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A FIXED POINT APPROACH TO MULTIAGENT ADAPTIVE CONTROL

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This article classifies and analyzes various approaches to the multiagent problem. Static, dynamic, and adaptive situations are considered, as is the possibility of a third party to facilitate the sending and receiving of signals. The objectives of the two agents might be in opposition, the same, other (but known), or other (but unknown). Three of the possible combinations of situations and objectives generally not discussed in the literature are given special attention here.

1. INTRODUCTION

The multiagent problem, in which an outcome is determined by the joint actions of two or more agents, is one that arises in a variety of contexts, including many economic and political situations. Because the problem is a complex one and because it has been studied by analysts in different disciplines using different approaches, many solution concepts have been proposed, some of which represent the same concept under different names. The purpose of this paper is to classify and analyze these various approaches. Only the two agent problem is treated in full recognition of the fact that, while this problem exhibits salient aspects of the more general multiagent problem it does not exhibit all aspects, e.g. coalition formation. The difficulty in going from the two agent case to the three or more agent case should not be underestimated.

2. A CLASSIFICATION OF MULTIAGENT SITUATIONS

A classification of various types of multiagent situations is presented in Table 1. The rows represent alternative treatments of time and information available to the agents. Thus the problem might be *static*, defined at a point in time, or *dynamic*, evolving over time. The *adaptive* case is one in which the problem evolves over time, but in which the agents learn from the past history of the system, with each communicating to and receiving signals from the other agents via the actions taken. Finally there is the *third party* case in which such a party, who is formally not one of the agents, facilitates the sending and receiving of signals via direct communication.

The columns of the table represent alternative assumptions concerning each agent's awareness of the objective(s) of the other agents. The *opposite* case is that in which the objectives of the agents are diametrically

TABLE 1

CLASSIFICATION OF MULTIAGENT SITUATIONS

Objectives / Time; Information	Opposite	Same	Other, But Known	Other, But Unknown
Static	Classical game theory	Classical team theory	Classical duopoly theory	Nash equilibrium
Dynamic	Differential games		Case A	
Adaptive		Learning by doing		Case B
Third party			Nash bargaining solution	Case C

opposed, the constant (or zero) sum case. The *same* case is that in which the objectives of the agents are identical, the constant difference case. The *other but known* case is where there may be elements of conflict, as in the "opposite" case, and elements of cooperation, as in the "same" case, but the objectives of every agent is known to all agents. The *other but unknown* case is that in which the objectives of other agents are unknown.

While not every case in Table 1 has been the subject of analysis, several have.[1] Thus classical game theory, e.g. two-person zero-sum games, would represent an example of the static/opposite case, classical team theory is an example of the static/same case, while classical duopoly theory exemplifies the static/other but known case.[2] The Nash equilibrium represents a solution concept for the static/other but unknown case. The dynamic case is illustrated in the opposite objectives case by differential games, while the adaptive case is illustrated in the same objectives case by learning by doing. The Nash bargaining game solution illustrates the third party/other but known case since it gives a third party referee or mediator a rule for allocating between the agents.[3]

The cases labelled A, B, and C in Table 1 are those of particular interest here. In these cases, particularly in B and C, where the objectives of the other agents are unknown, the agents typically formulate certain paradigms as a way of structuring information explaining the behavior and objectives of the other agents[4], then formulate theories as to the behavior of the other agents in the context of the paradigm. In the dynamic case the paradigm itself evolves over time, while in the adaptive case it evolves in response to actual behavior of the other agents and in the third-party case it is influenced by the third-party intervention.

3. Examples of Multiagent Situations

Many examples can be given from both economics and international relations of the cases of interest here—A, B, and C. In economics the problem of duopoly is, in reality, one in which agents are unaware of objectives of other agents, one in which they formulate paradigms as to behavior and objectives of the other firms (e.g. the leader-follower paradigm) and one in which they learn from both past behavior and third parties. Another example from economics is monetary and fiscal policy, involving conflict over the proper course for stabilization and control of the economy. Each agent has its own paradigm—monetarist or Keynesian—to explain the economy and the proper role for policy, with the

[1]See Intriligator (1971) for a treatment of several of these cases.

[2]On team theory see Marshak and Radner (1972).

[3]See Brito, Buoncristiani, and Intriligator (1977).

[4]Kuhn (1970) discusses such paradigms in science, which may be considered a multiagent situation in which one agent is "nature."

paradigms evolving over time so as to be consistent with the evidence. Yet a third example from economics is labor management negotiations, where, again, each side develops a paradigm. In this example a third party, such as an arbitrator or mediator, can play a significant role in modifying the paradigms used by both sides.

Other examples of multiagent situations arise in the area of international relations. One is an armament race, where each side formulates certain paradigms as to the objectives and behavior of the other, e.g. as to the question of superiority vs. sufficiency in levels of armaments.[5] Another example is overall postwar East-West relations, where each side has developed its own paradigm, e.g. the "lessons of Munich" leading the West to a policy of containment and the German invasion of Russia leading the East to large standing armies and the objective of a buffer zone in Europe. A third example is regional conflicts, such as the Middle East, where third parties have played highly influential roles via "shuttle diplomacy," international conferences, etc.

4. A Formal Model of the Two Agent Situation

In the formal model of the two agent situation, the agents, indexed by $i = 1, 2$, each seek to maximize the objective functional

$$(1) \qquad J_i = \int_{\tau}^{T} W_i[x(t), u_i(t)]dt + G_i[x(T)], \quad i = 1, 2,$$

by choice of $u_i(t)$ over the period from τ to T, where τ can assume all values between 0 and T. They are therefore continually revising their plans at each instant τ in $[0, T]$, maximizing J_i relative to the initial point τ. The resulting plan for each agent is called a "rolling plan" because of its continual revision in the light of new information. Rather than commit themselves in advance to a specific course of action the agents can revise their plans at each instant of time τ. It will be assumed here that $W_i(\cdot)$ and $G_i(\cdot)$ are twice differentiable concave functions.

The system as described by the vector of state variables $x \in X$ evolves over time according to the differential equation system

$$(2) \qquad \dot{x} = f[x(t), u_1(t), u_2(t)], \quad \tau \leq t \leq T,$$

defined over the finite interval $[0, T]$. Here $f(\cdot)$ is a concave and twice differentiable function and $u_i(t)$, under control of agent i, is an element of Ω^i, a compact subset of Euclidean r_i space.

The historical evolution of $u_i(t)$ in the interval $[0, \tau]$ is given by the

[5]See Intriligator and Brito (1976a, 1976b).

trajectory

(3) $$v_i \in V_i(\tau),$$

where $V_i(\tau)$ is the space of all continuous functions $u_i(t)$ defined over $[0, \tau]$, and where, at any time, $u_i(t) \in \Omega^i$. The planned evolution of $u_i(t)$ in the interval $[\tau, T]$ is given by the trajectory

(4) $$u_i \in U_i(\tau),$$

where $U_i(\tau)$ is the space of all continuous functions $u_i(t)$ defined over $[\tau, T]$, and where, at any time t, $u_i(t) \in \Omega^i$. The objective of agent i is then to select $u_i \in U_i(\tau)$, using information contained in $V_1(\tau) \times V_2(\tau)$, so as to maximize J_i in (1) subject to the differential equation system in (2).

5. The Dynamic Situation in Which the Other Agent's Strategy is Known (Case A).

Consider now Case A in Table 1 —the dynamic situation in which the other agent's strategy is known.

PROPOSITION 1. If the i^{th} agent knows the strategy of the j^{th} agent on the interval $(\tau, T]$ then there exists a continuous function F_i such that the i^{th} agent's optimal strategy is given by

(5) $$F_i: U_j \times X \rightarrow U_i$$

the mapping F_i can be interpreted as a reaction curve in function space.

Proof. The existence of the F_i mappings follows from the generalized Weierstrass theorem, assuming $W_i(\cdot)$ and $f(\cdot)$ are sufficiently smooth so that J_i is a continuous functional of the control trajectory u_i.[6] To show that F_i is continuous use the maximum principle and consider the arbitrary variation of the j^{th} agents strategy on $[\tau, T]$ u_j^* defined by

(6) $$u_j(t) = u_j^*(t) + \epsilon\eta^*(t).$$

Let $q^*(t)$ be the costate variable associated with (2). Then $u_i^*(t)$ is the solution to

(7) $$\frac{\partial W_i[x(t), u_i^*(t)]}{\partial u_i(t)} + q^*(t)\frac{\partial f[x(t), u_i^*(t), u_j^*(t) + \epsilon\eta(t)]}{\partial u_i(t)} = 0.$$

Clearly for all t in $[\tau, T]$ $u_i^*(t)$ is a continuous function of t, implying that F_i is a continuous function of u_j.

Given the F_i mappings there exist fixed point strategies via

[6]For the generalized Weierstrass theorem see Intriligator (1971).

PROPOSITION 2. There exist fixed point strategies u_i^*, u_j^* such that

(8) $$u_i^* = F_i(u_j^*), \qquad u_j^* = F_j(u_i^*)$$

and the pair of fixed point strategies (u_i^*, u_j^*) constitute a Nash equilibrium

(9) $$J_i(u_i^*, u_j^*) \geq J_i(u_i, u_j^*).$$

Proof. The set $U_i(\tau)$ is compact since $u_i(t)$ is uniformly bounded and $U_i(\tau)$ is a closed subset of the set of all paths defined over $[\tau, T]$, which is a complete space. Thus the mapping

(10) $$(u_i^*, u_j^*) \to (F_i(u_j^*), F_j(u_i^*)),$$

which exists since the mapping is a continuous one from a compact set into itself, represents a mapping from a compact set into itself. Such a mapping always has a fixed point. The fixed point is a Nash equilibrium by construction of the F_i.

This fixed point pair of strategies obtained here requires that each agent know the other agent's strategy (or objective function), a requirement which comprises more structure than is generally available. Without this information there is no guarantee that the Nash equilibrium will be attained—even asymptotically. It thus becomes important to modify the problem by not assuming knowledge of the other's strategy, leading to Case B in Table 1.

6. The Adaptive Situation in Which the Other Agent's Strategy is Not Known (Case B).

In Case B in Table 1 the agents do not know the strategy (or objective) of the other agent, but they obtain some information concerning the other agent from their knowledge of past strategies up to time τ, given by v_i and v_j. Specifically, it will be assumed that agent i uses the information contained in $V_i(\tau) \times V_j(\tau)$ to predict the response of agent j to its own action.[7] Thus it is assumed that there exist continuous functions P_i

(11) $$P_i : V_i(\tau) \times V_j(\tau) \times U_i(\tau) \to U_j(\tau)$$

mapping past strategies of both agents and the feasible set of current strategies (at time τ) into a prediction by agent i of the strategy to be employed by agent j. This prediction uses all information available on past actions up to time τ. The i^{th} agent's choice of strategy u_i on the interval

[7]This type of information is not taken into account in the classical models of duopolistic interaction that treat a dynamic process using a myopic framework in which information as to past behavior plays no role.

$[\tau, T]$ is then given by the composite mapping

(12) $$F_i \cdot P_i: V_i(\tau) \times V_j(\tau) \times U_i(\tau) \rightarrow U_i(\tau).$$

In this adaptive case the composite mappings yield fixed points strategies via

PROPOSITION 3. For all v_i, v_j in $V_i(\tau) \times V_j(\tau)$ there exist strategies u_i^{**}, u_j^{**} such that

(13) $$u_i^{**} = F_i \cdot P_i(v_i, v_j, u_i^{**}), \quad u_j^{**} = F_j \cdot P_j(v_i, v_j, u_j^{**}).$$

Proof. For fixed v_i, v_j the composite mapping $F_i \cdot P_i$ can be considered a mapping of U_i, which is compact, as noted above, into itself. $F_i \cdot P_i$ is continuous, however, so there exists a fixed point u_i^{**}.

It should be noted that the Nash equilibrium (or "rational expectations" solution) will be reached only if each of the agents correctly predicts the strategy of the other

(14) $$u_j^{**} = P_i(v_i, v_j, u_i^{**}).$$

This prediction is unlikely, however, so the dynamics of the process involves a study of the evolution of the rolling plan.[8]

The mapping given by (12) suggests that in the adaptive case the behavior of the two agents is coupled only by the information contained in $V_i(\tau) \times V_j(\tau)$. Thus, the obvious question occurs about the existence of "self-fulfilling equilibria", that is, equilibria in which the behavior of agents is such that their expectations are fulfilled. An example of such an equilibrium would be if every agent expected inflation and thus tried to avoid monetary assets. The consequences of such behavior may be such that there would indeed be inflation. In the next proposition it will be demonstrated that all self-fulfilling equilibria must be Nash equilibria and thus their existence is independent of how agents process information. This is not to suggest that research into the nature of such processes is not interesting, but it should be noted that such research addresses phenomena that inherently concern disequilibria. Although the proposition addresses the case of two agents it can be applied to markets with two classes of identical agents.

PROPOSITION 4. All equilibria that are self-fulfilling are Nash equilibria.

Proof. Define π_i as the projection of an element in $V_i(\tau) \times V_j(\tau) \times U_i(\tau)$ to $U_i(\tau)$

(15) $$\pi_i: V_i(\tau) \times V_j(\tau) \times U_i(\tau) \rightarrow U_i(\tau).$$

[8]See Brito (1972) for an example of such a process for the case of the dynamics of an arms race. There it was shown that the stability of the process depends on the agents' reactions to information about derivatives.

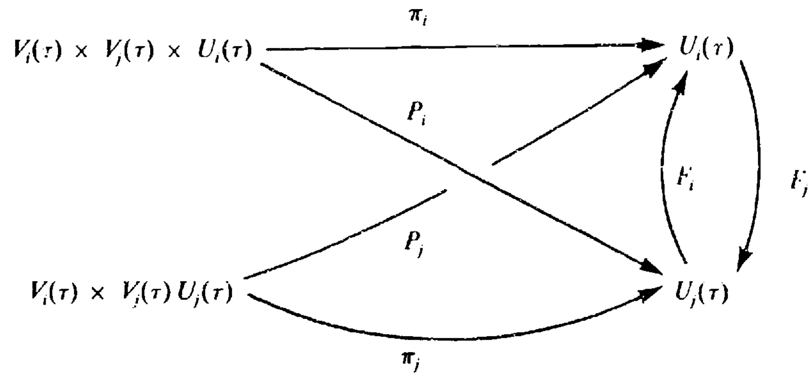

Figure 1 Interactions between the agents

Figure 1 depicts the interactions between the two agents.

Suppose that the information available to the two agents at time τ is $\bar{v}_i(\tau)$ and $\bar{v}_j(t)$ and that $\bar{u}_i(\tau)$ and $\bar{u}_j(\tau)$ are the equilibrium strategies chosen. Then if the equilibrium is self-fulfilling,

$$\bar{u}_j(\tau) = P_i[\bar{v}_i(\tau), \bar{v}_j(\tau), \bar{u}_i(\tau)] \tag{16}$$

and

$$\bar{u}_j(\tau) = F_j\{\pi_i[\bar{v}_i(\tau), \bar{v}_j(\tau), \bar{u}_i(\tau)]\}. \tag{17}$$

Equating these implies that

$$P_i[\bar{v}_i(\tau), \bar{v}_j(\tau), \bar{u}_i(\tau)] = F_j\{\pi[\bar{v}_i(\tau), \bar{v}_j(\tau), \bar{u}_i(\tau)]\} \tag{18}$$

which, of course, represents Nash equilibria.

7. The Third Party Situation in Which the Other Agent's Strategy is Not Known (Case C).

In Case C in Table 1 the agents do not know the strategy of the other agent, but they obtain some information concerning the other agent from third parties, such as mediators. In this case historical information is supplemented by information made available by a third party intermediary concerning the strategies or objective function of the other agent. This case is important because in the absence of such third party information it is frequently difficult or impossible to communicate a complete strategy. Rather, the agents can only communicate through their actions, which may lead to false signals or undesirable outcomes. Furthermore, in the absence of third party information the agents may very well choose to vary their initial strategies widely in order to map out the other agent's response curve, possibly leading to incorrect inferences or even disastrous outcomes. With third party information the need for such probing is reduced and, if a fixed point such as u_i^{**} and u_j^{**} in (13) is found, it will be stable. Examples of such third party interaction include mediators in labor-management negotiations and shuttle diplomacy in the Middle East.

8. Conclusion

This paper has classified multiagent situations and treated three important cases that have generally not been discussed in the literature. The first, Case A, is one in which the agents each know the strategies or objective function of the other, leading, in a dynamic framework, to fixed point strategies, which constitute a Nash equilibrium. The second, Case C, is one in which the agents do not know the strategy or objective functions of the other but can learn from the past history of the system. In this adaptive case each agent may formulate predictions of the strategy to be taken by the other, leading again to fixed point strategies. By contrast to the previous case, however, the fixed point in this case generally does not constitute a Nash equilibrium. However, if it is a self-fulfilling equilibrium it must be a Nash equilibrium. The final case, C, is one intermediate between the previous two, in which historical information is supplemented by information made available by a third party intermediary. Such information is generally helpful in reaching the fixed point strategy and in avoiding possibly disastrous outcomes engendered when the agents attempt to probe the system for information.

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