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MICROECONOMICS

STOCHASTIC MODELS OF PRICE ADJUSTMENT*

BY STEVEN BARTA AND PRAVIN VARAIYA

Some models of stochastic approximation are presented which seek to explain how several sellers in a single market adjust their prices and quantities in disequilibrium and when the demand for their product is imperfectly known. These adjustment schemes have the known property that they permit sellers to simultaneously learn *their demand function more accurately and to* search *for more satisfactory price levels with little computation. Some subtle effects of stochastic environments are discovered which have escaped informal discussions of the problem.*

1. INTRODUCTION

Several authors seeking to explain how sellers set prices or quantities outside of equilibrium simplify their analysis by "avoid[ing] the problem of what firms should do when they do not know their demand functions" [4, p. 186]. The simplification is achieved by assuming either that sellers know very little or ignore their monopoly power [2, 8], or that sellers know their demand functions exactly [1, 3]. These assumptions are made in spite of the fact that in discussing their models these authors often argue in terms of the uncertainty in demand.

When there is uncertainty about its demand function, the firm can experiment with its prices and observe the reactions of its customers, and with this additional information the firm may discover levels of profitable prices. The problem of finding the optimal sequence of prices can be posed as a problem in Bayesian decision theory, and this has been done for the case of a single firm in a very simple economic environment [4]. Such a formulation has two deficiencies. First, the computational effort necessary to calculate the optimal sequence is so great that even its normative significance is diminished if costs of computation are taken into account. Secondly, any attempt to extend along these directions the formulation to include several interacting firms appear to lead inevitably into the considerably more intractable theory of sequential stochastic games. (For some recent efforts in this area see [9, 10].)

It is the objective of this article to present a family of price-adjustment processes for firms in a single market which (a) are robust as well as computationally simple, (b) exhibit the fact that firms must experiment to discover profitable prices, and (c) possess orthodox convergence properties. From the viewpoint of economic theory it is interesting to note here that the convergence of these processes is determined largely by the convergence of corresponding rules (such as those studied in [1, 2, 31]) where the firms know their demand functions in advance. This is because the processes presented here converge if (i) the behavior of consumers is systematic enough even while it is random so that each firm can

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learn, through repeated trials, the demand for its product at any set of *fixed* prices, and if (ii) assuming the firms know their demand functions, the adjustment processes lead to prices which converge to an equilibrium. Since condition (ii) has already been well investigated in the economics literature, our main task is to investigate condition (i). As we shall see this condition captures certain subtle phenomena which elude informal discussions of price adjustment under uncertainty. As an example, we may mention here that prices can stabilize to an excess supply situation simply because firms react faster to excess demand than to excess supply.

From the mathematical viewpoint the proposed rules belong to the family known as "stochastic approximation" schemes following the pioneering paper of Robbins and Monro [7]. However, we shall follow the formulation due to Ljung [11] not only because it is considerably more general in several respects but also because it clearly points out the dual functions of learning and search mentioned above. Motivated by the same concerns as those mentioned above, Aoki [12, 13] has already used stochastic approximation methods to model some adjustment processes. The relation between his work and that presented here will be detailed in Section 4.

In the next section we state the main results of [11] in the form of an abstract adjustment model. In Section 3 this model is used to investigate stochastic versions of the more concrete processes proposed in [1, 2, 3].

2. An Abstract Adjustment Model

N firms produce and sell a homogeneous product. At the end of period t firm n sets certain instrumental variables (e.g. prices or quantities) denoted by the vector v_t^n. It is assumed that v_t^n belongs to an a priori fixed, compact set B^n. Let $B = B^1 x \ldots x B^N$. Let $v_t = (v_t^1, \ldots, v_t^N)$ be the distribution of these variables across the market. (In the following whenever a superscript is omitted from a variable name, it designates the vector whose components correspond to the various firms; thus $y = (y^1, \ldots, y^N)$ etc.) In period $t+1$ consumers search among firms and react to the distribution v_t. Their behavior as observed by the nth firm is formulated by it as the vector $y_t^n(v_t, \xi_{t+1})$ where $\{\xi_t\}$ is a random vector sequence defined on the probability space $(\Omega, \mathcal{F}, P)$. Note that y_t^n is determined by the action of all firms v_t and the random variable ξ_{t+1} which does not depend on v_t.

Based on this observation the firm adjusts its instruments at the end of period $t+1$ according to the rule

$$
\begin{aligned}
(2.1)\qquad v_{t+1}^n(\omega) &= [v_t^n(\omega) + \gamma_t H^n(y_t^n(v_t(\omega), \xi_{t+1}(\omega)))]^B \\
&= [v_t^n(\omega) + \gamma_t z_t^n(v_t(\omega), \xi_{t+1}(\omega))]^B.
\end{aligned}
$$

Here $\gamma_t > 0$ is a constant determining speed of adjustment, $H^n(\ )$ is a function which relates an observation to desirable directions of change in the variable v^n, $z_t^n(v, \xi) = H^n(y_t^n(v, \xi))$, and $[\ ]^B$ is any function satisfying

$$[x]^B \in B \text{ for all } x, [x]^B = x \quad \text{for } x \in B.$$

For each fixed v let $Z_t^n(v) = Ez_t^n(v, \xi_{t+1})$. The explicit dependence of $Z_t(v)$ is supposed to be transitory. That is, if consumers face a constant distribution v then their average behavior stabilizes, i.e., there is a function $Z(v)$ so that

$$\lim_{t\to\infty} Z_t(v) = Z(v) \quad \text{for each } v \in B.$$

Thus the environment is "stationary" in an important sense.

Next, it is assumed that as a result of firms seeking their goals, or as a hidden "aim" of market forces, the instrumental variables are directed to the v^* defined by the "equilibrium" condition

$$Z(v^*) = 0 \text{ i.e. } Z^n(v^*) = 0,\ n = 1, \ldots, N.$$

If each firm n could directly observe $Z^n(v)$ then v^* would be an equilibrium of the differential equation

$$\dot{v} = Z(v). \tag{2.2}$$

On the other hand, for any *fixed* v, $Z^n(v)$ can be estimated by

$$\zeta_{t+1}^n(v, \omega) = \zeta_t^n + \gamma_t[z_t^n(v, \xi_{t+1}(\omega)) - \zeta_t^n(v, \omega)],\ \zeta_0 = 0. \tag{2.3}$$

(For example, if $\gamma_t = t^{-1}$, then (2.3) yields $\zeta_t^n = t^{-1} \sum_{\tau\le t} z_\tau^n(v, \xi_{\tau+1})$ which is a robust estimator.) Thus the actual adjustment rule (2.1) can be seen as a way of combining simultaneously the "learning" process (2.3) and the equilibrium-seeking process (2.2).

We impose the following conditions.

(C1) $\qquad 0 \le \gamma_t \le 1, \qquad \gamma_t \to 0 \text{ as } t \to \infty, \qquad \sum_t \gamma_t = \infty.$

(C2a) For each fixed $v \in B$, the random sequence $\{\zeta_t(v)\}$ generated by (2.3) converges to $Z(v)$ a.s.

(C2b) For each fixed ξ, the function $z_t(v, \xi)$ is uniformly Lipschitz in v belonging to an open set $B^0 \supset B$, with Lipschitz constant $k_t(\xi)$. Furthermore, the random sequence $\{r_t\}$ generated by

$$r_{t+1}(\omega) = r_t(\omega) + \gamma_t[k_t(\xi_{t+1}(\omega)) - r_t(\omega)],\ r_0 = 0,$$

converges to a constant r a.s.

(C3a) The set B is defined by $B = \{v | \beta(v) \le b\}$ for some constant b where β is a twice continuously differentiable function, and there is a constant k so that

$$E[(z_t^n(v, \xi_{t+1}))' \beta_{vv}(u) z_t^n(v, \xi_{t+1})] \le k$$

for all t, and v, u in B. Here $\beta_{vv}(u)$ is the Hessian of β evaluated at u.

(C3b) For all $v \in \partial B$, the boundary of B,

$$(\beta_v(v))' Z(v) < 0.$$

(C3c) v^* is an asymptotically stable equilibrium of the differential equation (2.2), and its domain of attraction contains an open set $B^0 \supset B$.

We discuss these conditions after stating the main result.

T2.1. Consider the random sequence $\{v_t\}$ generated by (2.1). Suppose (C1)–(C3) hold. Then v_t converges to v^* a.s.

Proof. The assertion is an immediate consequence of Theorem 3.1, Theorem 5.2 and the subsequent remark in [11].

Consider the conditions in reverse order. (C3c) says that if $Z(v)$ were directly observable then all solutions of (2.2) which start in B converge to v^*. Since this case is well-studied in the literature, it need not detain us further. Since (C3b) is only slightly stronger than the statement that B is an invariant set of (2.2), it is usually satisfied whenever (C3c) is. (C3a) guarantees that the effect of the disturbances $\{\xi_t\}$ is not too large. For instance, it is easily verified if $z_t^n(v, \xi)$ is continuous in (v, ξ) uniformly in t and $\sup E|\xi_t|^2 < \infty$.

(C2a) guarantees that it is possible to estimate $Z(v)$ for any fixed v while (C2b) guarantees that estimates $\zeta_t(v)$ converges to $Z(v)$ uniformly for $v \in B$, and this implies in particular that $Z(v)$ is a Lipschitz function so that the differential equation (2.2) is well-behaved.

The requirement $\gamma_t > 0$ reflects the fact that in (2.1) z_t^n is a direction of desirable change in v_t^n, whereas the bound $\gamma_t \leq 1$ is merely a normalizing condition in light of the second condition $\gamma_t \to 0$. This latter condition is necessary if learning behavior is to be exhibited since then as time progresses new observations should have decreasing importance. The condition is discussed more fully in the next section in the context of a specific example. The divergence of $\sum \gamma_t$ is obviously essential.

The interesting conditions therefore are (C2a) and (C3c). Roughly speaking, the former guarantees that learning is possible in principle, while the latter guarantees convergence in the absence of uncertainty. The remaining conditions link these in such a way as to ensure that both functions can be carried out simultaneously.

We conclude this section by giving some simple conditions which guarantee (C2a) and (C2b).

L2.1. Suppose ξ_t satisfies (2.4) and γ_t satisfies (C1) and (2.5).

(2.4) $$\xi_s, \xi_t \text{ are independent if } |t-s| \geq M \text{ for some } M < \infty.$$

(2.5) $$\gamma_{t+1} \leq \gamma_t \text{ and } \gamma_{t+1} \geq \gamma_t(1-\gamma_{t+1}).$$

(i) If for each v in B there exists $\alpha > 1$ such that

(2.6) $$E|z_t(v, \xi_{t+1}) - Z_t(v)|^\alpha \leq \delta_t^\alpha$$

for some nondecreasing sequence δ_t and

(2.7) $$\sum_t \gamma_t^{\bar{\alpha}} \delta_t^\alpha < \infty, \text{ where } \bar{\alpha} = \min(\alpha, 1 + \tfrac{1}{2}\alpha),$$

then (C2a) holds.

(ii) If $Ek_t(\xi_{t+1}) = r_t$ converges to r, and if there exists $\alpha > 1$ such that

(2.8) $$E|k_t(\xi_{t+1}) - r_t|^\alpha \leq \delta_t^\alpha$$

for δ_t nondecreasing and (2.7) is satisfied, then (C2b) holds. Furthermore, if $1 < \alpha \leq 2$, then (2.5) is not needed.

Proof. See Appendix A.1.

Consider (2.4). Note that the disturbance term is a consequence of consumer search. If periods remote in the past do not affect their search and hence their behavior in the present period, or if their behavior varies independently between distant periods, or is a new "generation" of consumers replaces the previous generation every so often; in all such environments (2.4) may be reasonable.[1] (2.5) has the following meaning. From (2.5), $\zeta_t^n(v)$ can be expressed as a moving average

$$\zeta_t^n(v) = \sum_{\tau \leq t} c_{t,\tau} z_\tau^n(v, \xi_{\tau+1})$$

where $c_{t,\tau} = \gamma_{\tau-1} \prod_{s=\tau}^{t-1} (1-\gamma_s)$ if $\tau < t$ and $c_{t,t} = \gamma_{t-1}$, so that $c_{t,\tau+1} \geq c_{t,\tau}$ if and only if $\gamma_\tau \geq \gamma_{\tau-1}(1-\gamma_\tau)$. Thus (2.5) means that in the estimate $\zeta_t^n(v)$ recent observations are weighted more heavily than previous observations. Condition (2.7) is more interesting since it exhibits a trade-off between the efficiency of search and learning. Specifically, the slower γ_t converges to zero the greater is the effect of disturbances on the learning process, but the faster is the convergence of v_t to v^* if it converges at all, and (2.7) displays this conflict in the two functions.

3. Some Concrete Adjustment Processes

In this section we follow the abstract model introduced above to obtain stochastic versions of some adjustment processes studied in the literature.

3.1 *Fisher's Quasi-Competitive Adjustment*

For a discussion of this model the reader should consult [2]. At the end of period t, firm n ($n = 1, \ldots, N$), believing that it faces a flat demand curve, sets a price p_t^n and offers for sale the amount $S^n(p_t^n)$. $p_t^n \in B^n = [\underline{b}, \bar{b}] \subset R_+$, and $S^n(p^n)$ is just the inverse of the marginal cost curve.

In period $t+1$ consumers search among various firms and register the demand $d^n(p_t, \xi_{t+1})$ at firm n. $\{\xi_t\}$ is a stationary sequence of random vectors. Let $x^n(p_t, \xi_{t+1}) = d^n(p_t, \xi_{t+1}) - S^n(p_t^n)$ be the excess demand of firm n at end of period $t+1$. For each p fixed let $D^n(p) = E[d^n(p, \xi_t)]$, and let $X^n(p) = D^n(p) - S^n(p)$. At the end of period $t+1$ the firm adjusts its price according to the rule

$$p_{t+1}^n = [p_t^n + \gamma_t h^n x^n(p_t, \xi_{t+1})]^B \tag{3.1}$$

where $h^n > 0$ is constant.

Assume that ξ_t is bounded a.s. Let $B^0 \supset B$ be an open set such that for fixed ξ, $x^n(p, \xi)$ is Lipschitz in $p \in B^0$ with constant $k(\xi)$ and

$$k(\xi_t) \text{ is bounded a.s.} \tag{3.2}$$

Assume further that for each fixed p

$$x^n(p, \xi_t) \text{ is bounded a.s.} \tag{3.3}$$

[1] Also, as seen from [11], (2.4) can be replaced by weaker conditions which imply that v_t and ξ_{t+1} become independent as $t \to \infty$.

Finally, assume that $p^* \in B$ is a unique, globally stable equilibrium of the differential equation

$$\dot{p}^n = h^n X^n(p), n = 1, \ldots, N \tag{3.4}$$

with a domain of attraction containing B^0, and that

$$X^n(p) > 0 \quad \text{if } p^n \leq \underline{b}; \qquad X^n(p) < 0 \quad \text{if } p^n \geq \bar{b}. \tag{3.5}$$

T3.1. Suppose the assumptions made above hold. Suppose γ_t satisfies (C1), (2.5) and for some $\alpha > 1$

$$\sum_t \gamma_t^\alpha < \infty. \tag{3.6}$$

Suppose ξ_t, ξ_s are independent for $|t-s| \geq M$. Then the random-sequence $\{p_t\}$ generated by (2.1) converges to p^* a.s.

Proof. Because of T2.1 we only need to verify conditions (i), (ii) of L2.1. Because of (3.2), (3.3) and (3.6) these conditions hold for $\delta_t \equiv$ constant.

Since Fisher has extensively discussed the stability of (3.4), we need not consider it any further. (3.5) is reasonable in the partial equilibrium context of the model. Hence we shall only deal with the stochastic aspects of (3.1).

At first sight the stationary but myopic adjustment process

$$p_{t+1}^n = p_t^n + h^n x^n(p_t, \xi_{t+1}) \tag{3.7}$$

may appear more plausible than (3.1). Similar schemes have been studied for example in [6] and [15].[2] However (3.7) is incompatible with learning in the sense that if the firm knows it faces randomly fluctuating demand and if its intention is to discover constant levels of price and production which are compatible with average conditions of demand, then this intention cannot be realized through (3.7). For suppose for simplicity that $N = 1$, $S(p) = s_0 + sp$, $d(p, \xi_t) = d_0 - dp + \xi_t$ and ξ_t are independent with zero mean. Suppose further that h is so small that $|1 - h(s+d)| < 1$. Then $Ep_t \to \bar{p}$ and $E[p_t - Ep_t]^2 \to \sigma^2$ where $s_0 + s\bar{p} = d_0 + d\bar{p}$ and $\sigma^2 = h^2[1 - h(s+d)]^{-2} E\xi_t^2 > 0$ if $E\xi_t^2 > 0$. Thus, while the statistical average of the excess demand tends to vanish, actual prices and production levels fluctuate constantly with the demand.[3] On the other hand, if the firm is aiming to meet average demand then, as it gains information about this average demand, it must respond less and less to instantaneous fluctuations. This accounts for $\gamma_t \to 0$. A similar phenomenon occurs in [4] where the firm eventually stops adjusting its price even though demand continues to change randomly.

Instead of prices adjusting in proportion to excess demand, we could consider

$$p_{t+1}^n = p_t^n + \gamma_t H^n[x^n(p_t, \xi_{t+1})] \tag{3.8}$$

where H^n is a sign-preserving function. Suppose H^n has at most linear growth. Then $p_t \to \hat{p}$ a.s. where $EH^n[x^n(\hat{p}, \xi_t)] = 0$ so that $\hat{p}$ may not equal the competitive

[2] Our information regarding [15] is limited to the discussion in [5].

[3] Incidentally, a time-continuous version of this example shows that Theorem 3.3 of [6] is incorrect.

equilibrium p^*. Indeed, consider the specification of the preceding example with $n=1$, $H(x)=\alpha x$ if $x>0$, $H(x)=x$ if $x<0$. Then $\hat{p}$ is determined by $EH[x(\hat{p})+\xi_t]=0$ or $\alpha E[x(\hat{p})+\xi_t]^+=E[x(\hat{p})+\xi_t]^-$ where $f^+=fV0$ and $f^-=(-f)V0$. It follows that if $\alpha>1$, i.e., the firm reacts faster to positive excess demand, then $X(\hat{p})<0$ so that at the equilibrium price there is positive excess supply. Furthermore, the greater is the randomness in demand, the larger will be the value of $|X(\hat{p})|$. For instance, if ξ_t is uniformly distributed over $[-a, a]$, then $X(\hat{p})=-(1+\alpha)(\alpha-a)^{-1}a$. Thus if we interpret the sellers as workers supplying labor and if wages rise faster in conditions of excess demand than they fall in situations of unemployment, then wages will converge to an equilibrium where there is unemployment. Of course, the opposite tendency prevails if $\alpha<1$.

One final remark in connection with nonlinear functions H^n may be of interest. Under the conditions on $X(p)$ in [2], p^* is the competitive equilibrium and so, in particular, all of its components p^{*n} are equal. The equilibrium $\hat{p}$ is given by $EH^n[X^n(\hat{p})+\xi_t]=0$ $n=1,\ldots,N$. Even under the same conditions as in [2], it is, of course, no more generally the case that all the $\hat{p}^n$ are equal. A similar conclusion is reached in [4, p. 201] except that these differences in adjustment processes arise from different beliefs about the structure of the random demand.

3.2 *Diamond's Adjustment Model*

The reader should consult [1] for the model discussed here. Again there are several firms. In each period consumers search randomly among these firms but they do not discriminate between them on the basis of previous experience. Therefore each firm faces demand functions whose statistical properties are identical and so we need consider one firm only. A consumer who entered the market at some time $\tau\le t$ stays in the market until he encounters a price p_t which exceeds his own cutoff price q_t^τ. He then purchases the amount $d(p_t, \xi_{t+1})$. The cutoff price q_t^τ depends in some random way upon previous prices $p_\tau, \ldots, p_{t-1}$. In keeping with the spirit of the search process as described in [1], it is assumed here that $\{\xi_t\}$ is a sequence of stationary, independent random vectors.

Let $N_t^\tau(p)$ be the (random) number of consumers who entered the market at τ, who are still in the market at t, and whose cutoff price exceeds p. Then the demand function facing our firm in period t is $\sum_{\tau\le t} d(p, \xi_{t+1})N_t^\tau(p)$. The firm's unit production cost is constant, and we may assume it is zero, so that its profit function is $pd(p, \xi_{t+1})N_t(p)$ where $N_t(p)=\sum_{\tau\le t} N_t^\tau(p)$.

Let $r(p, \xi_{t+1})=pd(p, \xi_{t+1})$ and $R(p)=Er(p, \xi_{t+1})$ for each fixed p. Let $p_t\in B=[\underline{b}, \bar{b}]\subset R_+$ be the price set by the firm at the end of period t. Then in period $t+1$ it observes $N_t(p_t)$ and $r(p_t, \xi_{t+1})N_t(p_t)$ so that it knows $r(p_t, \xi_{t+1})$.

Suppose for the moment that the firm wishes to set its price at a level p^s at which $R(p^s)$ equals some "satisficing" level R^s.[4] Then if p_t is adjusted according to the rule

(3.9) $$p_{t+1}=[p_t+\gamma_t(R^s-r(p_t, \xi_{t+1}))]^B,$$

[4] We do not discuss the role of $N_t(p)$ any further since under the assumptions on the adjustments of the cutoff price given in [1], $N_t(p_t)\to N$, the total (fixed) number of customers in the market, whenever p_t converges.

it will converge to p^* a.s. under appropriate conditions which can be obtained from the results of the previous section.

Now suppose the firm wishes to maximize $R(p)$. Then if the maximum value of the profit is not known, a rule such as (3.9) is clearly inappropriate. In essence, the firm needs to obtain information from which it can infer whether or not it has reached a profit-maximizing position, and if it has not done so which direction of change would lead to an increase in profits. Such information is provided by the marginal revenue function $M(p)=(dR/dp)(p)$. Suppose momentarily that at each t the firm can obtain a sample $m_t(p_t, \eta_{t+1})$ where $\{\eta_t\}$ is a random sequence so that, for each fixed p, $Em_t(p, \eta_{t+1})=M_t(p)\to M(p)$ as $t\to\infty$. Then prices adjusted according to

$$p_{t+1}=[p_t+\gamma_t m_t(p_t, \eta_{t+1})]^B \tag{3.10}$$

would, under appropriate conditions, converge to p^* at which $M(p^*)=0$.

The sample $m_t(p_t, \eta_{t+1})$ may be obtained directly in some way not explicitly considered in the model, or it can be obtained by experimentation in the following way. Suppose each period t consists of two subperiods labeled $(t, 1)$ and $(t, 2)$. Suppose at the end of period $(t, 1)$, the firm's price is $p_{t,1}$ and in period $(t, 2)$ it observes $r(p_{t,1}, \xi_{t,2})$. At the end of period $(t, 2)$, it sets the price $p_{t,2}=p_{t,1}-a_t$ where $a_t>0$ is a predetermined sequence to be specified further. In period $(t+1, 1)$ the firm observes $r(p_{t,1}-a_t, \xi_{t+1,1})$. Define

$$m_t(p_{t,1}, \eta_{t+1})=\frac{1}{a_t}[r(p_{t,1}, \xi_{t,2})-r(p_{t,1}-a_t, \xi_{t+1,1})] \tag{3.11}$$

where $\eta_{t+1}=(\xi_{t,2}, \xi_{t+1,1})$. Suppose now that $p_{t+1,1}$ is adjusted according to

$$p_{t+1,1}=[p_{t,1}+\gamma_t m_t(p_{t,1}, \eta_{t+1})]^B \tag{3.12}$$

in a way quite similar to (3.10).

We can use T2.1 and L2.1 to obtain sets of conditions under any one of which the sequence $p_{t,1}$ generated by Diamond's process (3.12) converges to the profit-maximizing price. Here is one such result whose proof can be readily constructed using T2.1 and L2.1.

T3.2. Let $B^0\supset B$ be an open set such that

(i) $M(\underline{b})>0$, $M(\bar{b})<0$, M is monotonic on B^0,

(ii) $E[r(p, \xi_t)-r(p)]^2\le\sigma_r^2<\infty$ for $p\in B^0$,

(iii) $r(p, \xi)$ is twice continuously differentiable in p for fixed ξ and

$$E\left[\frac{\partial r}{\partial p}(p, \xi_t)-\frac{dR}{dp}\right]^2\le\sigma_m^2<\infty \quad \text{for} \quad p\in B^0$$

(iv) $0\le\gamma_t\le 1,\quad \gamma_t\to 0,\quad a_t\to 0,\quad \sum_t\gamma_t=\infty,\quad \sum_t\left(\frac{\gamma_t}{a_t}\right)^2<\infty.$

Then $p_{t,1}$ (and hence $p_{t,2}$ also) converges to P^* a.s. where $M(p^*)=0$.

Suppose the firm had a direct estimate of $M(p)$ so that it could use (3.10). Then, convergence of (3.10) to p^* would be guaranteed under conditions (i), (ii),

(iii) and

(iv′) $$0 \le \gamma_t \le 1,\ \gamma_t \to 0,\ \sum_t \gamma_t = \infty,\ \sum_t \gamma_t^2 < \infty .$$

Note that (iv′) is considerably weaker than (iv) because since $a_t \to 0$ a sequence $\{\gamma_t\}$ satisfying (iv) must decrease much faster than if it had to satisfy (iv′). In turn this means that convergence of p_t under (iv) is considerably slower than under (iv′). This is a reflection of the fact that obtaining an estimate of the marginal revenue function $M(p)$ from observations of random demand is a subtle process since M is not given in parametric form. Of course, if it were parametrized, say, it is known to be a linear function with unknown slope and intercept, then faster convergence is possible.

3.3 *Adjustment to a Nash Equilibrium*

For discussion of the model introduced here see [3]. In a certain sense this is an extension of Diamond's model discussed earlier. Consumer search behavior causes the demand faced by a firm to depend upon the other firms' prices. Each firm realizes that it faces a sloping demand curve, and its objective is to exploit this monopoly power.

If $p_t^n \in B^n = [\underline{b}, \bar{b}] \subset R_+$ is the price set by firm n at the end of period t, then the demand for its product during period $t+1$ is $d^n(p_t, \xi_{t+1})$ where $p_t = (p_t^1, \ldots, p_t^N)$ is the distribution of prices among the N firms and $\{\xi_t\}$ is a stationary random sequence. Let $D^n(p) = Ed^n(p, \xi_t)$. $C^n(q^n)$ is the cost function of firm n. We assume that D^n and C^n are twice continuously differentiable. Define the profit function $\pi^n(p) = p^n D^n(p) - C^n[D^n(p)]$. We assume with Fisher [3, p. 449] that for fixed values of p^i, $i \ne n$

(3.13) $$\pi^n(p) \text{ is strictly concave in } p^n \in B^n.$$

It follows from a result due to Rosen [14] that there exists a Nash equilibrium price vector $\tilde{p} \in B$, i.e.,

$$\pi^n(\tilde{p}^1, \ldots, p^n, \ldots, \tilde{p}^N) \le \pi^n(\tilde{p}) \quad \text{for} \quad p^n \in B^n, n = 1, \ldots, N.$$[5]

Next we assume (see [3, Theorem 3.1]) that

(3.14) $$\tilde{p} \text{ is the unique Nash equilibrium in } B$$

and for each n

(3.15) $$\frac{\partial \pi^n}{\partial p^n}(p) > 0 \quad \text{if} \quad p^n = \underline{b}, \frac{\partial \pi^n}{\partial p^n}(p) < 0 \quad \text{if} \quad p^n = \bar{b},$$

which is reasonable and self-explanatory.

[5] Fisher assumes the existence of the Nash equilibrium when in fact it is a consequence of the concavity assumption.

Now, for each $p=(p^1,\dots,p^N)\in B$, let $\bar{p}^n=\bar{p}^n(p)\in B^n$ be the price at which firm n maximizes profit when all the other firms' prices are fixed at p^i, $i\neq n$, i.e.,

$$\pi^n(p^1,\dots,\hat{p}^n,\dots,p^N)\leq\pi^n(p^1,\dots,\bar{p}^n,\dots,p^N)\quad\text{for}\quad\hat{p}^n\in B^n.$$

Equivalently, in view of (3.15), $\bar{p}^n$ is determined by

$$\frac{\partial\pi^n}{\partial p^n}(p^1,\dots,\bar{p}^n,\dots,p^N)=0. \tag{3.16}$$

Fisher imposes enough additional conditions to guarantee that the adjustment rule

$$\dot{p}^n=H^n[\bar{p}^n(p)-p^n], \tag{3.17}$$

where H^n is any sign-preserving function, has an asymptotically, globally stable equilibrium $\tilde{p}$ (see [3, Theorem 3.2]). Now for (3.17) to be a good description of an adjustment process, it presupposes that each firm n has an accurate knowledge of its profit function $\pi^n(p)$, so that it could solve for $\bar{p}^n$ from (3.16). If, however, it does not have this knowledge, the firm can still attempt to estimate $\partial\pi^n/\partial p^n$ from the observed random demand $d^n(p_t,\xi_{t+1})$ and its own cost function, as was proposed in regard to the Diamond model. For simplicity we assume that the firm has directly available to it the observation $m^n(p_t,\eta_{t+1})$ such that

$$Em^n(p,\eta_t)=\frac{\partial\pi^n}{\partial p^n}(p)\quad\text{for each fixed } p\in B, \tag{3.18}$$

and it uses this observation to adjust p^n according to the rule

$$p^n_{t+1}=p^n_t+\gamma_t m^n(p_t,\eta_{t+1}). \tag{3.19}$$

We can prove the following convergence result.

T3.3. Assume that (3.14), (3.15) hold and that $\tilde{p}$ is the unique asymptotically, globally stable equilibrium of (3.17) (for any sign-preserving H^n) with domain of attraction containing B. Assume that $\{\eta_t\}$ is a bounded, stationary process with η_t, η_s independent when $|t-s|\geq T$ for some $T<\infty$. Let $\{\gamma_t\}$ be a sequence satisfying

$$0\leq\gamma_t\leq 1,\ \gamma_t\to 0\quad\text{as}\quad t\to\infty,\ \gamma_{t+1}\geq\gamma_t(1-\gamma_{t+1}),\quad \textstyle\sum\gamma_t=\infty, \sum\gamma_t^{\alpha}<\infty \tag{3.20}$$

for some $\alpha>1$. Then the random sequence $\{p_t\}$ generated by (3.19) converges a.s. to $\tilde{p}$.

Proof. See Appendix A.2.

From our earliest discussion it is clear that if instead of (3.19) we consider the more general rule

$$p^n_{t+1}=p^n_t+\gamma_t H^n[m^n(p_t,\eta_{t+1})]$$

we still get convergence to $\tilde{p}$ if $H^n(x)=h^n x$ for h^n a positive constant, whereas, if H^n is a nonlinear sign-preserving function then the process is likely to converge to a different equilibrium. One additional remark which concerns the Fisher adjustment process (3.17) may be of interest. In the proof of T3.3 we show that the stability of (3.17) and the concavity assumption (3.13) together imply that the trajectories of

$$\dot{p}^n(t) = \frac{\partial \pi^n}{\partial p^n}(p(t)) \tag{3.21}$$

converge to $\bar{p}$. Now, from relatively abstract mathematical considerations (see [16]), we know that Nash equilibria are unlikely to be stable equilibria of systems with gradient dynamics as in (3.21). This consideration provides an argument (in addition to those made by Fisher himself) that the assumptions which guarantee stability of (3.17) are very restrictive.

4. Conclusions and Numerical Results

We have tried to show how disequilibrium adjustment processes which consider only "deterministic" environments can be modeled as stochastic approximation schemes so as to take into account uncertainties on the part of sellers. In doing so we have discovered a variety of subtle phenomena which have escaped informal discussions of the subject.

Aoki appears to be the first to have used stochastic approximation methods to model adjustment processes and he has been motivated by the same concerns. In [12] he has compared a Robbins–Monro scheme with three Bayesian formulations, all for a single firm whose demand function depends only on its own price, and shows that they are asymptotically equivalent. In [13] he considers several interacting firms in the same industry. In each period t each firm n adjusts its output rate q_t^n (subject to an exogenous disturbance), and then learns the common market clearing price p_t based on which it adjusts the next period's output rate. Thus in [13] the interacting firms are Marshallian quantity adjusting firms, somewhat similar to [17], unlike the price adjusting firms described here.

While we have given conditions which guarantee convergence to an equilibrium, the actual rate of convergence and the behavior of the price sequence outside of equilibrium depends critically upon the adjustment coefficients γ_t and the magnitude of the disturbances. While estimates of the asymptotic behavior of the random price sequence are available (see [11]–[13]), these estimates provide no understanding of the "transient" behavior. Therefore we present below the results of a numerical experiment of the scheme of Section 3.1 for a two-firm case. In terms of the notation introduced there $N = 2$, $d^n(p, \xi_t) = D^n(p) + \xi_t^n$ where ξ_t^n is

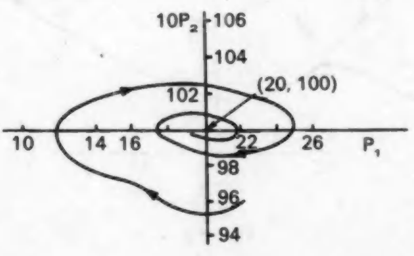

Figure 1

uniformly distributed over $[-a, a]$, $n=1,2$. The average demands $D^n(p)$ are taken to be

$$D^1(p)=\frac{p^2}{p^1+p^2}\cdot\frac{\beta}{p^1+p^2}+\delta^1 p^2,$$

$$D^2(p)=\frac{p^1}{p^1+p^2}\cdot\frac{\beta}{p^1+p^2}+\delta^2 p^1,$$

where δ^1, δ^2 and β are positive constants. The supply functions are taken as

$$S^n(p^n)=\alpha^n(p^n+c^n),^6 \qquad n=1,2$$

with α^n, c^n as positive constants. Equation (3.1) now reads

$$p^n_{t+1}=[p^n_t+\gamma_t(D^n(p^n_t)+\xi^n_t-S^n(p^n_t))]^B, \qquad n=1,2 \tag{4.1}$$

Nine sample paths of (4.1) are presented corresponding to three different values of $\{\gamma_t\}$ and three different values of the noise parameter a, as shown in Table 4.1 below. In Figures 2a, b, c, $\gamma_t\equiv 0.012$, a constant. Since γ_t is large, initial convergence is rapid but as t increases we do not get convergence since γ_t does not

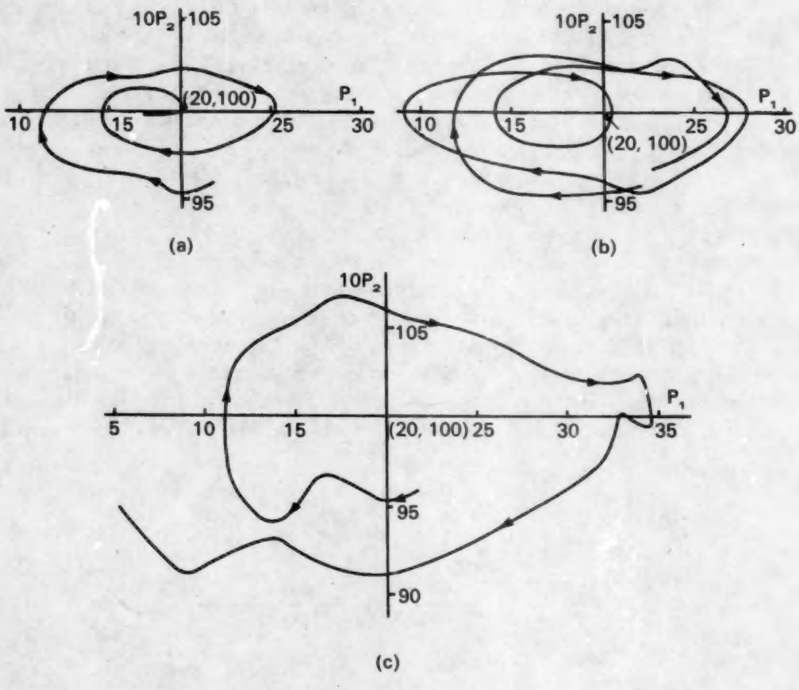

Figure 2

[6] The parameter values used in the numerical example are these: $\beta=100$, $\delta^1=27$, $\delta^2=0$, $\alpha^1=\alpha^2=1/18$, $c^1=4860$, $c^2=30$.

TABLE 4.1

a \ γ_t	0.012	$5(250+t)^{-1}$	$0.1t^{-1}$
1.0	2a	3a	4a
2.5	2b	3b	4b
5.0	2c	3c	4c

converge to zero. In Figures 3a, b, c we obtain convergence. In Figures 4a, b, c convergence is extremely small since γ_t is small. In all cases $t = 1, \ldots, 1250$. It is

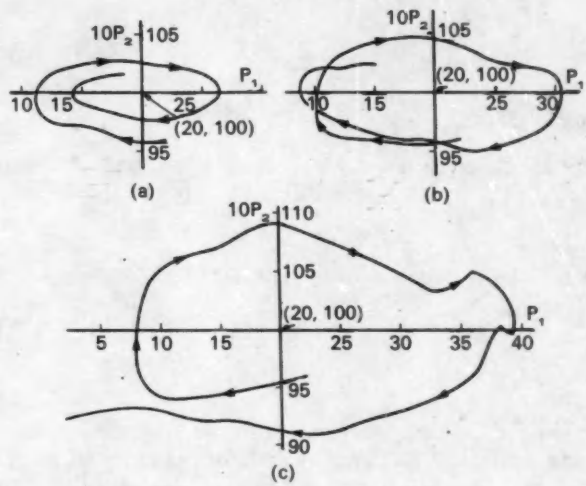

Figure 3

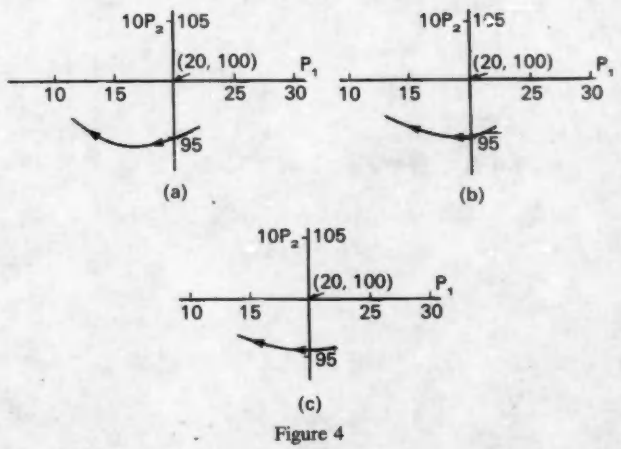

Figure 4

evident from these Figures that increased values of a leads to poorer convergence behavior. Ljung [11] has shown that asymptotically the behavior of the random sequence (4.1) is similar to the behavior of the trajectories of the differential equations

$$\dot{p}^n = D^n(p) - S^n(p) \qquad n = 1, 2 \tag{4.2}$$

and, for purposes of comparison, one trajectory of (4.2) is plotted in Figure 1.

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Appendix

A.1. *Proof of L2.1*

Since the proofs of (i) and (ii) are identical, we only prove (i). Let $\Delta_t = \zeta_t^n(v, \xi_{t+1}) - Z_t^n(v)$, $e_{t+1} = z_t^n(v, \xi_{t+1}) - Z_t^n(v)$. Then, by (2.3),

$$\Delta_{t+1} = \Delta_t + \gamma_t(e_{t+1} - \Delta_t)$$

and it must be shown that $\Delta_t \to 0$ a.s. For $m = 1, \ldots, M$ define the random sequence $\{e_t^m\}$ by

$$e_t^m = \begin{cases} e_t \text{ if } t = m \text{ modulo } M \\ 0 \text{ otherwise}. \end{cases}$$

Then $Ee_t^m = 0$ and e_t^m, e_s^m are independent for $t \neq s$ because of (2.4). By (2.6) $E|e_t^m|^\alpha \leq \delta_t^\alpha$ and we have (2.7), $\sum_t \gamma_t^{\alpha} \delta_t^\alpha < \infty$. It follows from Theorem 4.3 of [11] (where condition (2.5) is used) that the random sequence $\Delta_t^m \to 0$ a.s. where $\Delta_{t+1}^m = \Delta_t^m + \gamma_t(e_{t+1}^m - \Delta_t^m)$. But since $e_t = \sum_m e_t^m$ we have $\Delta_t = \sum_m \Delta_t^m$, so that $\Delta_t \to 0$ a.s.

A.2. *Proof of T.3*

The only difficulty in applying T2.1 and L2.1 is to show that under the hypothesis of the theorem $\bar{p}$ is a globally asymptotically stable equilibrium of

$$\dot{p}^n = \frac{\partial \pi^n}{\partial p^n}(p). \tag{A.1}$$

Now suppose $\bar{p}^n = \bar{p}^n(p) \neq p^n$ so that

$$\pi^n(p^1, \ldots, \bar{p}^n, \ldots, p^N) > \pi^n(p).$$

It is then a consequence of (3.13) and the definition (3.16) of $\bar{p}^n$ that

$$\frac{\partial \pi^n}{\partial p^n} \gtrless 0 \text{ according as } \bar{p}^n - p^n \gtrless 0.$$

Hence there is a sign-preserving function H^n (which depends on p) such that

$$\frac{\partial \pi^n}{\partial p^n} = H^n(\bar{p}^n - p^n)$$

and the stability of (A.1) follows.

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