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# ON THE THEORY OF INDUSTRIAL PRICE MEASUREMENT: OUTPUT PRICE INDEXES*

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*Two output price indexes are proposed to measure price change. Both include the effects of substitution caused by price change and exhibit desireable properties. The index numbers are developed from the theory of the firm, rather than from the viewpoint of how price changes affect consumers. The properties of output indexes, such as the WPI, are discussed in detail.*

Recent dramatic increases in the price level have stimulated discussion of how changes in the price level are measured. One conclusion of such discussions has been that the Wholesale Price Index (WPI) as currently constructed does not provide the best measure of price change.[1] This paper outlines a conceptual framework for constructing measures of price change in the universe presently covered by the WPI. This will be an exercise in the construction of economic index numbers. We feel that it is important to base index numbers upon well understood economic theory, and that several problems with the current WPI can be traced to the fact that it has no such basis.

A vast majority of the existing theory concerning price indexes concentrates upon measures of price change as they effect consumers.[2] The indexes introduced here are developed from the viewpoint of the firm. For a firm, price change comes in two forms, changes in input prices and changes in output prices. In this paper we focus on output price changes. Our objective is to find output price indexes which are consistent with the traditional theory of the firm and which exhibit properties that can reasonably be expected of price indexes.

It is important to clarify two points concerning our objectives. First, we concentrate on the construction of price indexes rather than quantity indexes. It is natural to deflate expenditure indexes by price indexes to obtain quantity indexes, but unfortunately, in most circumstances, a price index with desirable properties does not yield a well behaved quantity in-

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[1] As an example see William Nordhaus and John Shoven (1974).

[2] Early work is surveyed by Ragnar Frisch (1936). For more recent work see Paul Samuelson (1947), Robert Pollak (1971), Franklin Fisher and Karl Shell (Essay I) (1972) and Paul Samuelson and S. Swamy (1974).

dex. In this paper the properties of the quantity indexes implied by our price indexes will be a secondary concern. Secondly, we are not attempting to construct either a measure of overall price change or solely a measure of price change for a single firm. With varying degrees of accuracy, price indexes can be constructed for firms, industries or economies using our framework. The indexes should be considered independent of the level of aggregation.

The paper starts by introducing a model of firm behavior based upon competitive assumptions and considers building price indexes based upon profits. The second section focuses upon subindexes of the profits price index by defining three output price indexes. In the remainder of the paper we consider properties of these output price indexes and find that two of the indexes are acceptable measures of price change.

## I. The Basic Model

Our price indexes are based upon the traditional model of a perfectly competitive firm. Constructing output price indexes for a single product firm is trivial, thus we consider the firm (industry, economy) to have several outputs. The technology of the firm is summarized by its production function,

$$F(q_1, \ldots, q_m, x_1, \ldots, x_n) = 0 \tag{1}$$

where $q_1, \ldots, q_m$ represent the firm's $m$ outputs and $x_1, \ldots, x_n$ represent the $n$ inputs. We assume that $F$ is a twice differentiable function with the following neoclassical properties,

$$\frac{\partial q_i}{\partial x_j} > 0 \text{ and } \frac{\partial^2 q_i}{\partial x j^2} < 0 \, (i = 1, \ldots m; j = 1, \ldots, n)$$

The firm maximizes profits facing market determined prices for its inputs and outputs. Profits are given by,

$$\sum_{i=1}^{m} p_i q_i - \sum_{j=1}^{n} w_j x_j, \tag{2}$$

where $p_i$ represents the price of the $i_{th}$ output and $w_j$ the price of the $j^{th}$ input. We assume that all capital is rented by the firm.

Throughout our analysis we assume the existence of a unique profit maximizing vector of inputs and outputs which is non-negative and yields a positive profit. Specifically, this eliminates production functions which yield constant returns to scale, and also those which have flat portions on any production possibility frontier. Making these simplications greatly facilitates the analysis.

We now consider formulating price comparisons between two different firms or for the same firm at two points in time. Before embarking upon the construction of price indexes we would like to clearly establish the terminology to be used. Prices will be referred to as either reference situation or comparison situation prices. The term "base" is reserved to refer to variables related to the technology of the firm.

Initially we formulate price indexes for our firm assuming that we can observe prices, quantities and technology. The basic technique of price index construction will be illustrated by the example of the price index for profit. Define the profit function $\pi(p^a, w^b, F^c)$ to equal the maximum profit given prices $p^a$ and $w^b$ and technology $F^c$. A price index for profit is formed by comparing profits given differing prices. The price index for profits comparing reference situation and comparison situation prices using base situation technology is given by:

$$(3) \qquad \frac{\pi(p^c, w^c, F^b)}{\pi(p^r, w^r, F^b)}$$

Commonly the production function of the reference situation or comparison situation is used as the base technology. This choice is essentially arbitrary.[3] The properties of price indexes are very similar whatever base technology is chosen, and hereafter, unless otherwise noted, we will, for convenience sake, only discuss price indexes based upon the reference situation's technology.

The price index for profits given by (3) allows variable quantity weights as the firm substitutes inputs and outputs in response to price change. This distinguishes it from indexes such as the the WPI which use fixed-weight formulations. Commonly, indexes which allow such substitutions are called "true" price indexes, for example the "true cost of living index." We will avoid this terminology for, as the analysis of the next section demonstrates, the uniqueness implied by true is not present for output price indexes.

Our motivation for constructing separate output and input price indexes results from the fact that the price index for profit introduced above obscures important information about the details of price change. A change in this index has an ambiguous interpretation. For example, an increase in this index might correspond to any one of three situations—an increase in output prices and a decrease in input prices, a decrease in both output and input prices or an increase in both output and input prices. This type of difficulty can best be remedied by considering indexes which are subindexes of this price index for profits.[4] The remainder of this paper will focus on one such subindex, the output price index.

[3] Franklin Fisher and Karl Shell (Essay II) (1972) would take exception to this statement. See pages 56–57.

[4] For a discussion of subindexes see Robert Pollak (1975).

## II. Output Price Indexes Definitions

The example of the price index for profit demonstrates the techniques we use to isolate a measure of price change for output. For output price indexes we deal with revenue, the market valuation of output, rather than profit, and input prices as well as technology must be held constant. We present three alternative output price indexes.

### *A. $I^{\bar{w}}$ The Fixed Input Price Output Price Index (FIP)*

The definition of the FIP index is quite straight forward. This index is formed using a revenue function from the ordinary profit maximizing model of a firm.

*Definition 1.* The revenue function $R^*(p^a, w^b, F^c)$ is defined as:

$$R^*(p^a, w^b, F^c) = \sum_{i=1}^{m} p_i^a q_i^*$$

where the $q_i^*$ are found by solving:

$$\text{Maximize: } \sum_{i=1}^{m} p_i^a q_i - \sum_{j=1}^{n} w_j^b x_j$$

$$\text{Subject to: } F^c(q_i, x_j) = 0$$

The FIP output price index is defined as:

$$I^{\bar{w}}(p^r, p^c, w^r, F^r) \equiv \frac{R^*(p^c, w^r, F^r)}{R^*(p^r, w^r, F^r)}$$

This output price index holds both technology and input prices constant at their reference situation levels, but does not restrict substitution between either inputs or outputs. Alternative indexes are defined by restricting the substitution possibilities of the firm.

### *B. $I^{\bar{w},\bar{C}}$ The Fixed Cost Output Price Index (FC)*

This index is based upon a revenue function defined as follows.

*Definition 2.* The revenue function $\tilde{R}(p^a, w^b, C^c, F^d)$ is defined as:

$$\tilde{R}(p^a, w^b, C^c, F^d) = \sum_{i=1}^{m} p_i^a q_i$$

where the $\tilde{q}_i$ are found by solving:

$$\text{Maximize: } \sum_{i=1}^{m} p_i^a \tilde{q}_i - \sum_{j=1}^{n} w_j^b x_j$$

$$\text{Subject to: } F^d(q_i, x_i) = 0$$

$$\sum_{j=1}^{n} w_j^b x_j = C^c.$$

This fixed cost revenue function is based upon restricting a firm to substitute inputs in such a way that the total expenditure on inputs is constant.

Using this revenue function the FC output price index is defined as:

$$I^{\bar{w},\bar{C}}(p^r, p^c, w^r, C^r, F^r) \equiv \frac{\tilde{R}(p^c, w^r, C^r, F^r)}{\tilde{R}(p^r, w^r, C^r, F^r)}$$

$I^{\bar{w},\bar{C}}$ isolates price change by holding fixed input prices, input costs and technology. In comparison with the FIP index, $I^{\bar{w}}$, this index adds the restriction that input costs are constant. The third output price index we introduce is based upon a more stringent constraint on input substitution possibilities.

### C. $I^{\bar{x}}$ *The Fixed Input Quantity Output Price Index (FIQ)*

This index is based upon a revenue function in which output quantities are the only choice variable. The FIQ index has appeared previously in the index number literature under the guise of the national output deflator.[5] Here we do not wish to restrict ourselves to the output of a nation, and we derive the index as a conditional subindex of a price index for profits. Consider the following revenue function.

*Definition 3.* The revenue function $\hat{R}(p^a, x^b, F^c)$ is defined as:

$$\hat{R}(p^a, x^b, F^c) = \sum_{i=1}^{m} p_i^a \hat{q}_i$$

where the $\hat{q}_i$ are found by solving:

$$\text{Maximize: } \sum_{i=1}^{m} p_i^a q_i - \sum_{j=1}^{n} w_j x_j$$

$$\text{Subject to: } F^c(q_i, x_j) = 0$$

$$x_j = x_j^b (j = 1, \ldots, n).$$

The output price index $I^{\bar{x}}$, is formed as before

$$I^{\bar{x}}(p^r, p^c, x^r, F^r) \equiv \frac{\hat{R}(p^c, x^r, F^r)}{\hat{R}(p^r, x^r, F^r)}$$

[5]See Paul Samuelson (1950), Richard Moorsteen (1961), Franklin Fisher and Karl Shell (Essay II) (1972) and Paul Samuelson and S. Swamy (1974).

## III. Output Price Indexes Properties

Prior to formally introducing the properties we deem desirable for output price indexes, we can gain insight into the behavior of these indexes by considering the relationships between the indexes. It is important to note that comparisons can be made by only considering the numerators of the indexes. In all three cases the denominators represent the revenue observed in the references situation, and with $\sum_{j=1}^{n} w_j^r x_j^r = C^r$, the denominators are equal. Two theorems summarize the relationship between the indexes.

*Theorem 1.* $I^{\bar{w},\bar{C}} \geq I^{\bar{x}}$

Proof:

These two indexes deal with situations in which firms maximize revenue with fixed (identical) costs. The proposition to be proved is

$$\tilde{R}(p^c, w^r, C^r, F^r) \geq \hat{R}(p^c, x^r, F^r)$$

The revenue $\tilde{R}(p^c, w^r, C^r, F^r)$ is defined to be the maximum revenue given costs $C^r$, thus it is greater than or equal to any alternative such as $\hat{R}(p^c, x^r, F^r)$ with costs, $\sum_{j=1}^{n} w_j^r x_j^r = C^r$. Q.E.D.

*Theorem 2.* If $p^c \geq p^r$ then $I^{\bar{w}} \geq I^{\bar{w},\bar{C}} \geq I^{\bar{x}}$

Proof:

The second inequality simply repeats Theorem 1. The first inequality can be proved by utilizing a proof by contradiction. Consider

$$\tilde{R}(p^c, w^r, C^r, F^r) > R^*(p^c, w^r, F^r) \quad \text{if} \quad p^c \geq p^r \text{ i.e.}$$

$$\sum_{i=1}^{m} p_i^c \tilde{q}_i > \sum_{i=1}^{m} p_i^c q_i^*$$

if $p^c \geq p^r$ profits are never less for the unconstrained problem, i.e.

(5)
$$\sum_{i=1}^{m} p_i^c q_i^* - \sum_{j=1}^{n} w_j^r x_j^* \geq \sum_{i=1}^{m} p_i^c \tilde{q}_i - C^r$$

From the first order conditions of the maximization problem which defines the $R^*$ revenue function, the value of the marginal product must equal the input price.

$$w_j^r = p_i^c \frac{\partial q_i^*}{\partial x_j^*} \qquad (i = 1, \ldots, m, j = 1, \ldots, n).$$

and if $p_i^c > p_i^r (i = 1, \ldots, m)$, since we assume $\frac{\partial^2 q_i}{\partial x_j^2} < 0$, the following relationship holds.

$$\sum_{j=1}^{n} w_j^r x_j^* > C^r.$$

Substituting this into (4) yields.

$$\sum_{i=1}^{m} p_i^c q_i^* > \sum_{i=1}^{m} p_i^c \tilde{q}_i \quad \text{Q.E.D.}$$

Much of the literature on index numbers concerns listing desirable properties for index numbers.[6] We have chosen three properties as desirable for our indexes, and later we discuss other possible properties.

Property 1—Identity Test.

$$I(p^r, p^r) = 1$$

Property 2—Proportionality Test

$$I(p^r, \lambda p^r) = \lambda$$

Property 3—Monotonicity Test

$$\text{If } p^{c1} \geq p^c, I(p^r, p^{c1}) \geq I(p^r, p^c)$$

The Identity Test requires that the index should be unity if reference and comparison prices are equal. The Proportionality Test requires, for example, that if comparison prices are twice reference prices, the index equals two. Finally, if comparison prices are higher in one situation, by the Monotonicity Test the index should be higher.

By inspecting the definitions of our three indexes it can be verified that the Identity Test is satisfied in all cases. The Proportionality Test, on the other hand, is not satisfied by all three of our output price indexes.

*Theorem 3.* (a) $I^{\bar{w},\bar{C}}$ and $I^{\bar{x}}$ satisfy the Proportionality Test, while (b) $I^{\bar{w}}$ does not.

Proof:

The proof of (a) is omitted. For (b) we must show that the $R^*(p, w, F)$ revenue function is *not* homogenous of degree one in output prices.

Consider the case with $\lambda > 1$. If all output prices are multiplied by $\lambda$, output proportions will remain unchanged. However, in order to maintain the equality of input prices and the value of the marginal product,

$$w_j^r = \lambda p_i^r \frac{\partial q_i^*}{\partial x_i^*} \quad (i = 1, \ldots, m, j = 1, \ldots, n),$$

input levels will have to increase. With higher input levels, outputs will increase, yielding

[6]See Irving Fisher (1922), Ragnar Frisch (1936) and for a more recent discussion Paul Samuelson and S. Swamy (1974).

$$R^*(\lambda p^r, w^r, F^r) > \lambda R^*(p^r, w^r, F^r) \text{ if } \lambda > 1 \quad \text{Q.E.D.}$$

Given this result we shall drop $I^{\bar{\pi}}$, the FIP output price index from further consideration. This index allowed more flexibility in the choice of inputs than the other two indexes, and while this was its most appealing characteristic, it also led to the failure of the Proportionality Test. The other indexes restrict the substitution of inputs and outputs in some way.

Both the $I^{\bar{\pi},\bar{C}}$ and $I^{\bar{x}}$ indexes pass the Monotonicity Test. They are based upon maximizing profits with costs fixed, that is, maximizing revenue, and with perfectly elastic demand curves, higher prices result in higher total revenue.

Many other tests can and have been concocted for price indexes. We discuss two such tests commonly utilized, but which are not satisfied by our indexes.

Property 4 Point Reversal Test

$$I(p^r, p^c)\, I(p^c, p^r) = 1$$

Property 5 Circular Reversal Test

$$I(p^r, p^c)\, I(p^c, p^d) = I(p^r, p^d)$$

Any index which satisfies the Identity Test (Property 1) and the Circular Reversal Test will also satisfy the Point Reversal Test. Thus, for the indexes discussed above, we can limit our discussion to the Circular Reversal Test, Property 5. In terms of intercountry comparisons, it says, for example: A price index comparing Japan's prices with prices in the U.S. given U.S. technology (and inputs or costs) multiplied times an index comparing the prices in France with those in Japan using Japanese technology should equal an index comparing French prices to U.S. prices using U.S. technology. In our notation ($I^x$ index), this test requires

$$\frac{\hat{R}(p^c, x^r, F^r)}{\hat{R}(p^r, x^r, F^r)} \cdot \frac{\hat{R}(p^d, x^c, F^c)}{\hat{R}(p^c, x^c, F^c)} = \frac{\hat{R}(p^d, x^r, F^r)}{\hat{R}(p^r, x^r, F^r)}.$$

In both the intercountry example and the above equation, it is not clear that the Circular Reversal Test would or should hold. In terms of the example, the only circumstances under which it holds would be if U.S. and Japanese technologies are similar.

All of the output price indexes we have defined utilize the reference situation technology and inputs (or costs and input prices) as the base. As was mentioned in the discussion of the price index for profit, this choice is essentially arbitrary. If all indexes are defined with a base technology other than that of the reference or comparison situation, the index would pass the Circular Reversal Test. For example,

$$I^{\bar{x}}(p^r, p^c, x^b, F^b)\, I^{\bar{x}}(p^c, p^d, x^b, F^b) = I^{\bar{x}}(p^r, p^d, x^b, F^b).$$

The discussion of the properties of our two output price indexes continues in three subsections. First, we consider the relationship between our indexes and fixed-weight indexes. Secondly, we consider the implications of basing our indexes on homothetic production functions. Finally, a third subsection discusses the properties of our two indexes when used as deflators.

### *A. Fixed-Weight Price Indexes*

Initially we assumed that we could observe prices, quantities and technology. Given that techniques of estimating production relationships are imperfect, it is prudent to consider approximations to our index which do not rely on being able to observe technology. One such approximation is given by a Laspeyres fixed-weight price index. This index formulation is used by the Bureau of Labor Statistics for both the Consumer and Wholesale Price Indexes. A Laspeyres price index is defined as

$$L(p^r, p^c, q^r) \equiv \frac{\sum_{i=1}^{m} p_i^c q_i^r}{R^*(p^r, w^r, F^r)} = \frac{\sum_{i=1}^{m} p_i^c q_i^r}{\sum_{i=1}^{m} p_i^r q_i^r}$$

The revenue in the numerator used reference situation's quantities and comparison situation's prices. The Laspeyres index has the advantage that it requires only prices to be gathered for various comparison situations. Operationally this is an important point. For major aggregate indexes such as the CPI or the WPI the gathering of quantity weights is a time consuming and expensive process. The fact that this can be done infrequently using a Laspeyres fixed weight formula makes this type of index very attractive for many applications.

We can gain insight into the relationship between the Laspeyres index and our output price indexes by introducing a graphical framework. We consider the $I^x$ index, the output price index formed assuming input quantities are fixed, and a Laspeyres index.

Consider the comparison between the prices which generated points $R$ and $C$ in Figure 1. The $\hat{R}(p^a, x^b, F^b)$ revenue function gives the revenue where a price line with slope determined by prices $p^a$ is tangent to the production possibility curve determined by the pair $(x^b, F^b)$. The $I^x$ index is formed by dividing $\hat{R}(p^c, x^r, F^r)$, the revenue generated at point $A$, by $\hat{R}(p^r, x^r, F^r)$, the revenue represented by the solid line through $R$. The Laspeyres index has the same denominator but replaces the numerator with $\sum_{i=1}^{m} p^c \hat{q}_i^r$, the revenue represented by the dotted line through point $R$.

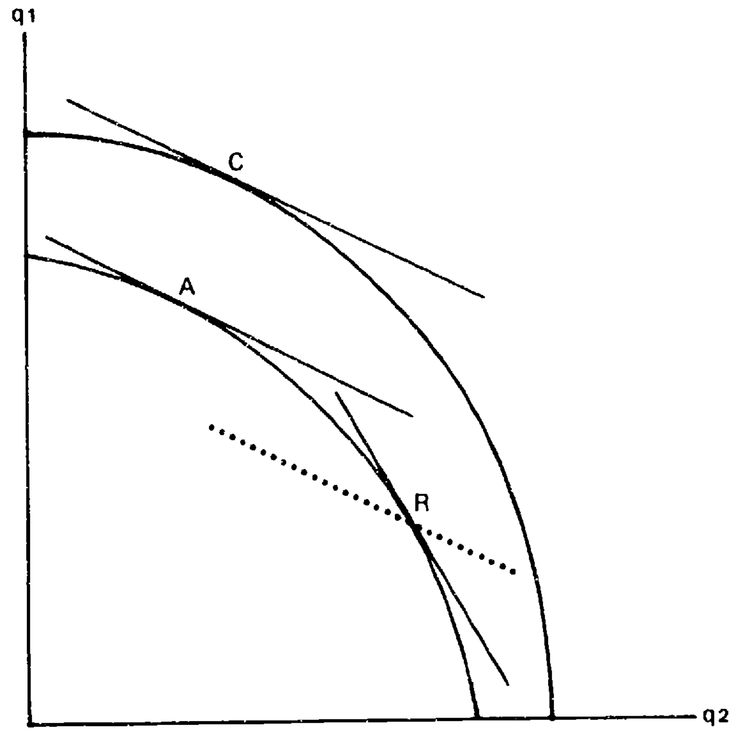


Figure 1

In the example in Figure 1, $I^{\bar{x}}$ is greater than the Laspeyres index. This result holds in general.

*Theorem 4.*

$I^{\bar{w},\bar{C}}(p^r, p^c, w^r, C^r, F^r) \geq I^{\bar{x}}(p^r, p^c, x^r, F^r) \geq L(p^r, p^c, q^r)$

Proof:

The first inequality repeats Theorem 1. The second is proved by contradiction. Consider

$$\sum_{i=1}^{m} p_i^c q_i^r > \hat{R}(p^c, x^r, F^r) = \sum_{i=1}^{m} p_i^c \hat{q}_i$$

$q^r$ is a feasible choice for $\hat{q}_i$ but $q_i$ are chosen because they obtain maximum revenue, hence the above statement cannot hold. Q.E.D.

It is interesting to note that the $I^{\bar{w},\bar{C}}$ index cannot be represented in this same graphical example. The production possibilities frontiers for $\tilde{R}(p^r, w^r, C^r, F^r)$ and $\tilde{R}(p^c, w^r, C^r, F^r)$ are not necessarily identical, because substitution of inputs is possible, and different production possibility frontiers are associated with different inputs.

The result from Theorem 4 that a Laspeyres index is a lower bound for our two output price indexes is different than the results for some other indexes. For both a cost of living index and an input price index a Laspeyres index is an upper bound. This difference results from the fact that sellers substitute into higher priced outputs and buyers substitute away from higher priced inputs.

The other commonly used fixed-weight index is the Paasche index, which is used, for example, to compute the deflators from the national income accounts. A Paasche index uses comparison situation weights and is defined as follows:

$$P(p^r, p^c, q^c) \equiv \frac{R^*(p^c, w^c, F^c)}{\sum_{i=1}^{m} p_i^r q_i^c}$$

In general there is no relationship between this index and the $I^{\bar{w},\bar{C}}(p^r, p^c, w^r, C^r, F^r)$ and $I^{\bar{x}}(p^r, p^c, x^r, F^r)$ indexes. However, the Paasche index bounds indexes which use comparison situation technology (and costs or inputs) as a base.

*Theorem 5.*

$$P(p^r, p^c, q^c) \geq I^{\bar{x}}(p^r, p^c, x^c, F^c) \geq I^{\bar{w},\bar{C}}(p^r, p^c, w^c, C^c, F^c)$$

We omit a formal proof. These indexes have identical numerators. The proof follows the proofs of Theorems 1 and 4, except it involves a comparison of denominators rather than numerators, and this reverses the inequalities.

With our results to this point we can only bound our economic indexes on one side with a fixed-weight index. A Laspeyres price index is a lower bound for output price indexes using reference situation technology, while a Paasche index is an upper bound for output price indexes using comparison situation technology. In the next section we discuss a special case in which Paasche and Laspeyres fixed-weight indexes form both upper and lower bounds for the $I^{\bar{x}}$ output price index.

### *B. Homothetic Production Functions*

The assumption of homotheticity plays a leading role in much of the analysis of both true cost of living indexes and national output deflators. The advantage of the assumption of homotheticity is that a homothetic production function yields a production possibility map in which all production possibility curves have identical shape. In such a case the choice of a particular production possibility curve as the base is not important. Thus the distinction between $I^{\bar{x}}(p^r, p^c, x^r, F^r)$ and $I^{\bar{x}}(p^r, p^c, x^c, F^c)$ disappears, and these FIQ output price indexes are bounded both above and below by a fixed-weight index. The remainder of this section formalizes these results.

*Definition 4.* A production function $F(q, x)$ is homothetic if it can be represented by $F(q, x) = G[H(q, x)]$ where $H(q, x)$ is a homogeneous function and $G$ represents any monotonically increasing transformation.

Thus any homogeneous function is homothetic, but homothetic functions are not necessarily homogenous. In particular we should mention that functions which are homogeneous of degree zero, i.e. ones which exhibit constant returns to scale, while homothetic were previously eliminated from consideration because of the requirement of a unique profit maximizing point.

For a fixed vector of inputs, $x^0$, the production possibility frontier is given by

$$F(q_1, \ldots, q_m, x^0) = 0,$$

and a production possibility map is derived by considering

$$F(q_1, \ldots, q_m, \mu x^0) = 0$$

where $\mu$ is a scalar varying from zero to infinity. Given a homothetic production function the production possibility frontiers will have constant slopes along any ray from the origin. This fact allows Fixed Input Quantity output price indexes to be formed independently of the base production possibility frontier.

Consider FIQ output price indexes based upon a homothetic production function, $F^H$.

*Theorem 6.* If $F^H$ is homothetic

$$I^{\bar{x}}(p^r, p^c, x^r, F^H) = I^{\bar{x}}(p^r, p^c, x^c, F^H)$$

Proof:

The proposition to be proved is that

$$\frac{\hat{R}(p^c, x^r, F^H)}{\hat{R}(p^r, x^r, F^H)} = \frac{\hat{R}(p^c, x^c, F^H)}{\hat{R}(p^r, x^c, F^H)}$$

The proof follows immediately from the fact that if $\sum_{i=1}^{m} p_i^r \bar{q}_i$ is $\hat{R}(p^r, x^r, F^H)$, $\hat{R}(p^r, x^c, F^H)$ will be $\sum_{i=1}^{m} p_i^r \lambda \bar{q}_i$ for some $\lambda > 0$. Similarly, if $\sum_{i=1}^{m} p_i^c \bar{\bar{q}}_i$ is $\hat{R}(p^c, x^r, F^H)$ then $\hat{R}(p^c, x^c, F^H)$ will be $\sum_{i+1}^{m} p_i^c \lambda \bar{\bar{q}}_i$.

This yields

$$\frac{\sum_{i=1}^{m} p_i^c \bar{\bar{q}}_i}{\sum_{i=1}^{m} p_i^r \bar{q}_i} = \frac{\lambda \sum_{i=1}^{m} p_i^c \bar{\bar{q}}}{\lambda \sum_{i=1}^{m} p_i^r \bar{q}_i} \text{ Q.E.D.}$$

*Corollary 1.*

$L(p^r, p^c, q^r) \leq I^{\bar{x}}(p^r, p^c, x^b, F^H) \leq P(p^r, p^c, q^c)$ for any $b$.

For the Fixed Cost output price index the assumption of homotheticity does not yield such convenient results. As mentioned above these price indexes do not compare revenue along a fixed production possibility frontier, since inputs respond to output price change. This means that even given homotheticity, the outputs of $\tilde{R}(p^c, x^r, C^r, F^H)$ and $\tilde{R}(p^c, w^c, C^c, F^H)$ do not lie on the same ray from the origin. The only gain from assuming homotheticity is that the FIC output price indexes can also be chosen as bounds for the unique $I^{\bar{x}}$ index.

*Corollary 2.*

$I^{\bar{w},\bar{C}}(p^r, p^c, w^r, c^r, F^H) \leq I^{\bar{x}}(p^r, p^c, x^b, F^H) \leq I^{\bar{w},\bar{C}}(p^r, p^c, w^c, c^c, F^H)$ for any $b$.

Another implication of homotheticity is that the FIQ output price index satisfies both the Point Reversal Test and the Circular Reversal Test, because it is independent of the base production possibility curve.

We will again discuss homotheticity as it effects the properties of the quantity indexes implied by our output price indexes, but in closing this section it is important to note that homotheticity only represents a special case.[7] In general the choice of the base input level will affect the price indexes, and they are only bounded on one side by Paasche or Laspeyres index.

## C. *Output Price Indexes as Deflators*

One important use of price indexes has always been to deflate series of total revenue to get "real output" or "output in constant dollars." This subsection explores the appropriateness of using the price indexes discussed above for this purpose.

Consider the deflation of the comparison situation revenue ($\hat{R}(p^c, x^c, F^c)$) by the Fixed Input Quantity output price index.

$$(5) \qquad \frac{\hat{R}(p^c, x^c, F^c)}{I^{\bar{x}}(p^r, p^c, x^r, F^r)} = \frac{\hat{R}(p^r, x^r, F^r)\hat{R}(p^c, x^c, F^c)}{\hat{R}(p^c, x^r, F^r)}$$

$$= \hat{R}(p^r, x^r, F^r) \cdot Q^{\bar{x}}(q^r, q^c, p^c)$$

[7]This same point is made very forcefully by Paul Samuelson and S. Swamy (1974) in their Concluding Warning (Page 592).

where $Q^{\bar{x}}(q^r, q^c, p^c)$ represents a quantity index. The quantity index comes from a decomposition of a revenue comparison which also yields the price index, i.e.

$$\frac{\hat{R}(p^c, x^c, F^c)}{\hat{R}(p^r, x^r, F^r)} = \frac{\hat{R}(p^c, x^c, F^c)}{\hat{R}(p^c, x^r, F^r)} \cdot \frac{\hat{R}(p^c, x^r, F^r)}{\hat{R}(p^r, x^r, F^r)},$$

where the first term on the right-hand-side is the quantity index and the second term is $I^{\bar{x}}(p^r, p^c, x^r, F^r)$. A similar quantity index can be defined corresponding to the $I^{\bar{w},\bar{C}}$ price index, i.e.

$$Q^{\bar{w},\bar{C}}(q^r, q^c, p^c) \equiv \frac{\tilde{R}(p^c, w^c, C^c, F^c)}{\tilde{R}(p^c, w^r, C^r, F^r)}$$

These quantity indexes compare quantities of output of the two situations using comparison situation prices as weights.

In order to give a reasonable interpretation to the deflation from equation (5) a quantity index should pass the same tests as a price index. If quantities do not change from one situation to another, the index should be unity (Identity Test); if quantities double, for example, the index should be two (Proportionality Test), and if quantities in one comparison situation are higher than in a second, the index should be higher for the first situation. (Monotonicity Test). Unfortunately, the quantity indexes defined above do not satisfy the requirements of all of these tests.

The difficulty can be illustrated by considering the $Q^{\bar{x}}(q^r, q^c, p^c)$ quantity index and the Proportionality Test in the following graphical framework. The comparison between point $R$ and $C$ is divided between a price index given by the revenue at $A$ (point $C$'s prices) divided by the revenue at $R$ and a quantity index given by the revenue at $C$ divided by the revenue at point $A$. The example in Figure 2 is constructed such that the comparison situation's quantities (point $C$) are a multiple, say $\lambda$, of the reference situation's quantities (point $R$). If the quantity index is to pass the proportionality test it must equal $\lambda$, but this is clearly not the case in this example.

$$(6) \qquad Q^{\bar{x}}(q^r, q^c, p^c) = \frac{\Sigma p^c q^C}{\Sigma p^c q^A} \neq \frac{\Sigma p^c q^C}{\Sigma p^c q^R} = \lambda$$

It is interesting to note that the third quantity in equation (6) is the quantity index implied by a Laspeyres index and that it passes the Proportionality Test. Another instance in which a well behaved quantity index is implied by our price index is the case of a homothetic production function. In this case, the quantity index $Q^{\bar{x}}(q^r, q^c, p^c)$ always compares quantities along a ray from the origin, and hence such a quantity index would pass the Proportionality Test.

The above discussion is intended primarily to illustrate the difficulties

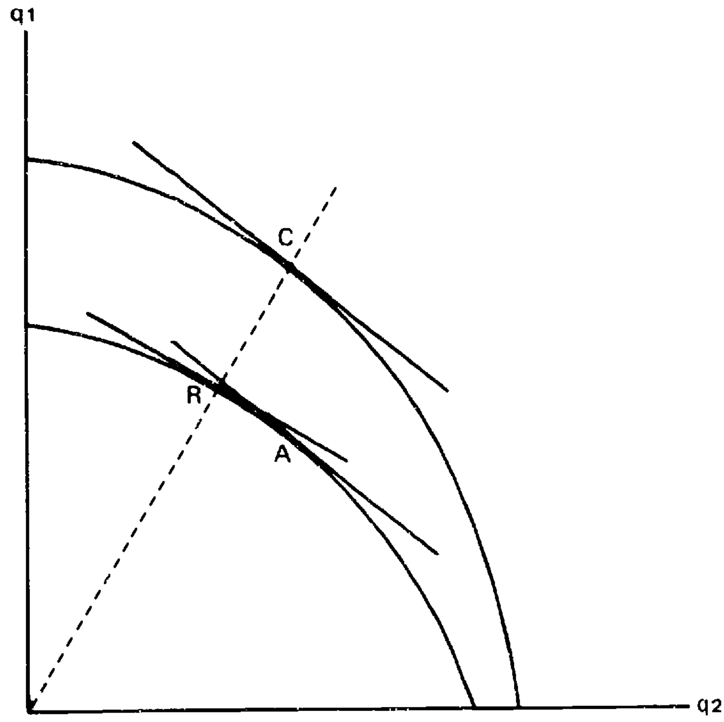


Figure 2

involved in using our measures of price change as deflators. In general, in a world in which input and output levels are altered by price change, a satisfactory price index does not imply a satisfactory quantity index. It is also true that a satisfactory quantity index does not imply a satisfactory price index. At the outset we took the position that we were constructing a measure of price change and not a deflator. If a quantity index is of primary importance for some purpose, it should, in fact, be constructed directly and any shortcomings of the implied price index should be noted.

## IV. Conclusions

This paper has proposed two output price indexes to measure price change. Both meet the stated criteria of including the effects of substitution caused by price change and including characteristics which are traditionally attributed to price indexes. Either of these indexes could provide the conceptual foundation for a program of industrial price measurement. It is important to realize that it is necessary to adopt such a conceptual foundation based upon clearly understood economic theory. If indexes are measuring identifiable economic constructs they can be combined into meaningful aggregates. Also, problems such as quality change which continue to perplex index construction can be analyzed with the measurement objectives of the index clearly in mind. Finally, any biases inherent in the index constructed should be clear from the underlying theory.

It might be unsettling to end with two different proposed output price indexes, and, in fact, it could be argued that the Fixed Cost output price index ($I^{\overline{w},\overline{C}}$) should be preferred since it allows greater substitution possibilities. We hesitate to make such a recommendation before further research has been conducted; specifically before attempts have been made to construct both indexes.

This paper only discusses one half of the problems involved in the measurement of price effects for a firm. Behavior is also significantly altered by changes in input prices, and by using the same type of analysis used above, input price indexes can also be formulated. Full description of the microeconomic impact of price change can be seen in a system of input as well as output price indexes. The subject of input price indexes and their properties is left to another paper.[8]

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## BIBLIOGRAPHY

Robert B. Archibald, "On the Theory of Industrial Price Measurement: Input Price Indexes," Bureau of Labor Statistics, Working Paper No. 48, May 1975.

Franklin Fisher and Karl Shell, *The Economic Theory of Price Indexes*, New York, 1972.

Irving Fisher, *The Making of Index Numbers*, Cambridge, 1922.

Ragnar, Frisch, "Annual Survey of General Economic Theory: The Problem of Index Numbers," *Econometrica*, January 1936, 4, pages 1–36.

Richard Moorsteen, "On Measuring Productive Potential and Relative Efficiency" *Quarterly Journal of Economics*, August 1961, 75, 3, pages 451–461.

John N. J. Muellbauer, "The Theory of the True Input Price Indexes" mimeo University of Warwick, November 1970.

William Nordhaus and John Shoven "Inflation 1973: The Year of Infamy" *Challenge*, May June 1974.

Robert A. Pollak, "The Theory of the Cost of Living Index," B.L.S. Working Paper No. 11, June 1971.

——— "Subindexes of the Cost of Living Index," *International Economic Review*, February 1975, 16, 1, pages 135–150.

Paul A. Samuelson, *Foundations of Economic Analysis*, Cambridge, 1947.

——— "Evaluation of Real National Income," *Oxford Economic Papers*, January 1950, 2, pages 1–29, reprinted in J. E. Stiglitz ed., *The Collected Scientific Papers of Paul A Samuelson*, Vol. II, Cambridge, 1966, pages 1044–72.

——— and S. Swamy, "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis," *American Economic Review* September 1974, LXIV, 4, pages 566–593.

[8]See Robert Archibald (1975) and also John Muellbauer (1970).