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BUDGETING, DECENTRALIZATION, AND AGGREGATION*

BY CHARLES BLACKORBY, DANIEL PRIMONT, AND R. ROBERT RUSSELL

The purpose of this article is to integrate the many results on the relationships among functional structure, consumer budgeting and decentralization, price aggregation, and demand analysis. The article then examines implications of these results for the empirical analysis of consumer expenditure data. New results concerning "weakly recursive" structures are also presented.

I. INTRODUCTION

Interest in functional structure, conceived independently by Leontief [1947] and Sono [1961], was rejuvenated by Strotz [1957, 1959] and Gorman [1959] in the context of consumer budgeting. Consumer budgeting was defined by Strotz [1957, p. 271] as follows: "A decision is first made as to how income should be allocated among the budget branches (given all prices). Each budget allotment is then spent optimally on the commodities in its branch, with no further references to purchases in other branches." Strotz and Gorman then went on to show that this type of behavior is rationalized by certain separability conditions regarding the consumer's utility function. Moreover, these restrictions on the consumer's preferences imply empirically refutable restrictions on the system of demand functions. The Strotz–Gorman analysis of the restrictions on demand functions implied by (symmetrically) structured direct utility functions was extended by Goldman and Uzawa [1964]. The demand implications of (symmetrically) structured *indirect* utility functions, first examined by Houthakker [1965] and Samuelson [1965], were extended by Lau (1969b].

Although separability is an inherently asymmetric concept, all of the above literature (except for Leontief) focuses on symmetrically structured direct and indirect utility functions. The demand implications of asymmetrically structured utility functions were developed by Lady and Nissen [1968] and Primont [1970]. Analysis of asymmetrically structured indirect utility functions can be found in Blackorby, Nissen, Primont, and Russell [1974].

It was first pointed out by Lau [1969a] that the Strotz–Gorman discussion of consumer budgeting, extended by Green [1964] and Blackorby, Lady, Nissen, and Russell [1970], confused two quite different notions corresponding to the two sentences in the above quote from Strotz. A similar observation can be found in Pollak [1970]. Indeed, a third notion—that of price aggregation—permeates the discussion of consumer budgeting.

It is the purpose of this paper to integrate the many results on the relationships between functional structure, consumer budgeting and decentralization, price

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aggregation, and demand analysis and to discuss the implications of these results for the empirical analysis of consumer expenditure data. As such, most of the results in this paper can be found in the literature cited above, although there are some new results (particularly concerning "weakly recursive" structures).[1] Although this paper attempts to integrate the literature on consumer budgeting and demand analysis in a way which will hopefully prove useful to demand analysts, an important caveat should be emphasized: this is not a historically complete survey of this area of consumer theory. Consequently, the omission of references to important works in the area merely reflects our own admittedly parochial education in this subject. This parochial perspective is also reflected in the focus and emphasis of the paper.[2]

The paper is organized as follows. Section II summarizes the relationships between various separability conditions and functional structure. There are few new results in this section. Section III discusses the relationship between functional structure and symmetric budgeting, decentralization, and price aggregation and Section IV examines the asymmetric counterparts of these concepts. Section V contains a few concluding remarks.

II. Separability, Functional Structure, and Demand

1. *Functional Separability*

Let Ω and Ω_+ be the nonnegative and strictly positive n-orthants, respectively. Denote commodity bundles by $X = [x_1, \ldots, x_n] \in \Omega$, and corresponding price vectors by $P = [p_1, \ldots, p_n] \in \Omega_+$. Letting the strictly positive scalar y represent consumer expenditure, $P/y \in \Omega_+$ is the normalized price vector.

The variable indices of X and P form the set, $\chi = [1, \ldots, n]$. Partition χ into m subsets or sectors, $\{\chi^1, \ldots, \chi^m\}$. Correspondingly, the vectors X and P have decompositions, $X = [X^1, \ldots, X^m]$, and $P = [P^1, \ldots, P^m]$. Similarly, Ω and Ω_+ have Cartesian decompositions $\Omega = \Omega^1 \times \ldots \times \Omega^m$ and $\Omega_+ = \Omega^1_+ \times \ldots \times \Omega^m_+$. When the k-th good (or price) is in the r-th sector, x_k is a component of $X^r \in \Omega^r$ and p_k is a component of $P^r \in \Omega^r_+$.

Let $U: \Omega \to R$ be a strictly quasi-concave, continuous, non-decreasing utility function, strictly increasing in one coordinate; and let $V: \Omega_+ \to R$, defined by

$$V(P/y) = \max_X \left\{ U(X) \,\middle|\, \frac{P}{y} \cdot X \leq 1 \right\},$$

be the corresponding indirect utility function. Given the properties of U, V is necessarily quasi-convex, continuous, nonincreasing and strictly decreasing in one coordinate. For our purposes, it is convenient to assume, in addition, that V is *strictly* quasi-convex.[3]

[1] In the interest of brevity, all proofs are omitted. Proofs of most of the propositions are contained in Appendix A to this paper. The appendix can be obtained from the authors.

[2] For a historical account of the separability literature, see Geary and Morishima [1973].

[3] In many, if not most, of the results of this paper, these conditions can be weakened considerably. This could be done, however, only at the cost of seriously complicating the exposition. See Diewert [1974].

Partition χ into χ^r and χ^c by letting $\chi^c = \bigcup_{s \neq r} \chi^s$ and define the correspondence, $\beta^r : \Omega \to \mathscr{P}(\Omega^r)$, by

$$\beta^r(X^r, X^c) = \{\hat{X}^r \in \Omega^r | U(\hat{X}^r, X^c) \geq U(X^r, X^c)\}.$$

This correspondence therefore defines a set of points in Ω^r for each fixed reference vector (X^r, X^c) such that each point in $\beta^r(X^r, X^c) \times \{X^c\}$ is "no worse than" (X^r, X^c).

The set of variables, χ^r, is said to be *separable* from the k-th variable in U if $\beta^r(X^r, X^c)$ is invariant with respect to the value of the k-th variable, x_k. This separability condition is equivalent to Gorman's [1968, p. 367] condition that "the conditional ordering on $[\Omega^r]$ is the same for all" values of x_k.

If U is continuously twice differentiable, the r-th sector, χ^r, is separable from the k-th variable if and only if

$$\frac{\partial}{\partial x_k}\left(\frac{\partial U/\partial x_i}{\partial U/\partial x_j}\right) \equiv 0$$

for all $i, j \in \chi^r$ and for some $k \notin \chi^r$. That is, marginal rates of substitution between goods in the r-th sector do not depend on the value of x_k.

Similarly, the set χ^r is separable from the k-th variable in V, $k \notin \chi^r$, if the correspondence, $\alpha^r : \Omega_+ \to \mathscr{P}(\Omega^r_+)$, defined by

$$\alpha^r\left(\frac{P^r}{y}, \frac{P^c}{y}\right) \equiv \left\{\frac{\hat{P}^r}{y} \in \Omega^r_+ \,\middle|\, V\left(\frac{\hat{P}^r}{y}, \frac{P^c}{y}\right) \leq V\left(\frac{P^r}{y}, \frac{P^c}{y}\right)\right\},$$

is independent of the value of the k-th normalized price p_k/y. If V is continuously twice differentiable, this separability condition is

$$\frac{\partial}{\partial(p_k/y)}\left(\frac{\partial V/\partial(p_i/y)}{\partial V/\partial(p_j/y)}\right) \equiv 0$$

for all $i, j \in \chi^r$ and for some $k \notin \chi^r$. Using Roy's Theorem,

$$x_i = \frac{-\partial\hat{V}/\partial p_i}{\partial\hat{V}/\partial y},$$

where $\hat{V}$ is defined by $\hat{V}(P, y) \equiv V(P/y)$, the separability condition can be rewritten as $\partial(x_i/x_j)/\partial p_k \equiv 0$, for all $i, j \in \chi^r$. That is, ratios of demand for goods in χ^r are independent of the value of the k-th price.

2. *Symmetric Structures*

Consumer preferences are said to be *directly strongly separable*[4] if every proper subset of the set of sectors, $\{\chi^1, \ldots, \chi^m\}$, is separable from its complement in U; i.e., the union of any number of sectors is separable from the variables in the remaining sectors. Strong separability implies, but is not implied by, a weaker structure, namely weak separability. Preferences are *directly weakly separable* if every sector, χ^r, is separable in U from the variables in all the other sectors.

[4] That is, strongly separable "in the indirected partition of χ." This phrase is implicitly included in all of our discussion of symmetric separability.

Results of Debreu [1959] (also proved by Gorman [1968] and Katzner [1969]), characterize the forms of the utility functions implied by these symmetric structures. If $m > 2$,[5] preferences are directly separable if and only if there exist continuous functions, $F, f^1, f^2, \ldots, f^m$, such that the utility function can be written as

$$U(X^1, \ldots, X^m) = F(f^1(X^1) + \ldots + f^m(X^m)),$$

where $F(\cdot)$ is strictly increasing in its single argument. Of course, U can be normalized so that

$$U(X) = f^1(X^1) + \ldots + f^m(X^m).$$

If each of the category satisfaction functions of a strongly separable structure is homothetic, the function is said to be *homothetically strongly separable*. If, in addition, each category felicity function is homogeneous of the same degree, the overall utility function is also homothetic and hence a member of the so called "Bergson" family of functions.[6]

Preferences are weakly separable if and only if there exist continuous functions, $\hat{F}, f^1, \ldots, f^m$, such that the utility function can be written as

$$U(X^1, \ldots, X^m) \equiv \hat{F}(f^1(X^1), \ldots, f^m(X^m))$$

where $F(\cdot)$ is strictly increasing in each of its m arguments. If a function is weakly separable and each of the category functions is homothetic the function is said to be homothetically weakly separable. The function itself need not be homothetic. Furthermore, in contrast to the homothetically strongly separable case, if the function is homothetically weakly separable but not strongly separable, each category felicity function may be trivially normalized to be positively linearly homogeneous (PLH).

The above representations can be adapted to account for unstructured sectors, called free sectors by Gorman [1968]. For example, each sector $\{\chi^r, \ldots, \chi^m\}$ may be separable from all other variables, in which case there exist functions, $\hat{F}, f^r, \ldots, f^m$, such that

$$U(X^1, \ldots, X^m) = F(X^1, \ldots, X^{r-1}, f^r(X^r), \ldots, f^m(X^m)).$$

This may not exhaust the structure, for each category function may itself be weakly or strongly separable in some partition of its variables.

Indirect weak and strong separability is defined analogously to direct weak and strong separability by replacing the direct utility function U with the indirect utility function V and X with P/y. Thus, if $m > 2$, indirect strong separability is equivalent to the existence of continuous, strictly quasi-convex, nonincreasing functions $v^1, \ldots, v^m$, and an increasing continuous function $\hat{V}$ such that

$$V(P/y) = \hat{V}\left(\sum_{r=1}^{m} v^r(P^r/y)\right).$$

[5] If $m = 2$, weak and strong separability coincide and the following additive representation does not go through.

[6] Bergson [1936] was in fact only concerned with a coordinate wise partition of χ, but the extension to other partitions is sufficiently similar to warrant the same name. If each of the category functions is homogeneous, but not necessarily of the same degree, $\sum_r \log f^r(x^r)$ is also in the Bergson family.

Indirect weak separability is equivalent to the existence of continuous, strictly quasi-convex, nonincreasing functions, $v^1, \ldots, v^m$, and a strictly increasing continuous quasi-concave function $\hat{V}$ such that

$$V(P/y) = \hat{V}(v^1(P^1/y), \ldots, v^m(P^m/y)).$$

The discussion of direct weak and strong separability applies equally to the corresponding indirect structures. (Of course, the monotonicity properties are inverted so that homotheticity and PLH in the direct corresponds to negative homotheticity and negative linear homogeneity (NLH)[7] in the indirect).

In general, direct and indirect separability do not imply one another. However, U is homothetically strongly (weakly) separable if and only if V is negatively homothetically strongly (weakly) separable (Lau [1969b] and Blackorby, Primont, and Russell [1975]).

3. *Asymmetric Structures*

It is convenient to introduce some additional notation. The *continuation* of the r-th sector is ${}_r\chi = \bigcup_{s=r}^m \chi^s$ and the vector corresponding to this continuation is

$${}_rX = (X^r, \ldots, X^m) \in {}_r\Omega = \prod_{s=r}^m \Omega^s.$$

Similarly, ${}_r\chi_t = \bigcup_{s=r}^m \chi^s$; $r \leq t = 1, \ldots, m$. The corresponding vector is

$${}_rX_t = (X^r, \ldots, X^t) \in {}_r\Omega_t = \prod_{s=r}^m \Omega^s.$$

The vectors ${}_rP$ and ${}_rP_t$ are defined analogously.

The direct or indirect utility function is *strongly recursive* in the ordered partition $\{\chi^1, \ldots, \chi^m\}$ if and only if each continuation ${}_r\chi = \bigcup_{s=r}^m \chi^s, r = 2, \ldots, m$, is separable from the variables in the prior sectors, $\chi^1, \ldots, \chi^{r-1}$ (i.e., variables in ${}_1\chi_{r-1}$). The direct or indirect utility function[8] is *weakly recursive* in the ordered partition $\{\chi^1, \ldots, \chi^m\}$ if and only if each sector, χ^r, is separable from the variables in the sectors, $\chi^1, \ldots, \chi^{r-1}$ (i.e., variables in ${}_1\chi_{r-1}$).

Note that the ordering of the sector indices, $1, \ldots, m$, is important in the definitions of asymmetric structures. For if a function were, say, strongly recursive for any permutation of the m indices, it is strongly separable. This statement is also true if "strongly" is replaced with "weakly."

The following results are due to Lady and Nissen [1968], Gorman [1968], and Primont [1970]. First, the utility function is strongly recursive if and only if there exist continuous, strictly quasi-concave, nondecreasing functions, $f^1(\cdot)$, $\ldots, f^m(\cdot)$, such that

$$U(X^1, \ldots, X^m) = f^1(X^1, f^2),$$

[7] A function $f(\cdot)$ is *NLH* if $\lambda^{-1}f(x) = f(\lambda x)$, for all $\lambda > 0$.

[8] It is the preferences which have structure, and are represented by functions which mirror that structure. This economizes somewhat an already awkward terminology.

where

$$f^r \equiv f^r(X^r, f^{r+1}), \qquad r = 1, \ldots, m-1,$$

and

$$f^m \equiv f^m(X^m);$$

and for $r = 1, \ldots, m-1, f^r(\cdot)$ is strictly increasing in f^{r+1}.

The utility function is weakly recursive if and only if there exist continuous, strictly quasi-concave functions, $f^1(\cdot), \ldots, f^m(\cdot)$, such that the utility function can be written as

$$F(X^1, \ldots, X^m) \equiv f^1(X^1, f^2, \ldots, f^m) \equiv f^1(X^1, {}_2f),$$

where

$$f^r \equiv f^r(X^r, f^{r+1}, \ldots, f^m) \equiv f^r(X^r, {}_{r+1}f), \qquad r = 2, \ldots, m-1,$$

and

$$f^m \equiv f^m(X^m),$$

where $f^1(\cdot)$ is strictly increasing in $f^2, \ldots, f^m$ and for $r = 1, \ldots, m, f^r(\cdot)$ is nondecreasing in X^r.

The analogous representation for indirect strong recursivity is

$$V(P/y) = v^1(P^1/y, v^2),$$

$$v^r = v^r(P^r/y, v^{r+1}), \qquad r = 2, \ldots, m-1,$$

and

$$v^m = v^m(P^m/y).$$

The indirect weakly recursive representation is

$$\begin{aligned} (1) \qquad V(P/y) &= v^1(P^1/y, v^2, \ldots, v^m), \\ &= v^1(P^1/y, {}_2v), \\ v^r &= v^r(P^r/y, v^{r+1}, \ldots, v^m), \qquad r = 2, \ldots, m-1, \\ &= v^r\left(\frac{P^r}{y}, {}_{r+1}v\right), \end{aligned}$$

and

$$v^m = v^m(P^m/y).$$

Blackorby, Nissen, Primont and Russell [1974] have shown that, if U is homothetic, each $f^r, r = 1, \ldots, m$, is homothetic and can be chosen *PLH* in the direct strongly recursive structure. A similar proof can be constructed for the direct weakly recursive structure and, of course for the indirect structures.

Although nonhomothetic direct and indirect strongly recursive functions are independent structures, Blackorby, Nissen, Primont, and Russell [1974] have shown that U is strongly recursive with a homothetic aggregator f^2 if and only if V is strongly recursive with a homothetic aggregator v^2. In fact, homotheticity of f^2

or v^2 implies homotheticity of f^r or v^r, $r = 3, \ldots, m$. Moreover, the f^r can be chosen PLH and the v^r can be chosen NLH. Unfortunately, homotheticity does not generate a dual equivalence relation between direct and indirect weak recursivity. However, indirect weak recursivity with negatively homothetic aggregator functions implies (but is not implied by) direct weak recursivity. To see this, recall that negative homotheticity of the aggregator function implies that there exists a representation in NLH aggregators. Let us therefore consider each v^r, $r = 2, \ldots, m$, in (1) to be NLH. Hence, $v^m(P^m/y) = y \cdot v^m(P^m)$ and, recursively,

$$v^1(P^1/y, {}_2v) = v^1(P^1/y, y \cdot v^2, \ldots, y \cdot v^m)$$

where

$$v^r(P^r/y, v^{r+1}, \ldots, v^m) = y \cdot v^r(P^r, y \cdot v^{r+1}, \ldots, y \cdot v^m), \qquad r = 2, \ldots, m-1.$$

Letting

$$\hat{v}^1(y, P^1, v^2, \ldots, v^m) \equiv v^1(P^1/y, y \cdot v^2, \ldots, y \cdot v^m),$$

we have a structure in which the v^r, $r = 1, \ldots, m$, contain nonnormalized prices as arguments.[9]

Recalling the nature of the separability condition in the indirect, this is equivalent to the fact that ratios of demand in the r-th sector, $r > 1$, depend only on prices in ${}_r\chi$:

$$\frac{x_i}{x_j} = \zeta^r_{ij}({}_rP), \qquad \forall i, \qquad j \in \chi^r, \qquad r = 2, \ldots, m.$$

These systems, together with the sector budget identities,

$$P^r \cdot X^r = y^r, \qquad r = 2, \ldots, m,$$

can be solved for

$$X^r = \phi^r({}_rP, y^r). \tag{2}$$

Of course, the demand functions for goods in χ^1 have no structure since V has no structure in χ^1.

Lady and Nissen [1968] and Primont [1970] have shown that the direct utility function is weakly recursive if and only if there exist conditional demand functions, ϕ^r, with images,

$$X^r = \phi^r({}_rP, {}_ry), \qquad r = 1, \ldots, m, \tag{3}$$

where ${}_ry = [y^r, \ldots, y^m]$ is the vector of optimum expenditures on the commodities in ${}_r\chi$.

Clearly, (2) implies (3) (but not conversely). As the demand functions for elements of χ^1 have no structure in either case, this proves that indirect weak recursivity with homothetic aggregators implies direct weak recursivity. These

[9] In fact this structure, implied by negative homotheticity of v^2, is equivalent to assuming that v is weakly recursive in prices and that prices in ${}_2\chi$ are separable in v from the expenditure variable.

results on conditional demand functions can be used to generate numerous differential restrictions on both the Marshallian and Hicksian demand function.[10]

III. Symmetric Budgeting, Decentralization, and Aggregation

1. *Introduction*

It was pointed out in the introductory section of this paper that there are actually two distinct concepts embodied in Strotz's [1957, 1959] description of the process of consumer budgeting. It is for this reason that this phenomenon has commonly been referred to as "two-stage optimization." That is, in the first stage the consumer allocates his income to budget categories and in the second stage the category incomes are allocated among the components of each category. Lau [1969a] has pointed out that the set of necessary and sufficient conditions for this two-stage optimization procedure to yield the correct demands are not necessarily conditions for either one of the two elements of this procedure taken separately. It is therefore instructive to deduce separately the necessary and sufficient conditions for each stage of the consumer budgeting procedure described by Strotz.

A third concept, which is intimately related to but distinguishable from the two stages of Strotz's budgeting procedure, is the notion of price aggregation. That is, can the initial income allocation be carried out knowing only price indices—such as the price of "food"—but not necessarily the individual food component prices? This is the fundamental issue addressed by Gorman [1959] in his elegant analysis of the Strotz consumer budgeting problem. There are actually two separate but related price aggregation issues. The first issue regards the existence of income allocation functions in which the price arguments are aggregate price indices for each of the budget categories. The second issue involves the existence of price aggregates which, when multiplied by the corresponding composite commodity, yield the optimal expenditure on the corresponding budget category. There are therefore four concepts embodied in the discussion of consumer budgeting or two-stage optimization, and each has a separate set of necessary and/or sufficient conditions (if known).

In fact, it is instructive to dichotomize each of these four concepts. As Poliak [1970] has pointed out, there is a sense in which normalization of prices (dividing price vectors by total expenditure) is not a free good when placed in the context of structured functions. We have already noted in Section II that indirect structure in normalized prices generally places weaker restrictions on preferences than do structural conditions with respect to nonnormalized prices. Consequently, it turns out that different necessary and sufficient conditions are needed for the rationalization of the four concepts discussed above[11] depending upon whether the income allocation or the allocation of category income is carried out in terms of normalized prices and normalized category income or in their nonnormalized counterparts.

[10] These restrictions and the appropriate set of references can be found in Appendix B, which can be obtained from the authors.

[11] Actually, it turns out that there are only three nontrivial dichotomizations since intercategory income allocation requires no structure.

The purpose of this section is to sort out all of these different concepts and to examine the necessary and sufficient conditions in terms of the structure of the consumer's preferences (i.e., the structure of the indirect and/or direct utility functions). Many, but not all, of the results in this section can be found in a diverse set of publications. As the literature in this area has not been entirely consistent with respect to definitions, the definitions that we adopt cannot be consistent with all of the papers. Our definitions are, however, internally consistent and as much as possible consistent with the salient literature.

The discussion is divided into two parts. The first examines the above concepts in the context of symmetric structures. The second half of the discussion concentrates on asymmetric structures, where additional issues are raised (particularly in the case of weakly recursive structures).

2. *Definitions*

a. *Budgetability*

Following Lau [1969], we say that a preference ordering is *budgetable* if there exist functions, θ^r, $r = 1, \ldots, m$, with images,

$$y^r = \theta^r(P, y),$$

where, it will be recalled, y^r is the optimal expenditure on the r-th group. Note that since θ^r is homogeneous of degree one in P and y, we can write

$$\frac{y^r}{y} = \theta^r(P/y, 1) = \zeta^r(P/y), \qquad r = 1, \ldots, m.$$

Thus, the preference ordering is budgetable if it is possible to find functions which permit the consumer to allocate income or income shares among the m budget categories in a nontrivial way, which is to say without first solving the entire (single-stage) optimization problem and then defining the appropriate function as the inner product of the category price vector and the overall demand function.[12]

b. *Price aggregation*

In the spirit of Gorman's paper [1959], we define *strong price aggregation* as the existence of PLH functions, $\Pi^1, \ldots, \Pi^m$, such that the income allocation functions, θ^r, can be written as follows:

$$y^r = \hat{\theta}^r(\Pi^1(P^1), \ldots, \Pi^m(P^m), y), \qquad r = 1, \ldots, m.$$

Similarly, we define *weak price aggregation* as the existence of functions, $\hat{\Pi}^1, \ldots, \hat{\Pi}^m$, such that the income share function can be written as follows:

$$\frac{y^r}{y} = \hat{\zeta}^r(\hat{\Pi}^1(P^1/y), \ldots, \hat{\Pi}^m(P^m/y)), \qquad r = 1, \ldots, m.$$

Thus, strong price aggregation is defined as the existence of a rule whereby income may be allocated among the m budget categories knowing only total expenditure

[12] In fact, as we shall see below, the existence of the above functions is as vacuous (in terms of empirical implications) as is the above inner product construction.

and the values of the category price indices. Weak price aggregation is equivalent to the existence of a rule for determining budget shares knowing only m price aggregates in normalized prices. It is clear that the homogeneity of degree one of $\hat{\theta}$ in P and y allows us to convert the strong aggregation function into the weak aggregation function. Thus, strong price aggregation implies weak price aggregation. The converse, however, is *not* true.

If, in addition to the existence of the above price indices, there exist quantity indices, $f^1, \ldots, f^m$, such that

$$\Pi^r(P^r) \cdot f^r(X^r) = y_r, \qquad r = 1, \ldots, m,$$

holds, we say that the preferences are characterized by *strong, additive price aggregation*. If this condition is satisfied, it is possible to formulate a first-stage optimization problem in which the consumer maximizes utility with respect to the composite commodities, $f^1, \ldots, f^m$, subject to the budget constraint,

$$\sum_r \Pi^r(P^r/y) f^r(X^r) = 1 \qquad r = 1, \ldots, m.$$

If there exist price aggregates in normalized prices, $\hat{\Pi}^1, \ldots, \hat{\Pi}^m$, such that

$$\hat{\Pi}^r(P^r/y) \cdot f^r(X^r) = y_r/y, \qquad r = 1, \ldots, m,$$

we say that the consumer's preferences are characterized by *weak, additive price aggregation*. If the consumer's preferences satisfy this condition, the sum of the above equation serves as a constraint in an optimization problem over the f^r. Clearly, the PLH of the Π^r means that strong additive price aggregation implies weak additive price aggregation, but, again, the converse is not true.

c. *Decentralizability*

If total expenditure is correctly allocated among the m budget categories, it does not follow that the consumer is able to allocate the category expenditures among the category components optimally without solving the entire optimization problem. It is clear that in general the allocation of expenditure among, say, clothing items is not independent of the way in which housing expenditures are allocated. If it is possible for the consumer to allocate optimally category expenditures knowing only intra-category prices, we say that the consumer's preferences are characterized by *strong decentralizability*. This concept is characterized as the existence of m vector valued functions ϕ^r such that

$$X^r = \phi^r(P^r, y^r), \qquad r = 1, \ldots, m.$$

Finally, if there exist m vector valued functions denoted $\hat{\phi}^r$ such that

$$X^r = \hat{\phi}^r(P^r/y, y^r/y),$$

we say that the consumer's preferences are characterized by *weak decentralizability*. Thus, under strong decentralizability, in order to allocate category expenditure correctly, the consumer must know absolute prices of category components and category expenditure whereas in the case of weak decentralizability the consumer only has to know normalized category prices and the budget share of each category. Thus, it is clear that strong decentralizability implies weak decentralizability but that the converse is not true.

d. *The Gorman polar form*

An important element in the proofs of Gorman [1959] is the class of preferences represented by an indirect utility function which can be written as

$$V(P, y) = \Psi\left(\frac{y}{\Pi(P)}\right) + \Lambda(P)$$

where Ψ is strictly increasing, Π is PLH, and Λ is homogeneous of degree zero. Throughout this paper, we refer to this indirect utility function as the "Gorman polar form." This indirect representation generates (see Gorman [1953]), as a special case where Ψ is the identity function, the class of orderings characterized by linear income consumption curves (which do not necessarily converge to a common point—much less the origin). If, in addition, $\Lambda(P) \equiv 0$, the Gorman polar form reduces to a representation of homothetic preferences. If $\Lambda(P)$ is linear, preferences are affinely homothetic.[13]

For each of the category satisfaction functions of a strongly or weakly separable utility function, define

$$h^r(P^r, y_r) = \max_{X^r} \{ f^r(X^r) | P^r \cdot X^r \leq y_r \}, \qquad r = 1, \ldots, m.$$

For our purpose, endowing each aggregator function, h^r, with the Gorman polar form property turns out to be most useful. Thus, when we refer to a directly strongly or weakly separable structure with the Gorman polar form, we mean that the aggregator functions have this form:

$$h^r(P^r, y_r) = \psi^r\left(\frac{y_r}{\Pi^r(P^r)}\right) + \Lambda^r(P^r), \qquad r = 1, \ldots, m.$$

3. *Necessary and/or Sufficient Conditions*

a. *Budgetability*

Lau [1969a] has shown that, even if the direct utility function has no structure but merely satisfies the maintained regularity properties (see page 46 above), the function is budgetable:

Proposition 1 (Lau [1969a]): If $U(\cdot)$ is continuous, strictly quasi-concave, nondecreasing, and strictly increasing in one coordinate, $U(\cdot)$ is budgetable.[14]

b. *Strong and weak price aggregation*

Strong and weak price aggregation are called strong and weak budgeting by Pollak [1970], who also provides an example of a preference ordering which is

[13] The linear expenditure system (Stone [1954], Geary [1949], Klein and Rubin [1948], and Samuelson [1948]) and the *S*-branch system (Brown and Heien [1972]) are generated by affinely homothetic preferences.

[14] Recall that proofs of the propositions stated throughout this paper can be found in Appendix A, which can be obtained from the authors.

amenable to strong price aggregation but has no separability properties. Hence, necessary conditions for either strong or weak price aggregation would not involve any separability conditions. Gorman [1959] proved, however, that if the direct utility function were weakly separable, direct homothetic separability or direct strong separability, where each category utility function has the Gorman polar form, would be necessary as well as sufficient. We state these relationships in the next three propositions.

Proposition 2 (Gorman [1959]): Direct homothetic separability implies strong price aggregation.

Proposition 3 (Gorman [1959]): Direct strong separability with the Gorman polar form implies strong price aggregation.

Proposition 4 (Gorman [1959]): If $U(\cdot)$ is weakly separable, strong price aggregation implies either direct homothetic separability or direct strong separability with category functions restricted by the Gorman polar form.

Weak price aggregation means that the share of the budget which is allocated to sector r can be written

$$\frac{y_r}{y} = \zeta^r\left(\hat{\Pi}^1\left(\frac{P^1}{y}\right), \ldots, \hat{\Pi}^m\left(\frac{P^m}{y}\right)\right), \qquad r = 1, \ldots, m.$$

Each $\hat{\Pi}^r$ is a price aggregate, useful in intersector allocation decisions, but does not have many of the nice properties which are desirable for price indices. In particular, it is *not* solely a function of prices, and it is *not* PLH in its arguments.

Proposition 5 (Pollak [1970]): If V is continuously differentiable, indirect strong separability implies weak price aggregation.

c. *Strong and weak additive aggregation*

Definitionally, in the case of additive aggregation, not only do there exist intercategory allocation functions with price aggregators as arguments, but in addition the price aggregates, when multiplied by quantity aggregators, add up to total expenditure.

Although necessary conditions for additive price aggregation are not known, it is well known that homothetic separability implies strong price aggregation. We have, however, been unable to discover a weaker set of sufficient conditions for weak additive price aggregation. Consequently, the distinction between weak and strong additive price aggregation may be vacuous.

Proposition 6 (Gorman [1959] and Blackorby, Lady, Nissen, and Russell [1970]): Homothetic weak separability implies strong additive price aggregation.

Although homothetic separability is not generally necessary for strong additive price aggregation, the following necessity theorem can be proved:

Proposition 7 (Blackorby, Lady, Nissen and Russell [1970]): If U is weakly separable, strong additive price aggregation implies homothetic weak separability.

d. *Strong and weak decentralizability*

Decentralizability refers to the ability of the consumer to make intracategory income allocations optimally and efficiently—that is, without requiring information on all prices and income. The information required in order to be able to allocate category income among the commodities in that category is, of course, exactly the information which is contained in the conditional demand functions.

Strong decentralizability is a case where only own prices and own category income is needed. In this case the conditional demand functions are

$$X^r = \phi^r(P^r, y_r), \qquad r = 1, \ldots, m.$$

Proposition 9 (Gorman [1971]): Weak separability of $U(\cdot)$ is both necessary and sufficient for strong decentralizability.

Weak decentralizability requires that knowledge of own normalized prices and the category budget share are sufficient information for intracategory allocations to be correct. That is, the intracategory allocation functions can be written as

$$X^r = \hat{\phi}^r\left(\frac{P_r}{y}, \frac{y_r}{y}\right), \qquad r = 1, \ldots, m.$$

Proposition 10 (Lau [1969a]): Indirect weak separability implies weak decentralizability.

Table 1 summarizes the results of this section on symmetric structures. Note that there are a large number of question marks in the "necessary conditions" column. For most empirical work, where the appropriate separability conditions are maintained hypotheses, this paucity of necessary conditions is not very important. Nevertheless, guided by the principle of Occam's razor, it would be useful to find the weakest set of structural conditions consistent with each of the budgeting-aggregation-decentralization hypotheses.

IV. Asymmetric Structures

1. *Direct and Indirect Strongly Recursive Structures*

Because strongly recursive structures are coordinate-wise weakly separable in all partitions to the left, there is a great deal of similarity to the symmetric case. It is primarily the existence of a free sector inside each category function which changes the results. In the asymmetric case there are *no* additive structures, which means that the Gorman polar form is of little importance. In particular, we call attention to the fact that the distinction between price aggregation and additive price aggregation virtually disappears.

a. *Recursive price aggregation*

Recursive structures do not, of course, generate (symmetric) price aggregation as defined in Part III. However, direct homothetically recursive utility functions give rise to recursive price aggregation; that is, expenditure on the r-th continuation, $z_r = {}_rP \cdot {}_rX$, can be written as

$$z_r = \theta^r(P^{r-1}, \Pi^r, z_{r-1}), \qquad r = 2, \ldots, m,$$

where

$$\Pi^r = \Pi^r(P^r, \Pi^{r+1}), \qquad r = 2, \ldots, m-1,$$

TABLE I

NECESSARY AND SUFFICIENT CONDITIONS FOR BUDGETABILITY, AGGREGATION, AND DECENTRALIZABILITY

Concept	Definitional Function Image	Sufficient Conditions	Necessary Conditions	Source
Budgetability	$y^r = \zeta^r(P, y)$ $\frac{y^r}{y} = \zeta^r(P/y)$	Regularity*	None	Lau [1969a]
Strong Price Aggregation	$y^r = \theta^r(\Pi^1(P^1), \ldots, \Pi^m(P^m), y)$	Direct Homothetic Separability *or* Direct Strong Separability with Gorman Polar Form	?†	Gorman [1959]
Weak Price Aggregation	$\frac{y^r}{y} = \hat{\zeta}^r\left(\hat{\Pi}^1\left(\frac{P^r}{y}\right), \ldots, \hat{\Pi}^m\left(\frac{P^m}{y}\right)\right)$	Indirect Strong Separability	?	Pollak [1970]
Strong Additive Price Aggregation	$\Pi^r(P^r) \cdot f^r(X^r) = y^r,\ \forall r$	Homothetic Separability	?†	Gorman [1959]
Weak, Additive Price Aggregation	$\hat{\Pi}^r(P^r/y) \cdot f^r(X^r) = \frac{y^r}{y}, \forall r$	Same as Strong	?	
Strong Decentralizability	$X^r = \phi^r(P^r, y^r)$	Direct Weak Separability	Direct Weak Separability	Gorman [1959, 1971]
Weak Decentralizability	$X^r = \hat{\phi}^r\left(\frac{P^r}{y}, \frac{y^r}{y}\right)$	Indirect Weak Separability	?	Lau [1969a]

* Regularity: U is continuous, nondecreasing, and strictly increasing in one coordinate.
† If $U(\cdot)$ is weakly separable, the sufficient conditions are also necessary.

and

$$\Pi^m = \Pi^m(P^m).$$

Moreover, the Π^r are PLH in their arguments. Thus, Π^2 is itself a homothetically strongly recursive structure. Finally, direct homothetic strong recursivity, also implies strongly recursive additive aggregation; i.e.,

$$z_r = \Pi^r \cdot f^r, \qquad r = 2, \ldots, m.$$

These results are summarized as

Proposition 11: Direct homothetically strong recursivity implies recursive strong price aggregation and recursive strong additive price aggregation.

b. *Recursive decentralizability*

Analogously to the symmetric case, recursive strong decentralizability is defined as the existence of vector valued functions,

$$X^r = \phi^r({}_rP, z_r), \qquad r = 2, \ldots, m.$$

Recursive weak decentralizability is defined by

$$X^r = \hat{\phi}^r\left(\frac{{}_rP}{y}, \frac{z_r}{y}\right), \qquad r = 2, \ldots, m.$$

Analogy to the symmetric case is completed by the following two propositions:

Proposition 12: Direct strong recursivity is necessary and sufficient for recursive strong decentralizability.

Proposition 13: Indirect strong recursivity implies recursive weak decentralizability.

2. *Direct Weak Recursivity*

In the case of weakly and strongly separable or strongly recursive direct utility functions, the specific aggregator functions have two rather natural (mutually consistent) interpretations. On the one hand, they can be thought of as category utility functions; on the other hand, they may be interpreted as amounts of surrogate commodities, each of which is the appropriate argument in the utility function. If, however, the only structure which is imposed upon the direct utility function is weak recursivity, the first of these two interpretations is inappropriate. It does not make sense to interpret the specific aggregator functions as category utility functions because the variables in ${}_{r+1}\chi$ are contained in other (higher numbered) aggregator functions as well as f^r. Hence, f^r cannot be interpreted as "the" (specific) utility function for ${}_r\chi$. The interpretation of the weakly recursive aggregators as surrogate commodities, however, becomes somewhat more interesting; in fact, it is suggestive of Lancaster's [1966] idea that satisfaction is derived from the characteristics of commodities, and hence only indirectly from the commodities themselves.

The following example might illuminate these interpretations. Let $U(\cdot)$ be weakly recursive. Then

$$U(X) = f^1(X^1, {}_2f),$$

where

$$f^r = f^r(X^r, {}_{r+1}f), \qquad r = 1, \ldots, m-1,$$

and

$$f^m = f^m(X^m).^{15}$$

Suppose we think of $f^r(\cdot)$ as being the surrogate commodity "warmth." Let X^r be the vector of different types of clothing, which clearly provide warmth. Let $f^s(\cdot)$ be the surrogate commodity "shelter," where X^s is the vector of housing services. If $s > r$, then f^s may be an argument of $f^r(\cdot)$ and "shelter" helps provide "warmth." Let $f^t(\cdot)$ and $f^u(\cdot)$ be "recreation" and "food" respectively. Dining out might be considered to add to the amount of "recreation;" hence, f^u is an argument of $f^t(\cdot)$. Furthermore vacations are "recreation" and the amount of vacation time which is taken may affect the amount of "shelter." If so, f^t would be an argument of $f^s(\cdot)$. As a result "food" affects the amount of "warmth" by affecting "recreation" which in turn affects "shelter" which is an argument of $f^r(\cdot)$. However, it may well be true that "food" affects the amount of "warmth" directly, and may itself be an argument of $f^r(\cdot)$. If nothing else contributed to "warmth," we could write $f^r(\cdot)$ as

$$f^r(\cdot) \equiv f^r(X^r, f^s(X^s, f^t(X^t, f^u(X^u, {}_{u+1}f), {}_{u+1}f), {}_uf), f^u(X^u, {}_{u+1}f)),$$

where, for notational convenience, we have assumed that $s = r + 1$, $t = r + 2$, and $u = r + 3$. The reader should refer to Chart 1 for an illustration of the above example of a weakly recursive structure.

The chart is arranged vertically by levels and horizontally by sectors. At the r-th level, if one reads across the chart, there appear all of the arguments of the function $f^r(\cdot)$. In our example, clothes (X^r), shelter (f^s), food (f^u) and (possibly) $f^{u+1}, \ldots, f^m$ are the arguments of $f^r(\cdot)$ (warmth). At the t-th level, vacations (X^t), food (f^u) and (possibly) $f^{u+1}, \ldots, f^m$ are the arguments of $f^t(\cdot)$ (recreation).

At the u-th sector, χ^u, by reading down the chart, one can find the set of functions (with indices less than u) of which f^u (food) is an argument. For example, food (f^u) is an argument of $f^u(\cdot)$ (warmth), since it is directly connected to $f^r(\cdot)$ (warmth) by a solid line. Food (f^u) is also an argument of $f^t(\cdot)$ (recreation) and (possibly) an argument of $f^s(\cdot)$ (shelter) for the same reason. On the other hand, $f^t(\cdot)$ (recreation), while an argument of $f^s(\cdot)$ (shelter), is *not* an argument of $f^r(\cdot)$ (warmth), since it is not directly connected to $f^r(\cdot)$ (warmth) but only indirectly through $f^s(\cdot)$ (shelter).

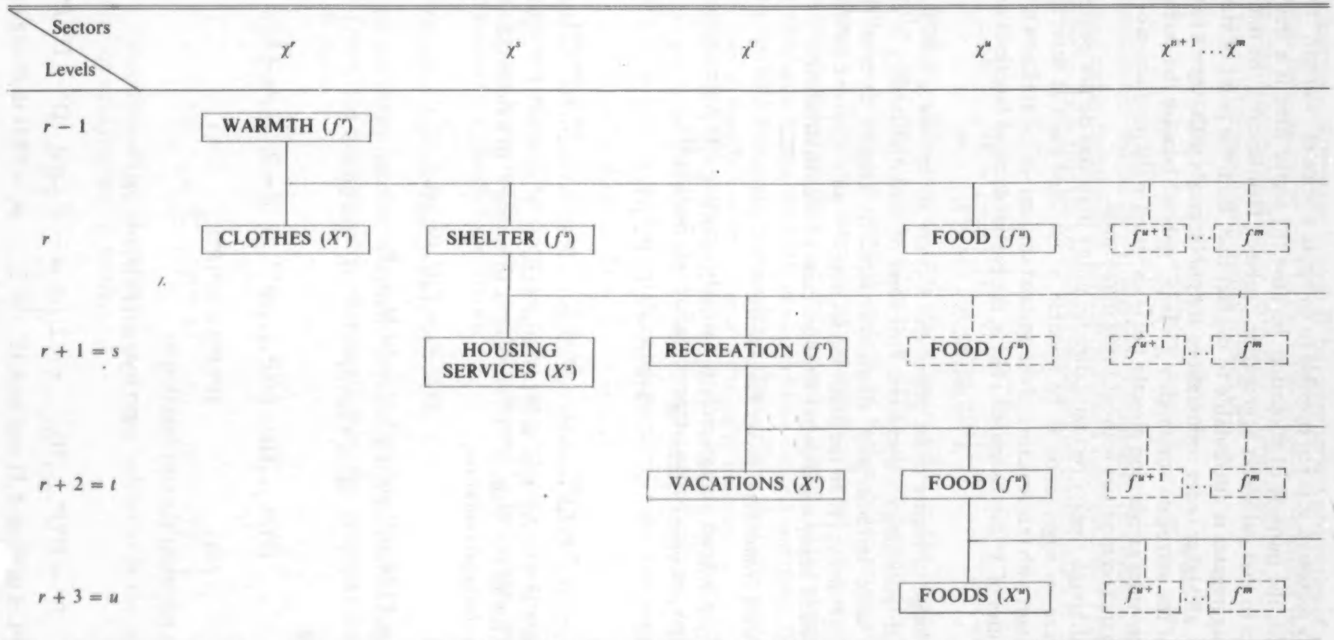
Finally, and perhaps most importantly, the chart is arranged so that each sector is separable from all sectors to its left but is not (necessarily) separable from sectors to its right. Thus, for example, χ^t is separable from χ^s and χ^r (but not necessarily their union) and from χ^q, $q = 1, \ldots, r-1$. It is this asymmetric separability that accounts for the asymmetric appearance of the chart.

Hopefully, the preceding example illustrates the richness of the weakly recursive structure. This very richness, however, presents difficulties for aggregation and decentralization. Surprisingly, even if the function is homothetically

[15] Note that $m \geq 3$, for if $m = 2$, there is no distinction between strong and weak recursivity.

CHART 1

A "UTILITY TREE" FOR A WEAKLY RECURSIVE UTILITY FUNCTION



weakly recursive, it is not possible to compute a price of "warmth" that could be used for the purpose of allocation, because the sector itself is a free sector with respect to that particular aggregation problem. Heuristically, the nonexistence of price aggregates is attributable to the fact that, from the point of view of the rth sector, all higher order commodities are public goods with respect to the production of the surrogate commodity f^r. It is "public" because the surrogate f^s not only provides satisfaction directly, but is also used in the production of lower order surrogate commodities without being diminished.

Although weakly recursive directly utility functions do not possess sufficient structure for aggregation to be possible, a restricted class of decentralization is consistent with this structure. If the optimal amounts of $({}_ry)$ are known, the optimal quantities of X^r can be found. That is, the partial demand functions are

$$X^r = \phi^r({}_rP, {}_ry), \qquad r = 1, \ldots, m.$$

Knowledge of higher order prices and income allocations is sufficient to make optimal intracategory decisions. This class of decentralizability is clearly quite weak, since intracategory allocation decisions require external information (viz., ${}_{r+1}P$ and ${}_{r+1}y$). In the following section, a weakly recursive indirect structure is shown to imply a somewhat stronger class of decentralizability.

3. *Indirect Homothetic Weak Recursivity*

If the indirect utility function is weakly recursive with homothetic aggregator functions, we can choose a representation (see Section II)

$$V(P/y) = \hat{v}^1(y, P^1, {}_2v)$$

where

$$v^r = v^r(P^r, {}_{r+1}v), \qquad r = 2, \ldots, m-1, \qquad v^m = v^m(P^m),$$

and each $v^r, r = 2, \ldots, m$, is NLH in ${}_rP$; i.e., $\lambda^{-1}v^r = v^r(\lambda P^r, \lambda^{-1}{}_{r+1}v), r = 2, \ldots, m-1$, for all $\lambda > 0$ and $\lambda^{-1}v^m = v^m(\lambda P^m)$. Inverting $\hat{v}^1$ in y yields the cost function C with the representation,

$$C(U, P) = \hat{C}(U, P^1, {}_2v).$$

As C is PLH in P and each v^r is NLH in ${}_rP$,

$$\lambda\hat{C} = \hat{C}(U, \lambda P^1, \lambda^{-1}\,{}_2v), \quad \text{for all } \lambda > 0.$$

Letting

$$\Pi^r(P^r, {}_{r+1}\Pi) = (v^r(P^r, {}_{r+1}v))^{-1}, \qquad r = 2, \ldots, m-1,$$

and

$$\Pi^m(P^m) = (v^m(P^m))^{-1},$$

we can represent the cost function by

$$C(U, P) = \Pi^1(U, P^1, {}_2\Pi),$$

where

$$\Pi^r = \Pi^r(P^r, {}_{r+1}\Pi), \qquad r = 2, \ldots, m-1, \quad \text{and} \quad \Pi^m = \Pi^m(P^m).$$

Π^1 is PLH in P^1 and ${}_2\Pi$ and each $\Pi^r, r = 2, \ldots, m$, is PLH in its arguments.

As PLH functions of prices only, each Π^r has appropriate properties of a price index. Although they cannot be used in the inter-category allocations properly, they do have interesting accountability properties. Interpreting $\Pi^r, r > 1$, as the unit price of the r-th surrogate commodity implies that

$$\left(\frac{\partial \Pi^1}{\partial \Pi^r}\right)_U = \sum_{s=2}^{r} \frac{\partial \Pi^1}{\partial \Pi^s}\left(\sum_{t=s+1}^{r-1} \frac{\partial \Pi^s}{\partial \Pi^t}\cdot\frac{\partial \Pi^t}{\partial \Pi^{t+1}}\cdots\frac{\partial \Pi^{r-1}}{\partial \Pi^r}\right), \qquad r = 2, \ldots, m,$$

is the compensated demand for the r-th surrogate commodity. In a straightforward but tedious and messy construction, exploiting Euler's theorem and PLH of the Π^r, it is possible to show that the optimal expenditure on commodities in the r-th sector is

$$y_r = \sum_{i \in \chi^r} p_i \phi_i(P^r, {}_{r+1}\Pi) = \left(\frac{\partial \Pi^1}{\partial \Pi^r}\right)_U \left(\Pi^r - \sum_{s=r+1}^{m} \frac{\partial \Pi^r}{\partial \Pi^s}\Pi^s\right), \qquad r = 1, \ldots, m.$$

Hence, the income expenditure on the r-th sector is equal to the compensated demand for the r-th surrogate commodity, $(\partial\Pi^1/\partial\Pi^r)|_U$, times the unit price of the r-th surrogate, Π^r, minus the optimal amount of higher order surrogates per unit of surrogate r, $(\partial\Pi^r/\partial\Pi^s)|_U$, times the unit price of the higher order surrogates, Π^s. In other words, the expenditure on the r-th sector is equal to the expenditure on the r-th surrogate minus the indirect expenditure on this surrogate through higher order (public) surrogate commodities. This means that the amount of "publicness" can at least be costed out in a meaningful manner.

In addition to this nice accountability property, rather stronger decentralization results are available. The application of Roy's Theorem generates the following image of the (vector valued) conditional demand function for the r-th category:

$$X^r = \hat{\phi}^r(P^r, {}_{r+1}\Pi, y_r), \qquad r = 2, \ldots, m.$$

This is much less information than is needed for decision making in the direct weakly recursive structure. In order to make intracategory allocations, own category prices and higher order price indices still need to be known, but only *own* category income, y_r, needs to be known.

4. *Weakly Recursive Indirect Utility Functions*

If the indirect utility function is weakly recursive in normalized but not non-normalized prices, the accountability characterized by the preceding structure disappears. Suppose

$$V\left(\frac{P}{y}\right) \equiv v^1\left(\frac{P^1}{y}, {}_2v\right),$$

$$V^r = v^r\left(\frac{P^r}{y}, {}_{r+1}v\right), \qquad r = 1, \ldots, m, \qquad v^m = v^m\left(\frac{P^m}{y}\right).$$

Only if $v^2(\,\cdot\,)$ were homothetic would there exist price indices as in the previous case. However, by applying Roy's theorem,

$$X^r = \hat{\phi}^r\left(\frac{P^r}{y}, {}_{r+1}v, \frac{y_r}{y}\right), \qquad r = 2, \ldots, m.$$

Hence, a limited form of decentralization is possible. The intracategory allocations can be made knowing own normalized prices, higher order aggregators, and only the budget share of that category. This is clearly a weaker form of decentralization than that characterized by a weakly recursive cost function.

V. Concluding Remarks

A rigid logical positivistic posture might lead to an interpretation of the foregoing results as no more than a derivation of the empirically refutable content of certain structural restrictions on preferences. While we agree that such exercises are an important part of economic research, we do not agree that this exhausts the useful content of economics. Empirical refutation of the implications of economic hypotheses is, in practice, very difficult to execute.[16] It is, perhaps, for this reason that much empirical work is buttressed by fairly strong maintained (untested) hypotheses. The structural restrictions which rationalize price aggregation and decentralization are especially useful as maintained hypotheses in empirical work. The two stage optimization problem rationalized by decentralizability is useful in reducing the scope of estimation problems to manageable proportions. If decision-making is decentralizable, the demand analyst can first estimate conditional demand functions for each category and then estimate the income allocation functions.[17] Moreover, if the specified utility function satisfies the appropriate structural restriction, theoretically consistent price indices can be used in the estimation of the income allocation functions.[18] Thus the two stage algorithms, while irrelevant to the derivation of empirically refutable implications, can nevertheless be useful to applied econometricians.[19] The fact that these structural restrictions can be associated with Strotz-type budgeting procedures, commonly observed by casual empiricists, should offer some Bayesian rationalization for the imposition of these structural restrictions in order to make the work of the applied econometrician a little easier.

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[16] The recent work of Christensen, Jorgenson, and Lau [1973a, 1973b] and Christensen and Manser [1972] has dramatically improved the potential for testing structural hypotheses.

[17] Of course, the estimates retain their optimality properties only if very severe restrictions are placed upon the overall variance–covariance matrix.

[18] For examples of such two stage estimation, see Heien [1973] and Russell, *et al.* [1974].

[19] The asymmetric structures legitimatize m-stage algorithms which might prove useful in estimating the choice functions generated by dynamic programming. This could be especially useful in examining intertemporal demand functions. Blackorby, Nissen, Primont, and Russell [1973] have shown that intertemporal decision-making is intertemporally consistent if and only if the intertemporal utility function is strongly recursive with a consistent (intertemporally stationary) representation.

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COMMENT

BY LOUIS PHLIPS

When R. Russell sent me the paper under discussion, he made the forecast that I would be "both impressed and distressed by the physical weight of the document". This forecast was wrong on all accounts: I am impressed by its quality, not its weight. This is indeed a useful and most interesting paper.

It is useful as a handy reference to and a complete exposition of the available literature on functional structure, the more so as it disentangles issues that are rather confused in the available *printed* literature. I am thinking especially of Section III.3 with its nice distinction between *budgetability*, *aggregation* and *decentralization*. Most interesting, of course, are the sections on "asymmetric structures" developing the properties of weak and strong recursivity and the possible implementations in terms of the characteristics of commodities.

ASYMMETRIC STRUCTURES OVER TIME

Recursivity obviously originated in the analysis of decision-making *over time*. It seems appropriate to indicate a few extensions of the analysis presented here to recursivity over time, in particular to *intertemporal utility functions*, on which the authors apparently wrote a paper forthcoming in *Econometrica*. (I wonder to what extent the impressive lists of authors of that paper, of the present one, and of the other ones listed in the bibliography under Blackorby, etc. . . . , are strongly or weakly recursive?).

The applicability of the theorems on recursivity to intertemporal decisions is the more obvious as the notation used is the same as in Koopmans (1960). However, interesting problems arise as a result of the fact that time runs only in one direction. In the partition of the commodity set, the ordering of the subjects or sectors has simply to be "appropriate", and the economist has to make sure that it is so. Over time, the ordering is determined by the passage of time.

The definition of recursivity given in the paper is based on separability with respect to prior time periods, and implies recursivity (of the intertemporal utility function) "forwards". It is of some interest, then, to introduce the concept of forwards separability (with respect to future time periods) which leads to a "backwards" recursive intertemporal utility function. That this is not an empty distinction can be illustrated by a number of considerations.

First, separability forwards is a weaker assumption than separability backwards, in terms of descriptive realism, precisely because time runs in one direction. Second, one can think of intertemporal utility functions that are separable forwards but *not* backwards. Consider the instantaneous dynamic utility function introduced by Houthakker and Taylor (19, Chapter V)

$$u = u[x(t); s(t)]$$

in which the state variables s, summarizing all past purchases, appear as additional parameters shifting marginal utilities upwards or downwards. Being strongly separable forwards, it is strongly recursive backwards ... on the assumption that it represents an intertemporal preference ordering!

Third, backwards recursivity leads to backwards decentralizability, just as forwards recursivity implies forwards decentralizability (see the proof of Proposition 12). In other words, "dynamic" models of the Houthakker–Taylor type (as discussed in the paper by Richard Boyce) imply backwards decentralizability, in the sense that only past (and present) prices and total expenditures have to be taken into account on the assumption that (a) the intertemporal utility function is strongly separable forwards *and* (b) given income (i.e. total expenditures) today is optimal from an intertemporal point of view. In fact, past prices and expenditures are taken into account through the state variable "s". However, the budget constraint used is based on *observed*, not optimal income, which is a strong assumption. Indeed, one may wonder why the consumer should be unable to allocate a given income without error among n commodities, while he would be capable of making a correct intertemporal income allocation (income being given without error!).

Finally, the preference inheritance mechanism implied in these dynamic models seems sufficient to guarantee intertemporal consistency (as defined by Strotz), although there is no strong recursivity forwards.

Of course, the assumption (made above) that $u(x;s)$ is a realistic representation of an *intertemporal* preference ordering is hard to swallow. I would prefer (see Philips (1974, Chapter X)) a functional such as

$$\int_0^{\infty} e^{-rt} u(x;s)\,dt.$$

But then there is no separability left, neither forwards nor backwards, and the Houthakker–Taylor approach appears as "myopic" and difficult to rationalize.

Symmetric Structures

As for the survey on symmetric structures and their implications, I would have welcomed some comments in the concluding section, on the empirical work done recently in the field. The authors seem to suggest that separability has been treated only as a maintained (untested) hypothesis. In fact, some fair amount of testing of the hypothesis itself is available, as exemplified by work by Barten (1967, 1969), Byron (1970a and b), Deaton (1972) and myself.

In most cases, these are tests on strong separability. Invariably they lead to rejection. Should we infer that weak separability is all right as a maintained hypothesis? I guess so. But I then wonder about the relevance of the conditions about aggregation. For budgetability and decentralization, weak separability is sufficient. But if we want to use price indices, theory comes up with a homotheticity condition on the branch utility functions. Many applied economists feel most uncomfortable about this, as we all know that no branches are homothetic. (So I was greatly surprised to discover in the paper by Jorgenson and Lau that branch homotheticity may show up!)

Should we, in the absence of homotheticity, set up our demand systems in such a way that price aggregation is not needed? If we do so, does not this amount to artificially supposing that the commodities under study cannot be disaggregated further down? Or could we say that the conditions for price aggregation refer to the practicability of budgeting and simultaneously stress that budgeting is an artifact imagined by economists but not carried out in practice by "real" consumers?

Strangely enough, tests on particular specifications of functional structure are very often accompanied (see also the Jorgenson–Lau paper), by tests on different "general" restrictions (homogeneity, symmetry, etc.) derived from utility maximization *per se*. This is understandable, as both the "particular" restrictions (derived from some form of separability) and the "general" restrictions (i.e. Slutsky conditions) lead to hypotheses which are nested in a less restricted model and therefore make likelihood ratio tests possible, technically speaking. However, I do not understand the economic rationale for tests of general restrictions such as, say, the symmetry of the cross-substitution effects. How could such a test, in the present state of the art, inform us about the "existence" of a utility function?

When we test weak against strong separability, we have two conflicting hypotheses, both imbedded in utility maximization. And to carry out the test, we use some specification of the utility function—hopefully one that is compatible with observed behavior (on this, see Basmann *et al.* (1973) and Byron (1973))—in order to have estimating equations to work with. On the contrary, when we test symmetry or any other Slutsky condition, we are in fact testing utility maximization itself, and even touching the basic axioms of the theory of value. But is not an axiom something in which one *believes*? And if, less dramatically, one has his doubts about utility *maximization* only, where is the alternative behavioral assumption against which to test it (in the framework of the same set of axioms about the preference ordering)? In the present state of the art, all we can do is to test symmetry against the absence of symmetry. But where is the *theory* that formally incorporates the absence of symmetry? In fact, one is hoping that the raw (unrestricted) data will give the answer, that "the data will speak". However, there are no unrestricted data (data never speak by themselves), as we always need some model specification to make computations possible. It is impossible, then, to make sure that the test is a test of the symmetry assumptions rather than of the underlying model specification (e.g. the utility function used to derive or implied in the estimating equations). That is why I am not too impressed (and certainly not distressed) when I hear that the theory of demand does not even get passing marks in the examination procedure set up in the Jorgenson–Lau paper.

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