all consumers. For example, using i as "the" discount rate should mean that the optimal combination of current and future consumption for the U_1 consumer has the highest present value to all consumers. However, the optimal combination of K_0 and K_1 for consumers with U_2 or U_3 will be different; the combination optimal for U_1 consumers must have a lower present value to U_2 or U_3 consumers than the combinations they actually prefer. In turn, the combination of K_0 and K_1 preferred by the consumer with U_2 will have the highest present value of any combination obtainable, if the equilibrium marginal time preference for U_2 is used as the discount rate. But different combinations are preferred by consumers with utility functions including U_1 andU_3.

The Fisher assumptions applied to Figure 3 would yield equilibrium solutions along the line XX', drawn tangent to $Q'X'P'$. If capital markets were perfect, an individual with OR of K_0 and OS of K_1 could be said

[4]The analysis here essentially reproduces that in Hirschleifer, "Optimum Investment Decision."

to have OQ' current resources.[5] If the individual has OQ' resources and an investment opportunity schedule like QWP, he can move along the line $Q'X'P'$, then borrow or lend at the going market rate. The best market opportunity thus involves investing to X', the point of tangency with i, then borrowing or lending if necessary to equate marginal time preference with i. The resulting market opportunity function XX' is obviously better than $T'WXZ$ except in the segment above X, where the two are necessarily the same. Individuals with preference functions like U_2 or U_3 will be able to reach higher indifference curves on these assumptions, but those with preferences like U_1 will not. The reason is that the original U_2 and U_3 equilibrium points reflect market imperfections—the existence of a higher market borrowing rate than the market lending rate—while at U_1 the individual is already at a lending rate equilibrium.

Although in Figure 3 we dropped the assumption of identical borrowing and lending rates, we retained the assumption that individuals are able to borrow any amount at the going rate. For the analysis of consumer decisions to purchase durable assets, it is helpful to examine cases in which selling future income (borrowing) is limited absolutely to a fixed amount. Also, we need to look at cases of "lump" investment, i.e., one cannot buy a portion of a new car. In addition, it is useful to analyze consumer decisions to use or refrain from using liquid assets in order to finance durable goods purchases.

Absolute Rationing, Continuous Investment Schedule

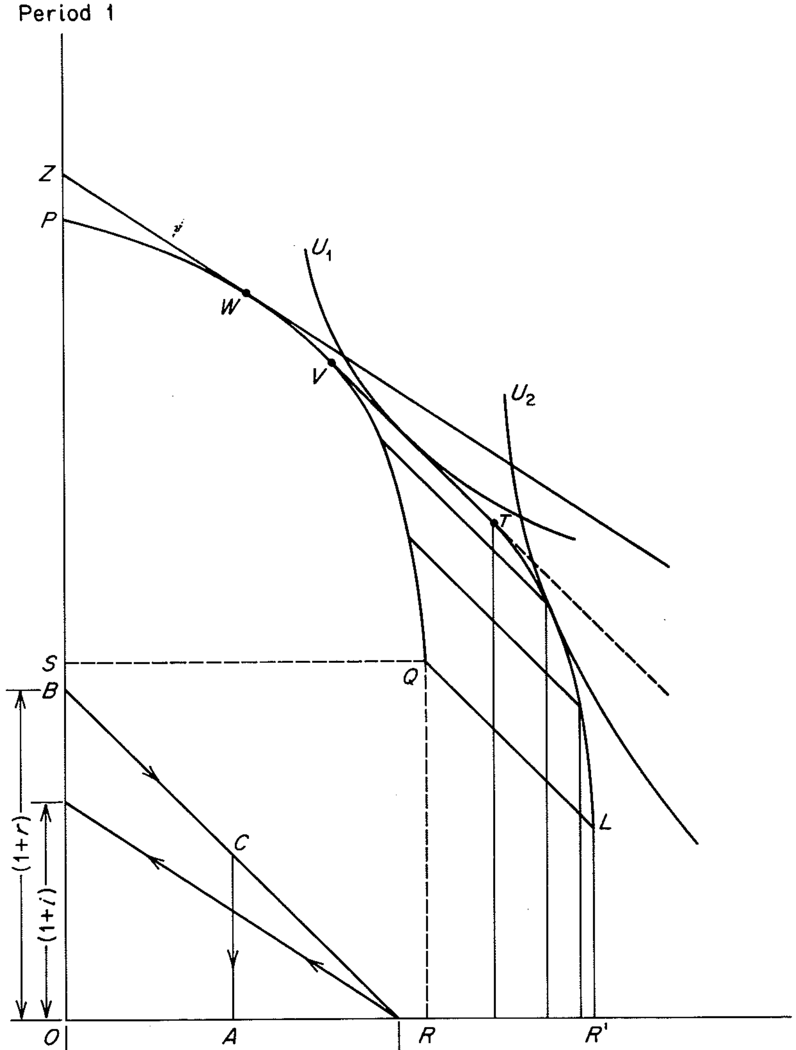
In Figure 4, we show a consumer with OR income in period 0, OS income expected in period 1, and an investment opportunity schedule QVP. The consumer can lend any amount at a rate equal to i; he can borrow an amount equal to OA at a rate of r. The borrowing rate for amounts in excess of OA is infinite, as indicated by the vertical line.

The market opportunities line can be readily constructed by the procedure used in Figure 3. The consumer can move downward from

[5] The discounted value of OS $(=QR)$ is RQ'; the discounted value is greater than the value at period O because we have portrayed a negative interest rate, as explained in footnote 3, above.

FIGURE 4

Optimal Consumer Choice with Imperfect Capital Markets: Absolute Limit to Borrowing, $r>i$



Q along his borrowing line or he can move upward along QVP, thence downward along the same borrowing function. It will not pay to invest beyond point V and then borrow because the marginal investment yield is less than the borrowing cost. Hence, one part of the market opportunities function is traced out by an envelope of locations that involve investing and borrowing, that is, by LTV. From V the best market opportunity involves investing (but no borrowing) until point W, where the marginal yield from investment falls to the same level as the lending rate. From W the lending rate determines the best set of market opportunities, WZ having the slope $1+i$. Thus the complete set of market opportunities is traced out by $R'LTVWZ$.

The equilibrium point depends, as usual, on preferences for present versus future consumption. If the utility function contains U_1, the equilibrium will take place at the market borrowing rate and rationing is not effective. Marginal time preferences, marginal yields from investment, and marginal borrowing costs are all equal. If the utility function includes U_2, rationing is effective. The marginal yield from investment and the marginal rate of time preference are equal, and both are higher than the market borrowing rate. A relaxation of rationing would permit the consumer to move along the dotted line extending beyond the segment VT and to a higher utility curve. The appropriate rate for discounting future yields is the marginal rate of time preference, which may or may not be equal to an observable market rate.

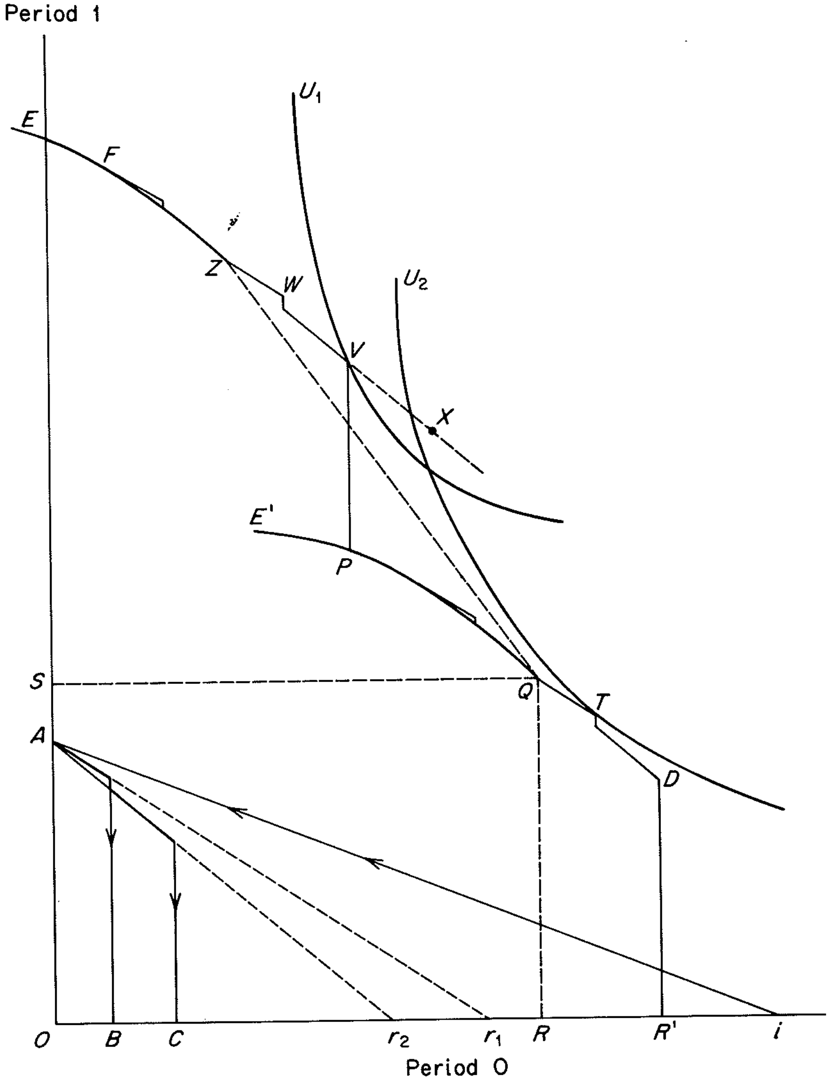
Absolute Rationing, Discontinuous Investment Schedule

In Figure 5, the market borrowing opportunities are pictured as a discontinuous stepped schedule. OB can be borrowed at a rate indicated by the slope of Ar_1 and an additional amount, BC, at the higher rate indicated by the slope of Ar_2; beyond OC no borrowing is possible at any rate. Any amount can be loaned at a rate indicated by the slope of Ai. The investment opportunity curve is QZE; the section QZ, shown as a dashed line, must be purchased as a whole or not at all. The consumer has the alternative investment opportunity line QE', which is simply

Appendix A

FIGURE 5

Optimal Consumer Choice with Imperfect Capital Markets: Stepped Borrowing Schedule, Absolute Limit to Borrowing, $r>i$, Investment Function Discontinuous



the segment of QZE above the point Z where the investment schedule becomes continuous.

The market opportunity line is constructed in the usual way. Starting at Q, the consumer can borrow along the line $QTDR'$. If he makes a lump investment he can reach points along $ZWVP$ by investing and borrowing. Note that some current-period consumption must be given up in order to make that investment, since the maximum allowable borrowing from future income is not enough to maintain current consumption at the same level as current income, given the cost of the investment. The consumer can move beyond Z and invest more than the amount required by the lump investment, or he can do without it and utilize the less profitable investment line QE'. If the lump investment is made, any of the points on the envelope associated with $FWVP$ can be reached; if it is not made, points on the envelope associated with TDR' can be reached. The consumer would not choose any point on the segment VP, which connects the two envelope sections, except point V; all other points involve smaller consumption in period 1 and the same amount in period 0.

Whether the consumer finds it better to make the lump investment or not depends on his preference function. If it includes U_1, he will do so; if it includes U_2, it will not be worth while. It is possible that he will be indifferent, since a utility indifference curve can evidently be tangent at both upper and lower sections of the market opportunity curve.

The marginal conditions for equilibrium cease to be meaningful in Figure 5 because of the assumed discontinuities, and the notion of a uniquely "correct" discount rate does also. For the consumer with U_1, the equilibrium marginal rate of time preference is higher than either of the market borrowing rates and is also higher than the yield from the lump investment. What constitutes the appropriate discount rate for comparing future and present returns is difficult to delineate. In Figure 5, the consumer with U_2 is shown to have a lower marginal rate of time preference at the optimal location than the consumer with U_1. However, an easing of the rationing restriction would permit the U_2 consumer to move along the dashed line extending from V. The widening of the range of choice may well make the optimal location one that includes the lump investment—for example, some point like X. At that location, the marginal rate of time preference in equilibrium would be greater than at point T, the equilibrium position given the original restraints. Thus marginal time preference at equilibrium is obviously inappropriate as a discount rate,[6] and there is no meaningful marginal borrowing cost or asset yield because of discontinuities in both functions.

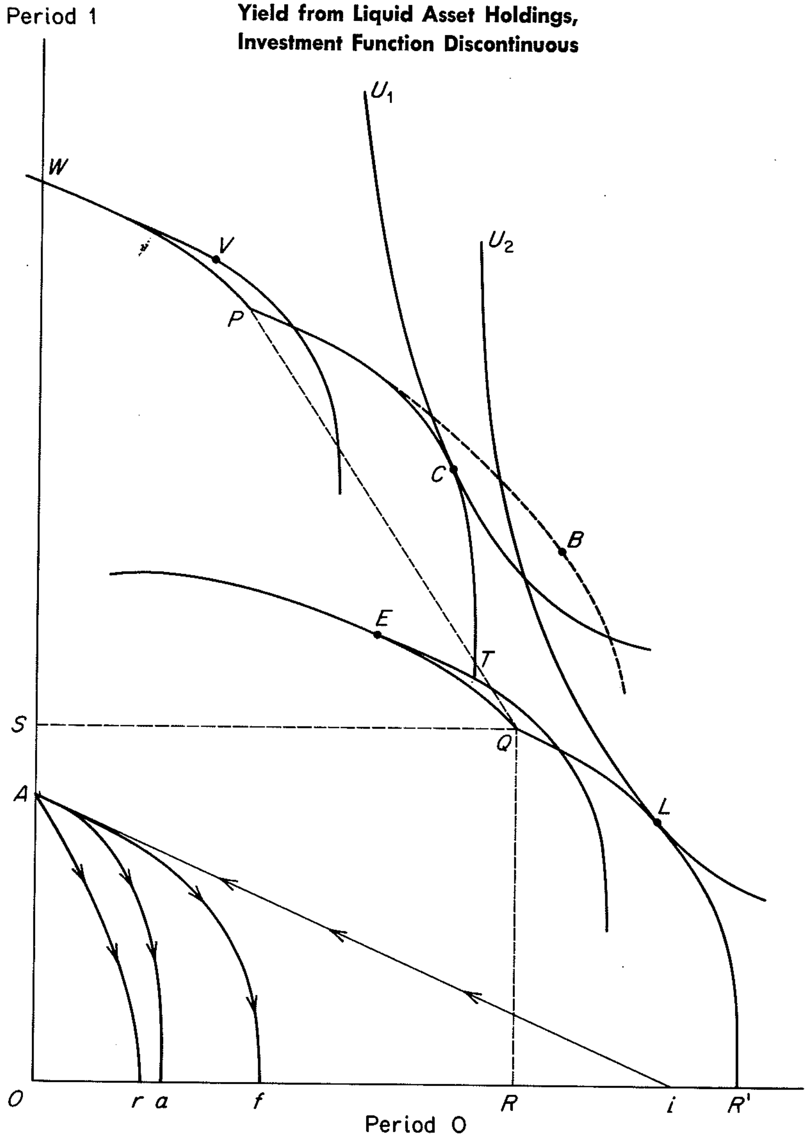
Absolute Rationing, Some Liquid Assets Available, Discontinuous Investment Schedule

To introduce liquid asset holdings into the analysis we show in Figure 6 the situation of a consumer with *OR* present income, *OS* income expected in period 1, and an investment opportunity schedule *QPW*, where *QP* is a lump investment. The consumer can lend at the rate indicated by the slope of *Ai*. He can borrow along the line *Ar* at increasing rates up to the maximum limit of *Or*. In addition, he has liquid assets equal to *Oa*. The assets may be thought of as having a subjective rate of return in addition to their market yield, since they provide both liquidity and a reserve against unforeseen contingencies. The marginal yield is very high for the first dollar of those assets, falling gradually as asset holdings become larger. At some point, asset holdings become sufficient to satisfy the need for liquidity and a contingency reserve, and the yield falls to the market return, i.e., to the lending rate. We have drawn *Aa* to reflect the subjective marginal yield from giving up liquid assets (potential consumption in period 1) in order to increase consumption in period 0. If the consumer holds *Oa*, he will give up a small amount at a rate of return equal to the lending rate; the rate of return gradually rises, eventually becoming infinite at *a*. The amount *Of* is the consumer's total liquid asset holdings plus his maximum permitted borrowing. The slope of *Af* measures the cost (in terms of period 1

[6] Figure 5, for example, makes clear that the equilibrium marginal rate of time preference for the consumer with U_1 cannot be used as the discount rate. Many combinations of K_0 and K_1 that do not include the lump investment will show higher *PV's* at that discount rate, and we know these combinations are inferior to the optimal one. Similarly, many combinations of K_0 and K_1 that include the lump investment will show higher *PV's* than the optimal combination for the consumer with U_2, using his marginal rate of time preference as a discount rate. The "correct" discount rate is evidently higher (lower) than the equilibrium marginal rate of time preference for U_2 (U_1). We see no formally correct solution to the problem.

FIGURE 6

Optimal Consumer Choice with Imperfect Capital Markets: Increasing Marginal Borrowing Costs, Increasing Subjective Yield from Liquid Asset Holdings, Investment Function Discontinuous



consumption) of adding to consumption in period 0 either by borrowing in the market or by drawing down assets. This can be thought of as a marginal borrowing cost function, including the partly subjective cost of borrowing from one's own current assets.[7]

A market opportunity curve can be drawn up in the same manner as in Figure 5. Starting at Q the consumer can borrow along the line Af ($= QR'$). Alternatively he can move to P by investing, then borrow along Af ($= PT$). We end with a "double" envelope curve, one around $WVCT$, if the lump investment is made, and another around ETL, if the lump investment opportunity is skipped. Given preferences for present and future consumption, the lump investment may or may not be made. In equilibrium, the marginal rate of time preference, the marginal cost of borrowing from the market, and the marginal yield on liquid assets will all be equal, but all three may be higher or lower than the yield from the lump investment because the latter involves a discontinuity. Again, we see that a relaxation of rationing may result in movement to a preferred location providing higher marginal time preferences, marginal borrowing costs, and marginal (subjective) asset yields. The consumer with U_2, for example, would prefer the point B on the dotted line to his present location at L, despite the fact that B would involve much less present consumption, hence a higher marginal rate of time preference in equilibrium. As in Figure 5, we see no formally correct discount rate that will always make the optimal pattern of K_0 and K_1 have the highest present value for any attainable combination.

[7] The discussion has been couched in terms of liquid assets. It applies equally well to any marketable asset held by households, including consumer durable assets that can be sold.