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METHODOLOGY

AN APPROACH TO THE FEEDBACK CONTROL OF NONLINEAR ECONOMETRIC SYSTEMS

BY GREGORY CHOW*

Using the method of dynamic programming, an approximately optimal feedback control solution is obtained to minimize the expectation of a quadratic loss function given a system of nonlinear structural econometric equations. Both the cases of known parameters and uncertain parameters are treated. The desirability of having a solution in feedback form is discussed. The Klein–Goldberger model serves as an illustration.

In this paper, I present an approach to perform approximately optimal feedback control to minimize the expectation of a quadratic loss function given a system of nonlinear structural econometric equations. The method is explained for simultaneous equation systems with given or unknown parameters (Sections 1 and 2). The usefulness of having a solution in feedback form is discussed (Section 3). The Klein–Goldberger model is used as an illustration (Section 4).

1. FEEDBACK CONTROL FOR KNOWN ECONOMETRIC SYSTEMS

The solution presented in this section for the feedback control of a nonlinear econometric system with known parameters has been obtained in Chow (1975, Chapter 12) and Chow (1976). The former reference applies the method of Lagrange multipliers while the latter applies the method of dynamic programming to the control of an econometric system with unknown parameters and deduces the solution as a by-product. The exposition in this section applies dynamic programming to the case of known parameters directly. It attempts to relate the theory of control for nonlinear systems to linear theory and emphasizes the computational aspects of the solution more than the previous references.

The i-th structural equation for the observation in period t is

$$y_{it} = \Phi_i(y_t, y_{t-1}, x_t, \eta_{it}) + \varepsilon_{it} \tag{1.1}$$

where y_{it} is the i-th element in the vector y_t of endogenous variables, x_t is a vector of control variables, η_{it} is a vector of parameters *and* exogenous variables not subject to control, and ε_{it} is an additive random disturbance with mean zero, variance σ_{ii} and distributed independently through time. In this section, the elements of η_{it} are treated as given, leaving ε_{it} to be the only random variables. Section 2 will deal with uncertainty in η_{it} which may also incorporate non-additive

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random disturbances if necessary. Lagged endogenous variables dated prior to $t-1$ will be eliminated by introducing identities of the form $y_{kt}=y_{j,t-1}$. Control variables will be incorporated in the vector y_t for two purposes. First, by defining $y_{kt}=x_{jt}$, one can write welfare loss as a function of y_t alone. Second, lagged control variables can be eliminated by identities of the form $y_{mt}=y_{k,t-1}=x_{j,t-1}$. The system of structural equations (1.1) can be written as

$$y_t=\Phi(y_t, y_{t-1}, x_t, \eta_t)+\varepsilon_t \tag{1.2}$$

with Φ denoting a vector function, and with $E\varepsilon_t\varepsilon_t'=\Sigma$.

We assume a quadratic loss function for a T-period control problem,

$$W=\sum_{t=1}^{T}(y_t-a_t)'K_t(y_t-a_t)=\sum_{t=1}^{T}(y_t'K_ty_t-2y_t'K_ta_t+a_t'K_ta_t) \tag{1.3}$$

where a_t are given targets, and K_t are known symmetric positive semidefinite matrices. The problem is to minimize the expectation E_0W conditioned on the information available at the end of period 0. Following the method of dynamic programming, we first solve the optimal control problem for the last period T by minimizing

$$V_T=E_{T-1}(y_T'K_Ty_T-2y_T'K_Ta_T+a_T'K_Ta_T)=E_{T-1}(y_T'H_Ty_T-2y_T'h_T+c_T) \tag{1.4}$$

with respect to x_T. In (1.4) we have defined

$$H_T=K_T; \qquad h_T=K_Ta_T; \qquad c_T=a_T'K_Ta_T \tag{1.5}$$

for the sake of future treatment of the multi-period control problem. Given past observations y_{T-1}, y_{T-2}, etc., the problem for period T is solved in the following steps.

(1) Starting with some trial value $\tilde{x}_T$ for the control, we set ε_T equal to zero and linearize the right hand side of (1.2) about $y_{T-1}=\overset{0}{y}_{T-1}$ (given), $x_T=\tilde{x}_T$ *and* $y_T=y_T^*$ which is the solution of the system

$$y_T^*=\Phi(y_T^*, \overset{0}{y}_{T-1}, \tilde{x}_T, \eta_T) \tag{1.6}$$

where y_T^* can be computed by some iterative method such as the Gauss-Seidel. The linearized version of the structure (1.2) is

$$y_T=y_T^*+B_{1T}(y_T-y_T^*)+B_{2T}(y_{T-1}-\overset{0}{y}_{T-1})+B_{3T}(x_T-\tilde{x}_T)+\varepsilon_T \tag{1.7}$$

where the j-th column of B_{1T} consists of the partial derivatives of the vector function Φ with respect to the j-th element of y_T, evaluated at the given values y_T^*, $\overset{0}{y}_{T-1}$, $\tilde{x}_T$ and η_T, and similarly for the j-th column of B_{2T} and B_{3T}. Computationally, if the structural functions Φ_i are listed in Fortran, each column of B_{1T} can be evaluated numerically as the rates of change in Φ_i with respect to a small change in the j-th element of y_T from y_T^*, and similarly for B_{2T} and B_{3T}. In econometric applications, B_{1T} is very sparse, each row typically consisting of very few elements corresponding to the other current endogenous variables in the equation.

(2) By solving (1.7), and without resorting to numerous iterative solutions of the nonlinear model in order to evaluate the required partial derivatives as is commonly practiced we obtain the linearized reduced-form

(1.8) $$y_T = A_T y_{T-1} + C_T x_T + b_T + u_T$$

where

(1.9) $$(A_T \quad C_T \quad u_T) = (I - B_{1T})^{-1}(B_{2T} \quad B_{3T} \quad \varepsilon_T),$$
$$b_T = y_T^* - A_T y_{T-1}^0 - C_T \tilde{x}_T.$$

Note that, since all the identities used to reduce a higher-order structure to first-order and to incorporate the current and lagged x's into y_t are already reduced-form equations, the matrix $I - B_{1T}$ takes the form

(1.10) $$I - B_{1T} = \begin{bmatrix} I - B_{1T}^* & 0 \\ 0 & I \end{bmatrix}$$

where the order of B_{1T}^* is the number of simultaneous structural equations excluding these identities. Thus only $I - B_{1T}^*$ has to be inverted for the computation of A_T, C_T and b_T in (1.8).

(3) We minimize (1.4) with respect to x_T, assuming that y_T is governed by (1.8). This is done by differentiating (1.4) with respect to x_T and interchanging the order of taking expectation and differentiation:

(1.11) $$\frac{\partial V_T}{\partial x_T} = 2E_{T-1}\left[\left(\frac{\partial y_T'}{\partial x_T}\right) H_T y_T - \left(\frac{\partial y_T'}{\partial x_T}\right) h_T\right]$$
$$= 2E_{T-1}[C_T' H_T (A_T y_{T-1} + C_T x_T + b_T + u_T) - C_T' h_T] = 0$$

where (1.8) has been used to substitute for $(\partial y_T'/\partial x_T)$ and y_T. The solution of (1.11) for x_T is

(1.12) $$\hat{x}_T = G_T y_{T-1} + g_T$$

where

(1.13) $$G_T = -(E_{T-1} C_T' H_T C_T)^{-1}(E_{T-1} C_T' H_T A_T)$$
$$g_T = -(E_{T-1} C_T' H_T C_T)^{-1}(E_{T-1} C_T' H_T b_T - E_{T-1} C_T' h_T).$$

By the linear approximation (1.8), A_T, C_T and b_T are not functions of ε_T and are thus nonrandom. Therefore, the expectation signs in (1.13) can be dropped, but we retain them for future discussion.

(4) Using the solution $\hat{x}_T$ of (1.12) to replace the initial guess $\tilde{x}_T$ in step (1), we repeat steps (1) through (4) till convergence in $\tilde{x}_T$. Observe that the solution, even when converging, is not truly optimal because we have used the approximate reduced form (1.8) with constant coefficients A_T, C_T and b_T. To obtain an exactly optimal solution, one would first compute $\tilde{y}_T$ as the solution of the stochastic structure (1.2) *with ε_T included*, rather than y_T^* as a solution of (1.6). Thus $\tilde{y}_T$ is a random vector depending on ε_T. Secondly, (1.7) would be replaced by

(1.14) $$y_T = \tilde{y}_T + B_{1T}(y_T - \tilde{y}_T) + B_{2T}(y_{T-1} - y_{T-1}^0) + B_{3T}(x_T - \tilde{x}_T).$$

The derivatives B_{1T}, B_{2T} and B_{3T} in (1.14) which are evaluated at $\tilde{y}_T$, and hence the matrices A_T, C_T and b_T in the resulting reduced form corresponding to (1.8), will be dependent on ε_T. The matrices G_T and g_T in the solution for $\hat{x}_T$ will be calculated by (1.13) with the expectation signs retained. Such a four-step iterative procedure would be optimal because when the solution $\hat{x}_T$ converges the value y_T given by the linearized structure (1.14) and its reduced form would be exactly equal to $\tilde{y}_T$, the solution value from the original structure (1.2); the second line of (1.11) would be exactly equal to the first line and not be merely an approximation. The earlier approximate solution amounts to replacing (1.14) by (1.7), i.e., linearizing the structure about the nonstochastic y_T^* rather than the stochastic $\tilde{y}_T$, thus making the derivatives B_{1T}, B_{2T} and B_{3T} nonstochastic. The first $\tilde{y}_T$ in (1.14), which equals $\Phi(\tilde{y}_T, \dots) + \varepsilon_T$ by (1.2), is replaced by $\Phi(y_T^*, \dots) + \varepsilon_T$ or $y_T^* + \varepsilon_T$ in (1.7). This approximate solution is the same as the certainty-equivalence solution obtained by minimizing (1.4) subject to the constraint (1.2) with $\varepsilon_T = 0$, as is shown in Chow (1975, Section 12.1).

(5) Using (1.8) for y_T and (1.12) for x_T, we compute the minimum expected loss for period T from (1.4), yielding

$$(1.15)\qquad \begin{aligned}\hat{V}_T &= y'_{T-1}E_{T-1}(A_T + C_TG_T)'H_T(A_T + C_TG_T)y_{T-1} \\ &\quad + 2y'_{T-1}E_{T-1}(A_T + C_TG_T)'(H_Tb_T - h_T) \\ &\quad + E_{T-1}(b_T + C_Tg_T)'H_T(b_T + C_Tg_T) \\ &\quad + E_{T-1}u'_TH_Tu_T - 2E_{T-1}(b_T + c_Tg_T)'h_T + E_{T-1}c_T.\end{aligned}$$

To generalize the solution to T periods, consider next the 2-period problem of choosing x_T and x_{T-1}. Since the optimal $\hat{x}_T$ and $\hat{V}_T$ have already been obtained, we apply the principle of optimality in dynamic programming and minimize with respect to x_{T-1} the expression

$$(1.16)\qquad \begin{aligned}V_{T-1} &= E_{T-2}(y'_{T-1}K_{T-1}y_{T-1} - 2y'_{T-1}K_{T-1}a_{T-1} + a'_{T-1}K_{T-1}a_{T-1} + \hat{V}_T) \\ &= E_{T-2}(y'_{T-1}H_{T-1}y_{T-1} - 2y'_{T-1}h_{T-1} + c_{T-1})\end{aligned}$$

where, after substitution of (1.15) for $\hat{V}_T$,

$$(1.17)\qquad \begin{aligned}H_{T-1} &= K_{T-1} + E_{T-1}(A_T + C_TG_T)'H_T(A_T + C_TG_T) \\ &= K_{T-1} + E_{T-1}(A'_TH_TA_T) + G'_T(E_{T-1}C'_TH_TA_T),\end{aligned}$$

the second line of (1.17) having utilized equation (1.13) for G_T,

$$(1.18)\qquad \begin{aligned}h_{T-1} &= K_{T-1}a_{T-1} + E_{T-1}(A_T + C_TG_T)'(h_T - H_Tb_T) \\ &= K_{T-1}a_{T-1} + E_{T-1}(A_T + C_TG_T)'h_T - E_{T-1}(A'_TH_Tb_T) \\ &\quad - G'_T(E_{T-1}C'_TH_Tb_T),\end{aligned}$$

$$(1.19)\qquad \begin{aligned}c_{T-1} &= E_{T-1}(b_T + C_Tg_T)'H_T(b_T + C_Tg_T) - 2E_{T-1}(b_T + C_Tg_T)'h_T \\ &\quad + a'_{T-1}K_{T-1}a_{T-1} + E_{T-1}u'_TH_Tu_T + E_{T-1}c_T.\end{aligned}$$

Since the second line of (1.16) has the same form as (1.4), we can repeat the steps in the solution for x_T with $T-1$ replacing T, yielding an optimal $\hat{x}_{T-1}$ in the form

(1.12) and the corresponding minimum 2-period loss ($\hat{V}_{T-1}$ from (1.16). The process continues backward in time until $\hat{x}_1$ and $\hat{V}_1$ are obtained.

Computationally, we suggest the following steps for the T-period optimal control problem. (1) Start with initial guesses $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_T$, solve the system (1.2) with $\varepsilon_t = 0$ for $y_1^0, y_2^0, \ldots, y_{T-1}^0$, using the Gauss–Seidel method. (2) For $t = T, T-1, \ldots, 1$, linearize the structural equations as in (1.6) and (1.7), noting that $y_t^* = y_t^0$ has been computed in step 1. Compute the reduced form coefficients A_t, C_t and b_t by (1.9). (3) Using (1.13) and (1.17) alternately, compute G_t and H_{t-1} for $t = T, T-1, \ldots, 1$. Use (1.18) to compute h_{t-1} and (1.13) to compute g_t backward in time. (4) Using the feedback control equations $\hat{x}_t = G_t y_{t-1} + g_t$ and the system (1.2) with $\varepsilon_t = 0$, compute successively $\hat{x}_1$, $y_1^0, \hat{x}_2$, y_2^0, etc. The $\hat{x}_t$ will serve as the initial guesses $\tilde{x}_t$ in step 1. The process can be repeated until the $\hat{x}_t$ converge. (5) Use (1.19) to compute c_{t-1} backward in time. $\hat{V}_1$ will be computed by (1.15) with 1 replacing T.

Recall that by our linearization of the structure about y_t^* (rather than about $\tilde{y}_t$ which depends on ε_t) all the coefficients A_t, C_t and b_t become constants, and the expectation signs in all calculations above can be dropped. We only retain the expectation $E_{t-1}u_t'H_tu_t = tr(H_tEu_tu_t')$ in the calculation of c_{t-1} by (1.19), which, by virtue of (1.9), equals tr $H_t(I-B_{1t})^{-1}\Sigma(I-B_{1t})^{-1'}$.

2. Feedback Control with Unknown Parameters

The exposition of Section 1 has paved the way for introducing randomness in the parameters η_t in the system (1.2). In principle, random η_t can be treated in the same way as random ε_t. To obtain an exact solution to the last-period control problem by the method of Section 1, it is necessary to linearize (1.2) about $\tilde{y}_T$, the solution value of y_T which depends on the random ε_T and η_T. Accordingly, the coefficients B_{1T}, B_{2T} and B_{3T} in (1.14) and A_T, C_T and b_T in the resulting reduced-form are all random functions of η_T. The approximate method we propose to solve the multiperiod control problem with unknown parameters also follows the 5 steps described at the end of Section 1, except that all the expectation signs have to be kept in the calculations.

To evaluate the expectations such as $E_{t-1}(A_t'H_tA_t)$ in (1.17), two approximations are made. First, all time subscripts of the expectation signs are replaced by zero. Thus information on the probability distribution of ε_t and η_t as of the beginning of the planning period is used for the calculation of the optimal $\hat{x}_1$; possible future learning about the unknown parameters is ignored. Second, we linearize the structure about y_t^* which is the solution of (1.2) with $\varepsilon_t = 0$ and η_t set equal to its mean $\bar{\eta}_t$, obtaining the structural coefficients $\bar{B}_{1t}$, $\bar{B}_{2t}$ and $\bar{B}_{3t}$; we then compute the $i-j$ element of expectation $E_0(A_t'H_tA_t)$ by the identity

$$(2.1) \qquad E_0(A_t'H_tA_t)_{ij} = (\bar{A}_t'H_t\bar{A}_t)_{ij} + \text{tr } H_tE_0(a_{jt}-\bar{a}_{jt})(a_{it}-\bar{a}_{it})'$$

where $\bar{A}_t = (I-\bar{B}_{1t})^{-1}\bar{B}_{2t}$, and the covariance matrix for any two columns a_{jt} and a_{it} of A_t can be approximated by the appropriate submatrix in D_t cov $(\eta_t)D_t'$, D_t being the matrix of the partial derivatives of the columns of A_t with respect to η_t. Numerically, the k-th column of D_t is computed as the rates of change of the columns of A_t with respect to a small change in the k-th element of η_t from $\bar{\eta}_t$. For a more thorough discussion of this method, the reader may refer to Chow (1976).

3. Usefulness of Feedback Control

If we treat the parameters η_t as known constants and set $\varepsilon_t = 0$, the method of Section 1 provides a solution to the optimal control of a nonlinear deterministic system. Currently, a popular way to solve such a deterministic control problem is to treat the multiperiod loss W as a function of $x_1, \ldots, x_T$ and minimize it by some gradient, conjugate-gradient or another standard computer algorithm, as in Fair (1974), Holbrook (1974), and Norman, Norman and Palash (1974). It may be useful to point out the possible advantages of the method of this paper as compared with this alternative approach.

(1) From the very narrow viewpoint of computing the optimal policy under the assumption of a deterministic model, the method of Section 1 compares favorably with the alternative method when the number of unknowns in the minimization problem is large. The number of unknowns equals the number T of planning periods times the number q of control variables. If we are dealing with 32 quarters and 4 control variables, there will be 128 variables, creating a formidable minimization problem. Our method, being based on the method of dynamic programming with a time structure, converts a problem involving T sets of control variables to T problems each involving only one set of control variables. Its computing cost increases only linearly with T. For each period t, we solve a minimization problem involving q controls; the matrix $C_t'H_tC_t$ to be inverted is $q \times q$. Also, if q is increased from 4 to 8, we have to solve an 8-variable problem 32 times, whereas the alternative method has to deal with 256 variables simultaneously.

On the other hand, our method is perhaps more constrained than the alternative method by the number of simultaneous equations (the order of the matrix $I - B_{1T}^*$ in equation 1.10) in the econometric system for our linearization requires the inversion of $I - B_{1t}^*$. However, by exploiting the bloc-diagonality and the sparseness of this matrix, it may be possible to deal with some 150 to 200 simultaneous equations. More computational experience is required to shed light on this question.

(2) Once we leave the realm of purely deterministic control, the advantages of our approach are numerous. First, after incorporating the random disturbances ε_t in an otherwise deterministic model, one can no longer regard as optimal the values of $x_2, \ldots, x_T$ obtained by solving the deterministic control problem. Only the value of x_1 for the first period constitutes an approximately optimal policy. In contrast with the method of deterministic control, the method of Section 1 yields the approximately optimal $\hat{x}_t$ $(t = 2, \ldots, T)$ as a function of the yet unobserved y_{t-1}. It provides analytically an estimate $\hat{V}_1$ of the minimum expected loss associated with the nearly optimal strategies. Using the alternative method, one would have to calculate y_1 from $\hat{x}_1$ and ε_1, solve a multiperiod control problem from period 2 to T to obtain $\hat{x}_2$, calculate y_2 from $\hat{x}_2$ and ε_2, etc., and repeat the T-period simulations many times to estimate the expected loss from such a strategy. Such computations are extremely costly, if not prohibitive.

(3) Our method yields a linearized reduced form at each period as a by-product. The reduced-form coefficients are extremely useful for computing the various dynamic multipliers of y_t with respect to current, delayed and cumulative changes of x_t, and for exhibiting how nonlinear the system is and how the various partial derivatives change through time.

(4) The feedback control equations are useful as a basis of policy recommendations. They can be used to compare different econometric models. They can be incorporated into the econometric model to study the dynamic properties of the system under control. Once the model is linearized, its dynamic properties can be deduced by spectral and auto-covariance methods, as described in Chow (1975, Ch. 3, 4, and 6). Not only the mean paths of the variables from periods one to T, but their variances, covariances, autocovariances and cross-covariances can be deduced.

(5) The value of having improved information (a smaller covariance matrix) for a subset of parameters can be ascertained by comparing the minimum expected losses computed by varying the covariance matrix of η_t using the method of Section 2. As a special case, the comparison of $\hat{V}_1$ computed by varying the covariance matrix of ε_t using the method of Section 1 helps to evaluate the importance of the stochastic disturbances in the expected welfare loss. In short, by our method, the rich theory of optimal control for linear systems can be applied to the control of nonlinear systems. Parts of this theory will be illustrated in Section 4.

4. A Numerical Example Using the Klein–Goldberger Model

To illustrate our method, the Klein–Goldberger model as adopted by Adelman and Adelman (1959, pp. 622–624) is used. The equations are listed below.

(4.1) Consumer expenditures in 1939 dollars $= C =$

$$y_1 = -22.26 + 0.55(y_6 + x_1 - y_{19}) + 0.41(y_{14} - y_{21} - y_3)$$
$$+ 0.34(y_9 + x_3 - y_{22}) + 0.26 y_{1,-1} + 0.072 y_{11,-1} + 0.26 z_2$$

(4.2) Gross private domestic capital formation in 1939 dollars $= I =$

$$y_2 = -16.71 + 0.78(y_{14} - y_{21} + y_9 + x_3 - y_{22} + y_5)_{-1}$$
$$-0.073\, y_{16,-1} + 0.14\, y_{12,-1}$$

(4.3) Corporate savings $= S_p =$

$$y_3 = -3.53 + 0.72(y_4 - y_{20}) - 0.028\, y_{17,-1}$$

(4.4) Corporate profits $= P_c =$

$$y_4 = -7.60 + 0.68\, y_{14}$$

(4.5) Capital consumption charges $= D =$

$$y_5 = 7.25 + 0.05(y_{16} + y_{16,-1}) + 0.044(y_{13} - x_1)$$

(4.6) Private employee compensation $= W_1 =$

$$y_6 = -1.40 + 0.24(y_{13} - x_1) + 0.24(y_{13,-1} - x_{1,-1}) + 0.29\, z_6$$

(4.7) Number of wage-and-slary earners = N_W =

$$y_7 = x_4 - (z_4 + z_5) \div 1.062 + (26.08 + y_{13} - x_1 - 0.08\, y_{16} - 0.08\, y_{16,-1} - 2.05\, z_6) \div (2.17 \times 1.062)$$

(4.8) Index of hourly wages = w =

$$y_8 = y_{8,-1} + 4.11 - 0.74(z_3 - y_7 - z_4 - z_5) + 0.52(y_{15,-1} - y_{23,-1}) + 0.54\, z_6$$

(4.9) Farm income = A =

$$y_9 = 0.054(y_6 + x_1 - y_{19} + y_{14} - y_{21} - y_3) + 0.012(z_1)(y_{10}) \div y_{15}$$

(4.10) Index of agricultural prices = p_A =

$$y_{10} = 1.39\, y_{15} + 32.0$$

(4.11) End-of-year liquid assets held by persons = L_1 =

$$y_{11} = 0.14(y_6 + x_1 - y_{19} + y_{14} - y_{21} - y_3 + y_9 + x_3 - y_{22}) + 76.03(1.5)^{-0.84}$$

(4.12) End-of-year liquid assets held by businesses = L_2 =

$$y_{12} = 0.26\, y_6 - 1.02(2.5) - 0.26(y_{15} - y_{15,-1}) + 0.61\, y_{12,-1}$$

(4.13) Gross national product = $Y + T + D$ =

$$y_{13} = y_1 + y_2 + x_2$$

(4.14) Nonwage nonfarm income = P =

$$y_{14} = y_{13} - y_{18} - y_5 - y_6 - x_1 - y_9 - x_3$$

(4.15) Price index of gross national product = p =

$$y_{15} = 1.062\, y_8(y_7) \div (y_6 + x_1)$$

(4.16) End-of-year stock of private capital = K =

$$y_{16} = y_{16,-1} + y_2 - y_5$$

(4.17) End-of-year corporate surplus = B =

$$y_{17} = y_{17,-1} + y_3$$

(4.18) Indirect taxes less subsidies = T =

$$y_{18} = 0.0924\, y_{13} - 1.3607$$

(4.19) Personal and payroll taxes less transfers = T_w =

$$y_{19} = 0.1549\, y_6 + 0.131\, x_1 - 6.9076$$

(4.20) Corporate income tax = T_c =

$$y_{20} = 0.4497\, y_4 + 2.7085$$

(4.21) Personal and corporate taxes less transfers = T_p =

$$y_{21} = 0.248(y_{14} - y_{20} - y_3) + 0.2695(y_{15,-1} \div y_{15})(y_{14} - y_{20} - y_3)_{-1} + 0.4497\, y_4 - 5.7416$$

(4.22) Taxes less transfers associated with farm income $= T_A =$

$y_{22} = 0.0512(y_9 + x_3)$

(4.23) $y_{23} = y_{15,-1}$

The control variables or instruments are

$x_1 = W_2 =$ Government employee compensation

$x_2 = G =$ Government expenditures for goods and services

$x_3 = A_2 =$ Government payments to farmers

$x_4 = N_G =$ Number of government employees.

The exogenous variables not subject to control are

$z_1 = F_A =$ Index of agricultural exports

$z_2 = N_p =$ Number of persons in the United States

$z_3 = N =$ Number of persons in the labor force

$z_4 = N_E =$ Number of nonfarm entrepreneurs

$z_5 = N_F =$ Number of farm operators

$z_6 =$ time $= 0$ for 1929 ($= 24$ for 1953).

In the control experiments reported below, 1953 was chosen as the first year of the planning period. Initial values of the endogenous variables y_0 and extrapolation formulas for the uncontrollable exogenous variables z_t (part of η_t in the notation of Section 1) are given by Adelman and Adelman (1959, p. 624). The four control variables have been listed in the last paragraph. When imbedded in the vector y_t in the notation of equation (1.2), they become respectively y_{24} to y_{27}. Three runs have been tried. Run 1 uses endogenous variables 7 (number of wage-and-salary earners), 13 (real GNP), 14 (real nonwage nonfarm income) and 15 (price index of GNP) as targets, with the value 1 specified for each of the corresponding 4 diagonal elements of the matrix K_t in the welfare function. These target variables are steered to grow at 2, 5, 5 and 1 percent per year respectively from their initial values at 1952. Run 2 uses variables 13, 15, 26 (government payments to farmers) and 27 (number of government employees) as target variables. The target for y_{26} is to remain at its historical 1952 value 0.1187, and for y_{27} is to grow 3 percent annually from its estimated 1952 value 9.393. Run 3 uses variables 7, 15, 26 and 27 as target variables. In effect, runs 2 and 3 tie up two instruments and uses the remaining two instruments to control real GNP and the price index, or employment of wage-and-salary earners and the price index.

A major motivation behind the above experiments is to find out whether the relationship between the general price index and real GNP (or employment) can be shifted at will by government policy according to the Klein–Goldberger model. The answer is definitely yes. The specified targets for the price index, real GNP, and/or employment of wage-and-salary earners are met exactly by the optimal control solutions of the above 3 runs, ignoring random disturbances. Thus the government can choose any price-GNP or price-employment combination at any period as it pleases by applying government employee compensation and government expenditures for goods and services as the control variables.

As pointed out by Chow (1975, pp. 167–8), if the number of target variables (the number of nonzero elements in the $p \times p$ diagonal matrix K_t) equals the number $q \leq p$ of control variables, the time path $\bar{y}_t$ generated by the deterministic system (which is obtained by ignoring the random disturbances in a linear econometric model) under optimal control will meet the targets exactly and the deterministic part W_1 of the minimum expected welfare loss will be zero, provided that the submatrix C_{1t} of the matrix C_t in the reduced form whose rows correspond to the target variables is of rank q. In the above three runs, the number of target variables equals the number of control variables, and the matrix C_{1t} for all t in the linearized reduced form has rank 4. Thus the targets are met exactly. This illustrates the application of control theory for linear systems to nonlinear econometric systems by the approach of this paper. Note that, in the theory for controlling known linear systems, Chow (1975, Chapters 7 and 8), it is useful to decompose the solution vector y_t into its deterministic part $\bar{y}_t$ (obtained by ignoring ε_t) and its stochastic part $y_t^* = y_t - \bar{y}_t$ due to the random disturbances. The same decomposition can now be achieved by our method for nonlinear systems. The autocovariance matrix of y_t^* provides the variances and covariances of the variables under control from their mean path $\bar{y}_t$. It can be derived analytically as in Chow (1975) once the system is linearized by the method of this paper.

To better appreciate the reason why government policy can shift the relationship between the general price index and real GNP (or employment), consider the "aggregate demand curve" and the "aggregate supply curve" implicit in the Klein–Goldberger model. The aggregate demand curve relating price to real GNP can be obtained by solving the aggregate demand sector consisting of 16 equations: (4.1)–(4.4), (4.9), (4.10), (4.13), (4.14), (4.17)–(4.22) of the IS sector and equations (4.11) and (4.12) of the LM sector. The aggregate supply curve is obtained by solving 6 equations: (4.5)–(4.8), (4.15) and (4.16). We refer to the short-run aggregate supply curve, holding all lagged dependent variables constant. (4.8) gives wage w as a linear function of employment N_W. (4.7) gives N_W as a function of real GNP, capital stock K, and government employee compensation W_2. Equations (4.16) and (4.5) explain K by capital consumption charges D (investment I being predetermined by equation 4.2) and D by K, GNP and W_2, yielding K as a function of GNP and W_2. Both w and N_W thus become functions of GNP and W_2. By (4.15) price $p = 1.062\, wN_W/(W_1 + W_2)$, where the private employee compensation W_1 is also a function of GNP and W_2 by virtue of (4.6). Hence the resulting aggregate supply curve relating p to GNP and W_2 can be shifted by manipulating the control variable W_2.

If the aggregate supply function relating price to real GNP or to employment contains no variables which are subject to government control, government policy can only shift aggregate demand and trace out the rigid relation between price and real GNP, but cannot achieve more real output or employment without inflation. A case in point is the relation between the wage rate and employment as given by (4.8). No government policy can shift this rigid relationship for the current period, given the predetermined variables. In terms of control theory, no two instruments can steer wage and employment to specified target values as they are linearly related by (4.8). The matrix C_{1t} has two linearly dependent rows and has rank smaller than the number of instruments.

We have computed the optimal control solutions for the three runs described above, and some other related runs, using $T=5$ and $T=10$ as the planning horizon. To start the iterations, we arbitrarily let the initial $\tilde{x}_{it}$ be the 3 percent annual growth path for each of the 4 control variables beginning from its historical value as of 1953; these initial paths are given in Table 1 for x_1 and x_2. For the first

TABLE 1

VALUES OF SELECTED VARIABLES AT THREE SUCCESSIVE PASSES FOR CONTROL BY THE KLEIN–GOLDBERGER MODEL—RUN 1 ($y_7, y_{13}, y_{14}, y_{15}$ AS TARGETS).

Variable	Pass	1953	1954	1955	1956	1957
x_1	0	15.70	16.17	16.66	17.16	17.67
(government	1	21.15	25.60	29.41	32.95	36.35
employee	2	21.21	26.03	30.63	35.35	40.28
compensation)	3	21.21	26.03	30.64	35.38	40.35
x_2	0	33.50	34.50	35.54	36.61	37.70
(government	1	39.96	45.42	49.68	53.59	57.60
expenditures	2	39.95	45.40	49.74	53.85	58.11
for goods	3	39.95	45.40	49.74	53.85	58.11
and services)						
y_{13}	0	171.24	171.85	174.41	178.12	182.31
(real GNP)	1	180.60	189.64	199.13	209.10	219.58
	2	180.60	189.63	199.11	209.07	219.52
	3	180.60	189.63	199.11	209.07	219.52
y_{15}	0	204.75	209.28	215.81	223.35	231.26
(price index)	1	204.52	207.10	210.23	213.80	217.68
	2	204.42	206.47	208.55	210.66	212.82
	3	204.42	206.47	208.53	210.62	212.72

period 1953, we use the values of the endogenous variables as of 1952 as starting values for the Gauss–Seidel iterations to solve for y^0_{1953}, given $\tilde{x}_{1953}$, and use y^0_{1953} as starting values to iterate for y^0_{1954}, given $\tilde{x}_{1954}$, etc. Table 1 shows the values of selected target and control variables for Run 1 at the three rounds of linearizations (three "passes" through step (1) of the method of Section 1) required for the convergence of the target variables to five significant figures. Note how rapidly these variables converge to the solution, the first pass already near the optimum.

In terms of computing time using the IBM 360-91 computer at Princeton University, each pass took slightly less than 4 seconds, and the total computer time for three passes was about 12 seconds. When we ran the experiments for 10 periods instead of 5, the time merely doubled, taking about 24 seconds for three passes to convergence. These would be minimization problems involving 40 variables in the alternative approach to deterministic control. Imagine a 120-variable problem with 4 controls and 30 periods using a quarterly model of similar size. The alternative approach would be almost prohibitive, but our method would take about 3×24 or 72 seconds. By our method, increasing the number of control variables from 4 to 5 would not require much more computing time, since a 5×5 $C_t'H_tC_t$ matrix is still easy to invert and the hard computing work is performed in obtaining the linearized reduced form. By the alternative method, a 120-variable problem would become a 150-variable problem. (For the same reasons, increasing the number of target variables from 4 to 5 or 6 while keeping the same 4 control variables in our example has produced almost no effect on the computing time.)

We next examine the coefficients G_t and g_t in the feedback control equations for the optimal solution of Run 1 with $T = 5$. Of the 27 variables in y_{t-1} (including 4 control variables), only 18 appear in the reduced form, the matrix A_t having 9 columns of zeros. Table 2 exhibits coefficients of selected lagged variables in the

TABLE 2

COEFFICIENTS OF SELECTED LAGGED VARIABLES IN THE FEEDBACK CONTROL EQUATIONS FOR GOVERNMENT EXPENDITURES—RUN 1 ($T = 5$)

	Lagged Variable								Intercept
Period	1	3	5	8	9	12	14	15	g
1	−0.260	−0.109	−0.768	−0.053	−0.768	−0.138	−0.659	−0.015	124.4
3	−0.260	−0.109	−0.768	−0.054	−0.768	−0.138	−0.659	−0.014	137.0
5	−0.260	−0.109	−0.768	−0.055	−0.768	−0.138	−0.659	−0.014	151.0

feedback control equations for government expenditures x_2. Note that the coefficients of the lagged expenditure, income and price variables are all negative, showing that government expenditures should respond negatively to recent signs of economic expansion. The feedback coefficients are practically identical for periods 1 through 5 for two reasons. First, since the number of instruments equals the number of target variables and the matrix C_{1t} has full rank, we have $K_t(A_t + C_tG_t) = 0$ and $H_t = K_t$, as shown in Chow (1975a, pp. 168–9). This means that the matrix H_t in the quadratic loss function V_t to be minimized in each future period is identical. Second, since the linearized reduced form coefficients A_t and C_t vary only slightly through time, the solution $G_t = (C_t'H_tC_t)^{-1}C_t'H_tA_t$ will also be stable through time. The intercept g_t, however, is increasing in order to meet the growing targets as we have specified.

It may be interesting to exhibit parts of the matrices A_t, C_t and b_t for $t = 1, 3, 5$ to show how time-varying they are. Table 3 shows selected coefficients of the

TABLE 3

REDUCED FORM COEFFICIENTS FOR CONSUMPTION FROM THE OPTIMAL SOLUTION—RUN 1

Period	a_{11}	a_{13}	a_{15}	a_{16}	c_{11}	c_{12}	b_1
1	0.3305	0.1425	0.2036	0	0.3005	0.2712	33.84
3	0.3311	0.1428	0.2053	0	0.2997	0.2736	35.58
5	0.3315	0.1429	0.2064	0	0.3005	0.2750	37.42
Goldberger	0.3219	0.0297	0.2834	0.1027	0.3355	0.2380	

reduced form equation for consumption expenditures y_1 from the optimal control solution of run 1. Their stability through time is apparent. The last row of Table 3 reproduces the corresponding coefficients from the study by A. S. Goldberger (1959, pp. 40–41) on impact multipliers of the Klein–Goldberger model, although for numerous reasons, including the differences between the two versions of the Klein–Goldberger model, the coefficients given by Goldberger should be different from ours.

If we were to pursue a dynamic policy analysis using the Klein–Goldberger model or any other nonlinear econometric model by the method of this paper, it would occupy a substantial volume. Once the model is linearized and the approximately optimal linear feedback control equations obtained, the methods of dynamic analysis as described in Goldberger (1959), Adelman and Adelman (1959), and Chow (1975a) can be applied to study numerous important and interesting questions of macroeconomic theory and policy. The main purpose of this paper has been to show that, using our method of feedback control, the theory and techniques for controlling linear econometric systems can be made applicable to nonlinear econometric systems. This paper has recommended the feedback approach, because it appears to be much more useful than the computation of optimal time paths for the deterministic version of a stochastic control problem and helps to tie together a significant part of stochastic control theory in economics.[1]

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[1] At the time of page proof for this paper (June, 1976), the method of section 1 has been successfully applied to control the Michigan Quarterly Econometric Model for 17 quarters, convergence being obtained in three "passes" as defined for Table 1 above.