Volume Title: Risk Aspects of Investment-Based Social Security Reform

Volume Author/Editor: John Y. Campbell and Martin Feldstein, editors

Volume Publisher: University of Chicago Press

Volume ISBN: 0-226-09255-0

Volume URL: http://www.nber.org/books/camp01-1

Publication Date: January 2001

Chapter Title: The Social Security Trust Fund, the Riskless Interest Rate, and Capital Accumulation

Chapter Author: Andrew B. Abel

Chapter URL: http://www.nber.org/chapters/c10594

Chapter pages in book: (p. 153 - 202)

5

The Social Security Trust Fund, the Riskless Interest Rate, and Capital Accumulation

Andrew B. Abel

The social security trust fund in the United States currently has about \$0.75 trillion in assets. Its assets are projected to grow to almost \$2 trillion (1998 dollars) in the year 2016. As the baby-boom generation begins to retire and collect social security benefits in the second decade of the twenty-first century, the social security trust fund will shrink, and it is projected to run out of assets in the year 2032.[1] The prospect that the social security system will run large deficits and exhaust the social security trust fund has given rise to a variety of proposals to "save social security." Some proposals are designed to exploit the equity premium, which is the excess of the rate of return on equity over the riskless interest rate. Since the equity premium has historically averaged several hundred basis points per year, it may be tempting to shift some of the assets of the social security trust fund (which currently holds only bonds) from bonds to equity. In this paper, I analyze the effects on the equilibrium equity premium and the equilibrium growth rate of the capital stock of such a portfolio change.

I have three goals in this paper. First, I want to develop a tractable stochastic dynamic general equilibrium model of social security and national capital accumulation with an endogenous equity premium. Second, although tractability dictates that the model be relatively simple, I want to calibrate the model numerically and would like the calibrated model to be

Andrew B. Abel is the Robert Morris Professor of Banking and professor of economics at the Wharton School of the University of Pennsylvania.

The author thanks Henning Bohn, John Campbell, Bill Dupor, Robert King, Deborah Lucas, Amir Yaron, and participants in the NBER preconference and conference for helpful comments. The author also thanks Farshad Mashayekhi for excellent research assistance.

1. Table III.B2 of Board of Trustees (1998) reports projections for the assets of the combined OASI and DI trust funds. The year-end projections based on intermediate cost assumptions in constant (1998) dollars are \$756.9 billion for 1998 and \$1,960.4 billion for 2016.

quantitatively plausible in some dimensions. In particular, I would like the model to be able to match the historical average equity premium and the historical average growth rate of capital. Third, I want to apply the model to analyze the effects on the equity premium and the growth rate of capital of investing some of the social security trust fund in risky capital.

A natural starting point for a model of social security and capital accumulation is Diamond's (1965) classic model of government debt in a neoclassical economy, which has been applied to analyze the effects of social security on national capital accumulation in a deterministic context.[2] In order to achieve the goals of this paper, I modify the Diamond model in two important ways. First, because the Diamond model is a deterministic one, the equity premium is identically zero in that model. Since I want to model the equilibrium equity premium, I introduce risk so that a positive average equity premium is a feature of equilibrium. Second, to help keep the analysis tractable, I replace the neoclassical production function with an *AK* model that is consistent with endogenous growth. I introduce risk in the model by assuming that productivity is stochastic.

I model four sets of economic actors—firms, individuals, the Treasury, and the social security system—and I describe the behavior of each of these sets of economic actors in the first four analytic sections of the paper. The behavior of firms is presented in section 5.1, where I present the stochastic *AK* technology and then derive the equilibrium wage and risky return on capital. With a stochastic *AK* technology, the rate of return on capital is stochastic but exogenous. The stochastic nature of the rate of return on capital allows for a positive equity premium in equilibrium. The exogenous nature of this risky rate of return keeps the model tractable. Although the risky rate of return is exogenous in this model, the riskless interest rate is endogenous, so the equity premium is also endogenous. Any change in the riskless interest rate is matched by a change in the equity premium of the same magnitude but in the opposite direction. Thus, I will focus attention on the behavior of the equilibrium riskless interest rate, recognizing that the results directly translate into results about the equity premium.

The consumption/saving and portfolio decisions of individuals are analyzed in section 5.2. My choice of a specification of the utility function reflects the tension between analytic tractability and quantitative realism. To achieve analytic tractability, I assume that the utility function is characterized by an intertemporal elasticity of substitution equal to one, as is the case, for example, with logarithmic utility. However, with logarithmic utility, the coefficient of relative risk aversion also equals one, and quantitative realism dictates a coefficient of relative risk aversion greater than one. Thus, I use a special case of the preferences introduced by Epstein and

2. For a textbook example, see Blanchard and Fischer (1989, 110–13).

Zin (1989) and Weil (1990) to allow for a coefficient of relative risk aversion greater than one and an intertemporal elasticity of substitution equal to one.

Although the behavior of firms and individuals is based on explicit maximization, I do not attempt to specify the objective functions of the Treasury and the social security system and then derive optimal policy. Instead, I specify policy functions for each of these fiscal institutions in sections 5.3 and 5.4. To prevent the amount of Treasury debt from becoming too large or too small in the face of stochastic shocks, I assume that the Treasury adjusts taxes and government purchases in response to deviations of the debt-GDP ratio from a target value. As for the social security system, I examine a pay-as-you-go defined-benefit system and allow social security taxes to adjust when the ratio of the social security trust fund to the aggregate capital stock deviates from its target value. In addition, I assume that the social security trust fund can choose how to allocate its portfolio to riskless bonds and risky capital.

Firms, individuals, the Treasury, and the social security system interact in capital markets to determine the riskless interest rate (and hence the equity premium) and the growth rate of the aggregate capital stock. This model has a convenient recursive structure. The riskless interest rate is determined by portfolio-allocation decisions of individuals and does not depend on the aggregate level of the capital stock. Then, given the value of the riskless interest rate, the saving decisions of individuals determine the growth rate of the capital stock. The presentation of results in section 5.5 reflects this recursive structure.

I examine the riskless interest rate in subsection 5.5.1. An increase in the amount of riskless bonds relative to the amount of capital causes the riskless interest rate to increase (equivalently, the equity premium to fall) because individuals must be induced to hold a higher share of riskless assets in their portfolios. In particular, if the social security trust fund sells some bonds to the public in exchange for risky capital, then, in the context of a pay-as-you-go defined-benefit system, the real interest rate must increase to induce individuals to increase the share of riskless assets in their portfolios.

After analyzing the equilibrium riskless interest rate in subsection 5.5.1, I analyze the equilibrium value of the growth rate of the capital stock in subsection 5.5.2. The growth rate of the capital stock is determined by the amount of saving in the economy. I show that, if the social security trust fund sells some bonds in exchange for risky capital, the capital stock in the following period will be higher than if the social security trust fund held only bonds. This effect arises because the change in the portfolio of the social security trust fund causes the riskless interest rate to increase, which reduces the present value of the social security benefits that current workers expect when they retire. In response to this reduction in the present value of lifetime income, current workers reduce their consumption and increase their saving. The effect on the saving of future generations involves additional effects operating through the adjustment of taxes to satisfy the budget constraints and policy functions of the Treasury and the social security system. I focus my analysis of saving by future generations by considering *constant growth paths,* which I define and analyze in subsection 5.5.2. Proposition 6 in this subsection presents a sufficient condition for the growth rate of the capital stock along a constant growth path to increase when the share of the social security trust fund invested in risky capital increases.

I explore the quantitative plausibility of the model in section 5.6, where I show that the endogenous riskless interest rate and growth rate of capital along a constant growth path can match the historical average values of these variables for reasonable values of the preference parameters. I also explore the sensitivity of these endogenous variables to various parameters and calibrated values of variables. In addition, I show that an increase in the share of the social security trust fund that is invested in risky capital will increase the growth rate of capital along a constant growth path because the sufficient condition in proposition 6 is satisfied in the baseline calibration and in the sensitivity analysis. Quantitatively, the model suggests that investing a modest fraction of the social security trust fund in risky capital will have only small effects on the riskless interest rate and the growth rate of the capital stock.

I present concluding remarks in section 5.7. Various technical derivations are relegated to appendixes A–E.

5.1 Factor Prices in General Equilibrium

The economy consists of overlapping generations of people who live for two periods. At the beginning of period t, a continuum of people with measure N_t is born. Each of these people inelastically supplies one unit of labor when young in period t and does not supply any labor when old in period $t + 1$.

Output in period t is produced using labor and capital. In period t, firm i uses labor, $N_{i,t}$, and capital, $K_{i,t}$, to produce output, $Y_{i,t}$, according to the production function

$$Y_{i,t} = A_t K_{i,t}^{\alpha}(N_{i,t}K_t)^{1-\alpha}, \tag{1}$$

where $A_t \geq A_L > 0$ is an independently and identically distributed (i.i.d.) productivity shock with mean $\overline{A}$,[3] K_t is the aggregate capital stock at the beginning of period t, and $0 < \alpha < 1$. The production function in equation

3. A_L is the greatest lower bound for A_t. In addition, I assume that there is a positive probability that A_t is within a small neighborhood of A_L. Specifically, $\mathrm{pr}\{A_t \geq A_L\} = 1$, and, for all $\varepsilon > 0$, $\mathrm{pr}\{A_L \leq A_t \leq A_L + \varepsilon\} > 0$.

(1) is consistent with endogenous growth (see, e.g., Barro and Sala-i-Martin 1995, 150).

Factor prices are determined in competitive markets, and the rental price of each factor equals its marginal product. Thus, the wage rate in period t, w_t, is

$$w_t = (1 - \alpha)A_t\left(\frac{K_{i,t}}{N_{i,t}}\right)^{\alpha} K_t^{1-\alpha}, \tag{2}$$

and the gross rate of return to capital in period t, R_t, is

$$R_t = \alpha A_t\left(\frac{N_{i,t}K_t}{K_{i,t}}\right)^{1-\alpha}. \tag{3}$$

In equilibrium, each firm will choose the same capital-labor ratio so that $K_{i,t}/N_{i,t} = K_t/N_t$ for all i. Now assume that the population is constant over time, adopt the normalization $N_t \equiv 1$, and substitute $K_{i,t}/N_{i,t} = K_t$ in equations (2) and (3) to obtain

$$w_t = (1 - \alpha)A_tK_t \tag{4}$$

and

$$R_t = \alpha A_t. \tag{5}$$

The gross rate of return on capital is random and has mean $\overline{R} = \alpha\overline{A}$.

5.2 Individual Optimization

Each person faces an optimization problem that includes a saving/consumption decision and a portfolio decision. I will solve the optimization problem of a person born in period t after first specifying the person's budget constraint and then specifying the person's utility function.

A representative person born at the beginning of period t supplies one unit of labor in period t and receives wage income equal to w_t. Also in period t, the person pays taxes T_t^T to the Treasury and pays social security taxes T_t^S. Both types of taxes are lump sum. I have distinguished taxes paid to the Treasury from taxes paid to the social security system so that I can keep track of the Treasury's outstanding debt and the amount of Treasury bonds held by the social security trust fund.

A young person in period t has disposable income of $w_t - T_t^S - T_t^T$, which can be used for consumption and the purchase of riskless bonds and risky capital. Riskless bonds purchased in period t pay a gross rate of return r_{t+1} in period $t + 1$. Let B_{t+1}^P be the value of riskless bonds purchased by a young person in period t (the superscript P denotes that the bonds are privately held, in contrast to bonds held by the social security trust fund). The person also purchases risky capital K^P_{t+1}, which pays a gross rate of return R_{t+1} in period $t + 1$. Since consumption when young, C_t, plus the purchases of bonds and risky capital equals disposable income,

$$C_t = w_t - T^S_t - T^T_t - B^P_{t+1} - K^P_{t+1}. \tag{6}$$

Let X_{t+1} be the consumption of an old person in period $t + 1$. This consumption is financed by the riskless bonds and risky capital purchased in period t and by social security benefits. Riskless bonds are worth $r_{t+1} B^P_{t+1}$, and risky capital is worth $R_{t+1}K^P_{t+1}$. Social security benefits consist of two components. One component is $\theta_1 w_{t+1} = \theta_1(1 - \alpha)A_{t+1}K_{t+1}$, which is proportional to the actual wage in period $t + 1$. The other component is $\theta_0\overline{w}_{t+1}$, where $\overline{w}_{t+1}$ is the expected value of w_{t+1} conditional on information available at the end of period t. Since the capital stock K_{t+1} is known at the end of period t, $\overline{w}_{t+1} = (1 - \alpha)\,\overline{A}K_{t+1}$. Taking account of both components of the social security benefits, the total amount of social security benefits, Q_{t+1}, received by an old person in period $t + 1$ is

$$Q_{t+1} = \theta_0(1 - \alpha)\overline{A}K_{t+1} + \theta_1(1 - \alpha)A_{t+1}K_{t+1}. \tag{7}$$

I assume that $\theta_0 \geq 0$ and $\theta_1 \geq 0$. It is convenient, although not strictly accurate, to refer to the parameters θ_0 and θ_1 as *replacement rates* for social security. Because the social security benefits received by an old person do not depend on the amount of social security taxes paid by that person or on any decision made by that person, I describe the system in this model as *a defined-benefit system.*

The solution of the person's optimization problem is facilitated by using equation (5) to rewrite the social security benefits in equation (7) as

$$Q_{t+1} = \theta_0(1 - \alpha)\overline{A}K_{t+1} + \theta_1\frac{1 - \alpha}{\alpha}R_{t+1}K_{t+1}. \tag{8}$$

Because w_{t+1} is perfectly correlated with R_{t+1} in this model, the claim on future social security benefits can be viewed as consisting of a riskless asset plus a risky asset with a payoff that is perfectly correlated with the rate of return on risky capital, as illustrated in equation (8).

I assume that individuals do not have a bequest motive and thus that they consume all available resources when they are old. Taking account of privately held bonds and risky capital as well as social security benefits, Q_{t+1}, yields

$$X_{t+1} = \left[B^P_{t+1} + \frac{\theta_0(1 - \alpha)\overline{A}K_{t+1}}{r_{t+1}}\right]r_{t+1} + \left(K^P_{t+1} + \theta_1\frac{1 - \alpha}{\alpha}K_{t+1}\right)R_{t+1}. \tag{9}$$

Suppose that each person born at the beginning of period t has the following utility function, which is a special case of the parametric class of preferences developed by Epstein and Zin (1989) and Weil (1990) and used by Bohn (1998a) to study intergenerational risk sharing:[4]

$$(10) \qquad U_t = \ln C_t + \frac{\delta}{1-\phi} \ln E_t\left\{X_{t+1}^{1-\phi}\right\} \quad \text{where } 0 < \phi \neq 1 \text{ and } \delta > 0.$$

For the utility function in equation (10), the intertemporal elasticity of substitution equals one, and the coefficient of relative risk aversion over second-period consumption is ϕ. I have chosen to specify a unitary intertemporal elasticity of substitution to simplify the consumption/saving decision and to help keep the general equilibrium analysis tractable. A standard time-separable utility function with a constant coefficient of relative risk aversion constrains the coefficient of relative risk aversion to equal the inverse of intertemporal elasticity of substitution, which equals one in this case. However, I do not constrain the coefficient of relative risk aversion to equal one because various studies of the equity-premium puzzle have shown that it is difficult, or perhaps impossible, to account for the large historical average value of the equity premium, $R_{t+1} - r_{t+1}$, with a coefficient of relative risk aversion as low as one.

The optimization problem of a young person in period t is to choose C_t, B^P_{t+1}, and K^P_{t+1} to maximize the utility function in equation (10) subject to the constraints in equations (6) and (9). The solution to this problem is easily expressed in term of Ω_t, the present value of lifetime resources, which is

$$(11) \qquad \Omega_t \equiv w_t - T^T_t - T^S_t + \frac{\theta_0(1-\alpha)\overline{A}K_{t+1}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}.$$

The present value of lifetime resources consists of disposable income, $w_t - T^T_t - T^S_t$, plus the present value of the social security benefits[5] to be received in period $t + 1$,

$$\frac{\theta_0(1-\alpha)\overline{A}K_{t+1}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}.$$

4. If $\phi = 1$, the utility function is $U_t = \ln C_t + \delta E_t\{\ln X_{t+1}\}$.

Individuals may also obtain utility from government purchases. I assume that any utility from government purchases is additively separable from utility of the consumer's own consumption.

5. In computing the present value of future social security benefits, the riskless component, $\theta_0(1 - \alpha)\overline{A}K_{t+1}$, is discounted by the riskless rate, r_{t+1}, and the risky component,

$$\theta_1\frac{1-\alpha}{\alpha}R_{t+1}K_{t+1},$$

is discounted by the risky rate, R_{t+1}.

Let a_{t+1} be the value of a young person's total assets at the end of period t. These assets consist of direct holdings of riskless bonds, B^P_{t+1}, and risky capital, K^P_{t+1}, plus the present value of future social security benefits,

$$\frac{\theta_0(1-\alpha)\bar{A}K_{t+1}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}.$$

Thus,

$$(12) \qquad a_{t+1} \equiv B^P_{t+1} + K^P_{t+1} + \frac{\theta_0(1-\alpha)\bar{A}K_{t+1}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}.$$

As shown in equation (A5) in appendix A, the optimal value of a_{t+1} is

$$(13) \qquad a_{t+1} = \frac{\delta}{1+\delta}\Omega_t.$$

To describe the optimal allocation of a young person's portfolio, let γ_{t+1} be the share of the total portfolio a_{t+1} devoted to risky assets, consisting of risky capital, K^P_{t+1}, and the present value of risky future social security benefits

$$\theta_1\frac{1-\alpha}{\alpha}K_{t+1}.$$

More precisely,

$$(14) \qquad \gamma_{t+1} \equiv \frac{K^P_{t+1} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}}{a_{t+1}}.$$

The definitions in equations (12) and (14) imply that $1-\gamma_{t+1}$ is the share of a young consumer's total portfolio devoted to riskless assets, consisting of riskless bonds, B^P_{t+1}, and the present value of riskless future social security benefits

$$\frac{\theta_0(1-\alpha)\bar{A}K_{t+1}}{r_{t+1}}.$$

Let $\gamma(r_{t+1})$ denote the optimal value of γ_{t+1}. This notation emphasizes that the optimal portfolio allocation depends on the riskless interest rate, r_{t+1}, which is an endogenous variable in this model. The optimal portfolio allocation also depends on the distribution of the risky rate of return, R_{t+1}, but this distribution is exogenous in this model, so the notation does not reflect this dependence. The optimal value of γ_{t+1} is characterized in appendix B, where it is shown that, if $\phi \leq 1$, then $\gamma'(r_{t+1}) < 0$. If $\phi > 1$, then $\gamma'(r_{t+1})$ may be negative, zero, or positive. Henceforth, I restrict attention to the case with $\gamma'(r_{t+1}) < 0$.[6]

The definition of γ_{t+1} in equation (14) and the optimal value of a_{t+1} in equation (13) imply the following expressions for optimal holdings of assets by a young person at the end of period t:

$$B^P_{t+1} + \frac{\theta_0(1 - \alpha)\overline{A}K_{t+1}}{r_{t+1}} = [1 - \gamma(r_{t+1})]\frac{\delta}{1 + \delta}\Omega_t, \tag{15}$$

and

$$K^P_{t+1} + \theta_1\frac{1 - \alpha}{\alpha}K_{t+1} = \gamma(r_{t+1})\frac{\delta}{1 + \delta}\Omega_t. \tag{16}$$

The riskless interest rate r_{t+1} affects the private demand for capital in two ways. Since I am restricting attention to the case in which $\gamma'(r_{t+1}) < 0$, an increase in the riskless interest r_{t+1} causes consumers to shift their portfolios toward the riskless asset and away from risky assets, thereby reducing the private demand for capital, for a given present value of lifetime resources Ω_t. In addition, if $\theta_0 > 0$, an increase in r_{t+1} reduces the present value of riskless social security benefits and thus reduces Ω_t, as shown in equation (11). This reduction in Ω_t reduces the private demand for capital. Thus, an increase in r_{t+1} reduces the private demand for capital both by changing the composition and (if $\theta_0 > 0$) by reducing the size of private portfolios.

5.3 The Treasury's Revenues, Expenditures, and Debt

The social security trust fund in the United States holds several hundred billion dollars of bonds issued by the Treasury. Because these bonds are liabilities of the Treasury and assets of the social security trust fund, it is important to treat the Treasury and the social security system separately. In this section, I specify the Treasury's behavior.

The budget constraint of the Treasury is

$$B_{t+1} = r_t B_t + G_t - T^T_t, \tag{17}$$

where B_t is the amount of Treasury debt outstanding at the end of period $t - 1$ (equivalently, the beginning of period t), r_t is the gross rate of return

6. In the calibration in sec. 5.6, the value of ϕ exceeds one so that, in principle, $\gamma'(r_{t+1})$ can be positive, negative, or zero. For all the cases examined in tables 5.1–5.3 below, $\gamma'(r_{t+1}) < 0$.

on these bonds, G_t is the Treasury's expenditure on purchases of consumption goods[7] during period t, and, as in section 5.2, T_t^T is the tax revenue collected from young consumers by the Treasury during period t.

A simple approach to modeling fiscal policy is to assume that government purchases, G_t, and Treasury taxes, T_t^T, are each proportional to aggregate output, A_tK_t, and then to let the stock of Treasury debt evolve according to equation (17). However, in the face of stochastic shocks to A_t, the stock of debt could become arbitrarily large or arbitrarily small (indeed, negative and large in absolute value). Therefore, I will modify the simple assumptions of proportional government purchases and taxes so that purchases are reduced and/or taxes increased if the stock of debt is above some target level. Similarly, if the stock of debt is below the target, then purchases are increased and/or taxes cut.

To measure the size of the Treasury's debt relative to the size of the economy, define $b_{t+1} \equiv B_{t+1}/K_{t+1}$ as the ratio of Treasury debt to the aggregate capital stock.[8] Let β be the "target" value of b_{t+1}. I have put the word *target* in quotation marks because the Treasury does not literally aim to set b_{t+1} equal to β. The Treasury moves the value of b_{t+1} toward β according to the following policy function:

$$b_{t+1} - \beta = \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \overline{A}), \tag{18}$$

where $\rho_B \geq 0$ and $\rho_A \geq 0$ are parameters governing the evolution of the debt-capital ratio b_t, and τ and g are parameters related to Treasury taxes and purchases, as described below.

If Treasury taxes were $T_t^T = \tau A_tK_t$, and if government purchases were $G_t = gA_tK_t$, then $(g - \tau)(A_t - \overline{A})K_t$ would be the amount by which net government expenditures (i.e., government purchases less taxes) in period t exceed the amount that was expected at the end of the previous period (when K_t was known). The Treasury can respond to unexpected net expenditures by increasing taxes, reducing government purchases, or increasing its outstanding debt. If the Treasury completely insulates the size of its debt from unexpected shocks by changing taxes and government purchases appropriately, then $\rho_A = 0$ in equation (18). Alternatively, if $\rho_A > 0$, then the Treasury finances at least part of unexpected net expenditures by increasing its debt.

Let D_t be the primary deficit in period t. Since the primary deficit is the amount by which government purchases (which do not include interest

7. I assume that all capital formation in the economy is done by the private sector so that all the Treasury's expenditure on goods is for consumption goods.

8. The size of a country's debt is often expressed as a debt-GDP ratio, B_t/Y_t, which is B_t/A_tK_t in this model. The measure that I use in this paper is proportional to $B_t/\overline{A}K_t$, which is the ratio of debt to "trend" GDP, $\overline{A}K_t$.

payments on government debt) exceed taxes, the Treasury's budget constraint in equation (17) implies

$$D_t = G_t - T_t^T = B_{t+1} - r_t B_t. \tag{19}$$

The Treasury policy function in equation (18) implies a value for the primary deficit. Multiplying both sides of equation (18) by K_{t+1}, substituting the resulting expression for B_{t+1} in equation (19), and recalling that $B_t = b_t K_t$ yield

$$D_t = [\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \bar{A})]K_{t+1} - r_t b_t K_t. \tag{20}$$

Given the value of the primary deficit in equation (20), the values of G_t and T_t^T still need to be determined. To the extent that D_t in equation (20) differs from $(g - \tau)A_t K_t$, government purchases and/or taxes need to be adjusted. I introduce a "tax responsiveness" parameter λ to determine how much of the required adjustment in $G_t - T_t^T$ is achieved by adjusting taxes. Whenever there is a gap between D_t and $(g - \tau)A_t K_t$, a fraction $\lambda(0 \leq \lambda \leq 1)$ of this gap is closed by changing taxes, and a fraction $1 - \lambda$ is closed by changing government purchases. Specifically,

$$T_t^T = \tau A_t K_t - \lambda[D_t - (g - \tau)A_t K_t], \tag{21}$$

and

$$G_t = gA_t K_t + (1 - \lambda)[D_t - (g - \tau)A_t K_t]. \tag{22}$$

The amount of taxes collected by the Treasury can be rewritten by substituting equation (20) into equation (21) to obtain[9]

$$\begin{aligned} T_t^T = {} & [(1 - \lambda)\tau + \lambda g]A_t K_t + \lambda r_t b_t K_t \\ & - \lambda[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \bar{A})]K_{t+1}. \end{aligned} \tag{23}$$

The expression for Treasury tax revenue in equation (23) can be simplified in special cases. For instance, if the tax-responsive parameter λ equals zero, the Treasury's tax revenue is simply $T_t^T = \tau A_t K_t$.[10] In this case, any gap between the primary deficit and $(g - \tau)A_t K_t$ is closed completely by adjusting government purchases.

9. Government purchases can be rewritten as

$$\begin{aligned} G_t = {} & [\lambda g + (1 - \lambda)\tau]A_t K_t - (1 - \lambda)r_t b_t K_t \\ & + (1 - \lambda)[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \bar{A})]K_{t+1}. \end{aligned}$$

10. One might think of this case as being one of complete tax smoothing by the Treasury, although it must be noted that, since all taxes are lump sum, the usual argument for tax smoothing does not apply here.

5.4 The Social Security System

The social security system collects taxes from young people and pays benefits to old people. Any excess of taxes over benefits is added to the social security trust fund, and any excess of benefits over taxes is paid from the social security trust fund.

Let $S_t \geq 0$ be the value of the social security trust fund at the beginning of period t. Currently in the United States, the social security trust fund is invested entirely in Treasury bonds, which pay a rate of return r_t. However, there are proposals to invest part of the social security trust fund in equities, which are modeled as risky capital in this paper. To account for this possible change, let K_t^S be the amount of risky capital held by the social security trust fund at the beginning of period t, and define $\gamma_{S,t} \equiv K_t^S/S_t \geq 0$ as the fraction of the social security trust fund invested in risky capital with a rate of return R_t. The condition $\gamma_{S,t} \geq 0$ rules out the possibility that the social security trust fund takes a short position in risky physical capital.

Let B_t^S be the value of riskless bonds held by the social security trust fund at the beginning of period t, and note that $1 - \gamma_{S,t} = B_t^S/S_t$ is the fraction of the social security trust fund invested in riskless bonds. The rate of return on the social security trust fund, R_t^S, is

$$R_t^S \equiv (1 - \gamma_{S,t})r_t + \gamma_{S,t}R_t. \tag{24}$$

The budget constraint of the social security trust fund is

$$S_{t+1} = R_t^S S_t + T_t^S - Q_t. \tag{25}$$

I have described the behavior of social security benefits, Q_t, and the rate of return, R_t^S. To complete the description of the behavior of the social security system, I must specify either the behavior of social security taxes, T_t^S, or the evolution of the size of the trust fund, S_t. I will specify the evolution of S_t, and thus T_t^S will be determined as a residual from equation (25).[11]

11. I have specified the social security system as a pay-as-you-go defined-benefit system, but the framework is flexible enough to model a fully funded defined-contribution system in which the social security taxes collected from workers are placed in the social security trust fund. A fully funded defined-contribution system can be modeled by specifying the amount of taxes collected from workers, T_t^S, and the fraction of the social security trust fund invested in risky capital, $\gamma_{S,t+1}$. The size of the social security trust fund at the beginning of period $t + 1$ is $S_{t+1} = T_t^S$, and the social security benefits in period $t + 1$ are

$$Q_{t+1} = R_{t+1}^S S_{t+1} = (1 - \gamma_{S,t+1})r_{t+1}S_{t+1} + \gamma_{S,t+1}R_{t+1}S_{t+1}.$$

Comparing this expression for social security benefits to the expression in eq. (8) and recalling that $\underline{S}_{t+1} = s_{t+1}K_{t+1}$ imply the following values for θ_0 and θ_1: $\theta_0 = (1 - \gamma_{S,t+1})r_{t+1}s_{t+1}/((1 - \alpha)A)$, and $\theta_1 = \alpha\gamma_{S,t+1}s_{t+1}/(1 - \alpha)$. A fully funded defined-contribution social security

Define $s_t \equiv S_t/K_t$ as the ratio of the social security trust fund to the aggregate capital stock, and let σ be the "target" value of s_t. The social security system does not aim to set s_t equal to σ in every period, but it tries to prevent s_t from wandering too far from σ by adhering to the following policy function:

$$s_{t+1} - \sigma = \rho_S(s_t - \sigma) + \rho_R(R_t^S - \overline{R}_t^S)s_t, \tag{26}$$

where $\overline{R}_t^S \equiv (1 - \gamma_{S,t})r_t + \gamma_{S,t}\overline{R}$ is the expected value of R_t^S conditional on information available at the end of period $t - 1$, and $\rho_S \geq 0$ and $\rho_R \geq 0$ are constants that parametrize the evolution of the social security trust fund relative to the capital stock. If $\rho_R = 0$, the size of the social security trust fund is completely insulated from shocks to the rate of return, R_t^S. In this case, the ratio of the social security trust fund to the capital stock, s_{t+1}, is always equal to the target value, σ. If $\rho_R > 0$, then, when the return on the social security trust fund, R_t^S, is higher than expected, at least part of the unexpected return is used to increase the size of the social security trust fund. The parameter ρ_S measures the persistence of changes in the ratio s_t.

The amount of social security taxes, T_t^S, is determined as a residual from equation (25). Substituting equation (26) into equation (25), using equation (24), and solving for T_t^S yield

$$T_t^S = Q_t - R_t^S s_t K_t + [\sigma + \rho_S(s_t - \sigma) + \rho_R \gamma_{S,t}(R_t - \overline{R})s_t]K_{t+1}. \tag{27}$$

5.5 General Equilibrium

Now that I have specified the behavior of firms, individuals, the Treasury, and the social security system, I will analyze the general equilibrium that arises when these economic actors interact in capital markets. The dynamic general equilibrium describes the equilibrium evolution of four endogenous variables: the riskless interest rate r_t, the aggregate capital stock K_t, the debt-capital ratio b_t, and the social security trust fund-capital ratio s_t. The dynamic behavior of these four variables is governed by a nonlinear difference equation system that is recursive. Given the values of r_t, K_t, b_t, s_t, and the exogenous variable A_t (and the implied value of R_t), the value of b_{t+1} is determined by the Treasury policy function in equation (18), and the value of s_{t+1} is determined by the social security policy function in equation (26). As I will show below, the equilibrium value of r_{t+1} is determined by the optimal portfolio shares using the values of b_{t+1} and

system has no effect on the equilibrium riskless interest rate (see n. 12 below) or on the growth rate of capital (see n. 17 below).

s_{t+1}. Finally, the value of K_{t+1} is determined using optimal saving behavior and the values of r_{t+1}, b_{t+1}, and s_{t+1}.

I consider a closed economy so that all the bonds issued by the Treasury, B_{t+1}, are held either by the domestic private sector, which holds the amount B^P_{t+1}, or by the social security trust fund, which holds the amount $(1 - \gamma_{S,t+1})S_{t+1}$. Therefore,

$$B^P_{t+1} = B_{t+1} - (1 - \gamma_{S,t+1})S_{t+1}. \tag{28}$$

Similarly, all the capital in the economy, K_{t+1}, is held either by the domestic private sector, which holds the amount K^P_{t+1}, or by the social security trust fund, which holds $\gamma_{S,t+1}S_{t+1}$. Therefore,

$$K^P_{t+1} = K_{t+1} - \gamma_{S,t+1}S_{t+1}. \tag{29}$$

I will restrict attention to equilibria in which the amount of Treasury debt outstanding, B_t, is positive, the social security trust fund, S_t, is nonnegative, and young consumers hold positive amounts of both bonds and capital in their portfolios. Since $\gamma_{S,t+1} \geq 0$, equation (28) implies that a sufficient condition for young consumers to have positive holdings of bonds is

$$b_{t+1} > s_{t+1}. \tag{30}$$

Equation (29) implies that young consumers will have positive holdings of risky capital in equilibrium if

$$\gamma_{S,t+1}s_{t+1} < 1. \tag{31}$$

Henceforth, I will assume that the conditions in equations (30) and (31) hold for all t.

5.5.1 The Equilibrium Riskless Interest Rate

The equilibrium riskless interest rate is determined by the optimal portfolio shares. It follows from equations (15) and (16) that

$$\gamma(r_{t+1})\left[B^P_{t+1} + \frac{\theta_0(1-\alpha)\bar{A}K_{t+1}}{r_{t+1}}\right] = [1 - \gamma(r_{t+1})]\left(K^P_{t+1} + \theta_1\frac{1-\alpha}{\alpha}K_{t+1}\right). \tag{32}$$

Using equations (28) and (29) and the definitions $b_{t+1} \equiv B_{t+1}/K_{t+1}$ and $s_{t+1} \equiv S_{t+1}/K_{t+1}$, equation (32) can be rewritten as

$$(33)\qquad 0 = \gamma(r_{t+1})\left[1 + b_{t+1} - s_{t+1} + \frac{\theta_0(1-\alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}\right] - 1 + \gamma_{S,t+1}s_{t+1} - \theta_1\frac{1-\alpha}{\alpha}.$$

Equation (33) determines the equilibrium riskless interest rate r_{t+1}.[12] Alternatively, it can be solved to obtain the equilibrium value of $\gamma(r_{t+1})$, which is

$$(34)\qquad \gamma(r_{t+1}) = \frac{1 + \theta_1\frac{1-\alpha}{\alpha} - \gamma_{S,t+1}s_{t+1}}{1 + \theta_1\frac{1-\alpha}{\alpha} + b_{t+1} - s_{t+1} + \frac{\theta_0(1-\alpha)\overline{A}}{r_{t+1}}}.$$

Equation (34), along with the conditions in equations (30) and (31), implies the following:

PROPOSITION 1. *In equilibrium,* $0 < \gamma(r_{t+1}) < 1$.

Proposition 1 states that, in equilibrium, young consumers hold positive amounts of risky assets in their portfolios. It is well known that an optimal portfolio will include positive holdings of risky assets only if the expected rate of return on risky assets is greater than the riskless rate of return.[13] Thus, proposition 1 implies the following corollary:

COROLLARY 1. *In equilibrium,* $r_{t+1} < \overline{R}$.

The following proposition, which is proved in appendix C, describes the properties of the equilibrium riskless rate of return defined implicitly in equation (33):

12. In a fully funded defined-contribution system

$$\frac{\theta_0(1-\alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha} = s_{t+1},$$

and

$$\theta_1\frac{1-\alpha}{\alpha} = \gamma_{S,t+1}s_{t+1},$$

as may be verified using the expressions for θ_0 and θ_1 in n. 11 above. In this case, the equilibrium condition for the riskless interest rate in eq. (33) becomes $0 = \gamma(r_{t+1})(1 + b_{t+1}) - 1$, which implies that the equilibrium riskless interest rate is independent of changes in the size or portfolio allocation of a fully funded defined-contribution social security system.

13. For any strictly increasing, strictly concave function $u(\quad)$, $u'[r + \gamma(R - r)](R - r) < u'(r)(R - r)$ if $R - r \neq 0$ and $\gamma > 0$. Thus, for nondegenerate distributions of R, $E\{u'[r + \gamma(R - r)](R - r)\} < u'(r)E\{R - r\}$. Therefore, if $E\{R\} \leq r$, then $E\{u'[r + \gamma(R - r)](R - r)\} < 0$, and the condition for the optimal value of γ, $E\{u'[r + \gamma(R - r)](R - r)\} = 0$, cannot hold. Thus, in order for the optimal value of γ to be positive, $E\{R\}$ must exceed r.

Proposition 2. *Suppose that* $\gamma'(r_{t+1}) < 0$.[14] *Let* $r(b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)$ *be the unique value of* r_{t+1} *that satisfies equation* (33). *Then*

$$\frac{\partial r}{\partial b_{t+1}} > 0,$$

$$\operatorname{sign}\left(\frac{\partial r}{\partial s_{t+1}}\right) = \operatorname{sign}[\gamma_{S,t+1} - \gamma(r_{t+1})],$$

$$\operatorname{sign}\left(\frac{\partial r}{\partial \gamma_{S,t+1}}\right) = \operatorname{sign}(s_{t+1}),$$

$$\frac{\partial r}{\partial \theta_0} > 0,$$

$$\frac{\partial r}{\partial \theta_1} < 0.$$

Before interpreting the various effects in proposition 2, it is worth recalling that, since the risky rate of return R_{t+1} is invariant to policy in this model, any change in the riskless interest rate is equivalent to a change in the equity premium of the same magnitude but in the opposite direction.

An increase in b_{t+1}, the ratio of Treasury bonds to the capital stock, increases the equilibrium interest rate in order to induce private investors to devote a larger share of their portfolios to riskless bonds. Similarly, if the social security trust fund has a positive balance ($s_{t+1} > 0$) and sells some riskless bonds in exchange for stock, thereby increasing $\gamma_{S,t+1}$, the equilibrium interest rate on bonds must increase in order for private investors to be willing to hold a higher ratio of bonds to stocks directly in their own portfolios.[15]

The effect of an increase in the size of the social security trust fund, represented as an increase in s_{t+1}, depends on the sign of $\gamma_{S,t+1} - \gamma(r_{t+1})$. If the share of the social security trust fund held in risky capital ($\gamma_{S,t+1}$) is smaller than the share of private portfolios held in risky capital, as is the case in the United States, where $\gamma_{S,t+1} = 0$, then an increase in the size of the social security trust fund, s_{t+1}, effectively reduces the ratio of riskless bonds to risky capital available to private investors. In order for these

14. As shown in app. B, the condition $\gamma'(r_{t+1}) < 0$ holds if $\phi \leq 1$. If $\phi > 1$, then $\gamma'(r_{t+1})$ may not be negative for some values of r_{t+1}, so the proof of uniqueness does not hold. Nevertheless, the effects on r_{t+1} of changes in b_{t+1}, s_{t+1}, $\gamma_{S,t+1}$, θ_0, and θ_1 in this proposition hold for any solution to eq. (33) for which $\gamma'(r_{t+1}) < 0$. As mentioned in n. 6 above, this condition holds for all cases in tables 5.1–5.3 below.

15. Recall that the social security system analyzed here is a defined-benefit system, so the change in the portfolio allocation of the social security trust fund does not affect the claims on future social security benefits held by workers.

investors to willingly reduce the ratio of riskless bonds to risky capital in their portfolios, the riskless interest rate must fall.

The "replacement rates" θ_0 and θ_1 have opposite effects on the equilibrium riskless interest rate. An increase in θ_0 increases the riskless component of the social security benefit that young people anticipate and effectively increases the holding of riskless assets by young people. In order for these people to be willing to increase their riskless holdings, the riskless interest rate must increase. However, an increase in θ_1 increases the risky component of the social security benefit that young consumers anticipate and effectively increases the holding of risky assets in private portfolios. The riskless interest rate must fall in order to induce consumers to be willing to hold an increased share of risky assets in their portfolios.

The equilibrium condition in equation (33) illustrates the extent to which the separate balance sheets of the Treasury and the social security trust fund can be consolidated for the purpose of determining the equilibrium riskless interest rate. Inspection of equation (33) implies the following proposition:

PROPOSITION 3. *For the purpose of determining the equilibrium riskless interest rate, all the information on the balance sheets of the Treasury and the social security trust fund is captured by*

$$v_{t+1} \equiv b_{t+1} - s_{t+1} + \frac{\gamma_{S,t+1}}{\gamma(r_{t+1})} s_{t+1}.$$

COROLLARY 2. *If* $\gamma_{S,t+1} = 0$, *then the effects of* b_{t+1} *and* s_{t+1} *on the riskless interest rate are captured entirely by* $b_{t+1} - s_{t+1}$.

If $\gamma_{S,t+1} = 0$, as is currently the case in the United States, the social security trust fund is held entirely in riskless bonds. In this case, for the purpose of determining the equilibrium riskless interest rate, the balance sheets of the Treasury and the social security trust fund can be consolidated. The only information needed from the separate balance sheets of these entities is the net amount of bonds, normalized by the aggregate capital stock, $b_{t+1} - s_{t+1}$, issued by the consolidated entity. However, even in this case, the balance sheets of the Treasury and the social security trust fund cannot, in general, be consolidated for the purpose of determining the growth rate of the capital stock. (For the more stringent conditions under which these balance sheets can be consolidated for the purpose of determining the growth rate of the capital stock, see proposition 4 below.)

COROLLARY 3. *If* $\gamma_{S,t+1} = \gamma(r_{t+1})$, *then the effects of* b_{t+1} *and* s_{t+1} *on the riskless interest rate are captured entirely by* b_{t+1}.

According to this corollary, if the social security trust fund maintains a risky portfolio share $\gamma_{S,t+1}$ equal to the share of risky assets in private portfolios,[16] $\gamma(r_{t+1})$, the equilibrium interest rate is independent of the size of the social security trust fund. In this case, changes in the size of the social security trust fund have no effect on the ratio of riskless assets to risky assets available to the private sector, and hence the equilibrium riskless interest rate is unaffected by such changes.

5.5.2 The Growth Rate of the Capital Stock

In this subsection, I use the optimal saving behavior of individuals, along with the saving behavior of the Treasury and the social security system, to determine how much capital is accumulated in the economy. Then I will analyze how the growth rate of the aggregate capital stock is affected by a change in the portfolio of the social security trust fund.

In a closed economy, the bonds and capital held by the private sector, $B^P_{t+1} + K^P_{t+1}$, plus the value of the social security trust fund, S_{t+1}, equal the aggregate capital stock, K_{t+1}, plus the value of bonds issued by the Treasury, B_{t+1}. This relation can be derived by adding equations (28) and (29) to obtain

$$B_{t+1} + K_{t+1} = B^P_{t+1} + K^P_{t+1} + S_{t+1}. \tag{35}$$

The size of the portfolio of the private sector, $B^P_{t+1} + K^P_{t+1}$, can be calculated from equations (11), (15), and (16) to obtain

$$B^P_{t+1} + K^P_{t+1} = \frac{\delta}{1+\delta}(w_t - T^T_t - T^S_t) - \frac{1}{1+\delta}\left[\frac{\theta_0(1-\alpha)\bar{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}\right]K_{t+1}. \tag{36}$$

The growth rate of the capital stock can be determined by substituting equation (36) into (35) and performing a tedious set of substitutions. To streamline the notation, define

$$\omega_t \equiv (s_t, b_t, A_t, \theta_0, \theta_1). \tag{37}$$

Appendix D shows that[17]

16. Recall that $\gamma(r_{t+1})$ is the share of risky assets—which include risky capital and the claim on risky future social security benefits—in the portfolios of young consumers, which consist of bonds, risky capital, and the riskless and risky components of future social security benefits.

17. In a fully funded defined-contribution social security system, $-(1-\alpha)(\theta_0\bar{A} + \theta_1 A_t) + [(1-\gamma_{S,t})r_t + \gamma_{S,t}R_t]s_t = (-\alpha A_t + R_t)\gamma_{S,t}s_t = 0$, where the first equality follows from the expressions for θ_0 and θ_1 in n. 11 above and the second equality follows from eq. (5). Therefore, $H_0(r_t, \gamma_{S,t}, \omega_t) = \delta\{[1-\alpha-(1-\lambda)\tau - \lambda g]A_t - \lambda r_t b_t\}$. Using the fact that

$$\frac{\theta_0(1-\alpha)\bar{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha} = s_{t+1}$$

(38) $$\frac{K_{t+1}}{K_t} = \frac{H_0(r_t, \gamma_{S,t}, \omega_t)}{H_1(r_{t+1}, \gamma_{S,t}, \omega_t)},$$

where

(39) $$H_0(r_t, \gamma_{S,t}, \omega_t) \equiv \delta\{[1 - \alpha - (1 - \lambda)\tau - \lambda g]A_t - (1 - \alpha)(\theta_0 \overline{A} + \theta_1 A_t) + [(1 - \gamma_{S,t})r_t + \gamma_{S,t}R_t]s_t - \lambda r_t b_t\}$$

and

(40) $$H_1(r_{t+1}, \gamma_{S,t}, \omega_t) \equiv 1 + \delta + \frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1 - \alpha}{\alpha} + (1 + \delta - \delta\lambda)[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \overline{A})] - [\sigma + \rho_S(s_t - \sigma) + \rho_R(R_t - \overline{R})\gamma_{S,t}s_t].$$

I will restrict attention to cases in which $H_0 > 0$ and $H_1 > 0$. Although H_0 and H_1 defy simple interpretations that are literally correct, for the sake of exposition I will offer loose interpretations. The term H_0 can be loosely interpreted as the disposable income of the young consumers,[18] and factors that increase the disposable income of the young consumers tend to increase capital accumulation. The term H_1 can be loosely interpreted as the ratio of noncapital wealth held by young consumers to the aggregate capital stock, where the noncapital wealth consists of Treasury bonds and claims on social security benefits.[19] Factors that increase this ratio tend to reduce capital accumulation.

(see n. 12 above) along with the social security policy function in eq. (26), $H_1(r_{t+1}, \gamma_{S,t}, \omega_t) = 1 + \delta + (1 + \delta - \delta\lambda)[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \overline{A})]$. Since both H_0 and H_1 are independent of the parameters of a fully funded defined-contribution social security system, the growth rate of the capital stock, $K_{t+1}/K_t = H_0/H_1$, is independent of the parameters of such a system.

18. More precisely, if $s_t = \sigma$, $A_t = \overline{A}$, and $b_t = \beta$, then $H_0K_t = \delta(w_t - T_t^T - T_t^S - \lambda\beta K_{t+1} + \sigma K_{t+1})$. If $\beta = \sigma = 0$, then H_0K_t is strictly proportional to the disposable income of young consumers, $w_t - T_t^T - T_t^S$.

19. More precisely, if $\lambda = 1$,

$$H_1 = 1 + \delta + \frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1 - \alpha}{\alpha} + b_{t+1} - s_{t+1},$$

where

$$\frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1 - \alpha}{\alpha}$$

is the present value of the claim on future social security benefits relative to the aggregate capital stock K_{t+1}, and if $\gamma_{S,t+1} = 0$, $b_{t+1} - s_{t+1}$ is the amount of Treasury bonds held by young consumers relative to the aggregate capital stock.

Inspection of equations (39) and (40) implies the following proposition about the consolidation of the balance sheets of the Treasury and the social security trust fund:

PROPOSITION 4. *If* $\gamma_{S,t} = 0$, $\lambda = 1$, *and* $\rho_S = \rho_B$, *then, for the purpose of determining the growth rate of the capital stock, all the information on the balance sheets of the Treasury and the social security trust fund is captured by* $b_t - s_t$, *and the information contained in the targets* β *and* σ *is captured by* $\beta - \sigma$.

Recall from corollary 2 that, if the social security trust fund holds only riskless bonds, then, for the purpose of determining the riskless interest rate, the balance sheets of the Treasury and the social security trust fund can be consolidated. For the purpose of determining r_t, the net indebtedness of the consolidated entity, $b_t - s_t$, is a sufficient statistic for b_t and s_t. However, for the purpose of determining the growth rate of the capital stock, a more stringent set of conditions is required to be able to consolidate the balance sheets of the Treasury and the social security trust fund. In addition to $\gamma_{S,t} = 0$, the parameter λ must equal one, and the persistence parameters ρ_B and ρ_S must be equal. The parameter λ must equal one because all adjustment in the net income of the social security system takes place through adjusting the taxes on the young. With $\lambda = 1$, all adjustment in the net income of the Treasury will also take place through adjusting taxes on the young.

The following two lemmas help prove and interpret the effects of changes in the portfolio of the social security system on the growth rate of the capital stock:

LEMMA 1. $$\frac{\partial H_0(r_t, \gamma_{S,t}, \omega_t)}{\partial r_t} = \delta[(1 - \gamma_{S,t})s_t - \lambda b_t].$$

Consider an increase in r_t that increases the amount of interest paid by the Treasury during period t by b_t dollars. This increase in r_t will increase the amount of interest received by the social security trust fund by $(1 - \gamma_{S,t})s_t$ dollars.[20] According to the social security policy function in equation (26), the social security system will not change the size of the trust fund in period $t + 1$ and thus will use the additional interest earnings to reduce social security taxes in period t by $(1 - \gamma_{S,t})s_t$ dollars. According to the Treasury policy function in equation (18), the Treasury will not change the size of its debt and thus will respond to the increased cost of debt service by increasing taxes by λb_t dollars. Taking account of both

20. The increase in interest payments by the Treasury is $B_t \Delta r_t = b_t$, which implies that $\Delta r_t = b_t / B_t = 1/K_t$. The increase in the interest earned by the social security trust fund is

$$(1 - \gamma_{S,t})S_t \Delta r_t = (1 - \gamma_{S,t})S_t \frac{1}{K_t} = (1 - \gamma_{S,t})s_t.$$

social security taxes and Treasury taxes, the total taxes paid by young consumers in period t fall by $(1 - \gamma_{S,t})s_t - \lambda b_t$ dollars, and the disposable income of young consumers increases by this amount. If $(1 - \gamma_{S,t})s_t - \lambda b_t > 0$, the increase in disposable income of young consumers increases H_0 and increases the size of portfolios held by young consumers.

A change in the riskless interest rate r_{t+1} affects H_1 as described in the following lemma:

LEMMA 2. $$\frac{\partial H_1(r_{t+1}, \gamma_{S,t}, \omega_t)}{\partial r_{t+1}} = -\frac{\theta_0(1-\alpha)\bar{A}}{r_{t+1}^2} \leq 0.$$

If $\theta_0 > 0$, an increase in r_{t+1} reduces the present value of the claim on future riskless social security benefits held by young consumers, thereby reducing H_1, the ratio of noncapital wealth to capital in the portfolios of young consumers.

Now consider a change in $\gamma_{S,t+1}$, the share of the social security trust fund that is held in risky capital. The following proposition applies to a change in the portfolio of the social security trust fund at the end of period t after b_t, s_t, r_t, and K_t have been determined:

PROPOSITION 5. *If* $\theta_0 > 0$ *and* $s_{t+1} > 0$, *then*

$$\left.\frac{dK_{t+1}}{d\gamma_{S,t+1}}\right|_{K_t, r_t, \omega_t} > 0.$$

Proof.

$$\left.\frac{dK_{t+1}}{d\gamma_{S,t+1}}\right|_{K_t, r_t, \omega_t} = -\frac{H_0}{H_1^2}\frac{\partial H_1}{\partial r_{t+1}}\frac{\partial r_{t+1}}{\partial \gamma_{S,t+1}} \cdot K_t > 0$$

because

$$\frac{\partial H_1}{\partial r_{t+1}} = -\frac{\theta_0(1-\alpha)\bar{A}}{r_{t+1}^2} < 0$$

(from lemma 2) and

$$\text{sign}\left(\frac{\partial r_{t+1}}{\partial \gamma_{S,t+1}}\right) = \text{sign}(s_{t+1})$$

(from proposition 2).

An increase in $\gamma_{S,t+1}$ has no direct effect on either $H_0(r_t, \gamma_{S,t}, \omega_t)$ or $H_1(r_{t+1}, \gamma_{S,t}, \omega_t)$. However, an increase in $\gamma_{S,t+1}$ increases the riskless interest rate r_{t+1}, provided that $s_{t+1} > 0$. The resulting increase in r_{t+1} reduces the present value of riskless social security benefits, which implies that the present value of lifetime resources, Ω_t, falls. In response to the fall in Ω_t, consumers reduce their consumption and increase their saving so that national capital accumulation increases.

COROLLARY 4. *If* $\theta_0 = 0$ *or* $s_{t+1} = 0$, *then*

$$\left.\frac{dK_{t+1}}{d\gamma_{S,t+1}}\right|_{K_t, r_t, \omega_t} = 0.$$

This corollary implies, for instance, that, if all social security benefits are risky so that $\theta_0 = 0$, then a change in the portfolio of the social security trust fund at the end of period t will have no effect on K_{t+1}.

Constant Growth Paths

I have shown that, given the capital stock K_t and the riskless interest rate r_t, an increase in the risky share of the social security trust fund, $\gamma_{S,t+1}$, at the end of period t increases the riskless interest rate r_{t+1} and the aggregate capital stock K_{t+1} in the following period. In this section, I focus on the long-run effects of a change in the portfolio allocation of the social security trust fund. I will focus on *constant growth paths,* which I define to be paths along which $\gamma_{S,t} = \gamma_S$ and $A_t = \overline{A}$ for all t so that $R_t = \overline{R}$, $b_t = \beta$, and $s_t = \sigma$ for all t.[21] Along such paths, the riskless interest rate and the growth rate of capital will be constant. Let r denote the constant value of the riskless interest rate along a constant growth path, and let η be the constant value of K_{t+1}/K_t along a constant growth path.

The equilibrium condition for the interest rate along a constant growth path is derived by substituting $b_{t+1} = \beta$ and $s_{t+1} = \sigma$ into equation (33) to obtain

$$\text{(41)} \quad \gamma(r)\left[1 + \beta - \sigma + \frac{\theta_0(1 - \alpha)\overline{A}}{r} + \theta_1\frac{1 - \alpha}{\alpha}\right] - 1 + \gamma_S\sigma - \theta_1\frac{1 - \alpha}{\alpha} = 0.$$

Similarly, the values of H_0 and H_1 along a constant growth path are derived by setting $\gamma_{S,t} = \gamma_S$, $A_t = \overline{A}$, $R_t = \overline{R}$, $b_t = \beta$, and $s_t = \sigma$ in equations (39) and (40), to obtain

$$\text{(42)} \quad H_0^*(r, \gamma_S, \sigma, \beta, \overline{A}, \theta_0, \theta_1) \equiv \delta\{[(1 - \alpha)(1 - \theta_0 - \theta_1) - (1 - \lambda)\tau - \lambda g]\overline{A} + [(1 - \gamma_S)r + \gamma_S\overline{R}]\sigma - \lambda r\beta\}$$

21. Of course, consumers do not know in advance that the realizations of A_t and R_t will always be equal to their respective expectations, so they take account of risk in making portfolio-allocation decisions.

and

$$(43)\quad H_1^*(r, \gamma_S, \sigma, \beta, \bar{A}, \theta_0, \theta_1) \equiv 1 + \delta + \frac{\theta_0(1-\alpha)\bar{A}}{r} + \theta_1 \frac{1-\alpha}{\alpha} + (1 + \delta - \delta\lambda)\beta - \sigma,$$

where the asterisks on H_0^* and H_1^* indicate that these terms are evaluated along a constant growth path.

The equilibrium condition in equation (41) for the riskless interest rate r along a constant growth path is identical (with appropriate relabeling of variables) to the equilibrium condition for the riskless interest rate in equation (33). Thus, the following corollary to proposition 2 describes the response of the riskless interest rate to various changes along a constant growth path:

COROLLARY 5. *Suppose that* $\gamma'(r) < 0$. *Let* $\tilde{r}(\beta, \sigma, \gamma_S, \theta_0, \theta_1)$ *be the unique value of r that satisfies equation (41). Then*

$$\frac{\partial \tilde{r}}{\partial \beta} > 0,$$

$$\text{sign}\left(\frac{\partial \tilde{r}}{\partial \sigma}\right) = \text{sign}[\gamma_S - \gamma(\tilde{r})],$$

$$\text{sign}\left(\frac{\partial \tilde{r}}{\partial \gamma_S}\right) = \text{sign}(\sigma),$$

$$\frac{\partial \tilde{r}}{\partial \theta_0} > 0,$$

$$\frac{\partial \tilde{r}}{\partial \theta_1} < 0.$$

Again, recall that any change in the riskless interest rate is matched by a change in the equity premium of equal size but in the opposite direction.

Now consider the effect on η, the growth rate of the capital stock along a constant growth path, of a permanent change in γ_S:

PROPOSITION 6. *If*

$$\delta[(1 - \gamma_S)\sigma - \lambda\beta] + \eta \frac{\theta_0(1-\alpha)\bar{A}}{r^2} \geq 0$$

along a constant growth path with $\sigma > 0$, *then* $d\eta/d\gamma_S > 0$.

Proof.

$$\frac{d\eta}{d\gamma_S} = \frac{1}{H_1^*}\left(\frac{\partial H_0^*}{\partial \gamma_S} + \frac{\partial H_0^*}{\partial r}\frac{\partial r}{\partial \gamma_S}\right) - \frac{\eta}{H_1^*}\left(\frac{\partial H_1^*}{\partial \gamma_S} + \frac{\partial H_1^*}{\partial r}\frac{\partial r}{\partial \gamma_S}\right)$$

$$= \frac{1}{H_1^*}\delta\left\{(\overline{R} - r)\sigma + [(1 - \gamma_S)\sigma - \lambda\beta]\frac{\partial r}{\partial \gamma_S}\right\}$$

$$+ \frac{\eta}{H_1^*}\left(\frac{\theta_0(1 - \alpha)\overline{A}}{r^2}\frac{\partial r}{\partial \gamma_S}\right)$$

$$= \frac{\delta}{H_1^*}(\overline{R} - r)\sigma$$

$$+ \frac{1}{H_1^*}\left\{\delta[(1 - \gamma_S)\sigma - \lambda\beta] + \eta\frac{\theta_0(1 - \alpha)\overline{A}}{r^2}\right\}$$

$$\times \frac{\partial r}{\partial \gamma_S} > 0.$$

since $\overline{R} > r$ (corollary 1) and $\partial r/\partial \gamma_S > 0$ (corollary 5).

A permanent increase in γ_S has a direct effect on H_0^* and indirect effects on both H_0^* and H_1^* operating through the riskless interest rate. The direct effect on H_0^* arises because an increase in γ_S increases the average earnings of the social security trust fund as it shifts its portfolio toward assets with a higher expected rate of return. The increase in the average portfolio earnings of the social security trust fund allows the social security tax on young consumers to be reduced, thereby increasing their disposable income. The increase in the disposable income of young consumers increases the amount of capital that they hold in their portfolios.

The indirect effects on H_0^* and H_1^* arise because an increase in γ_S increases the riskless interest rate (corollary 5). An increase in the riskless interest rate increases the interest earnings of the social security trust fund and increases the interest payments made by the Treasury. These changes in interest flows induce the social security system and the Treasury to change the amount of taxes collected from young consumers. As explained in the interpretation of lemma 1, these changes in taxes increase the disposable income of young consumers by an amount proportional to $(1 - \gamma_S)\sigma - \lambda\beta$, which is captured by the change in H_0^*. In addition, the increase in the riskless interest rate reduces H_1^* by reducing the present value of the future riskless social security benefits to be received by young consumers. This reduction in H_1^* increases the growth rate of the capital stock.

The direct effect on H_0^* and the indirect effect on H_1^* both increase the growth rate of the capital stock. However, the indirect effect on H_0^* can increase or decrease the growth rate of the capital stock depending on whether $(1 - \gamma_S)\sigma - \lambda\beta$ is positive or negative. Proposition 6 states a sufficient condition for the indirect effect on H_1^* to dominate the indirect effect on H_0^* so that an increase in γ_S unambiguously increases η. Corollary 6 below presents a condition for the indirect effect on H_0^* to increase η so that there is no conflict between the indirect effect on H_0^* and the indirect effect on H_1^*. In this case, of course, an increase in γ_S increases η. Corollary 7 presents an even stronger condition that guarantees that an increase in γ_S increases η. This condition, $\lambda = 0$, can be interpreted as complete tax smoothing. Thus, in the presence of complete tax smoothing, an increase in γ_S unambiguously increases the growth rate of the capital stock η.

COROLLARY 6. *If* $(1 - \gamma_S)\sigma - \lambda\beta \geq 0$ *along a constant growth path with* $\sigma > 0$, *then* $d\eta/d\gamma_S > 0$.

COROLLARY 7. *If* $\lambda = 0$ *along a constant growth path with* $\sigma > 0$, *then* $d\eta/d\gamma_S > 0$.

5.6 Calibration of the Model

In this section, I calibrate the model for a constant growth path. This calibration will serve two purposes. First, the calibration will shed light on the quantitative plausibility of the model. Second, the calibration will provide a quantitative measure of the effect of a change in γ_S on the growth rate of the capital stock. In particular, the calibration can be used to determine whether the condition in proposition 6 is satisfied so that an increase in γ_S increases the capital stock growth rate η along a constant growth path.

One approach to calibrating the model would be to specify values for the parameters of preferences and technology, the parameters of the social security policy function, the parameters of the Treasury policy function, and the distribution of the stochastic productivity variable A_t and then to compute the implied values of the riskless interest rate r and the growth rate of the capital stock η. I will make two modifications to this approach. The first modification, which is a trivial change, is to specify the distribution of the risky rate of return $R_t = \alpha A_t$ instead of specifying the distribution of A_t. The second modification is more fundamental. Instead of specifying the values of the preference parameters ϕ and δ and then computing the implied values of r and η, I will find the values of the preference parameters ϕ and δ for which the values of r and η implied by the model match the corresponding empirical values, which I denote as $\hat{r}$ and $\hat{\eta}$, respectively.

Because the coefficient of relative risk aversion ϕ affects only the portfolio-allocation decision while the time-preference discount factor δ affects only the saving/consumption decision, the values of these parameters that match $\hat{r}$ and $\hat{\eta}$ can be determined separately. Specifically, the coefficient of relative risk aversion ϕ is determined by the riskless-interest-rate equilibrium condition in equation (41) with $\hat{r}$ substituted for r. The empirical value $\hat{r}$, along with the values of the other parameters in this equation, implies a value for $\gamma(\hat{r})$. There is a unique value of the coefficient of relative risk aversion ϕ for which the optimal value of the portfolio share γ equals the value of $\gamma(\hat{r})$ implied by the equilibrium condition in equation (41).[22]

The value of the time-preference discount factor δ is chosen so that the growth rate of capital implied by the model, H_0^*/H_1^*, equals the empirical value of the growth rate, $\hat{\eta}$. Matching the implied and the empirical values of the growth rate of capital yields

$$H_0^* = \hat{\eta} H_1^*. \tag{44}$$

Since H_0^* and H_1^* are both linear functions of the time-preference discount factor δ, equation (44) is a linear function of δ that can be easily solved for the value of δ that allows the model to match the empirical growth rate of capital.

Because individuals are assumed to live for only two periods, each period in the model is half an adult lifetime. The length of a period is important for variables, such as rates of return, and parameters, such as the rate of time preference, that are expressed per unit of time. I will report annual values of these variables and parameters, and I will make adjustments to take account of the fact that a period is many years. Specifically, I will assume that a period in the model lasts N years. I will calculate the time-preference discount factor δ as $\delta = (1 + \nu)^{-N}$, where ν is the annual rate of time preference. Similarly, I will calculate the (gross) riskless interest rate per period r as $r = (1 + r_{\text{ann}})^N$, where r_{ann} is the (net) riskless interest rate per year. It will be convenient to define the following empirical values on a (net) annual basis: $\hat{r}_{\text{ann}} \equiv \hat{r}^{-N} - 1$ and $\hat{\eta}_{\text{ann}} \equiv \hat{\eta}^{-N} - 1$.

Converting the distribution of the (net) annual risky rate $R_{\text{ann},t}$ to a distribution of the risky rate per period R_t involves an additional consideration. Suppose that the distribution of the annual risky rate is a two-point distribution with $1 + R_{\text{ann},t} \in \{\mu + \chi, \mu - \chi\}$. If the annual risky rate were perfectly serially correlated over the N years of the period, then the (gross) risky rate per period would be a two-point distribution with $R_t \in \{(\mu + \chi)^N, (\mu - \chi)^N\}$. However, if the annual risky rate is not perfectly serially correlated, this two-point distribution would overstate the variance per N-year period. In fact, the presence of mean reversion in stock

22. $\lim_{\phi \to \infty} \gamma^* = 0$ and $\lim_{\phi \to 0} \gamma^* = \infty$, so there is at least one value of ϕ for which the optimal value γ^* equals the value implied by the equilibrium condition for the riskless interest rate. As shown in eq. (B11) in app. B, $d\gamma^*/d\phi < 0$ so that such a value of ϕ is unique.

prices—equivalently, negative serial correlation in stock returns—suggests that even assuming that the annual risky rate is i.i.d. over time would overstate the variance per N-year period. To allow for negative serial correlation in stock returns, I assume that (gross) annual risky returns follow a first-order two-point Markov process with

$$(45)\quad 1 + R_{\text{ann},t} \in \{\mu + \chi, \mu - \chi\} \quad \text{and} \quad \text{pr}\{R_{\text{ann},t+1} = R_{\text{ann},t}\} = 1 - \varphi.$$

Under the Markov process in equation (45), the (gross) annual risky return $1 + R_{\text{ann},t}$ has a mean equal to μ, a standard deviation equal to χ, and a first-order serial correlation equal to $1 - 2\varphi$.

The accumulation of annual returns over an N-year period is used in the portfolio-allocation decision of young consumers. As shown in equation (A4) in appendix A, the portfolio-allocation decision involves the choice of γ_{t+1} to maximize an expression containing

$$(46)\qquad E_t\left\{[(1 - \gamma_{t+1})r_{t+1} + \gamma_{t+1}R_{t+1}]^{1-\phi}\right\},$$

where the (gross) returns, r_{t+1} and R_{t+1}, are measured over an N-year period. Defining

$$(47)\qquad z_{\text{ann},t+j} \equiv [(1 - \gamma)(1 + r_{\text{ann}}) + \gamma(1 + R_{\text{ann},t+j})]^{1-\phi},$$

the expression in equation (46) can be written as $E_t\{\Pi_{j=1}^{N} z_{\text{ann},t+j}\}$. If the annual risky return $R_{\text{ann},t}$ follows a two-point Markov process, then $z_{\text{ann},t}$ will also follow a two-point Markov process. Lemma 3 in appendix E presents a simple method for computing $E_t\{\Pi_{j=1}^{N} z_{\text{ann},t+j}\}$.[23]

Table 5.1 contains the baseline values of the parameters used to calibrate the constant growth path. I use the moments of the annual risky rate of return reported by Mehra and Prescott (1985) for the period 1889–1978. Specifically, I set μ equal to 1.0698, which implies a 6.98 percent annual mean (net) risky return, and I set the standard deviation, χ, equal to 0.1654 per year. Fama and French (1988) report that, for return horizons of one year, the serial correlation of stock returns is negative but not significantly different from zero. The serial correlation of stock returns becomes significantly negative as the horizon is lengthened to two years and declines until the horizon is about three to five years. In order to capture this mean reversion in stock prices over longer periods, I specify an annual serial correlation of -0.1, which implies $\varphi = 0.55$.

Each period in the model represents half an adult lifetime. More specifically, the first period of a person's life corresponds to time in the labor force and the second period to retirement. I have chosen to set N, the

23. An alternative approach to calibrating N-period returns would be to calculate the empirical moments of N-period returns. The approach that I use in this paper is more flexible in that it allows N to be changed easily. In the baseline calibration, $N = 30$, and the sensitivity analysis reports results for $N = 25$ and $N = 35$.

Table 5.1 Baseline Calibration

Risky Rate (annual)			
Mean, μ (gross return per year)	1.0698	φ (serial correlation = $1 - 2\varphi$)	0.55
Std. dev., χ (per year)	0.1654	Number of years per period, N	30
Treasury Policy Function			
Government purchases parameter, g	0.2	Tax responsiveness, λ	0.1
Tax parameter, τ	0.2	Target bond ratio, β	0.25
Social Security Policy Function			
Riskless replacement rate, θ_0	0.15	Target trust fund ratio, σ	0.035
Risky replacement rate, θ_1	0.0	Risky trust fund share, γ_S	0.0
Share of capital in production function, α			0.375
Empirical Moments to Fit			
Riskless rate, $\hat{r}_{\text{ann}}$ (% per year)	0.80	Growth rate, $\hat{\eta}_{\text{ann}}$ (% per year)	1.3
Preference Parameter Values that Fit Empirical Moments			
Coefficient of relative risk aversion, ϕ	7.7577	Time preference, ν (% per year)	0.6874

number of years per period, equal to thirty, which is a compromise between the larger numbers of years in the workforce and the smaller number of years in retirement.

For the tax-policy function, I have set the government-purchases parameter g and the tax parameter τ both equal to 0.2, which is the approximate share of government purchases in GDP in the United States. It is difficult to pin down the value of the tax-responsiveness parameter λ, which is the fraction of the adjustment in the primary deficit that is achieved by changing taxes. Complete tax smoothing is represented by $\lambda = 0$. I set $\lambda = 0.1$ in the baseline simulation, and I explore the quantitative effect of λ in tables 5.2 and 5.3 below. I set the baseline value of the target bond-capital ratio, β, equal to 0.25. In 1997, the ratio of Treasury debt held by the public and by the social security trust fund to the stock of fixed private capital was 0.254.

In the baseline calibration, I treat the benefits in the pay-as-you-go social security system in the United States as riskless. Thus, I set the risky replacement rate θ_1 equal to zero. I set the riskless replacement rate θ_0 equal to 0.15. This value may seem low, but it is higher than the 12.4 percent social security tax rate, and it is almost twice as high as the ratio of social security benefits to compensation of employees in 1996 (which was 0.0787).[24] In 1997, the social security trust fund was 3.6 percent as large as the fixed private capital stock in the United States. I set the target value of the trust fund-capital ratio, σ, equal to 0.035 in the baseline. Since the social security trust fund is currently invested entirely in bonds, I set $\gamma_S = 0$ in the baseline calibration.

Over the past half century in the United States, the share of labor income in GDP has averaged 0.625, with a standard deviation of only 0.009.[25] Since the labor share is $1 - \alpha$, I set $\alpha = 0.375$ in the baseline calibration.

I calibrate the model to match two empirical moments: the average riskless interest rate, $\hat{r}_{\text{ann}}$, and the average growth rate of the capital stock, $\hat{\eta}_{\text{ann}}$. For $\hat{r}_{\text{ann}}$, I use 0.8 percent per year, which is the average value of the riskless interest rate reported by Mehra and Prescott (1985) for the period 1889–1978. As for the growth rate of the capital, it is important to note that the population and the labor force are constant across generations in the model. Thus, the appropriate empirical counterpart of the growth rate of capital in the model is the empirical growth rate of the capital-labor ratio. Over the period 1947–97, the fixed private capital stock in the United States grew by 3.17 percent per year, and employment grew by 1.65 percent per year, implying that the capital-labor ratio grew by approximately 1.52 percent per year. However, over the shorter period 1967–97, the annual growth of capital slowed to 2.90 percent per year, and the growth rate of employment increased to 1.87 percent per year, so the growth rate of the capital-labor ratio declined (relative to the longer time period) to 1.03 percent per year. I will use an intermediate value of 1.3 percent per year for $\hat{\eta}_{\text{ann}}$.

The last row of table 5.1 reports the values of the preference parameters for which the riskless interest rate r and the growth rate of the capital stock η calculated by the model match their empirical counterparts. Specifically, with a coefficient of relative risk aversion, ϕ, of 7.7577 and a rate of time preference, ν, of 0.6874 percent per year, the model matches the riskless interest rate and the growth rate of the capital stock. The values of these preference parameters are quite plausible.[26]

Table 5.2 reports the results of a sensitivity analysis that varies one pa-

24. Of course, this ratio in 1996 significantly understates the replacement ratio because the large population of baby-boom workers means that the ratio of workers to retirees is temporarily (for a few decades) high.

25. The labor share Λ is computed as the solution of the following equation:

$$\frac{\text{compensation of employees} + \Lambda(\text{proprietors' income})}{\text{GDP}} = \Lambda.$$

26. The literature on the equity-premium puzzle typically requires a coefficient of relative risk aversion ϕ well above ten to match moments of asset returns. However, one should not regard the relatively low and reasonable value of 7.7577 for ϕ as a resolution of the equity-premium puzzle because the model in the paper has not been calibrated to the variability of consumption.

Table 5.2 **Sensitivity Analysis: Changing One Parameter at a Time**

Parameter	Parameter Value	r_{ann}[a]	η_{ann}[a]	Parameter	Parameter Value	r_{ann}[a]	η_{ann}[a]
μ	1.05	−1.18	−0.62	μ	1.09	2.82	3.26
χ	0.12	2.75	1.96	χ	0.22	−1.01	0.44
φ	0.5	0.32	1.10	φ	0.6	1.28	1.48
N	25	0.27	0.55	N	35	1.25	1.84
θ_0	0.10	0.14	1.87	θ_0	0.20	1.27	0.76
θ_1	0.05	0.65	0.83	θ_1	0.10	0.51	0.34
σ	0.01	0.83	1.27	σ	0.06	0.77	1.33
γ_S	0.15	0.82	1.32	γ_S	0.30	0.83	1.35
g	0.15	0.80	1.35	g	0.25	0.80	1.25
τ	0.15	0.80	1.73	τ	0.25	0.80	0.81
λ	0.0	0.80	1.30	λ	0.2	0.80	1.30
β	0.20	0.75	1.36	β	0.30	0.85	1.24
α	0.32	1.20	2.11	α	0.43	0.43	0.48
ϕ	4.00	2.68	1.94	ϕ	10.00	0.05	0.97
ν[a]	0.0	0.80	1.79	ν[a]	1.40	0.80	0.77

[a]% per year.

rameter at a time. Each row of the table reports two values of a parameter that differ from the baseline value and also reports the implied (net) annual values of the riskless interest rate, r, and the growth rate of capital, η, along a constant growth path. For parameters that are not equal to zero in the baseline, table 5.2 reports results for one value larger than in the baseline and one value smaller than in the baseline. For parameters that equal zero in the baseline, table 5.2 reports results using two values larger than zero.

The most glaring result in table 5.2 arises when μ, the (gross) mean annual risky rate of return, is reduced to 1.05, which is a 5 percent average annual (net) rate of return. In this case, the model produces a riskless interest rate of −1.18 percent per year and a growth rate of capital of −0.62 percent per year. These results are far from their empirical counterparts. However, in judging the implications of these results for the empirical plausibility of the model, it is important to remember that the preference parameters ϕ and ν used in this calculation were calibrated under the assumption that the mean return μ is 1.0698. When $\mu = 1.05$, the model can match the empirical values of r and η by using the following values for the preference parameters: $\phi = 3.8186$, and $\nu = -1.35$ percent per year. Although a coefficient of relative risk aversion of 3.8 is very reasonable, the negative rate of time preference is a bit curious.[27]

27. Setting the rate of time preference equal to zero and using a coefficient of relative risk aversion ϕ equal to 3.8186 imply $r_{ann} = 0.80$ percent per year and $\eta_{ann} = 0.51$ percent per year when $\mu = 1.05$.

Table 5.3 **Risky Trust Fund When σ = 0.14**

	$\gamma_S = 0.0$	$\gamma_S = 0.15$	$\gamma_S = 0.30$
		$\lambda = 0.1$	
r_{ann} [a]	0.69	0.75	0.81
η_{ann} [a]	1.41	1.51	1.60
		$\lambda = 0.9$	
r_{ann} [a]	0.69	0.75	0.81
η_{ann} [a]	1.44	1.54	1.63

[a]% per year.

A primary issue motivating this paper is the effect of investing part of the social security trust fund in risky capital. The social security trust fund in the United States is currently invested entirely in riskless bonds, so I set $\gamma_S = 0$ in the baseline calibration. The sensitivity analysis in table 5.2 reports the results of increasing γ_S to 0.15 and to 0.3. The value of γ_S that is considered in current policy discussions is about 0.15. Increasing γ_S to 0.15 in the model increases the riskless interest rate to 0.82 percent per year (from 0.80 percent per year in the baseline) and increases the rate of growth of the capital stock to 1.32 percent per year (from 1.30 percent per year in the baseline). These effects are small because the social security trust fund is small relative to the capital stock.[28]

Along a constant growth path, the ratio of the social security trust fund to the capital stock equals its target value σ. The small value of σ used in the calculations reported above was chosen to match the current value of the ratio of the social security trust fund to the capital stock in the United States. However, the trust fund is projected to grow substantially over the next several years, reaching a peak value in the year 2016 that is 2.6 times as large as its current value.[29] Indeed, it is the prospect of a large trust fund that has fueled interest in investing part of the trust fund in equities. Table 5.3 presents the effects of investing part of the trust fund in risky capital along a constant growth path with $\sigma = 0.14$, which is four times

28. Formally, applying the implicit function theorem to equations (C3) and (C5) in app. C along a constant growth path implies

$$\frac{\partial r}{\partial \gamma_s} = -\frac{\sigma}{\gamma'(r)\left[1 + \beta - \sigma + \frac{\theta_0(1-\alpha)\bar{A}}{r} + \theta_1\frac{1-\alpha}{\alpha}\right] - \gamma(r)\frac{\theta_0(1-\alpha)\bar{A}}{r^2}},$$

so that $\lim_{\sigma\to 0} \partial r/\partial\gamma_S = 0$. Thus, when the trust fund-capital ratio σ is small, the effect of γ_S on the interest rate is small. The proof of proposition 6 indicates that $\partial\eta/\partial\gamma_S$ is the sum of a term that is proportional to σ and a term that is proportional to $\partial r/\partial\gamma_S$. Thus, $\partial\eta/\partial\gamma_S$ will be small if σ is small.

29. This projection, which is taken from the Board of Trustees (1998), is based on the assumption that the trust fund is invested entirely in bonds. If the trust fund earns a higher rate of return by investing in risky capital, then it would reach an even larger value.

as high as in the baseline calculations. Even with this much larger trust fund, investing 15 percent of the trust fund in risky capital has only modest effects on the riskless interest rate and the growth rate of capital. The riskless interest rate increases by only six basis points, and the growth rate of the capital stock increases by 0.1 percent per year.

The baseline calibration in table 5.1 above is based on a value of $\lambda = 0.1$. However, the value of λ, which measures the responsiveness of taxes to changes in the primary deficit needed to satisfy the Treasury policy function, is not well determined. In principle, it could be anywhere between zero and one. Recall that the sufficient condition in proposition 6 for an increase in γ_S to increase η depends on the value of λ. A higher value of λ makes this condition less likely to hold. To see whether a higher value of λ can violate this condition, suppose that $\lambda = 1$, which is its maximum admissible value. In this case, the sufficient condition in proposition 6 is

$$\delta[(1 - \gamma_S)\sigma - \beta] + \eta\frac{\theta_0(1 - \alpha)\overline{A}}{r^2} \geq 0.$$

In the baseline calculation,

$$\delta[(1 - \gamma_S)\sigma - \beta] = -0.175,$$

and

$$\eta\frac{\theta_0(1 - \alpha)\overline{A}}{r^2} = 1.729,$$

so the sufficient condition in proposition 6 is satisfied by a wide margin even when $\lambda = 1$. Thus, for any allowable value of λ, an increase in γ_S increases η, the growth rate of the capital stock along a constant growth path.

In addition, λ has a very small effect on the calculated responses of the interest rate and the growth rate of capital to a change in γ_S. The top panel of table 5.3 reports the values of the riskless interest rate and the growth rate of capital when $\lambda = 0.1$, and the bottom panel reports the results for $\lambda = 0.9$. The values of the riskless interest rate are identical in the top and bottom panels because the equilibrium condition for the riskless interest rate in equation (41) is independent of λ. Although the growth rate of capital is not independent of λ, the growth rates differ by only three basis points when λ increases from 0.1 to 0.9.[30]

30. Since

$$\delta[(1 - \gamma_S)\sigma - \beta] + \eta\frac{\theta_0(1 - \alpha)\overline{A}}{r^2}$$

is an increasing function of σ (provided that the trust fund is not entirely invested in risky capital), the sufficient condition holds for higher values of σ, such as $\sigma = 0.14$, as in table 5.3.

5.7 Concluding Remarks

I have shown that shifting some of the assets of the social security trust fund from bonds to risky capital increases the growth rate of the capital stock in the following period and along a constant growth path. This finding is virtually the opposite of the result in Abel (in press), where I show that such a portfolio shift of the social security trust fund reduces the amount of capital accumulation in the following period. Although there are various modeling differences between the two papers, the fundamental reason for the apparent difference in results is that the earlier paper analyzes a defined-contribution social security system and the current paper analyzes a defined-benefit social security system.[31] In both papers, when the social security trust fund moves into risky capital, the expected income of the trust fund increases, and this increase in expected income is passed along to individuals. In a defined-contribution system, a natural policy experiment is to hold the contribution fixed, and thus the gains from increased trust fund earnings accrue to individuals as increased retirement benefits when they are old. In response to increased retirement benefits, young consumers increase their current consumption and thus reduce capital accumulation. In a defined-benefit system, a natural policy experiment is to hold the benefit fixed, and thus the gains from increased trust fund earnings accrue to individuals in the form of lower taxes when they are young. In response to increased disposable income when young, consumers increase their saving when they are young, and thus capital accumulation increases.

Confining attention to defined-benefit social security systems, and holding social security benefits fixed as analyzed in the current paper, it might appear that the social security trust fund should invest in risky capital because this change in its portfolio allocation will increase the growth rate of the capital stock. However, there are still several questions that future research must address, even in the context of this model, to reach a strong policy recommendation about the allocation of the assets in the social security trust fund. First, the results about the effect on the growth rate of the capital stock are confined to constant growth paths along which all shocks take on their mean values. Of course, one of the concerns about

31. There is also a major modeling difference between the two papers. In the current paper, which examines a defined-benefit social security system, all individuals in a given cohort are identical, and they all hold portfolios with both bonds and risky capital. As shown analytically in nn. 12 and 17 above, a change in the portfolio of a defined-contribution social security system would have no effect on the riskless interest rate or the growth rate of capital in this sort of model because individuals would offset the effects of changes in the social security trust fund's portfolio by changing their own portfolios. The previous paper, which analyzes a defined-contribution system, introduces intracohort heterogeneity of earnings and fixed costs of investing in risky capital so that low-income individuals will not hold any risky capital directly in their portfolios. With this modification, changes in the portfolio of the social security trust fund are no longer neutral, even in a defined-contribution system.

investing some of the social security trust fund in risky capital is the risk that the rate of return may turn out to be very low. A normative welfare analysis would have to take into account the entire distribution of outcomes. In addition, a normative analysis would have to recognize that government purchases in this model are endogenous. To the extent that individuals obtain utility from government purchases, I have assumed that any utility from government purchases is additively separable from the utility of private consumption. Although this assumption is sufficient to analyze optimal private behavior and competitive equilibria, it does not address the welfare consequences of endogenous changes in the level of government purchases.

Intergenerational risk sharing is another important aspect of the welfare analysis of various social security policies. Bohn (1998b) analyzes the intergenerational sharing of various risks in considering the effects of including equities in the social security trust fund. While the framework that I have developed in this paper focuses on a narrower set of risks, it suggests the possibility of additional channels to share risks intergenerationally by allowing the Treasury's debt-capital ratio and the social security trust fund-capital ratio to vary across time and across generations in response to shocks. The various ρ_i parameters ($i = A, B, R, S$) reflect opportunities to share risks across time and across generations. Exploration of these opportunities is left for future research.

Appendix A

The Consumption and Portfolio Decision of an Individual

Using the definitions of Ω_t, a_{t+1}, and γ_{t+1} in equations (11), (12), and (14), respectively, it is convenient to rewrite the expression for consumption when young in equation (6) as

$$C_t = \Omega_t - a_{t+1} \tag{A1}$$

and consumption when old in equation (9) as

$$X_{t+1} = [(1 - \gamma_{t+1})r_{t+1} + \gamma_{t+1}R_{t+1}]a_{t+1}. \tag{A2}$$

The consumer's optimization problem can be rewritten by substituting equations (A1) and (A2) into equation (10) to obtain

$$\max_{a_{t+1},\gamma_{t+1}} \ln(\Omega_t - a_{t+1}) + \delta \ln a_{t+1} + \delta\psi(\gamma_{t+1}, r_{t+1}), \tag{A3}$$

where[32]

32. If $\phi = 1$, $\psi(\gamma_{t+1}, r_{t+1}) \equiv E_t\{\ln[(1 - \gamma_{t+1}) r_{t+1} + \gamma_{t+1}R_{t+1}]\}$.

(A4) $$\psi(\gamma_{t+1}, r_{t+1}) \equiv \frac{1}{1-\phi} \ln E_t\left\{[(1-\gamma_{t+1})r_{t+1} + \gamma_{t+1}R_{t+1}]^{1-\phi}\right\} \quad \text{if } 0 < \phi \neq 1.$$

The optimal level of saving and consumption is determined by differentiating the maximand in equation (A3) with respect to a_{t+1} and setting the derivative equal to zero to obtain

(A5) $$a_{t+1} = \frac{\delta}{1+\delta}\Omega_t.$$

The portfolio-allocation problem is solved by differentiating $\psi(\gamma_{t+1}, r_{t+1})$ with respect to γ_{t+1} and setting the derivative equal to zero to obtain

(A6) $$E_t\{[(1-\gamma_{t+1})r_{t+1} + \gamma_{t+1}R_{t+1}]^{-\phi}(R_{t+1} - r_{t+1})\} = 0.$$

Equation (A6) implicitly defines the optimal value of γ_{t+1}, which is the share of an individual's total portfolio devoted to risky assets. This equation holds for any $\phi > 0$, including the case of logarithmic utility, $\phi = 1$.

Let $\gamma(r_{t+1})$ be the value of γ_{t+1} that solves equation (A6). Appendix B derives the properties of $\gamma(r_{t+1})$.

Appendix B

Properties of $\gamma(r)$

If $E_t\{R_{t+1}\} - r_{t+1} = 0$, then $E_t\{r_{t+1}^{-\phi}(R_{t+1} - r_{t+1})\} = 0$, which implies that $\gamma_{t+1} = 0$ satisfies equation (A6). Therefore, if $E_t\{R_{t+1}\} - r_{t+1} = 0$, the optimal value of γ_{t+1} is zero, so

(B1) $$\gamma(E_t\{R_{t+1}\}) = 0.$$

As the gross riskless interest rate, r_{t+1}, approaches $R_L \equiv \alpha A_L$ from above, the optimal value of γ_{t+1} becomes arbitrarily large. That is,

(B2) $$\lim_{r_{t+1} \downarrow R_L} \gamma(r_{t+1}) = \infty.$$

To analyze the derivative of $\gamma(r_{t+1})$ with respect to r_{t+1}, define

(B3) $$F(\gamma, r, \phi) \equiv E\{x^{-\phi}(R - r)\},$$

where[33] $x \equiv r + \gamma(R - r) > 0$ and $r \geq R_L$. Observe from equation (A6) that the optimal value of γ, γ^*, solves

33. Both r and R are assumed to be positive. I restrict attention to values of γ for which $x > 0$ because optimality requires that $\text{pr}\{x > 0\} = 1$.

(B4) $$F(\gamma^*, r, \phi) = 0.$$

Differentiating equation (B3) with respect to γ yields

(B5) $$F_\gamma(\gamma, r, \phi) = -\phi E\{x^{-\phi-1}(R - r)^2\} < 0.$$

Since $F_\gamma(\gamma, r, \phi) < 0$, there is a unique value of γ that solves $F(\gamma, r, \phi) = 0$ for given values of r and ϕ.

The response of γ^* to a change in r is given by $\gamma'(r) = d\gamma^*/dr = -[F_r(\gamma^*, r, \phi)]/F_\gamma(\gamma^*, r, \phi)]$. Since $F_\gamma(\gamma^*, r, \phi) < 0$, the sign of $\gamma'(r)$ is the same as the sign of $F_r(\gamma^*, r, \phi)$. Differentiating equation (B3) with respect to r yields

(B6) $$F_r(\gamma, r, \phi) = E\{-\phi x^{-\phi-1}(1 - \gamma)(R - r) - x^{-\phi}\}.$$

Use the fact that $R - x = (1 - \gamma)(R - r)$ to rewrite equation (B6) as

(B7) $$F_r(\gamma, r, \phi) = E\{-\phi x^{-\phi-1}R + (\phi - 1)x^{-\phi}\}.$$

Inspection of equation (B7) reveals that, if $\phi \leq 1$, then $F_r(\gamma, r, \phi) < 0$ and hence $\gamma'(r) < 0$. In the case with $\phi > 1$, there is no general result for the sign of $\gamma'(r)$. Even when the distribution of R is a symmetrical two-point distribution, the sign of $\gamma'(r)$ is not determinate.

To determine the effect of a change in the coefficient of relative risk aversion, ϕ, differentiate $F(\gamma, r, \phi)$ with respect to ϕ to obtain

(B8) $$F_\phi(\gamma, r, \phi) = -E\{x^{-\phi}(R - r)\ln x\}.$$

If $R - r < 0$, then $\ln x < \ln r$, provided that $\gamma > 0$. Therefore, if $R - r < 0$, then $x^{-\phi}(R - r)\ln x > x^{-\phi}(R - r)\ln r$. Similarly, if $R - r > 0$, then $\ln x > \ln r$, provided that $\gamma > 0$. Therefore, if $R - r > 0$, then $x^{-\phi}(R - r)\ln x > x^{-\phi}(R - r)\ln r$. Thus, if the optimal value γ^* is positive, then, when $\gamma = \gamma^*$,[34]

(B9) $$E\{x^{-\phi}(R - r)\ln x\} > E\{x^{-\phi}(R - r)\ln r\} = 0.$$

Substituting equation (B9) into equation (B8) yields

(B10) $$F_\phi(\gamma, r, \phi) < 0.$$

Therefore, since $F_\gamma(\gamma, r, \phi) < 0$,

(B11) $$\frac{d\gamma^*}{d\phi} < 0,$$

which means that an increase in the coefficient of relative risk aversion, ϕ, leads to a reduction in γ, the share of the portfolio devoted to the risky asset.

34. I am assuming that the distribution of R is nondegenerate so that $\mathrm{pr}\{R \neq r\} > 0$.

Appendix C

Proof of Proposition 2

Define the function $f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)$ as the right-hand side of equation (33). This function is continuous in r_{t+1} for $\alpha A_L \equiv R_L < r_{t+1} \leq \overline{R} \equiv E_t\{R_{t+1}\}$. Equation (B1) and the condition in equation (31) imply

(C1) $$f(E_t\{R_{t+1}\}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1) < 0.$$

Equation (B2) and the condition in equation (30) imply

(C2) $$\lim_{r_{t+1} \downarrow R_L} f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1) = \infty.$$

Equations (C1) and (C2) and the fact that $f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)$ is continuous in r_{t+1} for $r_{t+1} \in (R_L, \overline{R}]$ imply that there is at least one value of $r_{t+1} \in (R_L, \overline{R}]$ for which $f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1) = 0$.

To prove that the equilibrium value of r_{t+1} is unique, differentiate $f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)$ with respect to r_{t+1}, and use the assumption in proposition 2 that $\gamma'(r_{t+1}) < 0$ to obtain

(C3) $$\frac{\partial f}{\partial r_{t+1}} = \gamma'(r_{t+1})\left[1 + b_{t+1} - s_{t+1} + \frac{\theta_0(1-\alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}\right] - \gamma(r_{t+1})\frac{\theta_0(1-\alpha)\overline{A}}{r_{t+1}^2} < 0.$$

Since $\partial f / \partial r_{t+1} < 0$, the value of r_{t+1} for which $f(r_{t+1}, b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1) = 0$ is unique.

To analyze the effects of various variables on the riskless interest rate, compute the following partial derivatives:

(C4) $$\frac{\partial f}{\partial b_{t+1}} = \gamma(r_{t+1}) > 0,$$

(C5) $$\frac{\partial f}{\partial \gamma_{S,t+1}} = s_{t+1}$$

(C6) $$\frac{\partial f}{\partial s_{t+1}} = \gamma_{S,t+1} - \gamma(r_{t+1}),$$

(C7) $$\frac{\partial f}{\partial \theta_0} = \gamma(r_{t+1})\frac{(1-\alpha)\overline{A}}{r_{t+1}} > 0,$$

(C8) $$\frac{\partial f}{\partial \theta_1} = -\frac{1-\alpha}{\alpha}[1 - \gamma(r_{t+1})] < 0,$$

where the inequalities in equations (C4), (C7), and (C8) follow from proposition 1.

The implicit-function theorem implies that

$$\frac{\partial r(b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)}{\partial z} = -\frac{\partial f/\partial z}{\partial f/\partial r_{t+1}}$$

$$\text{where } z \in \{b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1\}.$$

Therefore, sign $\{[\partial r(b_{t+1}, s_{t+1}, \gamma_{S,t+1}, \theta_0, \theta_1)]/\partial z\} = \text{sign}(\partial f/\partial z)$.

Appendix D

Derivation of the Growth Rate of the Capital Stock

Substitute equation (36) into equation (35), use the definitions $b_{t+1} \equiv B_{t+1}/K_{t+1}$ and $s_{t+1} \equiv S_{t+1}/K_{t+1}$, and multiply both sides of the resulting equation by $1 + \delta$ to obtain

(D1) $$(1 + \delta)(b_{t+1} + 1 - s_{t+1})K_{t+1} = \delta(w_t - T_t^T - T_t^S) - \left[\frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1 - \alpha}{\alpha}\right]K_{t+1}.$$

The amount of social security taxes can be rewritten by substituting equation (24) into equation (27) to obtain

(D2) $$T_t^S = [\sigma + \rho_S(s_t - \sigma) + \rho_R(R_t - \overline{R})\gamma_{S,t}s_t]K_{t+1} + Q_t - [(1 - \gamma_{S,t})r_t + \gamma_{S,t}R_t]s_tK_t.$$

Use equation (7) to substitute for Q_t in equation (D2) to obtain

(D3) $$T_t^S = \Phi(T_t^S, K_{t+1})K_{t+1} + \Phi(T_t^S, K_t)K_t,$$

where

(D4) $$\Phi(T_t^S, K_{t+1}) \equiv \sigma + \rho_S(s_t - \sigma) + \rho_R(R_t - \overline{R})\gamma_{S,t}s_t$$

and

(D5) $$\Phi(T_t^S, K_t) \equiv (\theta_0\overline{A} + \theta_1 A_t)(1 - \alpha) - [(1 - \gamma_{S,t})r_t + \gamma_{S,t}R_t]s_t.$$

Now rewrite the Treasury's tax revenue in equation (23) as

(D6) $$T_t^T = \Phi(T_t^T, K_{t+1})K_{t+1} + \Phi(T_t^T, K_t)K_t,$$

where

(D7) $$\Phi(T_t^T, K_{t+1}) \equiv -\lambda[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \overline{A})]$$

and

(D8) $$\Phi(T_t^T, K_t) \equiv [(1 - \lambda)\tau + \lambda g]A_t + \lambda r_t b_t.$$

Substitute equations (D3) and (D6) into equation (D1) and use equation (4) to obtain

(D9) $$\left[(1 + \delta)(b_{t+1} + 1 - s_{t+1}) + \frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}} + \theta_1\frac{1 - \alpha}{\alpha}\right]K_{t+1}$$
$$= \delta[(1 - \alpha)A_tK_t - \Phi(T_t^S, K_{t+1})K_{t+1} - \Phi(T_t^S, K_t)K_t$$
$$- \Phi(T_t^T, K_{t+1})K_{t+1} - \Phi(T_t^T, K_t)K_t].$$

Now use the definitions in equations (D4), (D5), (D7), and (D8) to rewrite equation (D9) as

(D10) $$H_1K_{t+1} = H_0K_t,$$

where

(D11) $$H_1 \equiv (1 + \delta)(1 + b_{t+1} - s_{t+1}) + \frac{\theta_0(1 - \alpha)\overline{A}}{r_{t+1}}$$
$$+ \theta_1\frac{1 - \alpha}{\alpha} + \delta[\sigma + \rho_S(s_t - \sigma) + \rho_R(R_t - \overline{R})\gamma_{S,t}s_t]$$
$$- \delta\lambda[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \overline{A})]$$

and

(D12) $$H_0 \equiv \delta\{[1 - \alpha - (1 - \lambda)\tau - \lambda g]A_t$$
$$- (1 - \alpha)(\theta_0\overline{A} + \theta_1A_t)$$
$$+ [(1 - \gamma_{S,t})r_t + \gamma_{S,t}R_t]s_t - \lambda r_t b_t\}.$$

Rewrite the expression for H_1 by substituting the Treasury policy function from equation (18) and the social security policy function from equation (26) into equation (D11) to obtain

(D13)
$$H_1 \equiv 1 + \delta + (1 + \delta - \delta\lambda)[\beta + \rho_B(b_t - \beta) + \rho_A(g - \tau)(A_t - \bar{A})] - [\sigma + \rho_S(s_t - \sigma) + \rho_R(R_t - \bar{R})\gamma_{S,t}s_t] + \frac{\theta_0(1-\alpha)\bar{A}}{r_{t+1}} + \theta_1\frac{1-\alpha}{\alpha}.$$

Appendix E

The Expectation of Multiyear Returns

LEMMA 3. *Let* $z_t \in \{Z(1), \ldots, Z(J)\}$, *and define the transition probabilities* $p(i, j) \equiv \mathrm{pr}\{z_{t+1} = Z(j) \mid z_t = Z(i)\}$ *and unconditional probabilities* $\pi(j) \equiv \mathrm{pr}\{z_t = Z(j)\}$. *If* $\Pi' \equiv [\pi(1), \ldots, \pi(J)]$, M *is the* $J \times J$ *matrix with* (i, j) *element* $m(i, j) \equiv p(i, j)\,Z(j)$ *and* $\mathbf{i}$ *is a* $J \times 1$ *vector of ones, then* $E\{z_{t+1}\cdots z_{t+N}\} = \Pi'(M^N)i$.

Proof.

(E1)
$$E\{z_{t+1}\cdots z_{t+N} \mid z_t = Z(i)\} = \sum_{j_1,\ldots,j_N}[p(i,j_1)p(j_1,j_2)\cdots p(j_{N-1},j_N)][Z(j_1)Z(j_2)\cdots Z(j_N)].$$

Use the definition $m(i, j) \equiv p(i, j)Z(j)$ to rewrite the conditional expectation as

(E2)
$$E\{z_{t+1}\cdots z_{t+N} \mid z_t = Z(i)\} = \sum_{j_1,\ldots,j_N} m(i,j_1)m(j_1,j_2)\cdots m(j_{N-1},j_N).$$

Define

(E3)
$$y(N,i,j) \equiv \sum_{j_1,\ldots,j_{N-1}} m(i,j_1)m(j_1,j_2)\cdots m(j_{N-1},j).$$

Equations (E2) and (E3) imply

(E4)
$$E\{z_{t+1}\cdots z_{t+N} \mid z_t = Z(i)\} = \sum_j y(N,i,j).$$

The unconditional expectation is

(E5)
$$E\{z_{t+1}\cdots z_{t+N}\} = \sum_{i,j}\pi(i)y(N,i,j).$$

Observe from equation (E3) that

(E6)
$$y(N+1,i,k) = \sum_j \sum_{j_1,\ldots,j_{N-1}} m(i,j_1)m(j_1,j_2)\cdots m(j_{N-1},j)m(j,k).$$

Equations (E3) and (E6) imply

$$\text{(E7)} \qquad y(N+1, i, k) = \sum_j y(N, i, j) m(j, k).$$

Rewrite equation (E7) in matrix form as

$$\text{(E8)} \qquad Y_{N+1} = Y_N M,$$

where the (i, j) element of Y_{N+1} is $y(N + 1, i, j)$ and the (i, j) element of Y_N is $y(N, i, j)$.

The solution of the matrix difference equation in (E8) is

$$\text{(E9)} \qquad Y_N = M^N.$$

Therefore, the definition of Y_N and equations (E5) and (E9) imply

$$\text{(E10)} \qquad E\{z_{t+1} \cdots z_{t+N}\} = \sum_{i,j} \pi(i) y(N, i, j) = \Pi'(M^N)\mathbf{i}.$$

References

Abel, Andrew B. In press. The effects of investing social security funds in the stock market when fixed costs prevent some households from holding stocks. *American Economic Review.*

Barro, Robert J., and Xavier Sala-i-Martin. 1995. *Economic growth.* New York: McGraw-Hill.

Blanchard, Olivier J., and Stanley Fischer. 1989. *Lectures on macroeconomics.* Cambridge, Mass.: MIT Press.

Board of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds. 1998. *1998 annual report.* Washington, D.C.: U.S. Government Printing Office.

Bohn, Henning. 1998a. Risk sharing in a stochastic overlapping generations economy. Department of Economics, University of California, Santa Barbara. Mimeo.

———. 1998b. Should the social security trust fund hold equities? An intergenerational welfare analysis. Department of Economics, University of California, Santa Barbara. Mimeo.

Diamond, Peter A. 1965. National debt in a neoclassical growth model. *American Economic Review* 55, no. 5:1126–50.

Epstein, Larry G., and Stanley E. Zin. 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica* 57, no. 4:937–70.

Fama, Eugene F., and Kenneth R. French. 1988. Permanent and temporary components of stock prices. *Journal of Political Economy* 96, no. 2:246–73.

Mehra, Rajnish, and Edward C. Prescott. 1985. The equity premium: A puzzle. *Journal of Monetary Economics* 15, no. 2:145–61.

Weil, Philippe. 1990. Nonexpected utility in macroeconomics. *Quarterly Journal of Economics* 105, no. 1:29–42.

Comment Deborah Lucas

Faced with a large projected social security deficit, a number of economists and policy makers have suggested moving a portion of the social security trust fund out of government securities and into the stock market. These proposals raise myriad questions. Would real gains be achieved, or would the government's actions be largely undone by private actions? How would such a change affect the relative return on safe and risky assets? How would it affect private savings and investment behavior? How would it affect risk sharing and income distribution within and across generations? Although progress can be made on some of these questions using partial equilibrium models, a complete analysis clearly requires a general equilibrium approach. This paper makes an important contribution to the policy debate by providing a general equilibrium framework that is amenable to a variety of interesting policy experiments.

Although essential for a complete analysis, one cost of the general equilibrium approach is a loss of transparency. My main goal in this discussion is to offer additional intuition for the main results and some thoughts on which results are likely to be robust. I will also discuss why some of the simplifying assumptions, although useful for analytic purposes, might hide some potentially large costs and benefits and might reduce the predicted general equilibrium effect. The rest of this Comment is organized as follows: First, I recap the main results and the intuition behind them. These issues are discussed in the context of a simple model that is useful for laying out the key sensitivities. Then I address some more specific aspects of the theoretical model and its calibration that bear on the robustness of the results. I conclude that, although the model is a useful tool for examining the qualitative effects of policy changes, the quantitative implications are less compelling.

General Equilibrium Revisited

Two main results emerge from the overlapping-generations production economy considered in this paper. The first is that, for most parameterizations, increasing the share of the trust fund invested in risky assets tends to increase the risk-free rate and thereby lower the equity premium (since the return on risky capital is fixed by the technology). The second is that, with some further restrictions on parameters, this policy change also increases capital accumulation. Calibration of the model suggests that, although these effects are present, they are small in economic terms. For

Deborah Lucas is the Donald C. Clark/Household International Distinguished Professor of Finance at the Kellogg Graduate School of Management, Northwestern University, and a research associate of the National Bureau of Economic Research.

instance, all else equal, changing the share of the trust fund in risky assets from 0 to 15 percent increases the risk-free rate from 0.8 percent per year to 0.82 percent per year. It also increases the average annual per capita growth rate of the capital stock from 1.3 to 1.32 percent.

Quantitatively, then, the model suggests that such a policy change will have a relatively small effect on equilibrium rates of return and aggregate output growth. It is natural to ask why this is the case in this model and whether it is likely to hold more generally. To think about this, consider the following simplified version of the model that abstracts from the details of the production technology, tax policy, and the evolution of the trust fund.

People live for two periods and have lifetime utility

$$U(c_0) + \beta E[V(c_1)].$$

In the first period of life, they receive income I_0 (e.g., from wages and any inheritance), pay taxes, τ, and save B^p in bonds and S^p in stocks. In the second half of life, consumption is financed out of any additional income I_1, the gross returns on stocks and bonds, R^s and R^b, and social security payments that have a present value of X. A fraction γ of social security payments is contingent on the realized return on stocks, and a fraction $(1 - \gamma)$ is risk free. In the first period of life, there is uncertainty about R^s and I_1. Then consumption each period can be written as

$$c_0 = I_0 - \tau - B^p - S^p,$$

$$c_1 = I_1 + B^pR^b + S^pR^s + \gamma XR^s + (1 - \gamma)XR^b.$$

Assuming no borrowing or short-sales restrictions, asset prices and investment policy are determined by combining market-clearing conditions with individual Euler equations. This results in two standard equations that characterize the equilibrium return on bonds and stocks, given the investment policy of individuals. Conversely, given returns, these equations characterize investment policy:

$$\frac{\beta E[V'(I_0 + B^pR^b + S^pR^s + \gamma XR^s + (1 - \gamma)XR^b)R^b]}{U'(I_0 - \tau - B^p - S^p)} = 1,$$

$$\frac{\beta E[V'(I_0 + B^pR^b + S^pR^s + \gamma XR^s + (1 - \gamma)XR^b)R^s]}{U'(I_0 - \tau - B^p - S^p)} = 1.$$

The reason to write down these equations is as a reminder of the factors that can potentially influence asset returns and investment policy in any standard model. These include changes in tax policy, changes in the incentive to work if I_j is endogenous, and borrowing or short-sales constraints that may prevent young people from offsetting changes in social security investment policy. This latter effect would turn the equalities given above into inequalities for constrained agents and result in market prices reflecting the marginal rate of substitution of unconstrained agents.

It is quite possible for a policy change to be neutral in this setting. Consider, for instance, a small increase in γ, $\Delta\gamma$, financed by a corresponding increase in stockholdings by the trust fund and a decrease in bondholdings. If there are no restrictions on B^p and S^p, then, holding all other policy variables constant, the following would be an equilibrium. People increase their bond holdings by $\Delta\gamma X$, buying what the trust fund sells, and conserving the portion of second-period income that is risk free. Similarly, they decrease their stockholdings by $\Delta\gamma X$, selling what the trust fund buys and conserving the portion of second-period income exposed to stock market risk. Asset returns remain unchanged since the first-order condition continues to hold. Investment in the first period, and hence growth, would also be unaffected because the aggregate investment in stocks is constant.

What breaks this neutrality in Abel's model? Since taxes are lump sum, they do not directly influence the incentive to work or invest. Borrowing or short-sales constraints are also not the answer because parameters are chosen such that, in the absence of the trust fund holding stocks, people choose to hold positive amounts of both assets. Furthermore, the policy experiments do not cause people to hit corners. This leaves the government's tax and expenditure policy, broadly defined, as the reason for nonneutrality. The way policy shows up is through adjustments in social security tax rates, federal tax rates, government expenditures, and government bonds outstanding. In Abel's model, these quantities move according to fixed reaction functions. The reaction functions are structured so that government policy partly offsets productivity shocks, effectively using the trust fund and fiscal policy to spread risks across generations in a way that would be impossible in an overlapping-generations model with only private-sector transactions.

If the government policy function is the cause of the nonneutrality, it remains to understand the precise direction of the nonneutrality. That is, why does the equity premium tend to fall, and why does investment tend to grow?

On the question of why the equity premium falls, the suggested intuition is that, if the government sells bonds, there must be an incentive for individuals to buy the bonds, and this tends to put upward pressure on the risk-free rate. Although broadly speaking this must be correct, we have seen that, if government policy is otherwise neutral, the change in government investments can occur with no change in equilibrium returns. An alternative way to interpret the drop in the predicted equity premium is that it is due to the reduced consumption risk in the second period that results from increased risk sharing via the government's reaction function.

More generally, if the net effect of any policy change is to decrease second-period consumption risk and particularly correlated risk, the equity premium will tend to fall. Conversely, the premium will rise if the policy results in riskier outcomes for stock market participants.

On the question of why investment increases, as suggested in the paper, a primary driver is that the policy experiment assumes that taxes on the young tend to fall. That is, rather than any gains from the higher average stock market returns being used to increase average benefits in the second period, it is assumed that expected gains to the government are used to reduce taxes. The wealth effect of this reduction in taxes tends to increase private savings, along with first-period consumption. As shown in his earlier analysis (Abel 1998), if instead the policy experiment were to use the higher average returns to increase the expected benefits for the old, the effect on saving would be the opposite since social security would partially crowd out private saving. The effect of a policy change on risk is also a major consideration for the level of savings. We know from other models that, for utility specifications exhibiting high risk aversion, a small increase in second-period consumption risk can result in a large increase in precautionary savings, and conversely for a risk reduction. For the parameterization considered, the effect of reduced risk was more than offset by the wealth effect, but, in general, the response of savings will be very sensitive to the details of the policy implementation and the utility specification.

Finally, it should be noted that the model is structured so that it is possible to consider two distinct aspects of social security interaction with the equity markets. It is possible to vary the portion of the trust fund invested in the market and independently vary the degree to which benefits are tied to market outcomes. Although this distinction is not emphasized, it is implicit in the parameterizations considered. Both aspects are important for evaluating the likely effect of the various proposals now being considered. For instance, the model implies that, if, as under one leading proposal, benefits continue to be guaranteed by the government but part of the trust fund is invested in stocks, the young will bear greater tax risk. It would be interesting to use the model to look more closely at the welfare implications of such proposals.

Modeling Assumptions and Alternatives

The comments presented above suggest that this model is indeed useful for thinking qualitatively about the general equilibrium effects of investing part of the social security trust fund in the market. It is more difficult, however, to have confidence in its quantitative implications. In this section, I focus on the assumptions in the theoretical model and calibration that might affect the quantitative results.

Market Structure

In general, a two-period overlapping-generations structure is convenient for computing analytic results because it makes each individual's decision problem relatively simple. For interpreting calibration results, however, it creates some complications. The most significant of these is the extremely limited private risk sharing that this structure allows owing to the minimal overlap between generations. This creates a large role for a benevolent government because that is the only entity that can enforce contracts across disconnected generations. Other analyses have shown that, as the assumed number of periods of life increases, this dichotomy between the effectiveness of government and private arrangements declines. Although not necessarily the case, the severe market incompleteness may magnify the effect of policy changes on savings and asset prices because it reduces the opportunities for offsetting private-sector contracts. If one were to look at welfare implications, the insurance gains from the existence of a larger trust fund would likely be overstated for the same reason.

The two-period overlapping-generations model has the further drawback that, if one must interpret each period as thirty years, it is not possible to capture the nuances of transition effects on intergenerational transfers that are likely to be critical. A related problem is the assumption of a constant population since many of the stresses in the current system arise from the changes in the relative size of different generations. In particular, this has implications for the incidence of tax liability.

Homogeneous Agents

The assumption of homogeneous agents, none of whom face binding borrowing constraints, seems likely to create a downward bias on the extent to which the policy change affects asset returns. Previous work (e.g., Geanakoplos, Mitchell, and Zeldes 1998) has pointed to the evidence that many young people are up against a borrowing constraint and that social security taxes are likely to crowd out private savings for this group. This suggests that investing more of the trust fund in the market on behalf of these constrained agents may be welfare improving. To the extent that stockholdings are highly concentrated owing to market frictions rather than to risk preferences, the improved risk sharing resulting from shifting stock market risk to nonstockholders would tend to result in a lower equity premium. On the other hand, if many people avoid risky investments simply because they are risk averse, exposing them to market risk via the trust fund may reduce welfare. Whether these effects are quantitatively important is unclear but worthy of consideration.

Heterogeneity in tax effects across generations and wealth classes may also be important in practice. Investment behavior and expected returns will be most sensitive to the effect of policies on the rich, while labor income is likely to be most sensitive to the tax and benefit implications for the middle class. The assumption of lump-sum taxes, combined with the assumption of homogeneity, abstracts from these effects.

Government Policy

As shown above, government fiscal policy is the main driver of general equilibrium effects. One might argue that, in the scenarios considered, the assumed government policy is in some respects overly benign and in others insufficiently benevolent. The assumption of lump-sum taxes is benign relative to the current tax structure, where social security taxes create a disincentive to work. At the same time, the proposed government-policy function is not optimal. The trust fund size and Treasury debt levels are targeted to be stationary relative to output, with gradual adjustments toward these goals over time. Although risk sharing could presumably be improved if benefits were designed to increase spanning, it is assumed throughout the analysis that they do not. As discussed earlier, some risk-sharing benefits are achieved by virtue of the government's tax and expenditure policies, but to what extent risk sharing is improved is unclear. A useful extension of the analysis would be to consider a variety of other policy functions.

Implications of the Equity-Premium Puzzle

The discussion in this section thus far has been largely cautionary, and it would be reasonable to take the calibration results as telling despite these potential biases or omissions. More problematic is the fact that the quantitative results are based on a model that cannot explain observed asset returns. As is noted in the paper, the only way to make the base case consistent with the historical equity premium is to assume an unrealistically volatile consumption process. The way in which this is accomplished is by choosing a technology in which wages and market returns have a correlation of one and in which the productivity shocks are chosen to match the observed volatility of stock market returns rather than the observed volatility of output. This is in contrast to the data, in which the volatility of output is a small fraction of the volatility of the stock market. The high correlation between wages and stock returns is also problematic on an annual basis, but perhaps less so over the thirty-year periods represented by the model.

The inflated volatility of consumption risk is important because the background level of this risk influences the elasticity of demand for risky assets. Hence, it influences the predictions about the effect of policy changes on asset returns and on the precautionary demand for savings. If neither this model nor any of its close relatives can explain the relation between asset returns and physical quantities, it is hard to be confident that it can reliably predict the change in asset returns resulting from a change in physical quantities.

Of course, the equity-premium puzzle is a problem, not only for the analysis in this paper, but for the entire debate over whether welfare would be improved by investing social security funds in the stock market. Until we can agree on whether the observed equity premium is really a puzzle or whether our theoretical models are misspecified, the economically right course of action will remain elusive.

References

Abel, Andrew B. 1998. The aggregate effects of including equities in the social security trust fund. Wharton School, University of Pennsylvania. Typescript.

Geanakoplos, John, Olivia Mitchell, and Steve Zeldes. 1998. Would a privatized social security system really pay a higher rate of return? Yale University. Typescript.

Discussion Summary

James Poterba noted that, although the model presented is already very involved, the next generation of models may call for some synthesis of the insights of this paper with features of John McHale's paper. In particular, it seems crucially important to understand what benefit and tax policies would result from a change in the portfolio allocation of the social security trust fund. We do not currently have empirical evidence to analyze this question, but the type of cross-country data that McHale uses in his work would be helpful in this regard. Ultimately, the goal should be to enrich the government sector or the social security sector of the model to shed light on the question of which generation would benefit from tax reductions or benefit increases when the portfolio composition of the social security trust fund is shifted toward equities. Poterba also followed up on a remark made by the discussant, Deborah Lucas, and noted that everyone in the model holds risky assets. This prediction is counterfactual. Limited stock market participation seems important to consider when thinking about investing the social security trust fund in equities, especially in the light of such potential market frictions as transaction costs.

Robert King wondered whether there was any empirical evidence supporting the prediction of the model that, probably even absent any social security system, the ratio of government debt to GDP or of government debt to capital affects rates of return. He also commented that, although he appreciated the model for its tractability, it probably features an inefficiently low rate of growth. Reasoning from the perspective of the endogenous-growth literature, the growth rate is too low because the market interest rate is below the social marginal product of capital owing to the presence of external effects of capital accumulation.

John McHale remarked that the paper yielded interesting results in terms of who gains from the shift in the portfolio allocation of the social security trust fund toward equities. From the paper's conceptualization of a defined-benefit system, it appears that the current generation stands to gain. Investing the social security trust fund in stocks allows them either to enjoy lower tax rates when young (as less prefunding is required), keeping the benefits unchanged, or to have higher benefits when old, keeping tax rates constant. This is not always what is envisioned in debates: the benefits are often expected to accrue mainly to future generations, not to the currently young.

Antonio Rangel noted that the conclusions of this paper contradict the ones obtained in a previous paper by the author (Abel 1998), owing to differences in the modeling assumptions concerning the government sector. He suggested adding political economy considerations to the analysis in order to resolve this issue.

David Cutler remarked that it is hard to think about a social security trust fund in the model as there seems to be no reason for having one. One justification for a social security system might be that people are myopic when saving for retirement in the sense that they discount the future more than the government does. This should have important implications for the effects and welfare gains of the experiments considered in the paper. He concluded that adding a discussion of why we have a social security trust fund would be useful.

In response to these remarks and suggestions, *Andrew Abel* made the following comments. First, with respect to the fact that the experiments are only ceteris paribus interventions, he agreed but added that all variables adjust endogenously. *Deborah Lucas* clarified that she was interested in an experiment in which taxes were changed simultaneously with the change in γ_s, the portfolio allocation of the social security trust fund.

With respect to Ricardian equivalence, *Abel* responded that it would obtain only for a defined-contribution social security system, not for a defined-benefit plan. He further acknowledged that analyzing the issues considered in the paper in a multiperiod overlapping-generations model was important, as are borrowing constraints, labor supply decisions, and distortionary taxation. Regarding the correlation between wages and interest rates, he pointed out that, although low at an annual frequency, the correlation coefficient grows to 0.8 or 0.9 over a thirty-year period, as documented in Jermann (1999).

Concerning the target σ for the social security trust fund, Abel replied that it could be zero or even negative. Relating to Cutler's comment, he noted that a utility-maximizing theory of the government was missing from the analysis.

Finally, in response to McHale's question, Abel emphasized the difference between the question of which generation benefits and at what stage in the life cycle a generation receives the benefit.

References

Abel, Andrew B. 1998. The aggregate effects of including equities in the social security trust fund. Wharton School, University of Pennsylvania. Typescript.

Jermann, Urban. 1999. Comment on Robert J. Shiller, "Social security and institutions for intergenerational, intragenerational, and international risk sharing." *Carnegie-Rochester Conference Series on Public Policy* 50 (June): 205–12.