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ON STOCHASTIC STABILITY OF COMPETITIVE EQUILIBRIUM

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Unpredictable changes in economic environment introduce random disturbances of competitive equilibrium. In the framework of stochastic stability in the mean, we establish a trade-off between the degree of stability and the size of the random fluctuations that can be absorbed by stable nonlinear and nonstationary equilibrium models. Since the obtained stability conditions are only sufficient, instability considerations of the proposed stochastic model are carried out to provide the necessity part in the stability criterion. Both inferior goods and the random disturbances are exposed as sources of instability in multiple market systems.

1. INTRODUCTION

There are at least two strong reasons for considering stochastic rather than deterministic models of competitive equilibrium. First, due to unpredictable changes in the tastes of consumers, changes in technology, etc., it is quite realistic to consider fluctuations of supply and demand schedules as external random disturbances of the equilibrium. Secondly, inaccuracies of the tâtonnement process are likely to produce random perturbations of the internal adjustment mechanism of a competitive equilibrium model. Both external and internal random disturbances introduce destabilizing effects into the market system, which cannot be estimated satisfactorily from deterministic market models. For these reasons, Turnovsky and Weintraub [1] initiated stochastic stability analysis of general equilibrium systems, and derived explicit conditions under which linear equilibrium models are stable with probability one.

The objective of this work is to introduce nonlinear stochastic models of competitive equilibrium as natural extensions of deterministic models analyzed by Arrow and Hahn [2]. This approach enables us to use the convenient diagonal dominance condition and derive explicit conditions for stochastic stability and instability of nonlinear equilibrium models under random disturbances. As shown in [3], the condition is ideally suited for establishing a trade-off between the degree of stability of the deterministic part of the model and the size of random disturbances that can be tolerated by a stable equilibrium.

Under the condition of diagonal dominance, stability is connective [4–6] and globally exponential in the mean. That is, it is stable despite structural perturbations of the market system whereby commodities or services can disappear from the market, can change from substitutes to complements to another group of commodities, etc. Such structural perturbations can be completely arbitrary in the sense that no stochastic nor deterministic description is required other than the usual continuity and boundedness of the corresponding excess demand functions. Furthermore, stability is both global and exponential, that is, it holds for all initial data, and the expected value of the prices tends to zero faster than an exponential, a property common to linear market models.

2. A Stochastic Nonlinear Model

Let us consider n interrelated markets of n commodities or services which are supplied from the same or related sources and that are demanded by the same or related industries. We assume that the changes in prices of commodities or services are governed by a stochastic equation of the Itô type,

$$dx = A(x)xdt + B(x)xdz, \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the vector of prices, and $z(t) \in \mathcal{R}$ is a random variable which is a normalized Wiener process with

$$\mathcal{E}\{[z(t_1) - z(t_2)]^2\} = |t_1 - t_2|, \tag{2}$$

where $\mathcal{E}$ denotes expectation. In the above stochastic equation (1), the $n \times n$ functional matrix $A:\mathcal{R}^n \rightarrow \mathcal{R}^{n^2}$ represents the deterministic interaction among prices, whereas the $n \times n$ functional diffusion matrix $B: \mathcal{R}^n \rightarrow \mathcal{R}^{n^2}$ describes the influence of stochastic disturbances on the price adjustment process. The functional matrices $A(x)$, $B(x)$ are sufficiently smooth so that the solution process $x(t; t_0, x_0)$ of (1) exists for all initial conditions $(t_0, x_0) \in \mathfrak{J} \times \mathcal{R}^n$ and all $t \in \mathfrak{J}_0$. The symbol $\mathfrak{J}$ represents the time interval $(\tau, +\infty)$, where τ is a number or $-\infty$, and $\mathfrak{J}_0$ is the semiinfinite time interval $[t_0, +\infty)$.

In the following analysis, we will consider stochastic stability of the equilibrium price $x^* = 0$ of the market model (1). We remember however, that $A(x^*)x^* = 0$, $B(x^*)x^* = 0$, for all $t \in \mathfrak{J}$, but $x^* \neq 0$. Then, we define the nonlinear functional matrices $\hat{A}(y)y \equiv A(y + x^*)(y + x^*)$, $\hat{B}(y)y \equiv B(y + x^*)(y + x^*)$ and consider the equation $dy = \hat{A}(y)ydt + \hat{B}(y)ydz$ instead of (1), where $y^* = 0$ represents the equilibrium price $x^* \neq 0$ under investigation.

In order to include the connective property of stochastic stability, we write the elements $a_{ij}(x)$, $b_{ij}(x)$ of the matrices $A(x)$, $B(x)$ as

$$a_{ij}(x) = \begin{cases} -\varphi_i(x) + e_{ii}\varphi_{ii}(x), & i = j \\ e_{ij}\varphi_{ij}(x), & i \neq j \end{cases}, \quad b_{ij}(x) = l_{ij}\psi_{ij}(x) \tag{3}$$

where φ_i, φ_{ij}, $\psi_{ij} \in C(\mathcal{R}^n)$.

The numbers $e_{ij}, l_{ij} \in [0, 1]$ in (3) are elements of the $n \times n$ interconnection matrices $E = (e_{ij})$, $L = (l_{ij})$. The element e_{ij} specifies the strength of the deterministic influence of the price x_j of the j-th market on the price x_i of the i-th market. The elements l_{ij} measure the influence of the stochastic disturbance z on the price x_i. The interconnection matrices were introduced in reference [4] to represent the structural changes in competitive equilibrium models. In particular, when k-th commodity or service "disappears" from the market, we set $e_{ik} = e_{kj} = 0$, for all $i, j = 1, 2, \ldots, n$. A less drastic case, when the price x_j does not influence the price x_i, is represented by $e_{ij} = 0$. In the following development, we also use the notion of the binary $n \times n$ fundamental interconnection matrices $\bar{E} = (\bar{e}_{ij})$ and $L = (\bar{l}_{ij})$ in which the elements are defined as

$$(4) \qquad \bar{e}_{ij} = \begin{cases} 1, & \varphi_{ij}(x) \not\equiv 0 \\ 0, & \varphi_{ij}(x) \equiv 0 \end{cases}, \qquad \bar{l}_{ij} = \begin{cases} 1, & \psi_{ij}(x) \not\equiv 0 \\ 0, & \psi_{ij}(x) \equiv 0. \end{cases}$$

Therefore, the matrix pair $(\bar{E}, \bar{L})$ represent the basic structure of the market system (1), and any pair of interconnection matrices (E, L) can be generated from the pair $(\bar{E}, \bar{L})$ by replacing the unit elements of $(\bar{E}, \bar{L})$ by corresponding elements e_{ij}, l_{ij} of (E, L). This fact is denoted by $(E, L \in (\bar{E}, \bar{L})$.

3. Stochastic Connective Stability

Stochastic stability of the equilibrium price $x^* = 0$ of the market system (1) is a convergence of the solution process $x(t; t_0, x_0)$ starting at time t_0 and the initial price $x_0 = x(t_0)$ towards the equilibrium. The convergence is measured in terms of "stochastic closeness" (e.g., in the mean, almost sure, in probability, etc.), which, in turn, generates various notions of stochastic stability. In case of the nonlinear model under consideration, we are interested in establishing conditions for globally exponential and connective stability in the mean [3]. That is, conditions under which the expected value of the distance between the price adjustment process $x(t; t_0, x_0)$ and the equilibrium price $x^* = 0$, which is denoted by $\mathcal{E}\{\|x(t; t_0, x_0)\|\}$, tends to zero exponentially as time increases for all initial data $(t_0, x_0) \in \mathfrak{J} \times \mathcal{R}^n$ and all interconnection matrices $(E, L) \in (\bar{E}, \bar{L})$. More precisely we have the following:

Definition 1. The equilibrium price $x^* = 0$ of the market system (1) is connectively and globally exponentially stable in the mean if and only if there exist two positive numbers Π and π independent of initial conditions (t_0, x_0) such that

$$(5) \qquad \mathcal{E}\{\|x(t; t_0, x_0)\|\} \leq \Pi \|x_0\| \exp[-\pi(t - t_0], \ \forall t \in \mathfrak{J}_0$$

for all $(t_0, x_0) \in \mathfrak{J} \times \mathcal{R}^n$ and all $(E, L) \in (\bar{E}, \bar{L})$.

To derive sufficient conditions for the kind of stability expressed by Definition 1, we impose certain bounds on the coefficients of the functional matrices $A(x)$ and $B(x)$. We assume that the functions in (3) satisfy the constraints

$$\varphi_i(x) \geq \alpha_i, \; |\varphi_{ij}(x)| \leq \alpha_{ij}, \; |\psi_{ij}(x)| \leq \beta_{ij}, \; \forall x \in \mathbb{R}^n \tag{6}$$

for some numbers $\alpha_i > \alpha_{ii} \geq 0$, $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$. Since $\alpha_i - \alpha_{ii} > 0$, from (3) and (6) it follows that the price on each individual market when decoupled from the rest of the other markets, obeys the law of supply and demand. From (3) and (6), it follows further that no sign restrictions are placed on the interactions among markets, so that the system (1) may represent a "mixed" market system with substitute-complement commodities or services.

We also need to define the constant $n \times n$ matrices $\bar{A} = (\bar{a}_{ij})$ and $\bar{B} = (\bar{b}_{ij})$ as

$$\bar{a}_{ij} = \begin{cases} -\alpha_i + \bar{e}_{ii}\alpha_{ii}, & i = j \\ \bar{e}_{ij}\alpha_{ij}, & i \neq j \end{cases}, \quad \bar{b}_{ij} = \bar{l}_{ij}\beta_{ij} \sum_{k=1}^{n} \bar{l}_{ik}\beta_{ik}. \tag{7}$$

The conditions for connective stochastic stability of the market system will be expressed in terms of the pair $(\bar{E}, \bar{L})$, but will be valid for all pairs $(E, L) \in (\bar{E}, \bar{L})$. This is important in that we show stochastic stability of a class of nonlinear market systems by proving stochastic stability of one member of that class which is a linear system described by the constant matrices $(\bar{A}, \bar{B})$ corresponding to $(\bar{E}, \bar{L})$. More importantly, this shows a wide tolerance of stable market systems to nonlinear and structural perturbations and, therefore, inherent robustness of general equilibrium models.

We say that an $n \times n$ matrix $M = (m_{ij})$ is a *negative dominant diagonal matrix* if

$$m_{jj} < 0, \quad |m_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |m_{ij}|, \quad j = 1, 2, \ldots, n. \tag{8}$$

Now, we define a matrix $\bar{M}$ as

$$\bar{M} = \bar{A} + \bar{A}^T + \bar{B}, \tag{9}$$

and prove the following:

Theorem 1. The equilibrium price $x^* = 0$ of the market system (1) is connectively and globally exponentially stable in the mean if the $n \times n$ matrix M defined by (9) is a negative dominant diagonal matrix.

Proof. Let us consider a decrescent, positive definite, and radially unbounded function $\nu: \mathcal{R}^n \to \mathcal{R}_+$,

(10)
$$\nu(x) = \sum_{i=1}^{n} x_i^2$$

as a candidate for Liapunov's function for the system (1). Using Itô's calculus, we examine the expression [7],

(11)
$$\mathcal{L}\nu(x) = \frac{\partial \nu(x)}{\partial x} A(x)x + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \nu(x)}{\partial x_i \partial x_j} s_{ij}(x),$$

where $\partial\nu/\partial x = (\partial\nu/\partial x_1, \partial\nu/\partial x_2, \ldots, \partial\nu/\partial x_n)$ is the gradient of $\nu(x)$, $\partial^2\nu/\partial x_i\partial x_j$ is the (i, j)-th element of the Hessian matrix related to $\nu(x)$, and s_{ij}'s are the elements of the $n \times n$ matrix $S = B(x)xx^T B^T(x)$. To conclude stability of the equilibrium, we observe that $\nu(x)$ is a positive definite function, and demonstrate that $\mathcal{L}\nu(x)$ is negative definite.

Let us calculate $\mathcal{L}\nu(x)$ as,

(12)
$$\begin{aligned}\mathcal{L}\nu(x) &= \sum_{i=1}^{n} 2x_i\left(\sum_{j=1}^{n} a_{ij}x_j\right) + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}x_j\right)^2 \\ &= \sum_{j=1}^{n} 2a_{jj}x_j^2 + \sum_{j=1}^{n} x_j \sum_{\substack{i=1\\ i\neq j}}^{n} 2a_{ij}x_i + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} b_{ij}x_j\right)^2\end{aligned}$$

The negative dominant diagonal property of the matrix $\overline{M}$ is equivalent to

$$2\bar{a}_{jj} + \bar{b}_{jj} < 0,$$

(13)
$$2\bar{a}_{jj} + \bar{b}_{jj} + \sum_{\substack{i=1\\ i\neq j}}^{n} (\bar{a}_{ij} + \bar{a}_{ji} + \bar{b}_{ij}) \leq -\bar{\pi}, \quad j = 1, 2, \ldots, n$$

where $\bar{\pi}$ is a positive number. From (12) and (13), we get the inequality

(14)
$$\mathcal{L}\nu(x) \leq -\bar{\pi}\,\nu(x),$$
$$\forall(t, x) \in \mathfrak{J} \times \mathcal{R}^n, \quad \forall(E, L) \in (\bar{E}, \bar{L})$$

By applying the stochastic comparison principle of [8] to (14), we obtain

(15)
$$\mathcal{E}\{\nu[x(t; t_0, x_0)]\} \leq \nu(x_0)\exp[-\bar{\pi}(t - t_0)],$$
$$\forall t \in \mathfrak{J}_0, \forall(t_0, x_0) \in \mathfrak{J} \times \mathcal{R}^n, \forall(E, L) \in (\bar{E}, \bar{L}).$$

From (10) and (15), we get (5) with $\Pi = 1$, $\pi = (1/2)\bar{\pi}$. The proof of Theorem 1 is completed.

Now, several comments regarding the above result are in order. The criterion of stochastic stability is the diagonal dominance of system matrices, which has important economic interpretations similar to those for deterministic markets [2]. The supply and demand functions for any given commodity will be more affected by the changes in its own price than by the sum of changes in other prices and the random disturbance combined together.

The negative dominant diagonal property is more conservative than the usual quasidominant diagonal property used to establish exponential connective stability of deterministic systems [6]. This is a consequence of the fact that we are limited to a choice of Liapunov function (10) that is twice differentiable as required by (11).

We should also notice an important fact that follows from above Theorem 1 regarding the nonlinearities in our model (1) of multiple markets. The shapes of nonlinear functions in the interactions among the markets, are not specified in any way but by simple boundedness conditions (6). This is what we meant by saying that stable market systems are robust and can tolerate a wide range of nonlinearities in markets interactions. By the fact that stability is also connective, we establish the robustness of the market system to variations in the interaction structure. Finally, we notice that our stability criterion is insensitive to the signs of interactions among the markets, thus allowing for changing of commodity from a substitute to a complement to another commodity.

Going through the proof of Theorem 1, we see that it can be immediately extended to the nonstationary models with a vector stochastic perturbations [3],

$$dx = A(t,x)x\,dt + \sum_{k=1}^{m} B_k(t,x)x\,dz_k, \tag{16}$$

where $z(t) \in \mathcal{R}^m$ is normalized Wiener process with

$$\mathcal{E}\{[z(t_1) - z(t_2)][z(t_1) - z(t_2)]^T\} = |t_1 - t_2|I, \tag{17}$$

where I is the $m \times m$ identity matrix. The elements $a_{ij}(t,x)$ and $b_{ij}^k(t,x)$ of the matrices $A(t,x)$, $B_k(t,x)$ satisfy conditions similar to those of (6). Specifically, the conditions on $b_{ij}^k(t,x)$ imply that there exist positive numbers $\bar{b}_{ij}$ so that

$$\sum_{k=1}^{m}\left[\sum_{j=1}^{n} b_{ij}^k(t,x)x_j\right]^2 \leq \sum_{j=1}^{n} \bar{b}_{ij}x_j^2. \tag{18}$$

By using the same Liapunov function $\nu(x)$ defined in (10) and following the proof of Theorem 1, we arrive at inequality (14). Then, diagonal dominance of the matrix $\overline{M} = \overline{A} + \overline{A}^T + \overline{B}$, where $\overline{B} = (\bar{b}_{ij})$ is an $n \times n$ matrix with elements defined in (18), implies globally exponential and connective stability of the equilibrium price $x^* = 0$ of the market model (16).

4. INSTABILITY

Since our conditions for stochastic stability are only sufficient, we are motivated to undertake instability considerations of competitive equilibrium models in randomly varying environment. Such considerations will provide sufficient conditions for stochastic instability which can be used to furnish a necessity part of our stability conditions, and expose possible causes of instability in the price adjustment mechanism of multiple markets [9].

Two distinct sources of instability will be considered: The presence of strongly inferior goods and the effect of random disturbances. Two sets of different instability conditions will be obtained depending on which one of the destabilizing effects is prevailing in the market.

The instability conditions we are about to derive, is established for the fundamental interconnection matrices $\bar{E}, \bar{L}$, but are valid for all interconnection matrices. Therefore, we introduce the following:

Definition 2. The equilibrium price $x^* = 0$ of the market system (1) is completely connectively unstable in the mean if and only if it is unstable in the mean for all interconnection matrices $(E, L) \in (\bar{E}, \bar{L})$.

We show that if one of the commodities is inferior, that is, an increase in price of that commodity (holding other prices constant) leads to an increase in the quantity demanded of that commodity (Giffen paradox [2]), then instability can be established. To this end, we prove the following:

Theorem 2. The equilibrium price $x^* = 0$ of the market system (1) is completely connectively unstable in the mean if for some $i = 1, 2, \ldots, n$, there exist numbers $\alpha_i > 0$, $\alpha_{ij} \geq 0$, $j = 1, 2, \ldots, n$, such that the coefficients $a_{ij}(x)$ of the $n \times n$ matrix $A(x)$ defined by (3), satisfy the conditions

$$\varphi_i(x) \geq \alpha_i, \varphi_{ij}(x)x_i x_j \leq -\alpha_{ij}x_j^2, \forall x \in \mathcal{R}^n. \tag{19}$$

Proof. Let us consider the following function

$$v_i(x_i) = x_i^2, \tag{20}$$

as a candidate for the Liapunov function of the i-th market. Computing $\mathcal{L}v_i(x_i)$ with respect to (1), we get

$$\begin{aligned} \mathcal{L}v_i(x_i) &= 2\varphi_i(x)x_i^2 - 2\sum_{j=1}^{n} e_{ij}\varphi_{ij}(x)x_i x_j + \left[\sum_{j=1}^{n} b_{ij}(x)x_j\right]^2 \\ &\geq 2\alpha_i v_i(x_i) + 2\sum_{j=1}^{n} e_{ij}\alpha_{ij}v_j(x_j) \\ &\geq 2\alpha_i v_i(x_i), \forall x \in \mathcal{R}^n, \forall (E, L) \in (\bar{E}, \bar{L}). \end{aligned} \tag{21}$$

By following the same argument as in the proof of Theorem 1, we obtain from (21),

$$\mathcal{E}\{|x_i(t; t_0, x_0)|\} \geq |x_{i0}| \exp[\alpha_i(t - t_0)], \forall t \in \mathfrak{J}_0 \tag{22}$$

where $x_{i0} = x_i(t_0) \neq 0$. This proves Theorem 2.

If we dispose of connective property of instability and ask simply that the system (1) is unstable for the fundamental interconnection matrices $\bar{E}, \bar{L}$, then we can relax further the conditions of Theorem 2. We assume that for some $i = 1, 2, \ldots, n$, the functions in (3) satisfy the following constraints

$$\varphi_i(x) \leq \alpha_i, 2\sum_{j=1}^{n} \varphi_{ij}(x)x_ix_j \geq \sum_{j=1}^{n} \alpha_{ij}(x_i^2 + x_j^2) \tag{23}$$

$$\left[\sum_{j=1}^{n} \psi_{ij}(x)x_j\right]^2 \geq \sum_{j=1}^{n} \beta_{ij}x_j^2, \forall x \in \mathfrak{R}^n$$

where $\alpha_{ij} \geq 0$, $\alpha_i > \alpha_{ii}$, $\beta_{ij} \geq 0$, and $\beta_{ii} > 2(\alpha_i - \alpha_{ii})$.

In Theorem 2, we required that at least one commodity is strongly inferior. Conditions $\beta_{ii} > 2(\alpha_i - \alpha_{ii})$ is weaker in that it does not require the presence of inferior goods. Instability if present, is introduced solely by the random disturbance. This shows that under the constraints (22), we have to have a certain degree of stability of the deterministic part of the system, in order for the system to absorb a limited amount of random fluctuations. We finally prove the following:

Theorem 3. The equilibrium price $x^* = 0$ of the market system (1) is unstable in the mean if for some $i = 1, 2, \ldots, n$, there exist numbers $\alpha_i > \alpha_{ii} \geq 0$, $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$, $j = 1, 2, \ldots, n$, such that the coefficients $a_{ij}(x)$, $b_{ij}(x)$ of the $n \times n$ matrices $A(x)$, $B(x)$ defined by (3), satisfy the conditions (23).

Proof. We start again with the Liapunov function of (20) and compute

$$\mathfrak{L}\, v_i(x_i) = -2\varphi_i(x)x_i^2 + 2\sum_{j=1}^{n} \varphi_{ij}(x)x_ix_j + \left[\sum_{j=1}^{n} \psi_{ij}(x)x_j\right]^2 \tag{24}$$

$$\geq [-2(\alpha_i - \alpha_{ii}) + \beta_{ii}]v_i(x_i), \quad \forall\, x \in \mathfrak{R}^n.$$

Using the same arguments as in the proof of Theorem 2, we obtain inequality (22) where α_i is replaced by $[-(\alpha_i - \alpha_{ii}) + (1/2)\beta_{ii}]$. In view of condition $\beta_{ii} > 2(\alpha_i - \alpha_{ii})$, Theorem 3 is established.

5. Conclusion

The usual dominant diagonal property is shown to be a suitable means to establish a trade-off between the degree of stability and the size of the random fluctuations of the supply and demand functions which can be absorbed by stable competitive equilibrium models. Furthermore, the diagonal dominance property was shown to imply exponential stability in the mean of the equilibrium price under structural perturbations.

The results obtained in this paper can be improved in a number of different ways. It is possible to relax the constraints on the interactions among the individual markets and establish the weaker global asymptotic property of stochastic stability [3]. By combining the results obtained in references [3] and [6], we can use the decomposition-aggregation analysis and consider multiple markets of composite commodities or services. Finally, it would be of interest to try various other kinds of stochastic stability [1] in the context of diagonal dominance, which could open up a possibility to include price expectations in our nonlinear equilibrium models.

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